THE MARCINKIEWICZ INTERPOLATION THEOREM AND ITS EXTENSIONS

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ABSTRACT

This thesis is primarily devoted to the study of the Marcinkiewicz interpolation theorem and its applications.

The Marcinkiewicz theorem is extended to function spaces that include both the Lebesgue-Orlicz and Lorentz spaces, namely the rearrangement invariant function spaces. Without imposing any additional hypotheses, weighted generalizations are obtained and applied to well known operators in Fourier analysis.

The Hardy spaces of analytic functions do not fall into the class of rearrangement invariant function spaces. However, following Igari's generalization of the Marcinkiewicz theorem to Hardy spaces, a variant and weighted extension are proved and applied to obtain a weighted integral estimate involving the Littlewood-Paley g-function.

To The Memory Of My Father

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TABLE OF CONTENTS

INTRODUCTION

The investigation of many mathematical problems is significantly simplified if it is possible to reduce it to equations involving linear operators in function spaces. The study of these problems is effectively divided into three independent parts: transformation to an integral equation; investigation of the corresponding integral expression as an operator acting on function spaces; and, finally, application of methods of functional analysis to obtain solutions to the original problem.

Central to the second of the above divisions is the theory of interpolation. This theory is based on two results: the theorem of Riesz[57] and Thorin[73], and the interpolation theorem of Marcinkiewicz[45].

One of the formulations of the Riesz-Thorin interpolation theorem is the following:

Theorem: Let $(\mathfrak{M}, \mu)$ and $(\mathfrak{N}, \nu)$ be σ-finite measure spaces, and let $1 \leq p_i, q_i \leq \infty$, $i=0,1$. Suppose for complex valued functions, the operator

$$T: L^{p_0}(\mathfrak{M}, d\mu) \cap L^{p_1}(\mathfrak{M}, d\mu) \to L^{q_0}(\mathfrak{N}, d\nu) \cap L^{q_1}(\mathfrak{N}, d\nu)$$

is linear and satisfies $\|Tf\|_{q_i} \leq M_i \|f\|_{p_i}$, $i=0,1$. Then for each $f \in L^{p_0} \cap L^{p_1}$, and $\alpha \in (0,1)$, $Tf \in L^q$ and

$$(1) \quad \|Tf\|_q \leq M_0^{\alpha} M_1^{1-\alpha} \|f\|_p,$$

where $p^{-1} = \alpha p_0^{-1} + (1-\alpha)p_1^{-1}$, $q^{-1} = \alpha q_0^{-1} + (1-\alpha)q_1^{-1}$. Moreover, if $p<\infty$, then T extends uniquely to all $f \in L^p(\mathfrak{M}, d\mu)$ preserving (1).

A profound result in the area of interpolation is the Marcinkiewicz theorem, as it turns "weak" information into "strong" information. It s basic special case is stated below.

Theorem: Let $(\mathfrak{M},\mu)$ and $(\mathfrak{N},\nu)$ be σ-finite measure spaces, and let $1 \le p_0, p_1 < \infty$, $p_0 \ne p_1$. If, for all $f \in L^{p_i}(\mathfrak{M}, d\mu)$, $i=0,1$, and $\lambda > 0$, the linear operator T satisfies

$$\nu\{x \in \mathfrak{N}: |Tf(x)| > \lambda\} \le M_i \lambda^{-p_i} \|f\|_{p_i,\mu}^{p_i}, \quad i=0,1.$$

Then $T: L^p(\mathfrak{M},\mu) \to L^p(\mathfrak{N},\nu)$, $p^{-1} = \alpha p_0^{-1} + (1-\alpha)p_1^{-1}$, $\alpha \in (0,1)$ and $\|Tf\|_p \le M_\alpha \|f\|_p$, where M_α is independent of f.

We summarize in Table 1 some of the most important classical L^p-inequalities and indicate which of the above interpolation theorems is applied. These inequalities are useful as they give conditions on the indices p and q for which the operators are bounded from L^p to L^q. The proofs typically require delicate application of one of the interpolation theorems.

In this thesis, we study generalizations and extensions of the Marcinkiewicz theorem. Specifically, we generalize the interpolation result to the n-dimensional analogue of Hardy spaces of analytic functions, to Lorentz- and Orlicz spaces, and to rearrangement invariant function spaces. The interpolation theorems are further significantly extended to weighted integral estimates without imposing additional assumptions on the operators. The weight functions considered in these results depend only on certain characteristics of the function spaces. These results are then used to give estimates on certain integral operators.

A brief outline of the development of the theory of interpolation is now given. A more detailed account may be found in Bergh and Löfström[6] or Aronszajin and Gagliardo[2].

In 1923, Hausdorff proved that the Fourier coefficients of an L^p function, $1<p<2$, are in $l^{p'}$, $p'=p(p-1)^{-1}$. The abstraction of this result due to Riesz[57] in 1926 was the first interpolation theorem. In 1939, Thorin[73], and, independently, Tamarkin and Zygmund[72] proved the theorem using complex variable techniques. Theirs is still the standard proof found in the literature. Generalizations to analytic families of operators was obtained by Stein[66], using a lemma of Hirschman[23], and extensions to weighted norm estimates by Stein and Weiss[69] in 1958. Further extensions to product spaces with applications to partial differential operators were given by Okikiolu[50] in 1975.

Riesz[58] has also shown that the Hilbert transform, H, is a bounded operator from L^p to L^p, $1<p<\infty$. For p=1, H is not bounded, but the transform does satisfy a "weak" boundedness condition, namely, for $\lambda>0$

[80 ,Vol.1, p.134]:

$$\text{measure}\{x\in\mathbb{R}: |(Hf)(x)| > \lambda\} \leq M\lambda^{-1}\|f\|_1.$$

The abstraction of this theory is the Marcinkiewicz Interpolation Theorem. It was first announced by Marcinkiewicz [45] in 1939, and a proof due to Zygmund [79] appeared in 1956. Calderón [12] and Hunt [25], [26] in 1966 (see also Oklander [51]) extended the Marcinkiewicz theorem to the L(p,q) spaces of Lorentz. Further extensions to Orlicz- and weighted Orlicz spaces were obtained by several authors, notably Torchinsky [75], Heinig and the author [22] and Milman [46]. Generalizations to Hardy spaces are due to Igari [28], [29], and to rearrangement invariant Banach function spaces by Calderón [10], Boyd [8], Bennett [4] and Semenov [62].

Significant abstractions of both the Riesz-Thorin and Marcinkiewicz interpolation theorems were initiated by Lions and Peetre [37] in 1964, and others. These results are based on the introduction of certain functionals. We define below one of these, namely the K-functional.

Let $(B_0, \|\cdot\|_{B_0})$ and $(B_1, \|\cdot\|_{B_1})$ be Banach spaces continuously imbedded in a Hausdorff topological vector space V. (B_0, B_1) is called an interpolation pair. The sum B_0+B_1 is defined as

$$B_0+B_1 = \{f\in V: f=f_0+f_1,\ f_i\in B_i,\ i=0,1\}$$

and Peetre's K-functional on B_0+B_1 is given by

$$K(t,f,B_0,B_1) = \inf(\|f_0\|_{B_0}+t\|f_1\|_{B_1}),$$

where $t>0$ and the infimum is taken over all decompositions $f=f_0+f_1$, with $f_i\in B_i$, $i=0,1$. The intermediate spaces $(B_0,B_1)_{\theta,q,K}$ are then defined to be the class of those f for which

$$\|f\|_{\theta,q} = \left(\int_0^\infty [t^{-\theta} K(t,f,B_0,B_1)]^q \frac{dt}{t}\right)^{1/q}$$

is finite.

A particular example of the K-functional is well known if $B_0=L^p$, $0<p<\infty$, $B_1=L^\infty$. In this instance, [6],

$$K(t,f,L^p,L^\infty) \approx \left(\int_0^{t^p} f^{*p}(x)dx\right)^{1/p}$$

with equality when $p=1$. Here f^* is the decreasing rearrangement of f. Other examples may be found, for example, in [63] and [76]. The intermediate spaces also are readily derived for L^{p_0} and L^{p_1}, $0<p_0<p_1<\infty$:

$$(L^{p_0},L^{p_1})_{\theta,q,K} = L(p,q)$$

the Lorentz spaces [6].

In Chapter I, we present basic definitions and elementary results, some without proof, about L^p and $L(p,q)$ spaces. Also, properties of rearrangements of functions, together with a number of integral inequalities are given. We introduce the Fourier series of a function, the Littlewood-Paley g-function, and, in general, quasi-linear operators. The chapter concludes with a comparison of the Riesz-Thorin and Marcinkiewicz interpolation theorems.

The main result of Chapter II is Igari's interpolation theorem and its weighted extension. For its proof a refinement of a decomposition theorem Igari [29] is required.

Chapter III contains interpolation theorems in rearrangement invariant Banach function spaces. We begin by introducing these function spaces, their fundamental function, the Lorentz spaces $\Lambda(X)$ and $M(X)$, where X is rearrangement invariant, and "weak type" operators. The main interpolation theorems are preceeded by some technical lemmas.

The last chapter contains applications to illustrate the main results of Chapter III. Specifically, the weighted Marcinkiewicz theorem obtained by Heinig[20] is given (Corollary 4.1) and extended (Corollary 4.2). It is shown (Lemma 4.7) that the Lorentz-Zygmund spaces introduced by Bennett and Rudnick[5] are special cases of these more general spaces, and one of their main results, Theorem B is generalized (Theorem 4.8) to the weighted case. In addition, weighted estimates involving the fractional integral operator, the Hilbert transform, as well as the Fourier transform are given.

Finally, we sketch the recent abstract generalizations developed in [21] and state a generalization of the Hausdorff-Young inequality.

TABLE I

NAME OF INEQUALITY	CONDITIONS	INEQUALITY	THEOREM APPLIED
CONVOLUTION OR YOUNG	$1\leq p,q,r\leq\infty$ $1/p + 1/q=1 +1/r$	$\|f*g\|_r \leq \|f\|_p\|g\|_q$	RIESZ-THORIN
HAUSDORFF-YOUNG	$1\leq p\leq 2$ $1/p + 1/q=1$	$\|\hat{f}\|_q \leq \|f\|_p$	RIESZ-THORIN
GENERALIZED HAUSDORFF-YOUNG	$1<p<2$	$(\int \|x\|^{p-2}\|\hat{f}(x)\|^p dx)^{1/p} \leq M\|f\|_p$	MARCINKIEWICZ
HARDY-LITTLEWOOD MAXIMAL	$1<p\leq\infty$	$\|Mf\|_p \leq M\|f\|_p$	MARCINKIEWICZ
M. RIESZ	$1<p<\infty$	$\|Hf\|_p \leq M\|f\|_p$	MARCINKIEWICZ
SOBOLEV	$0<\alpha<n$ $1<p,r<\infty$ $1/p + 1/r + \alpha/n=2$	$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\|f(x)g(y)\|}{\|x-y\|^\alpha}\,dxdy \leq M\|f\|_p\|g\|_r$	MARCINKIEWICZ

CHAPTER I

§1.1: Notation, definitions and basic theory.

For the convenience of the reader, and to make this thesis more self-contained, we introduce notations, definitions, standard results about L^p spaces, and certain integral inequalities. A more detailed account of these concepts may be found, for example, in the work of Hunt[26], Okikiolu[49], Stein and Weiss[71], and Zygmund[80].

$\mathbb{N}=\{1,2,3,\ldots\}$, $\mathbb{R}$ and $\mathbb{C}$ will denote the natural, real and complex numbers, respectively, with $\mathbb{T}=[-\pi,\pi]$ the circle group and $\mathbb{R}^+=[0,\infty)$ the non-negative reals. For $n\in\mathbb{N}$, $\mathbb{R}^n=\{x:x=(x_1,\ldots,x_n),\ x_i\in\mathbb{R},\ 1\leq i\leq n\}$ is the n-dimensional Euclidean space . For $x\in\mathbb{R}^n$, $|x|=[x_1^2+\ldots+x_n^2]^{1/2}$ is its distance to the origin, and for any Lebesgue measurable set $E\subseteq\mathbb{R}^n$, $|E|$ denotes its (Lebesgue) measure. All measure spaces considered throughout this thesis will be assumed to be complete, separable, σ-finite and positive. Also, M and A, together with any subscripts or superscripts will denote constants depending on the indicated parameters.

Definition 1.1: *Let $(\mathfrak{M},\mu)$ be a measure space. A complex valued μ-measurable function f defined on $\mathfrak{M}$ belongs to the Lebesgue space $L^p_\mu(\mathfrak{M})$, $0<p\leq\infty$, if and only if*

$$\|f\|_{p,\mu}=\begin{cases}\left(\int_{\mathfrak{M}}|f(x)|^p d\mu(x)\right)^{1/p} & 0<p<\infty\\ \operatorname{ess\,sup}_{x\in\mathfrak{M}}|f(x)| & p=\infty\end{cases}$$

is finite. Here

$$\operatorname*{ess\,sup}_{x\in\mathfrak{M}} |f(x)| = \inf\{y: \int_{\{x:\ |f(x)|>y\}} d\mu(x)=0,\ y>0\}.$$

If μ is the Lebesgue measure, L^p_μ is denoted by L^p, and $\|f\|_{p,\mu}=\|f\|_p$. We identify throughout functions which differ only on a set of μ-measure zero.

Theorem 1.2: (Hölder's Inequality) Let $1\leq p,q,r\leq\infty$ *satisfy* $1/p+1/q=1/r$, *where we use the convention* $1/\infty=0$. *If* $f\in L^p_\mu(\mathfrak{M})$ *and* $g\in L^q_\mu(\mathfrak{M})$, *then their pointwise product* $f\cdot g$ *is in* $L^r_\mu(\mathfrak{M})$ *and*

$$\|f\cdot g\|_{r,\mu}\leq\|f\|_{p,\mu}\|g\|_{q,\mu}.$$

If $p=q=2$, this inequality is the Cauchy-Schwartz-Bunyakowski inequality.

A partial converse to Theorem 1.2 is the following:

Theorem 1.3: If $f\in L^p_\mu(\mathfrak{M})$, $1\leq p<\infty$, *then*

$$\|f\|_{p,\mu}=\sup\left|\int_{\mathfrak{M}} f(x)g(x)d\mu(x)\right| = \sup\int_{\mathfrak{M}}|f(x)g(x)|\,d\mu(x),$$

where the supremum is taken over all simple functions $g\in L^q_\mu(\mathfrak{M})$, $1/q+1/p=1$, *such that* $\|g\|_{q,\mu}\leq 1$.

The triangle inequality in L^p_μ is Minkowski's Inequality:

Theorem 1.4: If $f,g\in L^p_\mu(\mathfrak{M})$, $1\leq p\leq\infty$, *then their pointwise sum,* $f+g$, *is in* $L^p_\mu(\mathfrak{M})$ *and*

$$\|f+g\|_{p,\mu}\leq\|f\|_{p,\mu}+\|g\|_{p,\mu}.$$

If $0<p<1$, then

$$\|f+g\|_{p,\mu}^p \leq \|f\|_{p,\mu}^p + \|g\|_{p,\mu}^p.$$

Theorems 1.2 and 1.4 show that, for $1\leq p\leq\infty$, $L_\mu^p(\mathcal{M})$ is a normed linear space with norm $\|\cdot\|_{p,\mu}$. Under this norm, L_μ^p is complete and hence a Banach space. If $0<p<1$, then $\|f-g\|_{p,\mu}^p$ defines a metric under which $L_\mu^p(\mathcal{M})$ is complete.

Next, we state a continuous version of Minkowski's inequality.

Theorem 1.5: *Let $(\mathcal{M},\mu)$ and $(\mathcal{N},\nu)$ be measure spaces, and let f be a $\mu\times\nu$-measurable function on $\mathcal{M}\times\mathcal{N}$. Then for $1\leq p<\infty$:*

$$\left(\int_{\mathcal{N}}\left(\int_{\mathcal{M}}|f(x,y)|\,d\mu(x)\right)^p d\nu(y)\right)^{1/p}\leq\int_{\mathcal{M}}\left(\int_{\mathcal{N}}|f(x,y)|^p d\nu(y)\right)^{1/p}d\mu(x).$$

The following result is called Fubini's theorem. It gives conditions under which the interchange of order of integration is justified.

Theorem 1.6: *Let $(\mathcal{M},\mu)$ and $(\mathcal{N},\nu)$ be measure spaces and $(\mathcal{M}\times\mathcal{N},\mu\times\nu)$ the product measure space. If f is a $\mu\times\nu$-measurable function on $\mathcal{M}\times\mathcal{N}$, and if any one of the integrals $\int_{\mathcal{M}\times\mathcal{N}}|f(x,y)|\,d\mu\times\nu$, $\int_{\mathcal{M}}(\int_{\mathcal{N}}|f(x,y)|\,d\nu)d\mu$ or $\int_{\mathcal{N}}(\int_{\mathcal{M}}|f(x,y)|\,d\mu)d\nu$ is finite, then so are the remaining two, and*

$$\int_{\mathcal{M}\times\mathcal{N}}f(x,y)\,d\mu\times\nu=\int_{\mathcal{M}}\int_{\mathcal{N}}f(x,y)\,d\nu d\mu=\int_{\mathcal{N}}\int_{\mathcal{M}}f(x,y)\,d\mu d\nu.$$

Definition 1.7: *A real valued function C defined on an interval $I\subseteq\mathbb{R}$ is said to be convex on I if for any $a,b\in I$ and numbers $\alpha_1,\alpha_2>0$ such that $\alpha_1+\alpha_2=1$, then*

$$C(a\alpha_1+b\alpha_2) \leq \alpha_1 C(a) + \alpha_2 C(b).$$

If this inequality is reversed, C is called concave.

Note that C is convex if and only if -C is concave. Also, if x is an interior point of I, and C is convex, then there exists a constant M, such that for all $y \in I$,

$$C(y) - C(x) \geq M(y-x).$$

For example, t^p is convex if $1 \leq p < \infty$, and concave if $0 < p \leq 1$.

<u>*Theorem 1.8 (Jensen's Inequality)*</u>: *Let $(\mathfrak{M}, \mu)$ be a finite positive measure space and C a convex function. Then*

$$C\left(\frac{1}{\mu(\mathfrak{M})}\int_{\mathfrak{M}} |f(x)|\,d\mu(x)\right) \leq \frac{1}{\mu(\mathfrak{M})}\int_{\mathfrak{M}} C(|f(x)|)\,d\mu(x),$$

provided the right side is finite. If C is concave, this inequality is reversed.

We close this section by proving a number of integral inequalities. These theorems are essential to the proofs of the interpolation theorems in later chapters.

The first theorem is a weighted form of Hardy's inequality [19]. It s proof is due to Torchinsky (personal communication).

<u>*Theorem 1.9*</u>: *Let $W(x)$, $f(x) \geq 0$ be defined on R^+, and let $1 \leq p < \infty$; $r > 0$. If (i) $W(x)$ is non-increasing, then*

$$(1) \quad \left\{\int_0^\infty W(x)x^{-r-1}\left(\int_0^x f(y)\,dy\right)^p dx\right\}^{1/p} \leq \frac{p}{r}\left\{\int_0^\infty W(x)x^{-r-1}[xf(x)]^p dx\right\}^{1/p}$$

or (ii) $W(x)$ is non-decreasing, then

(2) $\{\int_0^\infty W(x)x^{r-1}(\int_x^\infty f(y)dy)^p dx\}^{1/p} \le \frac{p}{r}\{\int_0^\infty W(x)x^{r-1}[xf(x)]^p dx\}^{1/p}$.

Proof: *(i):* Let $y=tx$, and apply Theorem 1.5. Set $x=\frac{z}{t}$ and use the fact that W decreases to yield

$$\{\int_0^\infty W(x)x^{-r-1}(\int_0^x g(y)dy)^p dx\}^{1/p} = \{\int_0^\infty W(x)x^{p-r-1}(\int_0^1 g(xt)dt)^p dx\}^{1/p}$$

$$\le \int_0^1 (\int_0^\infty W(x)x^{p-r-1}g^p(xt)dx)^{1/p}dt$$

$$= \int_0^1 (\int_0^\infty W(\tfrac{z}{t})(\tfrac{z}{t})^{p-r-1}g^p(z)\tfrac{dz}{t})^{1/p}dt$$

$$\le \int_0^1 (\int_0^\infty W(z)(z)^{p-r-1}g^p(z)dz)^{1/p}t^{r/p-1}dt$$

$$= \frac{p}{r}(\int_0^\infty W(z)z^{-r-1}(zg(z))^p dz)^{1/p}.$$

If we now consider case (ii), the same argument as that above is applied, only now the integral is over $[1,\infty)$ instead of $[0,1]$, and we note that $W(\frac{x}{t}) \le W(x)$ for $t \in [1,\infty)$.

If $W \equiv 1$, (1) and (2) are Hardy's inequalities.

If $0<p<1$ and f is assumed increasing in the above theorem, the same conclusion holds. For another extension of Hardy's inequality involving mixed norms, see Muckenhoupt[47] and Bradley[9].

Theorem 1.10 (Calderón's Inequality[11]): *If $f(x)\ge 0$ is monotone on $\mathbb{R}^+$ and α real, then for $0<p\le q$*

(3) $\{\int_0^\infty [x^\alpha f(x)]^q x^{-1}dx\}^{1/q} \le M(\alpha,p,q)\{\int_0^\infty [x^\alpha f(x)]^p x^{-1}dx\}^{1/p}$.

Proof: First assume that f is non-decreasing, and let $-\infty<k<\infty$. Integrating yields

$$\int_{2^{k-1}}^{2^k} t^{\alpha q-1}dt = \frac{1}{\alpha q}(2^{\alpha q k} - 2^{\alpha q(k-1)}), \qquad \alpha\ne 0,$$

$$= \log 2, \qquad \alpha=0,$$

so that $\int_{2^{k-1}}^{2^k} t^{\alpha q-1}dt = M_{\alpha,q} 2^{\alpha qk}$. Thus, since $\frac{p}{q} \leq 1$,

$$\left(\int_0^\infty [x^\alpha f(x)]^q x^{-1}dx\right)^{p/q} = \left(\sum_{k=-\infty}^{\infty} \int_{2^{k-1}}^{2^k} f^q(t) t^{\alpha q-1}dt\right)^{p/q}$$

$$\leq \left(\sum_{k=-\infty}^{\infty} f^q(2^k) \int_{2^{k-1}}^{2^k} t^{\alpha q-1}dt\right)^{p/q}$$

$$= \left(\sum_{\infty}^{\infty} f^q(2^k) M_{\alpha,q} 2^{\alpha qk}\right)^{p/q}$$

$$\leq M_{\alpha,q}^{p/q} \sum_{k=-\infty}^{\infty} f^p(2^k) 2^{\alpha pk}.$$

Next,

$$2^{\alpha pk} = \begin{cases} \dfrac{(\alpha p)}{(2^{\alpha p}-1)} \displaystyle\int_{2^k}^{2^{k+1}} t^{\alpha p-1}dt & \alpha \neq 0 \\ (1/\log 2) \displaystyle\int_{2^k}^{2^{k+1}} \frac{dt}{t} & \alpha=0 \end{cases}$$

so that $2^{\alpha pk} = M_{\alpha,p} \int_{2^k}^{2^{k+1}} t^{\alpha p-1}dt$, and therefore

$$\sum_{k=-\infty}^{\infty} f^p(2^k) 2^{\alpha pk} = \sum_{k=-\infty}^{\infty} f^p(2^k) M_{\alpha,p} \int_{2^k}^{2^{k+1}} t^{\alpha p-1}dt$$

$$\leq M_{\alpha,p} \sum_{k=-\infty}^{\infty} \int_{2^k}^{2^{k+1}} f^p(t) t^{\alpha p-1}dt$$

$$= M_{\alpha,p} \int_0^\infty f^p(t) t^{\alpha p-1}dt.$$

Combining the above observations yields

$$\left(\int_0^\infty [x^\alpha f(x)]^q x^{-1}dx\right)^{1/q} \leq M_{\alpha,p,q} \left(\int_0^\infty [x^\alpha f(x)]^p x^{-1}dx\right)^{1/p}$$

where

$$M_{\alpha,p,q} = \begin{cases} \left[\frac{(1-2^{-\alpha q})}{(\alpha q)}\right]^{1/q}\left[\frac{(\alpha p)}{(2^{\alpha p}-1)}\right]^{1/p}, & \alpha\neq 0 \\ (\log 2)^{(p-q)/pq} = \lim_{\alpha\to 0} M_{\alpha,p,q}, & \alpha=0. \end{cases}$$

If f decreases, we mimic the above proof, and the result follows.

Theorem 1.11 (Stampacchia [55]): Let f, g and h be non-negative functions on $\mathbb{R}^+$, and h increasing with inverse function γ. If $1\leq p<\infty$ then

$$(4)\quad \int_0^\infty f(x)\left(\int_0^{h(x)} g(y)dy\right)^p dx \leq \left(\int_0^\infty g(y)\left(\int_{\gamma(y)}^\infty f(x)dx\right)^{1/p}dy\right)^p$$

and

$$(5)\quad \int_0^\infty f(x)\left(\int_{h(x)}^\infty g(y)dy\right)^p dx \leq \left(\int_0^\infty g(y)\left(\int_0^{\gamma(y)} f(x)dx\right)^{1/p}dy\right)^p.$$

If, instead, h is assumed decreasing, then the right hand sides of (4) and (5) are interchanged.

Proof: Assume h is increasing. To show (4), first note that Theorem 1.3 allows us to write

$$\left(\int_0^\infty f(x)\left(\int_0^{h(x)} g(y)dy\right)^p dx\right)^{1/p} = \sup\left(\int_0^\infty f(x)\left(\int_0^{h(x)} g(y)dy\right)F(x)dx,\right.$$

where the supremum is taken over all simple functions $F(x)\geq 0$ such that, for $p'=p(p-1)^{-1}$,

$$\left(\int_0^\infty F^{p'}(x)f(x)dx\right)^{1/p'} \leq 1.$$

An interchange of order of integration and Hölder's inequality yields

$$\int_0^\infty f(x)\int_0^{h(x)} g(y)dyF(x)dx = \int_0^\infty g(y)\left(\int_{\gamma(y)}^\infty f(x)F(x)dx\right)dy$$

$$\leq \int_0^\infty g(y)\left(\int_{\gamma(y)}^\infty f(x)dx\right)^{1/p}\left(\int_{\gamma(y)}^\infty f(x)F^{p'}(x)dx\right)^{1/p'}dy$$

$$\leq \int_0^\infty g(y)\left(\int_{\gamma(y)}^\infty f(x)dx\right)^{1/p}dy.$$

Hence, taking the supremum and then p^{th} powers completes the proo .

The other cases are similar, and therefore their proofs are omitted.

§1.2 Fourier methods

In this section, the classical Hardy spaces and their extension are discussed. We commence with some fundamental facts about Fourier series.

Definition 1.12: *Let $f \in L^p(\mathbb{T})$, $1 \leq p \leq \infty$.* *The trigonometric series $\sum_{n=-\infty}^{\infty} a_n e^{inx}$ is called the Fourier series of f if the coefficients a_n are given by*

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} f(x)dx, \quad n=0, \pm 1, \pm 2, \cdots.$$

In this case, we write

$$f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{inx}.$$

If $f \in L^1(\mathbb{T})$, Kolmogorov[32] has shown that its Fourier series may diverge almost everywhere. Lusin[40] conjectured that the Fourier series of an L^2 function converges almost everywhere. This conjecture was proved to be true by Carleson[13] in 1966 and Hunt[27] showed that the Fourier series converges almost everywhere for f in L^p, $1<p<\infty$. For general orthogonal systems, this result fails even for p=2[52].

If $f \in L^p(\mathbb{T})$, $1 < p \leq \infty$, the conjugate function $\tilde{f}$ of f is defined by

$$\tilde{f}(x) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{\varepsilon < |x-y| < \pi} f(y)/[\tan(x-y)/2] \, dy.$$

The Fourier series of $\tilde{f}$, or conjugate series of f, is given by

$$\tilde{f}(x) \sim - \sum_{n=-\infty}^{\infty} i\,\mathrm{sgn}(n) a_n e^{inx},$$

where $\mathrm{sgn}(n) = \frac{n}{|n|}$ if $n \neq 0$, and $\mathrm{sgn}(0) = 0$.

Let $T(f) = \tilde{f}$, then this linear operator is not bounded from L^1 to L^1, but it is known that

$$|\{x \in \mathbb{T} \; |T(f)(x)| > y\}| \leq \frac{M}{y} \|f\|_1, \quad y > 0.$$

Let $n \in \mathbb{N} \cup \{0\}$ and define

$$S_n(x) = S_n(f,x) = \sum_{k=-n}^{n} a_n e^{inx},$$

and

$$\sigma_n(x) \equiv \sigma_n(f,x) = \frac{S_0(x) + \cdots + S_n(x)}{n+1},$$

the n^{th} partial sum and n^{th} Cesaro mean, respectively, of the Fourier series of f. Fejer's theorem shows that as n tends to infinity, σ_n converges to f almost everywhere, f in L^1. Contrast this with the result of Kolmogorov that shows the existence of an $f \in L^1(\mathbb{T})$ for which $S_n(f,x)$ diverges a. e. In fact [80], there is an $f \in L^1(\mathbb{T})$ for which $S_n(f,x)$ diverges everywhere as n tends to infinity.

We now define the Littlewood-Paley g-function by

$$g(f,x) = g(x) = \left(\sum_{n=1}^{\infty} \frac{|S_n(x) - \sigma_n(x)|^2}{n} \right)^{1/2}.$$

Note also that there exists an $f \in L^1(\mathbb{T})$ such that $g(x)$ diverges almost everywhere.

Definition 1.13: Let $D = \{z \in \mathbb{C}: |z| < 1\}$ be the open unit disc in $\mathbb{C}$. An analytic function f defined on D is said to belong to the Hardy space $H^p(D)$, $0<p\leq\infty$ if

$$\|f\|_{H^p} \equiv \begin{cases} \sup\limits_{r>0}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p} & 0<p<\infty \\ \sup\limits_{z\in D}|f(z)| & p=\infty \end{cases}$$

is finite.

If $1\leq p\leq\infty$, the Hardy spaces are normed under $\|\cdot\|_{H^p}$, and are in fact subspaces of $L^p(T)$. The space H^p can also be considered as the class of $L^p(T)$ functions whose Fourier series have the form $\sum\limits_{n=0}^{\infty} a_n e^{inx}$.

The n-dimensional analogue is as follows:

Definition 1.14: Consider the vectors $F(x_1,\ldots,x_n,y)=F(x,y)=$ $=(u_o(x,y),u_1(x,y),\ldots,u_n(x,y))$ *whose components are defined in*

$\mathbb{R}_+^{n+1}=\{(x,y): x\in\mathbb{R}^n, y>0\}$ *and satisfy the generalized Cauchy-Riemann equations*

$$\sum_{j=o}^{n} \frac{\partial u_j}{\partial x_j} = 0 \quad \text{and} \quad \frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j} \qquad 0\leq j,\ k\leq n$$

with $x_o=y$. *We say* $F\in H^p$, $0<p$, *if*

$$\|F\|_{H^p} \equiv \sup_{y>o}\left(\int_{\mathbb{R}^n}|F(x,y)|^p\, dx\right)^{1/p} < \infty .$$

For $p\geq1$, this defines a norm. Moreover, [68, p.220] if $1<p<\infty$, H^p is equivalent to L^p.

If $p=1$ the situation is different, now $H^1(\mathbb{R}^n)$ may be defined equivalently ([68]) to consist of those $f\in L^1(\mathbb{R}^n)$, such that, the Riesz transforms of f, $R_j f\in L^1(\mathbb{R}^n)$, $j=1,2,\ldots,n$ and

$$\|f\|_{H^1} \equiv \|f\|_1 + \sum_{j=1}^{n} \|R_j f\|_1 .$$

For $f\in L^p(\mathbb{R}^n)$ $1\le p<\infty$ the Riesz transforms are defined by

(2) $$(R_j f)(x) = \lim_{\varepsilon\to 0_+} C_n \int_{|y|>\varepsilon} \frac{y_j}{|y|^{n+1}} f(x-y)\,dx, \quad y=(y_1,\ldots,y_n)$$

where $C_n = \dfrac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}}$.

Let K_j be the kernel of R_j , then the vector kernel $K=(K_1,\ldots,K_n)$ is of Calderón-Zygmund type ([56]) and satisfies

(3) $$|\hat{K}(x)| \le M \text{ and } \int_{|x|\ge A|y|} |K(x-y)-K(x)|\,dx \le M_A$$

for all $y\in\mathbb{R}^n$ and some constant A. In fact ([17], [61]) $K*f$ is bounded in L^p, $1<p<\infty$, that is

(4) $$\|K*f\|_p \le M_p\|f\|_p$$

if K satisfies (3).

Now let $Hf = (f,Rf)\equiv(f,R_1 f,\ldots,R_n f)$, then for $1\le p<\infty$

$$\|Hf\|_p \le \|f\|_p + \sum_{j=1}^{n} \|R_j f\|_p ,$$

which shows that $f\in H^p$, $1\le p<\infty$ implies $Hf\in L^p$.

This observation is used in Chapter II, in fact we require only that the kernel of the convolution transform satisfies (3).

§1.3: Introduction to L(p,q) spaces

Let f be a complex valued function on a measure space $(\mathcal{M},\mu)$, and let $E_y\equiv E_y(f)=\{x\in\mathcal{M}: |f(x)|>y\}$, $y>0$. Then the distribution function f_* of f is defined by $f_*(y)=\mu(E_y)$. It is assumed throughout that $f_*(y)<\infty$ for $y>0$.

The (non-negative) non-increasing rearrangement f^* of f is given by

$$f^*(t) = \inf\{y: f_*(y) \leq t\},\ t>0,$$

and the integral average f^{**} of f^* on $\mathbb{R}^+$ is defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(x)\,dx,\ t>0.$$

Clearly, both f^* and f_* are non-increasing and continuous from the right. Also, f_*, f^* and f^{**} are related by the formula

$$xf^{**}(x) = xf^*(x) + \int_{f^*(x)}^{\infty} f_*(t)\,dt.$$

The upper and lower $*$-operations are not additive, or even subadditive. However, the following inequalities

$$(f+g)_*(t) \leq f_*(t/2) + g_*(t/2)$$

$$(f+g)^*(t) \leq f^*(t/2) + g^*(t/2)$$

hold.

$(\mathcal{M},\mu)$ is said to be atom free if for any μ-measurable sets E and F with $E\subset F$ and $\mu(E)<\mu(F)$, there exists a μ-measurable set G, $E\subset G\subset F$, such that $\mu(E)<\mu(G)<\mu(F)$. If μ is atom free, there exists a μ-measurable set E with $\mu(E)=t$, such that

$$f^{**}(t) = \frac{1}{t}\int_E |f(x)|\,d\mu(x);$$

specifically

$$\sup_E \left\{\int_E |f(x)|\,d\mu(x),\ \mu(E)=s\right\} = \int_0^s f^*(t)\,dt.$$

For any μ-measurable set E, we have

$$\int_E |f(x)|\,d\mu(x) \leq \int_0^{\mu(E)} f^*(t)\,dt.$$

Henceforth, we assume all the measure spaces to be atom free.

For any μ-measurable functions f and g,

$$\int_{\mathfrak{M}} |f(t)g(t)|\,d\mu(t) \leq \int_0^\infty f^*(t)g^*(t)\,dt$$

and $(f+g)^{**}(t) \leq f^{**}(t)+g^{**}(t)$. In addition, for $0<q<\infty$, $(f^*)^q = (|f|^q)^*$. These last two results are used to advantage in proving completeness of the $L(p,q)$ spaces defined below, and in showing that there are no non-trivial continuous linear operators between certain $L(p,q)$ spaces.

The following result indicates the usefulness of rearrangements in the study of function spaces.

Theorem 1.15 [49]: *Let $(\mathfrak{M},\mu)$ be a measure space, and f a measurable function on $\mathfrak{M}$. If ϕ is a real valued, differentiable function on $\mathbb{R}^+$ with $\phi(0)=0$, absolutely continuous and monotone increasing on $\mathbb{R}^+$, then*

$$\int_{\mathfrak{M}} \phi(|f(t)|)\,d\mu(t) = \int_0^\infty f_*(t)\phi'(t)\,dt$$
$$= \int_0^\infty \phi(f^*(t))\,dt.$$

For example, if $\phi(t)=t^p$, $0<t<\infty$, $0<p<\infty$, then

$$\int_{\mathfrak{M}} |f(x)|^p dx = p\int_0^\infty t^{p-1} f_*(t)\,dt = \int_0^\infty f^{*p}(t)\,dt.$$

Definition 1.16: *The Lorentz space $L(p,q)$ is the collection of all μ-measurable functions f defined on $(\mathfrak{M},\mu)$ such that $\|f\|^*_{p,q}<\infty$, where*

$$\|f\|^*_{p,q} = \begin{cases} \left(\int_0^\infty [t^{1/p} f^*(t)]^q \frac{dt}{t}\right)^{1/q}, & 0<p<\infty,\ 0<q<\infty \\ \sup_{t>0} t^{1/p} f^*(t), & 0<p\leq\infty,\quad q=\infty. \end{cases}$$

$\|\cdot\|^*_{p,q}$ in general is not a norm; however, if $1<p<\infty$, $1\leq q\leq\infty$ and setting

$\|f\|_{p,q}=\|f^{**}\|^*_{p,q}$, then $\|\cdot\|_{p,q}$ is a norm and $L(p,q)$ is a Banach space. Moreover,

$$\|f\|^*_{p,q} \leq \|f^{**}\|^*_{p,q} \leq \frac{1}{p-1}\|f\|^*_{p,q}.$$

If $p=q$, $\|\cdot\|^*_{p,q}=\|\cdot\|_p$, the usual L^p norm. If $0<p<\infty$, $L(p,\infty)$ is called the weak L^p space.

The following embedding theorem follows from Theorem 1.10:

Theorem 1.17: If $0<q_1\leq q_2\leq\infty$, then $L(p,q_1)\subseteq L(p,q_2)$ and

$$\|f\|^*_{p,q_2} \leq M\|f\|^*_{p,q_1},$$

where $M=(p/q_2)^{1/q_2}(q_1/p)^{1/q_1}$.

§1.4 Linear and quasi-linear operators.

Let X and Y be linear spaces of real or complex valued functions, and T a function from X to Y. T is called a quasi-linear operator if for all f and g in X, $|T(f+g)|\leq \kappa(|Tf|+|Tg|)$, for some $\kappa>0$. If $\kappa=1$, then T is said to be sublinear. T is defined to be linear if for all f and g in X, and $\alpha_1,\alpha_2\in\mathbb{R}$ or $\mathbb{C}$,

$$T(\alpha_1 f+\alpha_2 g) = \alpha_1 Tf + \alpha_2 Tg.$$

Clearly, every linear operator is quasi-linear.

Definition 1.18: Let $(\mathcal{M},\mu)$ and $(\mathcal{N},\nu)$ be measure spaces, and T a quasi-linear operator defined on simple functions on $(\mathcal{M},\mu)$ into ν-measurable functions on $(\mathcal{N},\nu)$. T is said to be of strong type (p,q), $0<p,q\leq\infty$, if there is some constant $M>0$, independent of f, such that,

$$\|Tf\|_{q,\nu} \leq M\|f\|_{p,\mu}.$$

The least such M is called the norm of T.

If $q<\infty$ and T satisfies

$$y[(Tf)_*(y)]^{1/q} \leq M\|f\|_{p,\mu}$$

or, equivalently,

$$\sup_{t>0} t^{1/q}(Tf)^*(t) \leq M\|f\|_{p,\mu},$$

then T is said to be of weak type (p,q). If $q=\infty$, weak and strong type are defined to coincide.

The concept of weak type (p,q;r,s) was defined by Bennett and Rudnick [5], and used to advantage in the study of interpolation in Lorentz-Zygmund spaces. We do not require this concept here since our approach is very different from theirs.

Suppose now that T is of strong type (p,q), $0<p,q<\infty$, then

$$\begin{aligned} x[(Tf)^*(x)]^q &\leq \int_0^x [(Tf)^*(t)]^q dt \\ &\leq \int_0^\infty [(Tf)^*(t)]^q dt \\ &= \|Tf\|_{q,\mu}^q \\ &\leq M^q\|f\|_{p,\mu}^q, \end{aligned}$$

so that strong type implies weak type. However, the converse is false, as shown by the following example:

Let $g\in L^1(0,\infty)$, $x>0$ and define T by

$$(Tg)(x) = \frac{1}{x}\int_0^x g(t)dt.$$

It is readily verified that T is of weak type (1,1). Now let $g(x)=\chi_{[0,1]}(x)$ be the characteristic function of the interval [0,1], then

$g \in L^1(\mathbb{R}^+)$. However

$$\int_0^\infty |(Tg)(x)|dx = \int_0^\infty (\frac{1}{x}\int_0^x \chi_{[0,1]}(t)dt)dx$$

$$= \int_0^1 dx + \int_1^\infty \frac{dx}{x} = \infty,$$

which shows that T is not of strong type (1,1). The operator $\widetilde{F}$ defined in §1.2 is another example that weak type (1,1) does not imply strong type (1,1). Also, it is not difficult to show that there are operators of weak type (p,p), $1<p<\infty$ which are not of strong type (p,p).

To close this section, we give an example that shows for certain indices, there does not exist a non-trivial continuous linear operator between L(p,q) spaces. This result appears to be new.

<u>*Theorem 1.19:*</u> *Let $0<p<1$, $p<q\leq\infty$, $0<r<\infty$ and $0<s\leq\infty$. Then there does not exist any non-trivial continuous linear operators from $L(p,r)$ to $L(q,s)$.*

<u>*Proof:*</u> Consider first the function f_α^{**}, $0<\alpha<1$, defined for a measurable function f by

$$f_\alpha^{**}(t) = \frac{1}{t}\int_0^t f^{*\alpha}(x)dx.$$

Clearly, $\frac{1}{t}\int_0^t f^{*\alpha}(x)dx \geq \frac{1}{t}\int_0^t f^{*\alpha}(t)dx = f^{*\alpha}(t)$, and, since $f^{*\alpha}=(f^\alpha)^*$, it follows that

$$(g+f)_\alpha^{**}(t) = \frac{1}{t}\int_0^t (g+f)^{*\alpha}(x)dx$$

$$= \frac{1}{t}\int_0^t (|g+f|^\alpha)^*(x)dx$$

$$\leq \frac{1}{t}\int_0^t (|g|^\alpha+|f|^\alpha)^*(x)dx$$

$$\leq \frac{1}{t}\int_0^t [(|g|^{\alpha})^*(x)+(|f|^{\alpha})^*(x)]dx$$

$$= g_{\alpha}^{**}(t) + f_{\alpha}^{**}(t),$$

where the last inequality follows from the sublinearity of the $**$-operation.

We consider first the case $q<\infty$, $s=\infty$, and assume that there exists an operator T, such that, for some $M>0$, and all $f\epsilon L(p,r)$,

$$\|Tf\|_{q,\infty}^* \leq M\|f\|_{p,r}^*.$$

Let E, $E_1,\cdots,E_n$ be measurable sets, such that, $0<\mu(E)<\infty$, $\mu(E_i)=\mu(E)/n$, $1\leq i\leq n$, $E_i\cap E_j=\phi$, $i\neq j$, $\bigcup_{i=1}^n E_i=E$. We note that

$$\left(\int_0^\infty (t^{1/p}(\chi_E)^*(t))^r \frac{dt}{t}\right)^{1/r} = \mu(E)^{1/p}\left(\frac{p}{r}\right)^{1/r}.$$

The theorem will follow if it is shown that T must vanish on χ_E, the characteristic function of E, for then T is zero on all $L(p,r)$ by the density of simple functions in $L(p,r)$ [20].

Fix $t>0$. Since $s=\infty$, $\|T\chi_E\|_{q,s}^* = \sup_{t>0} t^{1/q}(T\chi_E)^*(t)$ so that we need only prove $t^{1/q}(T\chi_E)^*(t)=0$. However, choosing $p<\alpha<\min(1,q)$ yields

$$t^{\alpha/q}(T\chi_E)^{*\alpha}(t) \leq t^{\alpha/q}\frac{1}{t}\int_0^t (T\chi_E)^{*\alpha}(x)dx$$

$$= t^{\alpha/q}\frac{1}{t}\int_0^t \left(\sum_{i=1}^n T\chi_{E_i}\right)^{*\alpha}(x)dx$$

$$\leq t^{\alpha/q}\sum_{i=1}^n \frac{1}{t}\int_0^t (T\chi_{E_i})^{*\alpha}(x)dx$$

$$= t^{\alpha/q}\sum_{i=1}^n \frac{1}{t}\int_0^t x^{\alpha/q}(T\chi_{E_i})^{*\alpha}(x)\frac{dx}{x^{\alpha/q}}$$

$$\leq t^{\alpha/q}M^{\alpha}\sum_{i=1}^n \frac{1}{t}\int_0^t \|\chi_{E_i}\|_{p,r}^{*\alpha}\frac{dx}{x^{\alpha/q}}$$

$$= t^{\alpha/q} M^{\alpha} (\tfrac{p}{r})^{\alpha/r} (\mu(E)/n)^{\alpha/p} n \frac{1}{t}\int_0^t x^{-\alpha/q} dx$$

$$= \frac{1}{1-\alpha/q} (\tfrac{p}{r})^{\alpha/r} M^{\alpha} t^{\alpha/q - 1} (\mu(E))^{\alpha/p} n^{1-\alpha/p} t^{1-\alpha/q}$$

$$= \frac{1}{1-\alpha/q} (\tfrac{p}{r})^{\alpha/r} M^{\alpha} (\mu(E))^{\alpha/p} n^{1-\alpha/p}.$$

If n tends to infinity, the result follows in this case.

If $s,q<\infty$ then Theorem 1.17 states that $L(q,s) \subset L(q,\infty)$, so that, if $\|Tf\|^*_{q,s} \leq M\|f\|^*_{p,r}$, then $\|Tf\|^*_{q,\infty} \leq M'\|f\|^*_{p,r}$. Thus the above argument shows that T=0 on $L(p,r)$.

Finally, if $q=\infty$, then $s=\infty$, and we have $L(\infty,\infty)=L^{\infty}$. Again, fixing $t>0$, it follows that for $p<\alpha$

$$(T\chi_E)^{*\alpha}(t) \leq \frac{1}{t}\int_0^t (T\chi_E)^{*\alpha}(x)dx$$

$$\leq \sum_{i=1}^{n} \frac{1}{t}\int_0^t (T\chi_{E_i})^{*\alpha}(x)dx$$

$$\leq M^{\alpha} \sum_{i=1}^{n} \frac{1}{t}\int_0^t \|\chi_{E_i}\|^{*\alpha}_{p,r} dx$$

$$= M^{\alpha} n(\mu(E)/n)^{\alpha/p},$$

from which the result follows.

The theorem is now complete.

Since for $p=r$, $L(p,r)=L^p$, then the last theorem shows that there does not exist non-trivial continuous linear operators from L^p to L^q, where $p<1$, $q>p$. This result is known and can be found in, e. g., Krasnoselskii et al. [34].

Note that there are no restrictions on r and s, that is, only the conditions on p and q are important. This is natural, in that, as we shall see in Chapter III, the function $t^{1/p}$ is the fundamental function of L(p,r), for any r.

§1.5 The Riesz-Thorin and Marcinkiewicz theorems.

In this section, the basic interpolation theorems are stated without proof, and we discuss briefly their differences. We also discuss their applicability in view of Theorem 1.19.

Theorem 1.20 (Riesz-Thorin): *Let* $L_\mu^{p_i}$ *and* $L_\nu^{q_i}$ *be Lebesgue spaces of complex valued functions;* $0<p_i,q_i\leq\infty$, *and let T be a sub-linear operator, such that, for* $M_0,M_1>0$,

$$\|Tf\|_{q_i,\nu} \leq M_i\|f\|_{p_i,\mu},\quad i=0,1.$$

Then for any $0<\alpha<1$, *T is continuous from* L_μ^p *to* L_ν^q, *where* $p^{-1}=(1-\alpha)p_0^{-1}+\alpha p_1^{-1}$, $q^{-1}=(1-\alpha)q_0^{-1}+\alpha q_1^{-1}$ *and*

$$\|Tf\|_{q,\nu} \leq M_0^{1-\alpha}M_1^{\alpha}\|f\|_{p,\mu}.$$

The extension of the Riesz-Thorin theorem to all $0<p_i,q_i\leq\infty$ was obtained by Calderón[11]. The theorem states that the set of points $(1/p,1/q)$ for which the operator is continuous is a convex set: more precisely, the set of continuity for T contains, along with any two points $(1/p_0,1/q_0)$ and $(1/p_1,1/q_1)$ the whole line segment joining them. For this reason, interpolation theorems are also called convexity theorems. Also, the norm of the operator is a logarithmically convex function, i.e. $\log M\leq(1-\alpha)\log M_0 + \alpha\log M_1$.

Theorem 1.21 (Marcinkiewicz): *If a quasilinear operator T is of weak types (p_i,q_i) with norms M_i, $i=0,1$, where $0<p_i\leq q_i\leq\infty$, $p_0<p_1$ and $q_0\neq q_1$, then T is of strong type (p,q) where $p^{-1}=(1-\alpha)p_0^{-1}+\alpha p_1^{-1}$, $q^{-1}=(1-\alpha)q_0^{-1}+\alpha q_1^{-1}$, $0<\alpha<1$.*

The theorem states that, as in the Riesz-Thorin theorem, the line segment joining $(1/p_0,1/q_0)$ and $(1/p_1,1/q_1)$ lies in the set of continuity of T. Note that the endpoints of the line segment are excluded, and that they must both lie on or below the line $1/p = 1/q$. The extension of this theorem to indices less than one is due to Hunt [25], where he extended this result to the L(p,q) spaces. In this paper, he also gives an example which shows that the result fails in the upper "triangle", i,e, $1/q > 1/p$.

Since weak type does not imply strong type, then Theorem 1.20 does not contain Theorem 1.21. Conversely, the bound of the norm of T in Theorem 1.21 tends to infinity as p tends to either p_0 or p_1, while it remains bounded in Theorem 1.20, so that Marcinkiewicz' theorem does not contain that of Riesz-Thorin. The constant in the conclusion of the Marcinkiewicz theorem remains the same whether real or complex valued functions are considered, whereas the logarithmic convexity of the constant is not preserved in the Riesz-Thorin theorem under this change.

Let us reexamine the hypotheses of the Marcinkiewicz and Riesz-Thorin theorems in view of Theorem 1.19. We shall use a series of diagrams to illustrate the conditions on the indices p and q. Since it is possible that either p or q or both are infinite, but neither may be zero, we consider the pairs $(1/p,1/q)$ instead of (p,q), with the convention $1/\infty = 0$. Also, the set $N=\{(1/p,1/q): 0 \leq 1/p,1/q < \infty\}$ is divided into the following mutually exclusive sets:

$$N_1 = \{(1/p,1/q): 1 < 1/p < \infty,\ 0 \leq 1/q < 1/p\}$$
$$N_2 = \{(1/p,1/q): 1/p > 1,\ 1/q \geq 1\}$$
$$N_3 = \{(1/p,1/q): 0 \leq 1/p \leq 1,\ 1/q \geq 1/p\}$$
$$N_4 = \{(1/p,1/q): 0 < 1/p \leq 1,\ 0 \leq 1/q < 1\}.$$

It is clear that $N = N_1 \cup N_2 \cup N_3 \cup N_4$.

First, the Marcinkiewicz Theorem is valid for all pairs $(1/p_i,1/q_i)$, $i=0,1$, for which $0 \leq 1/q_i \leq 1/p_i < \infty$, $q_0 \neq q_1$ and $p_0 < p_1$. This region of continuity is illustrated by the cross-hatched region of Figure I.

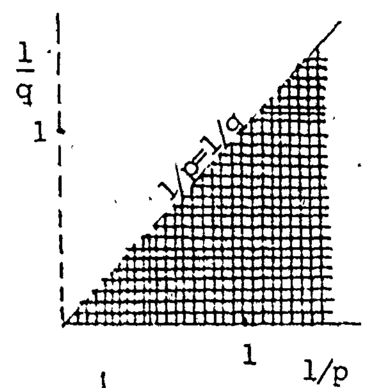
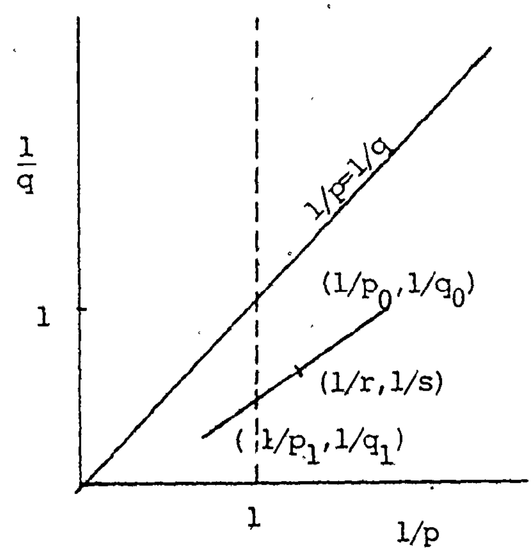


FIGURE I

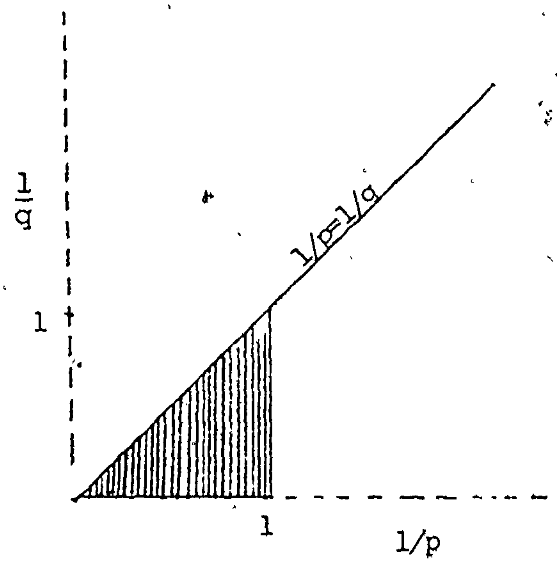
Let T be a linear operator satisfying the conditions of Theorem 1.21 when $1/p_0 > 1$ and $1/p_0 \neq 1/q_0$. Then part of the line segment L joining the points $(1/p_0,1/q_0)$ and $(1/p_1,1/q_1)$ must intersect the set N_1, that is, there exists at least one pair $(1/r,1/s)$ which is on L, distinct from L's endpoints, and is also in N_1. This is represented by Figure II.

FIGURE II



Since the Marcinkiewicz Theorem says that for every pair on L, $(1/p,1/q)$ say, not including the endpoints, the operator T is continuous from L^p to L^q, then in particular, T maps L^r continuously into L^s. However, Theorem 1.19 shows that only the zero operator is continuous from L^r to L^s, since $(1/r,1/s)\in N_1$. Thus T must be zero on the simple functions, since they are dense in any L^p space, $0<p<\infty$. Hence T must be the zero operator for any pair $(1/p,1/q)$ on L, $p<\infty$, again by the density of the simple functions. A similar argument holds if $p_0=q_0<1$ and $1\leq p_1<q_1$. Therefore, a necessary condition that there exist non-trivial linear operators satisfying the conditions of the Marcinkiewicz Interpolation Theorem is that either $1\leq p_i\leq q_i\leq\infty$, $p_0<p_1$, $q_0\neq q_1$ or $p_0=q_0$, $p_1=q_1$ and $0<p_0<p_1\leq\infty$. The pairs satisfying this condition are represented in Figure III.

FIGURE III



Next, we consider the Riesz-Thorin Theorem. For pairs $(1/p_i, 1/q_i)$ $i=0,1$, as described in the last paragraph, we apply the same reasoning as above to again show that there are no non-trivial continuous linear operators satisfying such conditions for such pairs. However, since the Riesz-Thorin Theorem is stated for all pairs $(1/p_i, 1/q_i)$, $i=0,1$, with $0 \leq 1/p_i, 1/q_i < \infty$, another possibility arises. Suppose, for example, $(1/p_0, 1/q_0) \in N_2$ and $(1/p_1, 1/q_1) \in N_4$, such that the line L joining them has non-empty intersection with N_1, as shown in Figure IV. Again, there must be some point $(1/r, 1/s) \in N_1$ and lying on L. The above argument shows that T again must be zero on the simple functions, since it is zero on L^r. Again we conclude, by the density of the simple functions in any L^p space, $p<\infty$, that T must be the zero operator.

From the discussion in the last paragraph we deduce that in order for there to exist non-trivial operators satisfying the conditions of the Riesz-Thorin Theorem, either

$$(1/p_0,1/q_0)\in N_3 \text{ and } (1/p_1,1/q_1)\in N_2\cup N_3\cup N_4;$$

$$(1/p_0,1/q_0)\in N_2 \text{ and } L\cap N_1 = \phi;$$

or

$$(1/p_0,1/q_0)\in N_4 \text{ and } L\cap N_1 = \phi.$$

We remark that if we now consider Hunt's and Calderón's extensions of the Marcinkiewicz and Riesz-Thorin Theorems to L(p,q) spaces, a similar argument to that above may be applied.

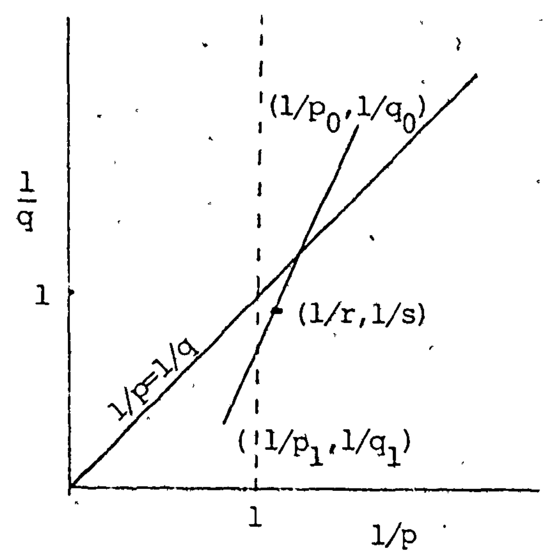


FIGURE IV

CHAPTER II

§2.1 Introduction.

Recall that the Littlewood-Paley g-function of a function f is defined as

$$g(x) = \left(\sum_{n=1}^{\infty} \frac{|S_n(x) - \sigma_n(x)|^2}{n}\right)^{1/2},$$

where S_n and σ_n are the n^{th} partial sum and n^{th} Cesaro mean of the Fourier series associated with f. Stein[67] has shown that if we define T by Tf=g, then T is sublinear and satisfies

$$|\{x: |g(x)|>y\}| \leq \frac{M}{y}\|f\|_{H^1}, \quad y>0$$

for all $f\varepsilon H^1[-\pi,\pi]$, while Zygmund [80, Vol. I,p.183] has shown that T is not of weak type (1,1). It is readily verified that T is of strong type (2,2) and hence one would expect to be able to interpolate to derive the known strong type (p,p), $1<p<2$, result.

Igari[29] has reformulated the hypotheses of the Marcinkiewicz theorem in terms of the n-dimensional analogue of Hardy spaces. The weak-type assumptions are stated in terms of distributions of functions, and the theorem is proved using the properties of the lower $*$-operation. We state and prove his theorem in terms of rearrangements, which then leads to a weighted version of Igari's result. The weighted interpolation theorem is then used to derive results for L(p,q) spaces.

In order to prove the interpolation theorems of this chapter, a modification of a decomposition theorem of Igari [29] is required. An estimate concerning the conjugate transform is then proved.

Lemma 2.1 is the required decomposition theorem.

Lemma 2.1: *Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, $n \in \mathbb{N}$, and let r, s, t, y be real numbers, such that, $1 \leq r, s \leq p$ and $t, y > 0$. Then there exist functions u, u', v, w and w_k, $k \in \mathbb{N}$ satisfying:*

(i) $f = u + u'$, $u' = v + w$, $w = \sum_{k=1}^{\infty} w_k$;

(ii) $u(x) = \begin{cases} f(x) & |f(x)| < t \\ 0 & \text{otherwise}; \end{cases}$

(iii) $|v(x)| \leq 2^{n/r} y$ a.e.;

(iv) $\int_{\mathbb{R}^n} |v(x)|^s dx \leq \int_{\mathbb{R}^n} |u'(x)|^s dx$;

(v) $\sum_{k=1}^{\infty} (\int_{\mathbb{R}^n} |w_k(x)|^s dx)^{1/s} \leq 2^{(s+1)/s} (\int_{\mathbb{R}^n} |u'(x)|^s dx)^{1/s}$;

(vi) *there exists a sequence of cubes $\{Q_k\}_{k=1}$ with disjoint interiors, such that the support of w_k is in Q_k and* $\sum_{k=1}^{\infty} |Q_k| \leq y^{-r} \int_{\mathbb{R}^n} |u'(x)|^r dx$;

(vii) $\int_{\mathbb{R}^n} w_k(x) dx = 0$, $k \in \mathbb{N}$.

Proof: Fix $t > 0$, define u by (ii) and set $u' = f - u$. Divide $\mathbb{R}^n$ into a mesh of cubes $\{Q\}$, parallel to the coordinate axes, such that, for each cube Q and fixed $y > 0$ and $r \leq p$

$$(1) \quad |Q| > y^{-r} \int_{\mathbb{R}^n} |u'(x)|^r dx.$$

Now subdivide each cube Q into 2^n congruent subcubes having sides parallel to those of Q, and denote by $\{Q_{1k}\}$ those cubes for which

$$(2) \quad |Q_{1k}|^{-1} \int_{Q_{1k}} |u'(x)|^r dx > y^r.$$

Let $\{Q'_{1j}\}$ be the complement of $\{Q_{1k}\}$, that is, those cubes satisfying

$$(3)\quad |Q'_{1j}|^{-1}\int_{Q'_{1j}} |u'(x)|^r dx \leq y^r.$$

From (2) and then (1) we obtain

$$y^r|Q_{1k}| < \int_{Q_{1k}} |u'(x)|^r dx \leq \int_Q |u'(x)|^r dx < y^r 2^n |Q_{1k}|.$$

Next, define v and w_{1k} on Q_{1k} by

$$(4)\quad v(x) = |Q_{1k}|^{-1}\int_{Q_{1k}} u'(t)dt, \quad x\varepsilon Q_{1k}$$

$$w_{1k}(x) = \begin{cases} u'(x) - v(x) & x\varepsilon Q_{1k} \\ 0 & x\notin Q_{1k}. \end{cases}$$

Consider each Q'_{1j}, and subdivide it in the same manner as Q, to obtain sequences of cubes $\{Q_{jk}\}$ and $\{Q'_{ji}\}$ as above. Now extend definition (4) for v and w_{jk} to the cubes Q_{jk}. Continuing this process of subdivision, and then relabelling the cubes Q_{jk} by Q_k and setting $v(x)=u'(x)$ if $x\varepsilon\mathbb{R}^n\setminus\Omega$, where $\Omega=\sum_{n=1}^{\infty} Q_k$, we obtain functions v and w_k defined on all of $\mathbb{R}^n$.

It is clear that (i) holds, and, since

$$\sum_{k=1}^{\infty} |Q_k| < y^{-r}\int_\Omega |u'(x)|^r dx \leq y^{-r}\int_{\mathbb{R}^n} |u'(x)|^r dx,$$

then (vi) holds. Also,

$$\int_{\mathbb{R}^n} w_k(x)dx = \int_{Q_k} [u'(x)-v(x)]dx$$

$$= \int_{Q_k} u'(x)dx - \int_{Q_k} |Q_k|^{-1}\int_{Q_k} u'(t)dtdx = 0,$$

so that (vii) holds.

For $x\varepsilon Q_k$, an application of Hölder's inequality yields

$$|v(x)|^s \leq |Q_k|^{-s}(\int_{Q_k}|u'(x)|dx)^s$$
$$\leq |Q_k|^{-s}|Q_k|^{s/s'}\int_{Q_k}|u'(x)|^s dx$$
$$= |Q_k|^{-1}\int_{Q_k}|u'(x)|^s dx$$

and then integrating we obtain

$$\int_{\mathbb{R}^n}|v(x)|^s dx = \int_{\mathbb{R}^n\setminus\Omega}|v(x)|^s dx + \int_{\Omega}|v(x)|^s dx$$
$$\leq \int_{\mathbb{R}^n\setminus\Omega}|u'(x)|^s dx + \sum_{k=1}^{\infty}\int_{Q_k}|Q_k|^{-1}\int_{Q_k}|u'(x)|^s dx dt$$
$$= \int_{\mathbb{R}^n}|u'(x)|^s dx$$

which is (iv). Note that the last integral is dominated by $\int_0^{f_*(t)} f^{*s}(x)dx$ and for $t>0$, is finite by Hölder's inequality. This fact will be used in the proof of Theorem 2.3.

Next,

$$\int_{\mathbb{R}^n}|w_k(x)|^s dx = \int_{Q_k}|w_k(x)|^s dx$$
$$= \int_{Q_k}|u'(x)-v(x)|^s dx$$
$$\leq 2^s\{\int_{Q_k}|u'(x)|^s dx + \int_{Q_k}|v(x)|^s dx\}$$
$$\leq 2^s\{\int_{Q_k}|u'(x)|^s dx + \int_{Q_k}|u'(x)|^s dx\}$$
$$= 2^{s+1}\int_{Q_k}|u'(x)|^s dx,$$

from which (v) follows.

Finally, if $x\varepsilon Q_k$, then

$$|v(x)|^r \leq |Q_k|^{-1}\int_{Q_k}|u'(x)|^r dx$$
$$\leq 2^n y^r.$$

On the other hand, if $x\varepsilon\mathbb{R}^n\setminus\Omega$ then $|v(x)|^r = |u'(x)|^r$ and there exist arbitrarily small cubes Q, such that, $x \in Q$ and

$$|Q|^{-1} \int_Q |u'(t)|^r dt \leq y^r.$$

This implies $|u'(x)|^r \leq y^r$ a.e. by Lebesgue's theorem [68]. This completes the proof of the lemma.

Lemma 2.2 [29]: Suppose w_k, u' *and* Q_k, $k \in \mathbb{N}$ *are as in Lemma 2.1. Let* $\tilde{Q}_k$ *be the cube obtained by expanding* Q_k *concentrically* A *times, where* A *is the constant for which the kernel of* §2.1 (3) *satisfies*

$$\int_{|x| \geq A|y|} |K(x-y)-K(x)| dx \leq M_A .$$

Let

$$\tilde{w}_k(y) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y) w_k(x) dx.$$

and $E = \bigcup_{k=1}^{\infty} \tilde{Q}_k$, *then there is a constant* $M>0$ *independent of* f, *such that*

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}^n \setminus E} |\tilde{w}_k(x)| dx \leq M \int_{\mathbb{R}^n} |u'(x)| \, dx .$$

If K is the Riesz kernel K_j of R_j then $\tilde{w}_k = (R_j w_k)$ the j^{th} Riesz transform. The conclusion then holds for each Riesz transform and since $|Rw_k| = (\sum_j |R_j w_k|^2)^{1/2}$, also for Rw_k.

Proof: Fix a cube Q_k with centre $a=(a_1,\ldots,a_n)$ and side length 2h, that is

$$Q_k = \{(x_1,\ldots,x_u) : |x_i - a_i| \leq h;\ 1 \leq i \leq n\}.$$

Let

$$\tilde{Q}_k = \{(x_1,\ldots,x_u) : |x_i - a_i| \leq Ah;\ 1 \leq i \leq n\}$$

then by (vi) and (vii) of Lemma 2.1

$$\int_{\mathbb{R}^n \setminus E} |\tilde{w}_k(x)| dx \leq \int_{\mathbb{R}^n \setminus \tilde{Q}_k} |\tilde{w}_k(x)| dx$$

$$= \int_{\mathbb{R}^n \setminus \tilde{Q}_k} \left| \int_{\mathbb{R}^n} (K(x-y) - K(x-a)) w_k(y)\, dy \right| dx$$

$$\leq \int_{Q_k} |w_k(y)| \int_{\mathbb{R}^n \setminus \tilde{Q}_k} |K(x-y) - K(x-a)|\, dx dy$$

$$\leq \int_{Q_{k,a}} |w_k(y+a)| \int_{\mathbb{R}^n \setminus \tilde{Q}_{k,a}} |K(x-y) - K(x)|\, dx dy,$$

where $Q_{k,a}$ and $\tilde{Q}_{k,a}$ are the translations of Q_k and $\tilde{Q}_k$ to the origin.

The last expression is dominated by

$$M_A \int_{Q_{k,a}} |w_k(y+a)|\, dy = M_A \int_{Q_k} |w_k(y)|\, dy.$$

Hence

$$\int_{\mathbb{R}^n \setminus E} |\tilde{w}_k(x)|\, dx \leq M_A \int_{Q_k} |w_k(y)|\, dy;$$

and using Lemma 2.1, (v), with s=1

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}^n \setminus E} |\tilde{w}_k(x)|\, dx \leq M_A \sum_{k=1}^{\infty} \int_{Q_k} |w_k(x)|\, dx$$

$$\leq M_A 2^2 \int_{\mathbb{R}^n} |u'(x)|\, dx.$$

This completes the proof.

§2.2 Igari's interpolation theorem and weighted extension.

Theorem 2.3: *Let* $1 \leq p_i \leq q_i < \infty$, $p_0 < p_1$, $q_0 \neq q_1$, $i=0,1$. *Suppose that a quasi-linear operator* $T: H^{p_i} \to S$ *acting from* H^{p_i} *to μ-measurable functions satisfies*

$$(1) \quad \sup_{t>0} t^{1/q_i} (T\phi)^*(t) \leq M_i \|\phi\|_{H^{p_i}}$$

for all $\phi \in H^{p_i}$, $i=0,1$. *Let* $1/p = (1-\alpha)/p_0 + \alpha/p_1$ *and* $1/q = (1-\alpha)/q_0 + \alpha/q_1$ *for* $0<\alpha<1$. *Then* T *extends to an operator on* H^p *satisfying*

$\|T(f)\|_q \leq$ [illegible], [illegible] $\in H^p(\mathbb{R}^n)$.

Proof: We only prove the theorem when T is sublinear, the general case being analogous. Since

$$|Hf(x)| = \{|f(x)|^2 + \sum_{j=1}^{n} |(R_j f)(x)|^2\}^{1/2},$$

it suffices to show the result for $R_j f \equiv \tilde{f}$. Also we prove the theorem for $p_0=1$ only, since the case $p_0>1$ follows from Theorem 1.10, §1.2(4) and the Marcinkiewicz theorem. In addition for the sake of brevity we interprete $\|\phi\|_{H^{p_i}}$ of (1) as $\|\tilde{\phi}\|_{p_i}$. The validity of the result is clearly not affected by this interpretation.

Let $f(x)=u(x)+u'(x)$, where

$$u(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq f^*(y^\sigma),\ y>0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\sigma = \frac{1/q - 1/q_0}{1/p - 1/p_0} = \frac{1/q - 1/q_1}{1/p - 1/p_1} = \frac{1/q_0 - 1/q_1}{1/p_0 - 1/p_1},$$

then

$$u^*(t) \leq \begin{cases} f^*(y^\sigma) & \text{if } 0<t\leq y^\sigma \\ f^*(t) & \text{if } y^\sigma \leq t. \end{cases}$$

We assume first $q_0<q_1$, that is, $\sigma>0$.

Let $u'(x)=v(x)+w(x)$, then $u'^*(t)\leq v^*(t/2)+w^*(t/2)$ where v and w are defined in Lemma 2.1, with y replaced by $f^{**}(y^\sigma)$, and

$$u'^*(t) \leq \begin{cases} 0 & y^\sigma \leq t \\ f^*(t) & 0<t\leq y^\sigma. \end{cases}$$

Now a change of variable and Minkowski's inequality yields

$$\|T(\tilde{f})\|_q = \{\int_0^\infty (T(\tilde{f}))^{*q}(t)dt\}^{1/q}$$

$$\leq 3^{1/q}\{[\int_0^\infty (T(\tilde{u}))^{*q}(t)dt]^{1/q} + [\int_0^\infty (T(\tilde{v}))^{*q}(t)dt]^{1/q} + [\int_0^\infty (T(\tilde{w}))^{*q}(t)dt]^{1/q}\}$$

$$\equiv 3^{1/q}\{I_1 + I_2 + I_3\}.$$

Considering I_1, we obtain by the Riesz inequality [§1.2,(4)], and (1)

$$I_1^{p_1} \leq M_1^{p_1}\left(\int_0^\infty y^{-q/q_1}\|\tilde{u}\|_{p_1}^q dy\right)^{p_1/q}$$

$$\leq M_1^{p_1}M_{p_1}^{p_1}\left(\int_0^\infty y^{-q/q_1}\left(\int_{\mathbb{R}^n}|u(t)|^{p_1}dt\right)^{q/p_1}dy\right)^{p_1/q}$$

$$= M_1^{p_1}M_{p_1}^{p_1}\left(\int_0^\infty y^{-q/q_1}\left(\int_0^\infty u^{*p_1}(t)dt\right)^{q/p_1}dy\right)^{p_1/q}$$

$$= M_1^{p_1}M_{p_1}^{p_1}\left(\int_0^\infty y^{-q/q_1}\left[\int_0^{y^\sigma}u^{*p_1}(t)dt + \int_{y^\sigma}^\infty u^{*p_1}(t)dt\right]^{q/p_1}dy\right)^{p_1/q}.$$

If $q \geq p_1$, Minkowski's inequality applies, so that

$$I_1^{p_1} \leq M_1^{p_1}M_{p_1}^{p_1}\left[\left\{\int_0^\infty y^{-q/q_1}\left(\int_0^{y^\sigma}u^{*p_1}(t)dt\right)^{q/p_1}dy\right\}^{p_1/q}\right.$$

$$\left. + \left\{\int_0^\infty y^{-q/q_1}\left(\int_{y^\sigma}^\infty u^{*p_1}(t)dt\right)^{q/p_1}dy\right\}^{p_1/q}\right]$$

$$\equiv M_1^{p_1}M_{p_1}^{p_1}[J_0 + J_1].$$

By a change of variable

$$J_0 \leq \left\{\int_0^\infty y^{-q/q_1}[f^{*p_1}(y^\sigma)y^\sigma]^{q/p_1}dy\right\}^{p_1/q}$$

$$= \sigma^{-p_1/q}\left\{\int_0^\infty y^{q/p_1 + 1/\sigma - q/q_1\sigma - 1}f^{*q}(y)dy\right\}^{p_1/q}$$

$$= \sigma^{-p_1/q}\left\{\int_0^\infty y^{q/p - 1}f^{*q}(y)dy\right\}^{p_1/q}$$

$$\leq \sigma^{-p_1/q}\left\{\int_0^\infty f^{*p}(y)dy\right\}^{p_1/p}M^{p_1},$$

where the last inequality follows from Theorem 1.10, since $q>p$. Hence $J_0 \leq \sigma^{-p_1/q}\|f\|_p^{p_1}M^{p_1}$.

Next, by Theorems 1.11 and 1.10

$$J_1^{q/p_1} \leq \int_0^\infty y^{-q/q_1}\left(\int_{y^\sigma}^\infty f^{*p_1}(t)dt\right)^{q/p_1}dy$$

$$\leq \left\{\int_0^\infty f^{*p_1}(t)\left(\int_0^{t^{1/\sigma}} y^{-q/q_1}dy\right)^{p_1/q}dt\right\}^{q/p_1}$$

$$= q_1/(q_1-q)\left\{\int_0^\infty f^{*p_1}(t)\, t^{(1-q/p_1)p_1/\sigma q}\,dt\right\}^{q/p_1}.$$

$$= q_1/(q_1-q)\{\int_0^\infty f^{*p_1}(t)t^{p_1/p\ -1}dt\}^{q/p_1}$$

$$\leq q_1/(q_1-q)\{\int_0^\infty f^{*p}(t)dt\}^{q/p}M^q ,$$

so that, for $q\geq p_1$,

$$I_1 \leq MM_1M_{p_1}[\ \left(\frac{q_1q_0(p_0-p_1)}{p_1p_0(q_0-q_1)}\right)^{p_1/q} + \left(q_1/(q_1-q)\right)^{p_1/q}]^{1/p_1}\|f\|_p .$$

If $q<p_1$, then

$$I_1^q \leq M_1^qM_{p_1}^q\{\int_0^\infty y^{-q/q_1}(\int_0^{y^\sigma}u^{*p_1}(t)dt)^{q/p_1}dy$$

$$+ \int_0^\infty y^{-q/q_1}(\int_{y^\sigma}^\infty u^{*p_1}(t)dt)^{q/p_1}dy\}$$

$$= M_1^qM_{p_1}^q\{J_2 + J_3\}.$$

Now

$$J_2 \leq \{\int_0^\infty y^{-q/q_1}(\int_0^{y^\sigma}f^{*p_1}(y^\sigma)dt)^{q/p_1}dy$$

$$= \sigma^{-1}\{\int_0^\infty f^{*q}(y)y^{q/p\ -1}dy\}$$

$$\leq \sigma^{-1}\{\int_0^\infty f^{*p}(y)dy\}^{q/p}M^q$$

by a change of variable and Theorem 1.10. Also, by Theorems 1.10 and 1.11

$$J_3 = \int_0^\infty y^{-q/q_1}(\int_{y^\sigma}^\infty u^{*p_1}(t)dt)^{q/p_1}dy$$

$$\leq \int_0^\infty y^{-q/q_1}(\int_{y^\sigma}^\infty f^{*p_1}(t)dt)^{q/p_1}dy$$

$$\leq M^q\int_0^\infty y^{-q/q_1}(\int_{y^\sigma}^\infty f^{*p}(t)t^{p/p_1\ -1}dt)^{q/p}$$

$$\leq M^q(\int_0^\infty t^{p/p_1\ -1}f^{*p}(t)(\int_0^{t^{1/\sigma}} y^{-q/q_1}dy)^{p/q}dt)^{q/p}$$

$$= M^qq_1/(q_1-q)(\int_0^\infty t^{p/p_1\ -1}f^{*p}(t)t^{(1/\sigma\ -\ q/\sigma q_1)p/q}dt)^{q/p}$$

$$= M^\sigma(q_1/(q_1-q)(\int_0^\infty f^{*p}(t)dt)^{q/p}$$

so that, in this case,

$$I_1 \leq [\frac{q_1 q_0(p_0-p_1)}{p_1 p_0(q_0-q_1)} + q_1/(q_1-q)]^{1/q} M M_1 M_{p_1} \|f\|_p.$$

We now consider I_2. Since $p<q$, we obtain by Theorem 1.10,(1), and the Riesz inequality

$$I_2 = (\int_0^\infty (T(\tilde{v}))^{*q}(y)dy)^{1/q}$$

$$\leq M (\int_0^\infty (T(\tilde{v}))^{*p}(y) y^{p/q-1} dy)^{1/p}$$

$$\leq M M_1 (\int_0^\infty y^{p/q - p/q_1 - 1} \|\tilde{v}\|_{p_1}^p dy)^{1/p}$$

$$\leq M M_1 M_{p_1} (\int_0^\infty y^{p/q - p/q_1 - 1} (\int_{\mathbb{R}^n} |v(x)|^{p_1} dx)^{p/p_1} dy)^{1/p}$$

$$= M M_1 M_{p_1} (\int_0^\infty y^{p/q - p/q_1 - 1} (\int_{\mathbb{R}^n} |v(x)|^{p_1-1} |v(x)| dx)^{p/p_1} dy)^{1/p}.$$

With y replaced by $f^{**}(y^\sigma)$ in Lemma 2.1, and then (iii) of that same lemma implies that the last expression is dominated by

$$M M_1 M_{p_1} 2^{n/rp_1'} (\int_0^\infty y^{p/q - p/q_1 - 1} f^{**(p_1-1)p/p_1}(y^\sigma) \times (\int_{\mathbb{R}^n} |v(x)| dx)^{p/p_1} dy)^{1/p}$$

$$\leq M M_1 M_{p_1} 2^{n/rp_1'} (\int_0^\infty y^{p/q - p/q_1 - 1} f^{**(p_1-1)p/p_1}(y^\sigma) \times (\int_{\mathbb{R}^n} |u(x)| dx)^{p/p_1} dy)^{1/p}$$

$$\leq M M_1 M_{p_1} 2^{n/rp_1'} (\int_0^\infty y^{p/q - p/q_1 - 1} f^{**(p_1-1)p/p_1}(y^\sigma) \times (\int_0^{y^\sigma} f^*(t)dt)^{p/p_1} dy)^{1/p}$$

$$= M M_1 M_{p_1} 2^{n/rp_1'} \sigma^{-1/p} (\int_0^\infty y^{p(1/p - 1/p_1)-1} f^{**(p_1-1)p/p_1}(y) \times (\int_0^y f^*(t)dt)^{p/p_1} dy)^{1/p}$$

$$= MM_1 M_{p_1} 2^{n/rp} \hat{1}_\sigma^{-1/p} \left(\int_0^\infty y^{-p/p_1} y^{-(p_1-1)p/p_1} \times \left(\int_0^y f^*(t)dt\right)^{(p_1-1)p/p_1 + p/p_1} dy\right)^{1/p}$$

$$= MM_1 M_{p_1} 2^{n/rp} \hat{1}_\sigma^{-1/p} \left(\int_0^\infty y^{-p} \left(\int_0^y f^*(t)\right)^p dy\right)^{1/p}$$

$$\leq MM_1 M_{p_1} 2^{n/rp} \hat{1}_\sigma^{-1/p} \cdot (p/(p-1)) \|f\|_p,$$

where the last inequality is from Theorem 1.9.

Next, we estimate I_3. By (1) with $p_0=1$

$$I_3 = \left(\int_0^\infty (T(\tilde{w})^{*q}(y)dy\right)^{1/q}$$

$$\leq M_0 \left(\int_0^\infty y^{-q/q_0} \|\tilde{w}\|_1^q dy\right)^{1/q}$$

$$\leq M_0 \left\{ \left(\int_0^\infty y^{-q/q_0} \left(\int_E |(\tilde{w})(x)| dx\right)^q dy\right)^{1/q} + \left(\int_0^\infty y^{-q/q_0} \left(\int_{\mathbb{R}^n \setminus E} |(\tilde{w}(x)| dx\right)^q dy\right)^{1/q} \right\}$$

$$= M_0 [J_4 + J_5],$$

where E is the set of Lemma 2.2. By Lemma 2.2, a change of variable and Theorems 1.9 and 1.10

$$J_5 \leq M \left(\int_0^\infty y^{-q/q_0} \left(\int_{\mathbb{R}^n} |u(x)| dx\right)^q dy\right)^{1/q}$$

$$\leq M \left(\int_0^\infty y^{-q/q_0} \left(\int_0^{y^\sigma} f^*(t)dt\right)^q dy\right)^{1/q}$$

$$= M\sigma^{-1/q} \left(\int_0^\infty y^{q(1/p-1)-1} \left(\int_0^y f^*(t)dt\right)^q dy\right)^{1/q}$$

$$\leq M\sigma^{-1/q} (q/q(1-1/p)) \left(\int_0^\infty f^{*q}(t) t^{q/p-1} dt\right)^{1/q}$$

$$\leq M\sigma^{-1/q} (p/(p-1)) \left(\int_0^\infty f^{*p}(t) dt\right)^{1/p}.$$

Finally, with $1<r<p$, by Hölder's inequality, the Riesz inequality and Lemma 2.1, (vi) and (v), we have

$$J_4 = \{\int_0^\infty y^{-q/q_0}(\int_E |(\tilde{w})(x)|dx)^q dy\}^{1/q}$$

$$\leq \{\int_0^\infty y^{-q/q_0}(\int_E |(\tilde{w})(x)|^r dx)^{q/r}(|E|)^{q/r'} dy\}^{1/q}$$

$$\leq M_r A^{n/r'}\{\int_0^\infty y^{-q/q_0}(\int_0^\infty w^{*r}(t)dt)^{q/r}(\sum_{k=1}^\infty |Q_k|)^{q/r'} dy\}^{1/q}$$

$$\leq M_r A^{n/r'}\{\int_0^\infty y^{-q/q_0}(\int_0^\infty w^{*r}(t)dt)^{q/r} f^{**-rq/r'}(y^\sigma)(\int_0^\infty u'^{*r}(x)dx)^{q/r'} dy\}^{1/q}$$

$$\leq M_r A^{n/r'} 2^{(r+1)/r}\{\int_0^\infty y^{-q/q_0}(\int_0^\infty u'^{*r}(x)dx)^q f^{**q(1-r)}(y^\sigma)dy\}^{1/q}$$

$$\leq M_r A^{n/r'} 2^{(r+1)/r}\sigma^{-1/q}\{\int_0^\infty y^{1/\sigma -1 - q/q_0\sigma + q(r-1)}\frac{(\int_0^y f^{*r}(t)dt)^q}{(\int_0^y f^*(x)dx)^{q(r-1)}}dy\}^{1/q}.$$

If we take $r=1+1/s<1+1/p'$, i.e. $0<1/s<1/p'<1$, then $1+1/p'=1+(p-1)/p =2-1/p<p$, so that $1<r<p$. Also, $s>p'=p/(p-1)$, hence $ps>p+s$ or $p>s/(s-1) =s'$, that is $q/s'>1$. Therefore, the last expression above is, by Hölder's inequality, Theorems 1.9 and 1.10, dominated by

$$M_r A^{n/r'} 2^{(r+1)/r}\sigma^{-1/q}\{\int_0^\infty y^{q(1/p + 1/s -1) -1}\frac{(\int_0^y f^*(t)f^{*1/s}(t)dt)^q}{(\int_0^y f^*(x)dx)^{q/s}} dy\}^{1/q}$$

$$\leq M_r A^{n/r'} 2^{(r+1)/r}\sigma^{-1/q}\{\int_0^\infty y^{q(1/p + 1/s -1) -1}(\int_0^y f^{*s'}(t)dt)^{q/s'}$$

$$\times \frac{(\int_0^y f^*(t)dt)^{q/s}}{(\int_0^y f^*(x)dx)^{q/s}} dy\}^{1/q}$$

$$= M_r A^{n/r'} 2^{(r+1)/r}\sigma^{-1/q}\{\int_0^\infty y^{q(1/p + 1/s -1) -1}(\int_0^y f^{*s'}(t)dt)^{q/s'} dy\}^{1/q}$$

$$\leq M_r A^{n/r'} 2^{(r+1)/r}\sigma^{-1/q}\{p/(p-s')\}^{1/s'}\{\int_0^\infty y^{q(1/p + 1/s -1) -1}$$

$$\times (yf^{*s'}(y))^{q/s'} dy\}^{1/q}$$

$$= M_r A^{n/r'} 2^{(r+1)/r} \sigma^{-1/q} [p/(p-s')]^{1/s'} \{\int_0^\infty y^{q/p-1} f^{*q}(y)dy\}^{1/q}$$

$$\leq M M_r A^{n/r'} 2^{(r+1)/r} \sigma^{-1/q} [p/(p-s')]^{1/s'} \{\int_0^\infty f^{*p}(t)dt\}^{1/p}.$$

Collecting terms, we have for $q \geq p_1$,

$$\|T(\tilde{f})\|_q \leq 3^{1/q} \{M_1 M_{p_1} M ([(\frac{q_1 q_0 (p_0-p_1)}{p_1 p_0 (q_0-q_1)})^{p_1/q} + (q_1/(q_1-q))^{p_1/q}]^{1/p_1}$$

$$+ (p/(p-1)) \quad M_1 M_{p_1} M 2^{n/rp_1'} (\frac{q_1 q_0 (p_0-p_1)}{p_1 p_0 (q_0-q_1)})^{1/p}$$

$$+ M_0 \quad M ((\frac{q_1 q_0 (p_0-p_1)}{p_1 p_0 (q_0-q_1)})^{-1/q} (p')$$

$$+ M_r A^{n/r'} 2^{(r+1)/r} (\frac{q_1 q_0 (p_0-p_1)}{p_1 p_0 (q_0-q_1)})^{-1/q} (\frac{p}{p-s'})^{1/s'})\} \|f\|_p,$$

and if $q < p_1$,

$$\|T(\tilde{f})\|_q \leq 3^{1/q} \{M_1 M_{p_1} M [\frac{q_1 q_0 (p_0-p_1)}{p_1 p_0 (q_0-q_1)} + (q_1/(q_1-q))]^{1/q}$$

$$+ M M_1 M_{p_1} 2^{n/rp_1'} (\frac{q_1 q_0 (p_0-p_1)}{p_1 p_0 (q_0-q_1)})^{1/p} (p/(p-1))$$

$$+ M_0 (\frac{q_1 q_0 (p_0-p_1)}{p_1 p_0 (q_0-q_1)})^{-1/q} (M(p') + M_r A^{n/r'} 2^{(r+1)/r} (p/(p-s'))^{1/s'})\} \|f\|_p.$$

If $q_1 < q_0$, the theorem is proved in a similar manner, except now $\sigma < 0$, and in this case we apply the dual forms of Theorems 1.10 and 1.11. This completes the proof.

The previous theorem extends to weighted inequalities. The weight functions satisfy the following:

Definition 2.4: Let W be a non-negative function defined on $(0,\infty)$. For $\alpha > 0$, $W \in \mathcal{W}(\alpha)$ if W is non-increasing and if it satisfies

$$\int_0^x t^{\alpha-1} W(t)dt \leq c x^\alpha W(x).$$

If $\alpha < 0$, we write $W \in \mathcal{W}_0(\alpha)$ if W is non-decreasing and

$$\int_x^\infty t^{\alpha-1}W(t)dt \leq Mx^\alpha W(x).$$

The main theorem of this chapter is now given. σ is the slope of the line through the points $(1/p_0, 1/q_0)$ and $(1/p_1, 1/q_1)$, that is

$$\sigma = \frac{1/q - 1/q_0}{1/p - 1/p_0} = \frac{1/q - 1/q_1}{1/p - 1/p_1} = \frac{1/q_0 - 1/q_1}{1/p_0 - 1/p_1}.$$

Theorem 2.5: *Let* $1 \leq p_i \leq q_i < \infty$, $q_0 \neq q_1$, $p_0 < p_1$, $i=0,1$. Suppose that a *quasilinear operator T, acts from* H^{p_i} *to* μ*-measurable functions with*

$$(2) \quad \sup_{t>0} t^{1/q_i}(T\phi)^*(t) \leq M_i \|\phi\|_{H^{p_i}}$$

for all $\phi \in H^{p_i}$, $i=0,1$. *Let* $0<\alpha<1$ *and* $1/p = (1-\alpha)/p_0 + \alpha/p_1$, $1/q = (1-\alpha)/q_0 + \alpha/q_1$. *Then T satisfies*

$$\{\int_0^\infty [W(t)(T\tilde{f})^*(t)]^q dt\}^{1/q} \leq M\{\int_0^\infty [W(t^{1/\sigma})f^*(t)]^p dt\}^{1/p},$$

where $W \in W^0(1/q - 1/q_1)$ *if* $q_0 < q_1$, *or* $W \in W_0(1/q - 1/q_1)$ *if* $q_1 < q_0$.

Proof: As in the previous theorem, we consider only sublinear operators, and assume $p_0 = 1$. As before, we write $\|\phi\|_{H^1} = \|\tilde{\phi}\|_1$.

We first consider the case $q_0 < q_1$, and again $\sigma > 0$, σ as defined above. Let $f = u + u'$ where

$$u(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq f^*(y^\sigma),\ y>0, \\ 0 & \text{otherwise} \end{cases}$$

and $u' = f - u$. Thus

$$u^*(t) \leq \begin{cases} f^*(y^\sigma) & 0<t\leq y^\sigma \\ f^*(t) & y^\sigma \leq t \end{cases}$$

and

$$u'^*(t) \leq \begin{cases} f^*(t) & \text{if } 0<t\leq y^\sigma. \\ 0 & y^\sigma<t \end{cases}$$

Let $u'(x)=v(x)+w(x)$, where v and w are defined as in Lemma 2.1 with y replaced by $f^{**}(y^\sigma)$. Since $u'^*(t)\leq v^*(t/2)+w^*(t/2)$, the sublinearity of T, a change of variable and Minkowski's inequality yields

$$\{\int_0^\infty [W(t)(T(\check{f}))^*(t)]^q dt\}^{1/q} \leq M\{[\int_0^\infty W^q(t)(T(\tilde{u}))^{*q}(t)dt]^{1/q}$$

$$+ [\int_0^\infty W^q(t)(T(\tilde{v}))^{*q}(t)dt]^{1/q} + [\int_0^\infty W^q(t)(T(\tilde{w}))^{*q}(t)dt]^{1/q}\}$$

$$\equiv M\{I_1 + I_2 + I_3\}.$$

By (2) and the Riesz inequality

$$I_1^{p_1} = \{\int_0^\infty W^q(x)(T(\tilde{u}))^{*q}(x)dx\}^{p_1/q}$$

$$\leq M_1^{p_1}\{\int_0^\infty W^q(x)x^{-q/q_1}\|\tilde{u}\|_{p_1}^q dx\}^{p_1/q}$$

$$\leq M_1^{p_1}M_{p_1}^{p_1}\{\int_0^\infty W^q(x)x^{-q/q_1}(\int_0^\infty u^{*p_1}(t)dt)^{q/p_1}dx\}^{p_1/q}.$$

If $q\geq p_1$, apply Minkowski's inequality to obtain

$$I_1^{p_1} \leq M_1^{p_1}M_{p_1}^{p_1}\{[\int_0^\infty W^q(y)y^{-q/q_1}(\int_0^{y^\sigma} u^{*p_1}(t)dt)^{q/p_1}dy]^{p_1/q}$$

$$+ [\int_0^\infty W^q(y)y^{-q/q_1}(\int_{y^\sigma}^\infty u^{*p_1}(t)dt)^{q/p_1}dy]^{p_1/q}\}$$

$$= M_1^{p_1}M_{p_1}^{p_1}[J_0 + J_1].$$

By a change of variable and Theorem 1.10

$$J_0 \leq \{\int_0^\infty W^q(y)y^{-q/q_1}(f^{*p_1}(y^\sigma)y^\sigma)^{q/p_1}dy\}^{p_1/q}$$

$$= \sigma^{-p_1/q}\{\int_0^\infty W^q(x^{1/\sigma})x^{q/p-1}f^{*q}(x)dx\}^{p_1/q}$$

$$\leq \sigma^{-p_1/q}\{\int_0^\infty W^p(x^{1/\sigma}) f^{*p}(x)dx\}^{p_1/p} M^{p_1}$$

so that

$$J_0 \leq \sigma^{-p_1/q}\{\int_0^\infty W^p(x^{1/\sigma}) f^{*p}(x)dx\}^{p_1/p} M^{p_1}$$

Next, by Theorems 1.11 and 1.10, the fact that $W \in \bar{W}^0(1/q - 1/q_1)$ and then Theorem 1.10 again, we have

$$J_1^{q/p_1} \leq \int_0^\infty W^q(x) x^{-q/q_1} (\int_{x^\sigma}^\infty f^{*p_1}(t)dt)^{q/p_1} dx$$

$$\leq \{\int_0^\infty f^{*p_1}(t) (\int_0^{t^{1/\sigma}} x^{-q/q_1} W^q(x)dx)^{p_1/q} dt\}^{q/p_1}$$

$$\leq M^q \{\int_0^\infty f^{*p_1}(t) (\int_0^{t^{1/\sigma}} x^{1/q - 1/q_1 - 1} W(x)dx)^{p_1} dt\}^{q/p_1}$$

$$\leq M^q\{\int_0^\infty f^{*p_1}(t) t^{1/\sigma(1/q - 1/q_1)p_1} W^{p_1}(t^{1/\sigma})dt\}^{q/p_1}$$

$$= M^q\{\int_0^\infty t^{p_1/p - 1} W^{p_1}(t^{1/\sigma}) f^{*p_1}(t)dt\}^{q/p_1}$$

$$\leq M^q\{\int_0^\infty W^p(t^{1/\sigma}) f^{*p}(t)dt\}^{q/p}.$$

Hence,

$$I_1 \leq MM_1 M_{p_1}\{\sigma^{-p_1/q} + 1\}^{1/p_1} (\int_0^\infty W^p(t^{1/\sigma}) f^{*p}(t)dt)^{1/p}.$$

If $q < p_1$, then

$$I_1^q \leq M_1^q M_{p_1}^q \{\int_0^\infty y^{-q/q_1} W^q(y) (\int_0^{y^\sigma} u^{*p_1}(t)dt)^{q/p_1} dy$$

$$+ \int_0^\infty y^{-q/q_1} W^q(y) (\int_{y^\sigma}^\infty u^{*p_1}(t)dt)^{q/p_1} dy\}$$

$$= M_1^q M_{p_1}^q (J_0^{q/p_1} + J_1^{q/p_1})$$

and the estimate is obtained as in the previous case, and the constant is similar to that above.

Next, we estimate I_2. By Theorem 1.10, (2), the Riesz inequality, Lemma 2.1 and Theorem 1.9, it follows that

$$I_2 = (\int_0^\infty w^q(x)(T(\tilde{v}))^{*q}(x)dx)^{1/q}$$

$$\lesssim M(\int_0^\infty w^p(x)(T(\tilde{v}))^{*p}(x)x^{p/q-1}dx)^{1/p}$$

$$\lesssim MM_1(\int_0^\infty w^p(x)x^{p/q-1-p/q_1}\|\tilde{v}\|_{p_1}^p dx)^{1/p}$$

$$\lesssim MM_1M_{p_1}(\int_0^\infty w^p(x)x^{p/q-p/q_1-1}\|v\|_{p_1}^p dx)^{1/p}$$

$$= MM_1M_{p_1}(\int_0^\infty w^p(x)x^{p/q-p/q_1-1}(\int_{\mathbb{R}^n}|v(t)|^{p_1-1}|v(t)|dt)^{p/p_1}dx)^{1/p}$$

$$\lesssim MM_1M_{p_1}2^{n(1-1/p_1)}(\int_0^\infty w^p(y)y^{p(1/q-1/q_1)-1}f^{**(1-1/p_1)p}(y^\sigma)$$

$$\times(\int_{\mathbb{R}^n}|v(t)|dt)^{p/p_1}dy)^{1/p}$$

$$\lesssim MM_1M_{p_1}2^{n(1-1/p_1)}(\int_0^\infty w^p(y)y^{p(1/q-1/q_1)-1}f^{**p(1-1/p_1)}(y^\sigma)$$

$$\times(\int_{\mathbb{R}^n}|u(t)|dt)^{p/p_1}dy)^{1/p}$$

$$\lesssim MM_1M_{p_1}2^{n(1-1/p_1)}(\int_0^\infty w^p(y)y^{p(1/q-1/q_1)-1-\sigma p(1-1/p_1)}$$

$$\times(\int_0^{y^\sigma}f^*(t)dt)^p dy)^{1/p}$$

$$= MM_1M_{p_1}2^{n(1-1/p_1)}\sigma^{-1/p}(\int_0^\infty w^p(x^{1/\sigma})x^{p(1/p-1/p_1)}\cdot 1/\sigma$$

$$\times x^{-p(1-1/p_1)}(\int_0^x f^*(t)dt)^p x^{1/\sigma-1}dx)^{1/p}$$

$$= MM_1M_{p_1}2^{n(1-1/p_1)}\sigma^{-1/p}(\int_0^\infty w^p(x^{1/\sigma})x^{-p}(\int_0^x f^*(t)dt)^p dx)^{1/p}$$

$$\lesssim MM_1M_{p_1}2^{n(1-1/p_1)}\sigma^{-1/p}(p/(p-1))(\int_0^\infty w^p(x^{1/\sigma})f^{*p}(x)dx)^{1/p}.$$

Finally, to estimate I_3, apply (2), Lemma 2.2 and Minkowski's inequality to obtain

$$I_3 = (\int_0^\infty (T(\tilde{w}))^{*q}(x)W^q(x)dx)^{1/q}$$

$$\leq M_0(\int_0^\infty W^q(x)x^{-q/q_0}\|\tilde{w}\|_1^q dx)^{1/q}$$

$$\leq M_0[(\int_0^\infty W^q(x)x^{-q/q_0} \quad (\int_E|\tilde{w}(t)|dt)^q dx)^{1/q}$$

$$+ (\int_0^\infty W^q(x)x^{-q/q_0}(\int_{\mathbb{R}^n\setminus E}|\tilde{w}(t)|dt)^q dx)^{1/q}]$$

$$= M_0[J_2 + J_3],$$

where E is as in Lemma 2.2.

By Lemma 2.2, a change of variable, Theorems 1.9 and 1.10

$$J_3 \leq M(\int_0^\infty y^{-q/q_0}W^q(y)(\int_{\mathbb{R}^n}|u(t)|dt)^q dy)^{1/q}$$

$$\leq M(\int_0^\infty y^{-q/q_0}W^q(y)(\int_0^{y^\sigma} f^*(t)dt)^q dy)^{1/q}$$

$$\leq M\sigma^{-1/q}(\int_0^\infty x^{q(1/p-1)-1}W^q(x^{1/\sigma})(\int_0^x f^*(t)dt)^q dx)^{1/q}$$

$$\leq M\sigma^{-1/q}(p/(p-1))(\int_0^\infty x^{q(1/p-1)-1}W^q(x^{1/\sigma})x^q f^{*q}(x)dx)^{1/q}$$

$$\leq M\sigma^{-1/q}(p/(p-1))(\int_0^\infty W^p(x^{1/\sigma})f^{*p}(x)dx)^{1/p}.$$

Choose r, such that, $1<r<p$. Then, by Hölder's inequality, the Riesz inequality, Lemma 2.1, (vi) and (v) and a change of variable, we have

$$J_2 = \{\int_0^\infty W^q(x)x^{-q/q_0}(\int_E|\tilde{w}(t)|dt)^q dx\}^{1/q}$$

$$\leq \{\int_0^\infty W^q(x)x^{-q/q_0}|E|^{q/r'}(\int_E|\tilde{w}(t)|^r dt)^{q/r}dx\}^{1/q}$$

$$\leq M_r A^{n/r'} \{\int_0^\infty W^q(x) x^{-q/q_0} (\sum_{k=1}^{\infty} |Q_k|)^{q/r'} (\int_0^\infty w_*^r(t)dt)^{q/r} dx\}^{1/q}$$

$$\leq M_r A^{n/r'} \{\int_0^\infty W^q(y) y^{-q/q_0} (\int_0^\infty w_*^r(t)dt)^{q/r} f^{**q(1-r)}(y^\sigma) (\int_0^\infty u'^{*r}(x)dx)^{q/r'} dy\}^{1/q}$$

$$\leq M_r A^{n/r'} 2^{(r+1)/r} \{\int_0^\infty y^{-q/q_0} W^q(y) (\int_0^\infty u'^{*r}(t)dt)^q f^{**q(1-r)}(y^\sigma) dy\}^{1/q}$$

$$\leq M_r A^{n/r'} 2^{(r+1)/r} \sigma^{-1/q} \{\int_0^\infty x^{1/\sigma - 1 - q/q_0\sigma + q(r-1)} W^q(x^{1/\sigma}) \frac{(\int_0^y f^{*r}(t)dt)^q}{(\int_0^y f^*(t)dt)^{q(r-1)}} dx\}^{1/q}.$$

If we take $r=1+1/s<1+1/p'$ as before, so that $1<r<p$, $q/s'>1$, then the last expression above is, by Hölder's inequality and Theorems 1.9 and 1.10,

$$M_r A^{n/r'} 2^{(r+1)/2} \sigma^{-1/q} \{\int_0^\infty W^q(x^{1/\sigma}) x^{q(1/p + 1/s - 1) - 1} \frac{(\int_0^x f^*(t) f^{*1/s}(t)dt)^q}{(\int_0^x f^*(t)dt)^{q/s}} dx\}^{1/q}$$

$$\leq M_r A^{n/r'} 2^{(r+1)/r} \sigma^{-1/q} \{\int_0^\infty W^q(x^{1/\sigma}) x^{q(1/p + 1/s - 1) - 1} \frac{(\int_0^x f^{*s'}(t)dt)^{q/s'}}{(\int_0^x f^*(t)dt)^{q/s}}$$

$$\times (\int_0^x f^*(t)dt)^{q/s} dx\}^{1/q}$$

$$= M_r A^{n/r'} 2^{(r+1)/r} \sigma^{-1/q} \{\int_0^\infty W^q(x^{1/\sigma}) x^{q(1/p + 1/s - 1) - 1} (\int_0^x f^{*s'}(t)dt)^{q/s'} dx\}^{1/q}$$

$$\leq M_r A^{n/r'} 2^{(r+1)/r} \sigma^{-1/q} (p/(p-s'))^{1/s'} \{\int_0^\infty W^q(x^{1/\sigma}) x^{q(1/p + 1/s - 1) - 1}$$

$$\times (x f^{*s'}(x))^{q/s'} dx\}^{1/q}$$

$$= M_r A^{n/r'} 2^{(r+1)/r} \sigma^{-1/q} (p/(p-s'))^{1/s'} \{\int_0^\infty W^q(x^{1/\sigma}) x^{q/p - 1} f^{*q}(x) dx\}^{1/q}$$

$$\leq M_r A^{n/r'} 2^{(r+1)/r} \sigma^{-1/q} (p/(p-s'))^{1/s'} (\int_0^\infty W^p(x^{1/\sigma}) f^{*p}(x) dx)^{1/p},$$

where s' is the conjugate index of s. Thus

$$I_3 \leq M_0 (M \sigma^{-1/q} (p/(p-1)) + M_r A^{n/r'} 2^{(r+1)/r} \sigma^{-1/q} (p/(p-s'))^{1/s'})$$

$$\times (\int_0^\infty W^p(x^{1/\sigma}) f^{*p}(x) dx)^{1/p},$$

so that

$$\left(\int_0^\infty [W(x)(T(\tilde{f}))^*(x)]^q dx\right)^{1/q} \leq M\left(\int_0^\infty W^p(x^{1/\sigma}) f^{*p}(x)dx\right)^{1/p},$$

where M has the same form as that in Theorem 2.3.

If $q_1<q_0$ and $p_0=1$, the theorem is proved in a similar manner, except now $\sigma<0$, $W\varepsilon W_0(1/q - 1/q_1)$ and the dual forms of Theorems 1.9 and 1.11 are applied. If $p_0>1$, and $q_1>q_0$, then by Theorem 1.10 and the Riesz inequality,

$$\left(\int_0^\infty (W(x)(T(\tilde{f}))^*(x))^q dx\right)^{1/q} \leq 2^{1/q}\left[\left(\int_0^\infty [W(x)(T(\tilde{u}))^*(x)]^q dx\right)^{1/q} + \left(\int_0^\infty (T(\tilde{u}'))^{*q}(x)W^q(x)dx\right)^{1/q}\right]$$

$$\leq M\left[\left(\int_0^\infty W^p(x)x^{p/q-1}(T(\tilde{u}))^{*p}(x)dx\right)^{1/p} + \left(\int_0^\infty W^p(x)x^{p/q-1}(T(\tilde{u}'))^{*p}(x)dx\right)^{1/p}\right]$$

$$\leq M\left[M_1\left(\int_0^\infty W^p(x)x^{p/q-1-p/q_1}\|\tilde{u}\|_{p_1}^p dx\right)^{1/p} + M_0\left(\int_0^\infty W^p(x)x^{p/q-1-p/q_0}\|\tilde{u}'\|_{p_0}^p dx\right)^{1/p}\right]$$

$$\leq M\left[M_1M_{p_1}\left(\int_0^\infty W^p(x)x^{p/q-p/q_1-1}\|u\|_{p_1}^p dx\right)^{1/p} + M_0M_{p_0}\left(\int_0^\infty x^{p/q-p/q_0-1}W(x)\|u'\|_{p_0}^p dx\right)^{1/p}\right]$$

and the remainder of the proof is as above and hence omitted. If $p_0>1$, $q_1<q_0$ then

$$\left(\int_0^\infty [W(x)(T(\tilde{f}))^*(x)]^q dx\right)^{1/q} \leq 2^{1/q}\left\{\left(\int_0^\infty W^q(x)x^{-q/q_1}M_1^q\|\tilde{u}'\|_{p_1}^q dx\right)^{1/q} + \left(\int_0^\infty W^q(x)x^{-q/q_0}M_0^q\|\tilde{u}\|_{p_0}^q dx\right)^{1/q}\right\}$$

$$\leq 2^{1/q}\left\{\left(\int_0^\infty W^q(x)x^{-q/q_1}M_1^qM_{p_1}^q\|u'\|_{p_1}^q dx\right)^{1/q} + \left(\int_0^\infty W^q(x)x^{-q/q_0}M_0^qM_{p_0}^q\|u\|_{p_0}^q dx\right)^{1/q}\right\}$$

and the proof proceeds as above (for details, see [20]).

§2.3 Applications.

As a corollary to Theorem 2.5 we have the following extension to L(p,q) spaces.

Corollary 2.6: Under the conditions of Theorem 2.5 and setting $1/r=\beta/p_1 + (1-\beta)/p$, $1/s=\beta/q_1 + (1-\beta)/q$, $0\leq\beta<1$, *then*

$$\|T(\tilde{f})\|^*_{s,q} \leq M\|f\|^*_{r,p}.$$

Proof: Assume $q_0<q_1$. The case $q_1<q_0$ is similar and hence omitted. If $W(t)=t^\tau$, $\tau=\beta/q_1 - \beta/q$, $0\leq\beta<1$, then $W\in W^0(1/q - 1/q_1)$ and

$$\left(\int_0^\infty (W(t)(T(\tilde{f}))^*(t))^q dt\right)^{1/q} = \left(\int_0^\infty (t^{\beta/q_1 + (1-\beta)/q}(T(\tilde{f}))^*(t))^q \frac{dt}{t}\right)^{1/q}$$

$$\leq M\left(\int_0^\infty (t^{1/\sigma(\beta/q_1 \quad -\beta/q)}f^*(t))^p dt\right)^{1/p}$$

$$= M\left(\int_0^\infty (t^{\beta/p_1 - \beta/p + 1/p}f^*(t))^p \frac{dt}{t}\right)^{1/p}$$

$$= M\left(\int_0^\infty (t^{1/r}f^*(t))^p \frac{dt}{t}\right)^{1/p}$$

$$= M\|f\|^*_{r,p}.$$

Since

$$\|T(\tilde{f})\|^*_{s,q} = \left(\int_0^\infty (t^{1/s}(T(\tilde{f}))^*(t))^q \frac{dt}{t}\right)^{1/q}$$

$$= \left(\int_0^\infty (t^{\beta/q_1 + (1-\beta)/q - 1/q}(T(\tilde{f}))^*(t))^q dt\right)^{1/q}$$

$$= \left(\int_0^\infty (W(t)(T(\tilde{f}))^*(t))^q dt\right)^{1/q}$$

the result follows.

Corollary 2.7: *Let $f\in L^p[-\pi,\pi]$, $1<p<2$, and g be the Littlewood-Paley g-function of f. If $W\in W^0(1/p - 1/2)$ then*

$$\{\int_0^\infty [W(t)g^*(t)]^p dt\}^{1/p} \leqslant M\{\int_0^\infty [W(t)f^*(t)]^p dt\}^{1/p}.$$

Proof: From §2.1 we know that the map $T(f)=g$ satisfies (1) of §2.2 for $p_0=q_0=1$ and $p_1=q_1=2$, so that the result follows from Theorem 2.5.

Corollary 2.8: *Let $0\leqslant\beta<1$ and $1<p<2$. If $1/r=(1-\beta)/p+\beta/2$ then for the Littlewood-Paley g-function*

$$\|g\|^*_{r,p} \leqslant M\|f\|^*_{r,p}.$$

It is clear from Corollary 2.8 that for $\beta=0$, we have the usual Littlewood-Paley inequality

$$\|g\|_p \leqslant M\|f\|_p, \quad 1<p<2.$$

As a final example, let $f\sim \sum_{k=-\infty}^{\infty} a_k e^{ikx}$ and suppose $h(x)$ is any integral valued measurable function. Define $(Tf)(x) = S_{h(x)}(x)$.

Corollary 2.9: *With notation as in Corollary 2.8, we have*

$$\|\sup_{k>0} |S_k(x)/(\log(k+2))^{1/p}|\|^*_{r,p} \leqslant M\|f\|^*_{r,p}$$

for all $f\in L(r,p)$.

Proof: (Zygmund [80 Vol. II, Chapter XIII, p.ps 161-167). The operator T satisfies

$$\|T\phi\|_{1,\mu} \leqslant M\|\phi\|_{H^1}$$

for $d\mu(x) = \frac{dx}{\log(|h(x)|+2)}$ and is of strong type (2,2) for $f \in L^2$, that is

$$\|Tf\|_{2,\mu} \leq M\|f\|_2.$$

Thus

$$\|S_{h(\cdot)}(\cdot)\|^*_{r,p,\mu} \leq M\|f\|^*_{r,p,\lambda},$$

where μ and λ indicate that the rearrangements are with respect to $d\mu$ and the Lebesgue measure, respectively. Since h is arbitrary, the result follows.

If in Corollary 2.9 we set $\beta=0$, then

$$\int_0^\infty \left(\sup_{k>0} |S_k(\cdot)| / (\log(k+2))^{1/p}\right)^{*p}(x)\,dx = \int_{-\pi}^{\pi} \sup_{k>0} \frac{|S_k(x)|^p}{\log(k+2)}\,dx$$

and the usual Littlewood-Paley type inequality holds, i.e., for any $f \in L^p$,

$$\left\|\sup_{k>0} \frac{|S_k(\cdot)|}{(\log(k+2))^{1/p}}\right\|_p \leq M\|f\|_p$$

(see Zygmund [80], Vol II p.167).

CHAPTER III

§3.1 Introduction.

Corresponding to each Orlicz space X, there exist unique Lorentz spaces $\Lambda(X)$ and $M(X)$, such that, $\Lambda(X) \subsetneq X \subsetneq M(X)$. Alternately, for each concave function ϕ_X, consider the Lorentz spaces Λ and M with fundamental function ϕ_X. There exists a unique Orlicz space L_X whose fundamental function is ϕ_X and so $\Lambda(X) \subsetneq L_X \subsetneq M(X)$. Sharpley [63], raised the following question: Is L_X an interpolation space for $\Lambda(X)$ and $M(X)$, that is, is $L_X = L_\phi$, where $\phi(s) = (\phi_X^{-1}(1/s))^{-1}$? This result would generalize classical fractional integration theorems for $L(p,q)$ and Orlicz spaces simultaneously. Torchinsky [75] and Sharpley [64] have obtained interpolation theorems in this direction, and in this chapter, these results are generalized to certain weighted estimates.

In order to define what is meant by weak type for general function spaces, we need the definition of a rearrangement invariant space. Throughout, the measure space $(\mathfrak{M},\mu)$ is assumed σ-finite, positive and atom free.

Definition 3.1: *Let $(X, \|\cdot\|_X)$ be a Banach or a complete metric space of complex valued functions on $(\mathfrak{M},\mu)$. X is called a function space if*

(i) $f \in X$, g measurable with $|g(t)| \leq |f(t)|$, almost everywhere, then

$g\varepsilon X$ and $\|g\|_X\leq\|f\|_X$;

and

(ii) $\{f_n\}_{n=1}^{\infty}\subset X$ *with* $0\leq f_n\leq f_{n+1}$ *for all* $n\in\mathbb{N}$, *such that,* f_n *converges almost everywhere to* f, *then either* $f\varepsilon X$ *and* $\|f_n\|_X$ *converges to* $\|f\|_X$, *or* $\|f_n\|_X$ *increases to infinity (the Fatou property).*

<u>Definition 3.2</u>: *Let X be a function space containing the characteristic function of any measurable set E, such that* $\mu(E)<\infty$. *X is rearrangement invariant (r.i.) if there exists an (equivalent) function space Z of Lebesgue measurable functions, defined on the interval* $[0,l)$, $0<l\leq\infty$, *with the property that for every* $g\varepsilon Z$, $g^*\varepsilon Z$ *and* $\|g\|_Z=\|g^*\|_Z$, *then* $f\varepsilon X$ *implies* $f^*\varepsilon Z$ *and* $\|f\|_X=\|f^*\|_Z$.

<u>Example 1</u>: Let $X=L^p_\mu(\mathbb{R}^n)$, $0<p\leq\infty$, then $Z=L^p(\mathbb{R}^+,dt)$.

<u>Example 2</u>: If $L_\mu(p,q)$ denotes the Lorentz space on $(\mathfrak{M},\mu)$, then $Z=L(p,q)$, the Lorentz space on $(\mathbb{R}^+,dt)$.

We consider only the case $l=\infty$ in Definition 3.2, while pointing out that the results below still hold if $l<\infty$.

§3.2 Fundamental functions.

In developing an abstract theory of interpolation, it is necessary to find some quantity that characterizes the spaces involved. For example, there is the Peetre K-functional as defined in the Introduction, and also Boyd's indices for rearrangement invariant spaces [8]. For our purposes, we need what is generally called the fundamental function, or scale function, of an r.i. space.

The complete metric spaces considered are those Lorentz spaces $L(p,q)$, such that, $0<p<1$ and $0<q\leq\infty$ or $p=1$ and either $0<q<1$ or $1<q\leq\infty$. From Hunt [26], we know that these spaces are metrizable, with metric generating the same topology as $\|\cdot\|^*_{p,q}$.

Definition: Let X be an r.i. Banach space, and let $\chi_{[0,t)}$ *be the characteristic function of the interval* $[0,t)$. *Then the fundamental function* ϕ_X *of X is defined by*

$$\phi_X(t) = \|\chi_{[0,t)}\|_Z,$$

where Z is the equivalent space of X.

If X is an L(p,q) space, then it's fundamental function is defined by

$$\phi_X(t) = \|\chi_{[0,t)}\|^*_{p,q}.$$

It is clear that ϕ_X is well defined, since μ is atom free.

Example 1: If $X=L(p,q)$, $0<p\leq\infty$, $0<q\leq\infty$, then $\phi_X(t)=t^{1/p}$, $0<p<\infty$, and $\phi_X(t)=1$ if $p=\infty$.

Example 2: We define the Orlicz spaces as follows: Let C be a convex, non-trivial, left continuous function defined on $[0,\infty]$ with $0\leq C(t)\leq\infty$, $C(0)=0$ and $C(t)/t$ increasing. C is called a Young's function. The inverse of C is given by

$$C^{-1}(t) = \inf\{s: C(s)\geq t\},$$

with the infimum of the empty set equal to infinity. Define the Orlicz space L_C by

$$L_C = \{f: \|f\|_{L_C} = \inf\{K>0: \int_{\mathfrak{M}} C(\frac{|f(x)|}{K})d\mu \leq 1\} < \infty\}.$$

L_C is a Banach space under the norm $\|\cdot\|_{L_C}$.

The fundamental function of L_C is given by

$$\phi_{L_C}(t) = 1/C^{-1}(1/t) \quad [53].$$

To illustrate some of the properties of ϕ_X when X is Banach, it is convenient to introduce the associate space X' of X. This and related spaces are studied, e.g., in Zippin [78], Semenov [62], Luxemburg [42], and Luxemburg and Zaanen [43].

If X is an r.i. Banach space, then the associate X' of X is the collection of all measurable functions g on $\mathbb{R}^+$, such that,

$$\|g\|_{X'} = \sup(|\int_0^\infty f(s)g(s)ds|)$$

is finite, where the supremum is taken over all $f \in Z$, such that, $\|f\|_Z \leq 1$, with Z the equivalent space of X.

From this point on, we suppress the role of Z. It will be understood that when X is an r.i. space and $f \in X$ and we write $f^* \in X$ or $\|f^*\|_X$ it is meant that $f^* \in Z$ and $\|f^*\|_Z$, respectively.

If X is spanned by the simple functions, then Luxemburg [41] has shown that $X'' = (X')' = X$. Also, X' and X'' are r.i. spaces, and if $\phi_{X'}$ is the fundamental function of X' then

$$(1) \quad \phi_X(t)\phi_{X'}(t) = t.$$

ϕ_X is increasing, so that (1) implies ϕ_X is continuous. Since

$$\frac{d}{dt}(\phi_X(t)\phi_{X'}(t)) = 1,$$

then

$$\frac{d}{dt}(\phi_X(t))\phi_{X'}(t) + \phi_X(t)\frac{d}{dt}\phi_{X'}(t) = 1.$$

Multiplying both sides by $\phi_X(t)/t$, and using the fact that $\phi_{X'}$ is non-decreasing, we have

$$(2) \quad \frac{d}{dt}\phi_X(t) \leq \phi_X(t)/t.$$

By (1) and (2), it follows that $\phi_X(t)/t$ is non-increasing and hence, for $0<\alpha<1$,

$$\phi_X(\alpha t)/\phi_X(t) \geq \alpha.$$

For any r.i. Banach space X, with norm $\|\cdot\|_X$, there is an equivalent r.i. norm, say $\|\cdot\|_{X_0}$, such that, ϕ_{X_0} is concave, and for $0<t<\infty$

$$\phi_X(t) \leq \phi_{X_0}(t) \leq \phi_X(2t) \quad [78].$$

§3.3 The Lorentz Λ,M spaces.

In this section, we define the Lorentz spaces Λ and M, and introduce the concept of weak type operators on r.i. spaces.

Definition 3.4: *Let X be an r.i. space with fundamental function ϕ_X with $\phi_X(0)=0$. The Lorentz class $\Lambda(X)$ is the collection of all measurable f, such that,*

$$\|f\|_{\Lambda(X)} = \int_0^\infty f^*(t)\phi_X(t)\frac{dt}{t} < \infty.$$

For any r.i. space X and fundamental function ϕ_X, f belongs to the Lorentz class M(X) if

$$\|f\|_{M(X)} = \sup_{t>0} f^{**}(t)\phi_X(t) < \infty.$$

If $X=L^\infty$, i.e. $\phi_X(t)=1$, then we define $\Lambda(X) = X = M(X)$.

Throughout, we assume that the fundamental functions satisfy either $\phi_X(t)=1$ or

$$(3) \qquad \frac{d}{dt}(\phi_X(t)) \approx \phi_X(t)/t$$

and

$$(4) \qquad \begin{cases} \phi_X(\alpha t)/\phi_X(t) \geq \alpha^r, \text{ for some } r\in\mathbb{R}^+, & 0<\alpha<1 \\ \phi_X(t)/t^s \text{ decreases for some } s\geq 1. \end{cases}$$

If (3) holds and X is an r.i. Banach space, then $\Lambda(X) \subsetneq X \subsetneq M(X)$. Observe that (4) holds with r=1 if X is a Banach space.

Note that if $X=L^p$, $0<p<\infty$, then $\Lambda(L^p)=L(p,1)$ and $M(L^p)=L(p,\infty)$, the "weak" L^p spaces.

Definition 3.5: *For X, Y r.i. spaces, a quasilinear operator T from $\Lambda(X)$ into M(Y) is said to be of weak type (X,Y), if there exists a constant M>0, such that,*

$$\sup_{t>0} (Tf)^*(t)\phi_Y(t) \leq M\|f\|_{\Lambda(X)}.$$

§3.4 Technical lemmas.

Before proving interpolation theorems in the setting of r.i. spaces, some technical lemmas are required. To simplify notation, let X_0, X_1, Y_0, Y_1, X and Y be r.i. spaces and define

$$(5)\quad \eta(t) = \phi_{X_0}(t)/\phi_{X_1}(t),$$

$$(6)\quad \xi(t) = \phi_{Y_0}(t)/\phi_{Y_1}(t),$$

$$(7)\quad \psi(t) = \eta^{-1}(\xi(t)),$$

and assume for Y

$$(8)\quad \phi_Y(t) = \phi_X(\psi(t))\phi_{Y_0}(t)/\phi_{X_0}(\psi(t)).$$

It is assumed throughout the rest of this chapter that C is a non-negative, increasing convex function on $[0,\infty)$ with $C(0)=0$, and that there exists $p>1$, such that $C(t)/t^p$ is non-increasing.

Observe that if $X_i = L^{p_i}$, $Y_i = L^{q_i}$, $i=0,1$ and $X=L^p$, $Y=L^q$ with $p^{-1}=(1-\alpha)p_0^{-1} + \alpha p_1^{-1}$, $q^{-1}= (1-\alpha)q_0^{-1} + \alpha q_1^{-1}$, for $0<\alpha<1$. If σ is the slope of the line between the points $(1/p_0,1/q_0)$ and $(1/p_1,1/q_1)$, then $\eta(t) = t^{1/p_0 - 1/p_1}$, $\xi(t) = t^{1/q_0 - 1/q_1}$, $\psi(t) = t^\sigma$ and

$$\phi_Y(t) = t^{\sigma/p + 1/q_0 - \sigma/p_0} = t^{(1/p - 1/p_0)\sigma + 1/q_0} = t^{1/q - 1/q_0 + 1/q_0}$$
$$= t^{1/q}.$$

Let A and B be Young's functions, and suppose T is a linear operator simultaneously of weak types (p_0,q_0), (p_1,q_1), $0<p_i<q_i<\infty$. If $X=L_A$, then $Y=L_B$, where $\phi_Y(t) = 1/B^{-1}(1/t)$, $\phi_X(t) = 1/A^{-1}(1/t)$, so that

$$\phi_Y(t) = \frac{\phi_X(\psi(t))\phi_{Y_0}(t)}{\phi_{X_0}(\psi(t))} = \phi_X(t^\sigma)t^{1/q_0 - \sigma/p_0}$$

or

$$B^{-1}(t) = A^{-1}(t^\sigma)t^{1/q_0 - \sigma/p_0}.$$

Lemma 3.6: *Let* η, ξ *and* ϕ_X/ϕ_{X_1} *increase from 0 to* ∞*, and suppose* ϕ_X/ϕ_{X_0} *is decreasing. Let* θ_0*,* θ_1*,* $\theta_2 > 0$ *and*

$$(9) \quad \int_0^t \phi_{X_0}(s)/\phi_X(s)\frac{ds}{s} \leq \theta_0\phi_{X_0}(t)/\phi_X(t);$$

$$(10) \quad \int_t^\infty \phi_{X_1}(s)/\phi_X(s)\frac{ds}{s} \leq \theta_1\phi_{X_1}(t)/\phi_X(t);$$

$$(11) \quad \int_0^t W(s)\phi_Y(s)/\phi_{Y_1}(s)\frac{ds}{s} \leq \theta_2 W(t)\phi_Y(t)/\phi_{Y_1}(t)$$

be satisfied for W a non-negative, non-increasing function defined on $\mathbb{R}^+$*. Then*

$$\int_0^t f^*(s)\phi_{X_0}(s)\frac{ds}{s} \quad \text{and} \quad \int_t^\infty f^*(s)\phi_{X_1}(s)\frac{ds}{s}$$

are finite whenever

$$\int_0^\infty W(\psi^{-1}(s))C(f^*(s)\phi_X(s))\frac{ds}{s} < \infty.$$

Proof: ψ^{-1} and C increase, and if $0<y<t$, we have

$$F(t) \equiv \int_0^t W(\psi^{-1}(x))C(f^*(x)\phi_X(x))\frac{dx}{x}$$

$$\geq W(\psi^{-1}(t))\int_{y/2}^{y} C(f^*(x)\phi_X(x))\frac{dx}{x}$$

$$\geq W(\psi^{-1}(t))C(f^*(y)\phi_X(y/2))\ln 2.$$

Hence

$$f^*(y)\phi_X(y/2) \leqslant C^{-1}(F(t)/W(\psi^{-1}(t))\ln 2)$$

and therefore, by (4)

$$(12) \quad f^*(y)\phi_X(y) \leqslant MC^{-1}(F(t)/W(\psi^{-1}(t))\ln 2).$$

Since ϕ_{X_0}/ϕ_{X_1} increases, then (9) and Jensen's inequality for $p>1$ yields

$$\left(\int_0^t f^*(y)\phi_{X_0}(y)\frac{dy}{y}\right)^p \leqslant \theta_0^{p-1}\left(\frac{\phi_{X_0}(t)}{\phi_X(t)}\right)^{p-1}\int_0^t (f^*(y)\phi_X(y))^p \frac{\phi_{X_0}(y)}{\phi_X(y)}\frac{dy}{y}$$

$$\leqslant \theta_0^{p-1}\left(\frac{\phi_{X_0}(t)}{\phi_X(t)}\right)^p\int_0^t (f^*(y)\phi_X(y))^p \frac{dy}{y}.$$

If we define $B(t) = t^p/C(t)$, $t>0$, then B increases, and by (12), it follows that

$$\theta_0^{p-1}\left(\frac{\phi_{X_0}(t)}{\phi_X(t)}\right)^p\int_0^t (f^*(y)\phi_X(y))^p\frac{dy}{y} = \theta_0^{p-1}\left(\frac{\phi_{X_0}(t)}{\phi_X(t)}\right)^p\int_0^t B(f^*(y)\phi_Y(y))$$

$$\times\, C(f^*(y)\phi_X(y))\frac{W(\psi^{-1}(y))}{W(\psi^{-1}(y))}\frac{dy}{y}$$

$$\leqslant \theta_0^{p-1}\left(\frac{\phi_{X_0}(t)}{\phi_X(t)}\right)^p B(MC^{-1}(F(t)/W(\psi^{-1}(t))\ln 2))\int_0^t C(f^*(y)\phi_X(y))\frac{W(\psi^{-1}(y))}{W(\psi^{-1}(y))}\frac{dy}{y}$$

$$\leqslant \theta_0^{p-1}\left(\frac{\phi_{X_0}(t)}{\phi_X(t)}\right)^p B(MC^{-1}(F(t)/W(\psi^{-1}(t))\ln 2))F(t)/W(\psi^{-1}(t)).$$

Since all the quantities on the right hand side are non-zero and finite, then $\int_0^t f^*(y)\phi_{X_0}(y)\frac{dy}{y} < \infty$.

Next, note that (11) implies

$$G(y) \equiv \int_0^{\psi^{-1}(y)} W(z)\phi_Y(z)/\phi_{Y_1}(z)\frac{dz}{z}$$

$$\leqslant \theta_2 W(\psi^{-1}(y))\phi_Y(\psi^{-1}(y))/\phi_{Y_1}(\psi^{-1}(y)).$$

But, since

$$\phi_Y(\psi^{-1}(y)) = \phi_X(y)\phi_{Y_0}(\psi^{-1}(y))/\phi_{X_0}(y)$$

we obtain

$$G(y)\phi_{X_0}(y)\phi_{Y_1}(\psi^{-1}(y))/\phi_{Y_0}(\psi^{-1}(y)) \leq \theta_2 W(\psi^{-1}(y))\phi_X(y).$$

C is convex, so that (10) and Jensen's inequality yield for $\theta_1 \geq 1$:

$$C\left(\frac{\phi_X(t)}{\phi_{X_1}(t)}\int_t^\infty f^*(y)\phi_{X_1}(y)\frac{dy}{y}\right) \leq \frac{\phi_X(t)}{\theta_1\phi_{X_1}(t)}\int_t^\infty C(\theta_1 f^*(y)\phi_X(y))\frac{\phi_{X_1}(y)}{\phi_X(y)}\,\frac{dy}{y}$$

$$\leq \frac{\theta_2\phi_X(t)\theta_1^{p-1}}{\phi_{X_1}(t)}\int_t^\infty \frac{W(\psi^{-1}(y))C(f^*(y)\phi_X(y))\phi_{X_1}(y)\phi_{Y_0}(\psi^{-1}(y))}{\phi_{X_0}(y)\phi_{Y_1}(\psi^{-1}(y))G(y)y}\,dy$$

$$\leq \frac{\theta_2\phi_X(t)\theta_1^{p-1}}{\phi_{X_1}(t)G(t)}\int_t^\infty W(\psi^{-1}(y))C(f^*(y)\phi_X(y))\xi(\psi^{-1}(y))/\eta(y)\,\frac{dy}{y}$$

and, since $\psi^{-1}(y) = (\eta^{-1}(\xi(y)))^{-1} = \xi^{-1}(\eta(y))$, that is $\xi(\psi^{-1}(y))/\eta(y) = 1$, we have

$$\int_t^\infty f^*(y)\phi_{X_1}(y)\frac{dy}{y} \leq \frac{\phi_{X_1}(t)}{\phi_X(t)}\,C^{-1}\left\{\frac{\theta_2\phi_X(t)\theta_1^{p-1}}{\phi_{X_1}(t)G(t)}\int_t^\infty W(\psi^{-1}(y))C(f^*(y)\phi_X(y))\frac{dy}{y}\right\}.$$

This completes the proof of the lemma.

<u>Lemma 3.7:</u> *Suppose η and ϕ_X/ϕ_{X_1} are increasing from 0 to ∞ and ϕ_X/ϕ_{X_0}, ξ (and hence ψ) are decreasing from ∞ to 0. If (9), (10) and, for non-negative, non-decreasing W and constant $\theta_3>0$*

$$(13)\quad \int_t^\infty W(y)\phi_Y(y)/\phi_{Y_1}(y)\,\frac{dy}{y} \leq \theta_3 W(t)\phi_Y(t)/\phi_{Y_1}(t),$$

then

$$\int_0^t f^*(y)\phi_{X_0}(y)\,\frac{dy}{y} \quad \text{and} \quad \int_t^\infty f^*(y)\phi_{X_1}(y)\,\frac{dy}{y}$$

are finite, whenever,

$$\int_0^\infty W(\psi^{-1}(y))C(f^*(y)\phi_X(y))\,\frac{dy}{y} < \infty.$$

Proof: Since ψ decreases, $W(\psi^{-1}(t))$ decreases, so that, for $0<y<t$,

$$F(t) = \int_0^t W(\psi^{-1}(z))C(f^*(z)\phi_X(z))\,\frac{dz}{z}$$

$$\geq W(\psi^{-1}(t))C(f^*(y)\phi_X(y/2))\ln 2.$$

Therefore, the argument of Lemma 3.6 yields $\int_0^t f^*(y)\phi_{X_0}(y)\,\frac{dy}{y} < \infty$.

Note that (13) implies

$$G(y) \equiv \int_{\psi^{-1}(y)}^\infty W(z)\phi_Y(z)/\phi_{Y_1}(z)\,\frac{dz}{z}$$

$$\leq \theta_3 W(\psi^{-1}(y))\phi_X(y)/\phi_{X_1}(y)$$

because

$$\phi_Y(\psi^{-1}(y))/\phi_{Y_1}(\psi^{-1}(y)) = \frac{\phi_X(y)\phi_{Y_0}(\psi^{-1}(y))}{\phi_{X_0}(y)\phi_{Y_1}(\psi^{-1}(y))}$$

$$= \phi_X(y)\xi(\psi^{-1}(y))/\phi_{X_1}(y)\eta(y)$$

$$\doteq \phi_X(y)/\phi_{X_1}(y).$$

Also, since ψ^{-1} decreases, $G(y)$ increases, so that, by (10) and Jensen's inequality $p\geq 1$

$$C\left(\frac{\phi_X(t)}{\phi_{X_1}(t)}\int_t^\infty f^*(y)\phi_{X_1}(y)\,\frac{dy}{y}\right) \leq \frac{\phi_X(t)\,\theta_1^{p-1}}{\phi_{X_1}(t)}\int_t^\infty C(f^*(y)\phi_X(y))\frac{\phi_{X_1}(y)}{\phi_X(y)}\,\frac{dy}{y}$$

$$\leq \frac{\phi_X(t)\theta_1^{p-1}}{\phi_{X_1}(t)G(t)} \int_t^\infty W(\psi^{-1}(y))C(f^*(y)\phi_X(y)) \frac{dy}{y}$$

$$< \infty.$$

It follows from this that

$$\int_t^\infty f^*(y)\phi_{X_1}(y) \frac{dy}{y} < \infty.$$

In this last lemma, we consider concave functions. It s proof may be found in Torchinsky [75, Lemma 3.11].

Lemma 3.8: Suppose f is a decreasing, non-negative function on $\mathbb{R}^+$, *and* ϕ *a fundamental function. If D is concave, increasing on* $[0,\infty)$ *with* $D(0)=0$, *then for* $t>0$,

$$(14) \quad \int_0^t f(y)\phi(y) \frac{dy}{y} \leq MD^{-1}\left(M'\int_0^t D(f(y)\phi(y)) \frac{dy}{y}\right)$$

and

$$(15) \quad \int_t^\infty f(y)\phi(y) \frac{dy}{y} \leq MD^{-1}\left(M'\int_{t/2}^\infty D(f(y)\phi(y)) \frac{dy}{y}\right).$$

The constants M and M' in (14) and (15) are not the same.

§3.5 Interpolation in r.i. spaces.

With the preliminary results of §3.2, 3.3 and 3.4, we can now state and prove our main interpolation theorems.

Theorem 3.9: Let T be a quasi-linear operator, simultaneously of weak types (X_i, Y_i), *with norms* M_i, $i=0,1$. *Suppose* W, ξ, η *and* ψ *are* *as in Lemma 3.6, and (9), (10), (11) and, for* $\theta_4, \theta_5 > 0$,

$$(16) \quad \int_t^\infty \phi_Y(y)/\phi_{Y_0}(y)\,\frac{dy}{y} \leq \theta_4 \phi_Y(t)/\phi_{Y_0}(t)$$

$$(17) \quad \int_0^t \phi_{X_1}(y)\,\frac{dy}{y} \leq \theta_5 \phi_{X_1}(t), \text{ if } X_1 \neq L^\infty,$$

hold. Then

$$\int_0^\infty W(t)C[(Tf)^*(t)\phi_Y(t)]\,\frac{dt}{t} \leq M\int_0^\infty W(\psi^{-1}(t))C[f^*(t)\phi_X(t)]\,\frac{dt}{t},$$

provided the right hand side is finite.

<u>*Proof:*</u> Let $u>0$ and $f=f_u + f^u$, where $f_u=f$ if $|f|\leq u$ and zero otherwise. Then

$$(f^u)^*(t) \leq \begin{cases} f^*(t), & \text{if } t\leq f_*(u) \\ 0, & \text{if } t>f_*(u) \end{cases}$$

and

$$(f_u)^*(t) \leq \begin{cases} u, & \text{if } t\leq f_*(u) \\ f^*(t), & \text{if } t>f_*(u). \end{cases}$$

It follows from Lemma 3.6 that $f^u \varepsilon \Lambda(X_0)$ and $f_u \varepsilon \Lambda(X_1)$, so that $T(f^u)$ and $T(f_u)$ are well-defined.

Since T is of weak type (X_0,Y_0) with norm M_0, we obtain with $u=f^*(\psi(t))$

$$(Tf^u)^*(t)\phi_Y(t) \leq M_0\,\frac{\phi_Y(t)}{\phi_{Y_0}(t)}\int_0^\infty (f^u)^*(y)\phi_{X_0}(y)\,\frac{dy}{y}$$

$$\leq M_0\int_0^{\psi(t)} [f^*(y)\phi_X(y)\phi_Y(t)\phi_{X_0}(y)/\phi_{Y_0}(t)\phi_X(y)]\,\frac{dy}{y}$$

$$\equiv M_0 I(t).$$

By (8)

$$\phi_{X_0}(\psi(t))\phi_Y(t)/\phi_X(\psi(t))\phi_{Y_0}(t) = 1,$$

so that (9) implies

$$\int_0^{\psi(t)} [\phi_{X_0}(y)\phi_Y(t)/\phi_X(y)\phi_{Y_0}(t)] \frac{dt}{t} \leq \theta_0.$$

Hence, Jensen's inequality, an interchange of order of integration, (16) and the fact that $C(t)/t^p$ decreases yields

$$\int_0^\infty W(t)C((Tf^u)^*(t)\phi_Y(t)) \frac{dt}{t}$$

$$\leq \theta_0^{-1}\int_0^\infty W(t)\int_0^{\psi(t)} C(\theta_0 M_0 f^*(y)\phi_X(y))\phi_Y(t)\phi_{X_0}(y)/\phi_{Y_0}(t)\phi_X(y) \frac{dy}{y}\frac{dt}{t}$$

$$= \theta_0^{-1}\int_0^\infty C(\theta_0 M_0 f^*(y)\phi_X(y))\phi_{X_0}(y)/\phi_X(y)\left(\int_{\psi^{-1}(y)}^\infty W(t)\phi_Y(t)/\phi_{Y_0}(t)\frac{dt}{t}\right)\frac{dy}{y}$$

$$\leq \theta_0^{-1}\theta_4\int_0^\infty W(\psi^{-1}(y))C(\theta_0 M_0 f^*(y)\phi_X(y))\phi_{X_0}(y)\phi_Y(\psi^{-1}(y))/\phi_X(y)\phi_{Y_0}(\psi^{-1}(y))\frac{dy}{y}$$

$$= \theta_0^{-1}\theta_4\int_0^\infty W(\psi^{-1}(y))C(\theta_0 M_0 f^*(y)\phi_X(y))\frac{dy}{y}$$

$$\leq \theta_0^{-1}\theta_4(\theta_0 M_0)^p\int_0^\infty W(\psi^{-1}(y))C(f^*(y)\phi_X(Y))\frac{dy}{y},$$

where we have assumed, without loss of generality, that $\theta_0 M_0 \geq 1$.

Next, since T is of weak type (X_1, Y_1) with norm M_1, we can write, for $X_1 \neq L^\infty$,

$$(Tf_u)^*(t)\phi_Y(t) \leq M_1 \frac{\phi_Y(t)}{\phi_{Y_1}(t)}\int_0^\infty f_u^*(y)\phi_{X_1}(y)\frac{dy}{y}$$

$$\leq M_1\left\{\frac{\phi_Y(t)}{\phi_{Y_1}(t)}\cdot\int_0^{\psi(t)} f^*(\psi(t))\phi_{X_1}(y)\frac{dy}{y}\right.$$

$$\left.+ \int_{\psi(t)}^\infty f^*(y)\phi_X(y)\frac{\phi_{X_1}(y)}{\phi_X(y)}\frac{dy}{y}\right\}.$$

$$\equiv M_1[I_1(t) + J(t)].$$

By (17) and (4)

$$I_1(t) \lesssim \theta_5\phi_Y(t)f^*(\psi(t))\phi_{X_1}(\psi(t))/\phi_{Y_1}(t)$$

$$= \theta_5\phi_Y(t)f^*(\psi(t))\phi_{X_0}(\psi(t))/\phi_{Y_0}(t)$$

$$\lesssim \theta_5[\phi_Y(t)/\phi_{Y_0}(t)\ln 2]\int_{\psi(t)/2}^{\psi(t)} f^*(y)\phi_{X_0}(2y)\,\frac{dy}{y}$$

$$\lesssim \theta_5 M[\phi_Y(t)/\phi_{Y_0}(t)\ln 2]\int_0^{\psi(t)} f^*(y)\phi_{X_0}(y)\,\frac{dy}{y}$$

$$= M\theta_5 I(t)/\ln 2$$

and the estimate follows as before, only now the constant is $\theta_4(M\theta_0\theta_5M_1/\ln 2)^p/\theta_0$.

By (10), (5), (6), (7) and (8), we have

$$\frac{\phi_Y(t)}{\phi_{Y_1}(t)}\int_{\psi(t)}^{\infty} \phi_{X_1}(y)/\phi_X(y)\,\frac{dy}{y} \lesssim \theta_1,$$

so that an application of Jensen's inequality, an interchange of order of integration and (11) yields

$$\int_0^{\infty} W(t)C(M_1J(t))\,\frac{dt}{t} \lesssim \theta_1^{-1}\int_0^{\infty} W(t)\int_{\psi(t)}^{\infty} C(\theta_1M_1f^*(y)\phi_X(y))\frac{\phi_{X_1}(y)\phi_Y(t)}{\phi_X(y)\phi_{Y_1}(t)}\,\frac{dy}{y}\,\frac{dt}{t}$$

$$= \theta_1^{-1}\int_0^{\infty}C(\theta_1M_1f^*(y)\phi_X(y))\phi_{X_1}(y)/\phi_X(y)\int_0^{\psi^{-1}(y)} W(t)\phi_Y(t)/\phi_{Y_1}(t)\,\frac{dt}{t}\,\frac{dy}{y}$$

$$\lesssim \theta_2/\theta_1\int_0^{\infty} W(\psi^{-1}(y))C(M_1\theta_1f^*(y)\phi_X(y))\,\frac{dy}{y}$$

$$\lesssim M_1^p\theta_2\theta_1^{p-1}\int_0^{\infty} W(\psi^{-1}(y))C(f^*(y)\phi_X(y))\,\frac{dy}{y}$$

where $\theta_1 M_1$ is assumed to be greater than 1.

If $X_1 = L^\infty$, then $\phi_{X_1}(t)=1$ and T is of weak type (X_1, Y_1) means that

$$(Tf_u)^*(t)\phi_{Y_1}(t) \lesssim M_1 \sup_x f_u^*(x) = M_1 f^*(\psi(t)).$$

Thus

$$(Tf_u^*(t)\phi_Y(t) \lesssim M_1 \frac{\phi_Y(t)}{\phi_{Y_1}(t)} f^*(\psi(t))$$

$$= M_1 \phi_Y(t) f^*(\psi(t)) \phi_{X_0}(\psi(t))/\phi_{Y_0}(t)$$

$$\lesssim M_1 [\frac{\phi_Y(t)}{\phi_{Y_0}(t)\ln 2}] \int_{\psi(t)/2}^{\psi(t)} f^*(y)\phi_{X_0}(2y) \frac{dy}{y}$$

$$\lesssim M_1 M [\frac{\phi_Y(t)}{\phi_{Y_0}(t)\ln 2}] \int_0^{\psi(t)} f^*(y)\phi_{X_0}(y) \frac{dy}{y}$$

$$= M_1 M \, I(t)/\ln 2,$$

and the estimate follows as above.

Finally, the sublinearity of T, (4), the convexity of C and the fact that $C(t)/t^p$ decreases show that

$$\int_0^\infty W(t) C((Tf)^*(t)\phi_Y(t)) \frac{dt}{t} \lesssim \int_0^\infty W(2t) C[(Tf^u)^*(t) + (Tf_u)^*(t)]\phi_X(2t)] \frac{dt}{t}$$

$$\lesssim M \int_0^\infty W(t) [C(4(Tf^u)^*(t)\phi_Y(t))$$

$$+ C(4(Tf_u)^*(t)\phi_Y(t))] \frac{dt}{t}$$

$$\leq 2^{2p} M\{\int_0^\infty W(t) C((Tf^u)^*(t)\phi_Y(t)) \frac{dt}{t}$$

$$+ \int_0^\infty W(t) C((Tf_u)^*(t)\phi_Y(t)) \frac{dt}{t}\}$$

so that the above estimates yield the result.

In the remaining theorems, M will be a constant, possibly different at different occurences. The bound of the norm of the operators will have the same form as in Theorem 3.9.

__Theorem 3.10:__ Let T be a sublinear operator, simultaneosly of weak types (X_i, Y_i) with norms M_i, $i=0,1$. Suppose η, ξ, ψ and W are as in Lemma 3.7, and (9), (10), (13), (17) and for $\theta_6>0$

$$(18) \quad \int_0^t \phi_Y(y)/\phi_{Y_0}(y) \cdot \frac{dy}{y} \leq \theta_6 \phi_Y(t)/\phi_{Y_0}(t)$$

are satisfied. Then the conclusion of Theorem 3.9 holds.

__Proof:__ Let u, f^u and f_u be as defined in Theorem 3.9. We consider only the case $X_1 \neq L^\infty$, as the proof of the theorem when $X_1 = L^\infty$ is analogous to that of Theorem 3.9. Thus, since T is of weak type (X_0, Y_0), and by Jensen's inequality, an interchange of order of integration and (17), we have

$$\int_0^\infty W(t) C[(Tf^u)^*(t)\phi_Y(t)] \frac{dt}{t} \leq \theta_0^{-1} \int_0^\infty W(t) \int_0^{\psi(t)} C[\theta_0 M_0 f^*(y)\phi_X(y)]$$

$$\times \frac{\phi_Y(t)\phi_{X_0}(y)}{\phi_{Y_0}(t)\phi_X(y)} \frac{dy}{y} \frac{dt}{t}$$

$$= \theta_0^{-1}\int_0^\infty C[\theta_0 M_0 f^*(y)\phi_X(y)]\frac{\phi_{X_0}(y)}{\phi_X(y)}\int_0^{\psi^{-1}(y)} W(t)\phi_Y(t)/\phi_{Y_0}(t)\ \frac{dt}{t}\ \frac{dy}{y}$$

$$\lesssim \theta_0^{-1}\theta_6\int_0^\infty W(\psi^{-1}(y))C[\theta_0 M_0 f^*(y)\phi_X(y)]\frac{\phi_{X_0}(y)\phi_Y(\psi^{-1}(y))}{\phi_X(y)\phi_{Y_0}(\psi^{-1}(y))}\ \frac{dy}{y}$$

$$\lesssim \theta_0^{-1}\theta_6(\theta_0 M_0)^p\int_0^\infty W(\psi^{-1}(y))C(f^*(y)\phi_X(y))\ \frac{dy}{y},$$

where again we assume $\theta_0 M_0 \geq 1$.

Since T is of weak type (X_1, Y_1) with norm M_1, we can imitate the the proof of Theorem 3.9 for f_u, to show that

$$(Tf_u)^*(t)\phi_Y(t) \lesssim M_1[I_1(t) + J(t)]$$

and $I_1(t) \lesssim M\theta_5 I(t)/\ln 2$, so that the estimate for I_1 follows as before.

Finally, since $J(t) = \int_{\psi(t)}^\infty f^*(y)\phi_X(y)\phi_{X_1}(y)/\phi_X(y)\ \frac{dy}{y}$,

then Jensen's inequality, (13), an interchange of order of integration, and the fact that $C(t)/t^p$ decreases yields

$$\int_0^\infty W(t)C(M_1 J(t))\frac{dt}{t} \lesssim \theta_1^{-1}\int_0^\infty W(t)\int_{\psi(t)}^\infty C(\theta_1 M_1 f^*(y)\phi_X(y))\frac{\phi_{X_1}(y)\phi_Y(t)}{\phi_X(y)\phi_{Y_1}(t)}\ \frac{dy}{y}\ \frac{dt}{t}$$

$$= \theta_1^{-1}\int_0^\infty C(\theta_1 M_1 f^*(y)\phi_X(y))\frac{\phi_{X_1}(y)}{\phi_X(y)}\left(\int_{\psi^{-1}(y)}^\infty W(t)\frac{\phi_Y(t)}{\phi_{Y_1}(t)}\ \frac{dt}{t}\right)\ \frac{dy}{y}$$

$$\lesssim \theta_1^{-1}\theta_3\int_0^\infty W(\psi^{-1}(y))C(\theta_1 M_1 f^*(y)\phi_X(y))\frac{\phi_{X_1}(y)\phi_Y(\psi^{-1}(y))}{\phi_X(y)\phi_{Y_1}(\psi^{-1}(y))}\ \frac{dy}{y}$$

$$\lesssim \theta_1^{p-1}\theta_3 M_1^p\int_0^\infty W(\psi^{-1}(y))C(f^*(y)\phi_X(y))\ \frac{dy}{y},$$

where $\theta_1 M_1 \geq 1$.

The remaining portions of the proof are the same as in Theorem 3.9, and are therefore omitted.

Theorem 3.11: *Let T, η, ξ, ψ and ϕ_Y be as in Lemma 3.6. Suppose $\theta_7, \theta_8, \theta_9, s, t>0$ and D is concave, satisfying*

(19) $D(st) \le \theta_7 D(s)D(t)$

(20) $\int_t^\infty D(\phi_Y(y)/\phi_{Y_0}(y))\,\frac{dy}{y} \le \theta_8/D(\phi_{Y_0}(t)/\phi_Y(t))$

(21) $\int_0^t W(y)D(\phi_Y(y)/\phi_{Y_1}(y))\,\frac{dy}{y} \le \theta_9 W(t)/D(\phi_{Y_1}(t)/\phi_Y(t))$

where $W(t)\ge 0$ and decreases. Then there exists $M>0$, independent of f, such that,

$$\int_0^\infty W(t)D((Tf)^*(t)\phi_Y(t))\,\frac{dt}{t} \le M\int_0^\infty W(\psi^{-1}(t))D(f^*(t)\phi_X(t))\,\frac{dt}{t}.$$

Proof: Define f^u, f_u and u as usual, and then it is not difficult to show that Tf^u and Tf_u are well defined. Thus, since T is of weak type (X_0, Y_0), we obtain by (14) and (19)

$$\int_0^\infty W(t)D[(Tf^u)^*(t)\phi_Y(t)]\,\frac{dt}{t} \le \int_0^\infty W(t)D(M_0\int_0^{\psi(t)} f^*(s)\phi_{X_0}(s)\frac{\phi_Y(t)}{\phi_{Y_0}(t)}\,\frac{ds}{s})\,\frac{dt}{t}$$

$$\le \int_0^\infty W(t)D(M_0M\frac{\phi_Y(t)}{\phi_{Y_0}(t)}D^{-1}[M'\int_0^{\psi(t)} D(f^*(s)\phi_{X_0}(s))\frac{ds}{s}])\frac{dt}{t}$$

$$\le M\int_0^\infty W(t)D(\frac{\phi_Y(t)}{\phi_{Y_0}(t)})\int_0^{\psi(t)} D[f^*(s)\phi_X(s)]D(\frac{\phi_{X_0}(s)}{\phi_X(s)})\frac{ds}{s}\,\frac{dt}{t}.$$

Interchanging the order of integration, and using (20) yields

$$M\int_0^\infty D(f^*(s)\phi_X(s))D(\frac{\phi_{X_0}(s)}{\phi_X(s)})\int_{\psi^{-1}(s)}^\infty W(t)D(\phi_Y(t)/\phi_{Y_0}(t))\,\frac{dt}{t}\,\frac{ds}{s}$$

$$\le M\int_0^\infty W(\psi^{-1}(s))D(f^*(s)\phi_X(s))D(\frac{\phi_{X_0}(s)}{\phi_X(s)})/D(\frac{\phi_{Y_0}(\psi^{-1}(s))}{\phi_{Y_1}(\psi^{-1}(s))})\,\frac{ds}{s}$$

$$= M\int_0^\infty W(\psi^{-1}(s))D(f^*(s)\phi_X(s))\ \frac{ds}{s}.$$

Similarly, since T is of weak type (X_1,Y_1) the subadditivity of D shows that

$$\int_0^\infty W(t)D((Tf_u)^*(t)\phi_Y(t))\ \frac{dt}{t} \leq \int_0^\infty W(t)D\{M\frac{\phi_Y(t)}{\phi_{Y_1}(t)}[\int_0^{\psi(t)} f^*(\psi(t))\phi_{X_1}(s)\ \frac{ds}{s}$$

$$+ \int_{\psi(t)}^\infty f^*(s)\phi_X(s)\frac{\phi_{X_1}(s)}{\phi_X(s)}\ \frac{ds}{s}]\}\ \frac{dt}{t}$$

$$\leq M\{\int_0^\infty W(t)D(\frac{\phi_Y(t)}{\phi_{Y_1}(t)}\int_0^{\psi(t)} f^*(\psi(t))\phi_{X_1}(s)\ \frac{ds}{s})\ \frac{dt}{t}$$

$$+ \int_0^\infty W(t)D(\frac{\phi_Y(t)}{\phi_{Y_1}(t)}\int_{\psi(t)}^\infty f^*(s)\phi_X(s)\frac{\phi_{X_1}(s)}{\phi_X(s)}\ \frac{ds}{s})\ \frac{dt}{t}$$

By (20), (17) and (21), the second integral is dominated by

$$M\int_0^\infty W(t)D(\phi_Y(t)/\phi_{Y_1}(t))D\{M'D^{-1}[M''\int_{\psi(t)/2}^\infty D(f^*(s)\phi_X(s))D(\phi_{X_1}(s)/\phi_X(s))\ \frac{ds}{s}]\}\ \frac{dt}{t}$$

$$\leq M\int_0^\infty W(t)D(\phi_Y(t)/\phi_{Y_1}(t))\int_{\psi(t)/2}^\infty D(f^*(s)\phi_X(s))D(\phi_{X_1}(s)/\phi_X(s))\ \frac{ds}{s}\ \frac{dt}{t}$$

$$\leq M\int_0^\infty D(f^*(s)\phi_X(s))D(\phi_{X_1}(s)/\phi_X(s))\int_0^{\psi^{-1}(2s)} W(t)D(\phi_Y(t)/\phi_{Y_1}(t))\ \frac{dt}{t}\ \frac{ds}{s}$$

$$\leq M\int_0^\infty D(f^*(s)\phi_X(s))D(\phi_{X_1}(s)/\phi_X(s))[W(\psi^{-1}(s))/D(\phi_{Y_1}(\psi^{-1}(2s))/\phi_Y(\psi^{-1}(2s))]\ \frac{ds}{s}$$

$$\leq M\int_0^\infty W(\psi^{-1}(s))D(f^*(s)\phi_X(s))\ \frac{ds}{s}.$$

Similarly, the first integral is dominated by

$$M\int_0^\infty W(\psi^{-1}(s))D(f^*(s)\phi_X(s))\ \frac{ds}{s},$$

and the result follows from the subadditivity of D.

Our last interpolation theorem is the following:

Theorem 3.12: *Let T, η, ξ and ψ be as in Theorem 3.10 and let D satisfy (19) and, for $\theta_{10},\theta_{11}>0$,*

$$(22)\quad \int_0^t D(\phi_Y(s)/\phi_{Y_0}(s))\,\frac{ds}{s} \leq \theta_{10}/D(\phi_{Y_0}(t)/\phi_Y(t))$$

$$(23)\quad \int_t^\infty W(s)D(\phi_Y(s)/\phi_{Y_1}(s))\,\frac{ds}{s} \leq \theta_{11}W(t)/D(\phi_{Y_1}(t)/\phi_Y(t))$$

for $W\geq 0$ and non-decreasing. Then the conclusion of Theorem 3.11 holds.

As a closing remark, if X is an r.i. Banach function space, then $\phi_X(t)/t$ decreases, from which it follows that

$$\int_0^t \phi_X(s)/\phi_{X_i}(s)\,\frac{ds}{s} \geq \phi_X(t)/t \int_0^t \frac{1}{\phi_{X_i}(s)}\,ds$$

$$\geq \frac{\phi_X(t)}{t}\,\frac{1}{\phi_{X_i}(t)t}\,t$$

$$= \phi_X(t)/\phi_{X_i}(t).$$

For example, the constant in (9) must read $\theta_0\geq 1$. If X is not a Banach space, $\theta_0>0$ is sufficient. A similar argument can be made for (17) and (18).

CHAPTER IV

§4.1: Interpolation results for L^p and $L(p,q)$ spaces.

In this section, we apply the theorems of Chapter III to obtain results concerning the L^p and $L(p,q)$ spaces. Our first corollary yields a result of Heinig [20]:

Corollary 4.1: Suppose T is a sublinear operator of weak types (p_i,q_i), $0<p_i\leq q_i\leq\infty$, $i=0,1$, $p_0<p_1$, $q_0\neq q_1$. *If either* $q_0<q_1$, *W decreases and satisfies*

$$\int_0^t W^p(s)s^{1/q-1/q_1-1}ds \leq \theta t^{1/q-1/q_1}W^p(t),$$

or $q_1<q_0$, *W increasing with*

$$\int_t^\infty W^p(s)s^{1/q-1/q_1-1}ds \leq \theta t^{1/q-1/q_1}W^p(t),$$

then

$$\{\int_0^\infty [W(t)(Tf)^*(t)]^q dt\}^{1/q} \leq M\{\int_0^\infty [W(t^{1/\sigma})f^*(t)]^p dt\}^{1/p},$$

where σ, *p and q are given in* §3.4.

Proof: Take X_0, X_1, Y_0, Y_1, X and Y as L^{p_0}, L^{p_1}, L^{q_0}, L^{q_1}, L^p and L^q respectively. Theorem 3.9 (resp. 3.11) for $q_0<q_1$ imply for $C(t)=t^p$ $(D(t)=t^p)$

$$\int_0^\infty [W(t)(Tf)^*(t)t^{1/q}]^p \frac{dt}{t} \leq M\int_0^\infty [W(t^{1/\sigma})f^*(t)]^p dt.$$

Since $q>p$, the left side of this last inequality dominates

$$\left(\int_0^\infty [W(t)(Tf)^*(t)]^q dt\right)^{p/q},$$

by Theorem 1.10, from which the result follows.

If $q_1<q_0$, σ is negative. Let $C(t)=t^q$ in Theorem 3.10 and $D(t)=t^q$ in Theorem 3.12 so that

$$\int_0^\infty [W(t)(Tf)^*(t)]^q dt \leq M\int_0^\infty W^q(t^{1/\sigma})[f^*(t)t^{1/p}]^q \frac{dt}{t}$$

$$\leq M\{\int_0^\infty [W(t^{1/\sigma})f^*(t)]^p dt\}^{q/p},$$

where the last inequality follows from Theorem 1.10.

__Corollary 4.2:__ *Let X_0, X_1, Y_0, Y_1 and T be as in Corollary 4.1. For any $0<r<\infty$, we have*

$$\int_0^\infty W(t)[t^{1/q}(Tf)^*(t)]^r \frac{dt}{t} \leq M\int_0^\infty W(t^{1/\sigma})[t^{1/p}f^*(t)]^r \frac{dt}{t}, \quad 0<r<\infty,$$

where W satisfies the conditions of Theorem 3.9 or 3.10 if $r\geq 1$, and Theorem 3.11 or 3.12 if $r<1$.

__Corollary 4.3:__ *Under the conditions of Corollary 4.1, and with $1/r = \beta/p_1 + (1-\beta)/p$, $1/s = \beta/q_1 + (1-\beta)/q$, $0\leq\beta<1/p$ if $q_0<q_1$, or $0\leq\beta<1/q$ if $q_1<q_0$, then*

$$\|Tf\|^*_{s,q} \leq M\|f\|^*_{r,p}.$$

__Proof:__ If $q_0<q_1$, set $C(t)=t^p$ in Theorems 3.9 and 3.11, and $W(t)=t^\delta$ with $\delta=\beta/q_1-\beta/q$, then

$$\int_0^\infty [(Tf)^*(t)t^{1/q+\delta}]^p \frac{dt}{t} \leq M\int_0^\infty (f^*(t)t^{1/p+\delta/\sigma})^p \frac{dt}{t}.$$

$1/q + \delta = 1/s$ and $1/p + \delta/\sigma = 1/p + (1/p_1 - 1/p)\beta = 1/r$ (so that $W(t)$ does satisfy the conditions of Corollary 4.1), so that this case follows as in Corollary 4.1.

The case $q_1 < q_0$ is proved similarly.

<u>Lemma 4.4</u>: *Let C and D be increasing on $[0,\infty)$ with $C(0)=D(0)=0$, $D(t)/t^p$ decreasing for some $p>1$ and D convex. Suppose DC^{-1} is concave increasing (or, equivalently, CD^{-1} convex increasing). If ϕ is a fundamental function and g positive decreasing,*

$$C^{-1}\left(\int_0^\infty C(\phi(t)g(t))\,\frac{dt}{t}\right) \leq M(\ln 2)^{-1}D^{-1}\left(\int_0^\infty D(\phi(t)g(t))\,\frac{dt}{t}\right).$$

<u>Proof:</u> Since DC^{-1} is subadditive and $DC^{-1}(t)/t$ decreases and ϕ increases

$$\begin{aligned} DC^{-1}\left(\int_0^\infty C(\phi(t)g(t))\,\frac{dt}{t}\right) &= DC^{-1}\left(\sum_{k=-\infty}^{\infty}\int_{2^{k-1}}^{2^k} C(\phi(t)g(t))\,\frac{dt}{t}\right) \\ &\leq DC^{-1}\left(\sum_{k=-\infty}^{\infty} C(\phi(2^k)g(2^{k-1}))\ln 2\right) \\ &\leq \sum_{k=-\infty}^{\infty} D(\phi(2^k)g(2^{k-1})) \\ &\leq (\ln 2)^{-1}\sum_{k=-\infty}^{\infty}\int_{2^{k-2}}^{2^{k-1}} D(\phi(4\times 2^{k-2})g(t))\,\frac{dt}{t} \\ &\leq M(\ln 2)^{-1}\int_0^\infty D(\phi(t)g(t))\,\frac{dt}{t} \end{aligned}$$

by §3.3, (4). Since $D^{-1}(t)/t$ decreases, the result follows.

Let $C(ab)\approx C(a)C(b)$ and $D(ab)\approx D(a)D(b)$. If $g(t)=f(t)v(t)$, where

$v(t)$ is decreasing, then

$$C^{-1}\left(\int_0^\infty C(v(t))C(\phi(t)f(t))\frac{dt}{t}\right) \leq MD^{-1}\left(\int_0^\infty D(v(t))D(\phi(t)f(t))\frac{dt}{t}\right).$$

Corollary 4.5: *If $C(v)=W$, then under the conditions of Theorem 3.9*

$$C^{-1}\left(\int_0^\infty C(v(t))C((Tf)^*(t)\phi_Y(t))\frac{dt}{t}\right) \leq MD^{-1}\left(\int_0^\infty D(v(\psi^{-1}(t)))D(f^*(t)\phi_X(t))\frac{dt}{t}\right).$$

If, in Corollary 4.5, $C(t)=t^q$, $D(t)=t^p$, $1\leq p\leq q$, then

$$\left(\int_0^\infty ((Tf)^*(t)\phi_Y(t)v(t))^q \frac{dt}{t}\right)^{1/q} \leq M\left(\int_0^\infty (f^*(t)\phi_X(t)v(\psi^{-1}(t)))^p \frac{dt}{t}\right)^{1/p}.$$

This implies Corollary 4.1 for $1\leq p_1\leq q_1\leq\infty$.

§4.2: Interpolation theorems for Lorentz-Zygmund and Orlicz spaces.

Results for the Lorentz-Zygmund and Orlicz classes are now derived. First, we give the definition of the Lorentz-Zygmund spaces. Properties and further results may be found in Bennett and Rudnick [5].

Definition 4.6: *The Lorentz-Zygmund spaces $L^{pq}(Log)^\delta$, $0<p,q\leq\infty$, $\delta\in\mathbb{R}$, consist of those measurable functions f for which*

$$\|f\|^*_{L^{pq}(Log)^\delta} = \begin{cases} \left\{\int_0^\infty [t^{1/p}(1+|\log t|)^\delta f^*(t)]^q \frac{dt}{t}\right\}^{1/q}, & q<\infty, \\ \sup_t t^{1/p}(1+|\log t|)^\delta f^*(t), & q=\infty, \end{cases}$$

is finite.

Lemma 4.7: *If $X=L^{pq}(Log)^\delta$, then*

$$\|\chi_{[0,t)}\|^*_{L^{pq}(Log)^\delta} \approx t^{1/p}(1 + |logt|)^\delta.$$

Proof: By definition

$$\phi_X^q(t) = \int_0^t s^{q/p-1}(1 + |\log s|)^{\delta q} ds.$$

Consider the case $0<t\leq 1$. If $\delta\geq 0$, then for $s\leq t$

$$(1 - \log s)^{\delta q} \geq (1 - \log t)^{\delta q},$$

so that

$$(p/q)t^{q/p}(1 - \log t)^{\delta q} \leq \int_0^t s^{q/p-1}(1 - \log s)^{\delta q} ds.$$

To obtain the reverse inequality, we integrate by parts which yields

$$\phi_X^q(t) = (p/q)t^{q/p}(1 - \log t)^{\delta q} + (\delta p)\int_0^t s^{q/p-1}(1 - \log s)^{\delta q-1} ds.$$

Continuing this n times until $-1<\delta q-n\leq 0$, then if $\delta q=n$, we obtain the result directly since $1-\log s\geq 1$. Otherwise,

$$\int_0^t s^{q/p-1}(1 - \log s)^{\delta q-n} ds \leq Mt^{q/p}(1 - \log t)^{\delta q-n}$$

$$\leq Mt^{q/p}(1 - \log t)^{\delta q}.$$

If $\delta<0$, $(1-\log s)^{\delta q}\leq(1-\log t)^{\delta q}$, $s\leq t\leq 1$, which implies

$$\phi_X^q(t) \leq (p/q)t^{q/p}(1 - \log t)^{\delta q}$$

and also

$$\int_0^t s^{q/p-1}(1 - \log s)^{\delta q} ds \geq \int_{t/2}^t s^{q/p-1}(1 - \log s)^{\delta q} ds$$

$$\geq (p/q)(1 - \log t/2)^{\delta q} t^{q/p}(1 - 2^{-q/p}),$$

which proves the lemma if $0<t\leq 1$.

If $N>1$ and $t\varepsilon[1,N]$, $\phi_X(t)$ and $t^{1/p}(1+|\log t|)^{\delta}$ attain maxima and non-zero minima, from which the result follows.

Finally, since

$$\lim_{t\to\infty}\frac{\int_0^t s^{q/p-1}(1+|\log s|)^{\delta q}ds}{t^{q/p}(1+|\log t|)^{\delta q}} = p/q$$

the result follows for $t>N$, which completes the proof of the lemma.

Lemma 4.7 shows that the Lorentz-Zygmund spaces are particular examples of the general spaces discussed in Chapter III. The next theorem generalizes Theorem B of Bennett-Rudnick[5].

Theorem 4.8: *Suppose* $1\leq p_i\leq r_i\leq\infty$, $p_0<p_1$, $r_0\neq r_1$, $\delta\varepsilon\mathbb{R}$ *and* T *is sublinear, of weak types* (p_i,r_i), $i=0,1$. *For* $0<\alpha<1$, *set* $1/p=\alpha/p_0+(1-\alpha)/p_1$, $1/r=\alpha/r_0+(1-\alpha)/r_1$ *and let* σ *be the slope of the line between the points* $(1/p_0,1/r_0)$ *and* $(1/p_1,1/r_1)$. *If* $W\geq 0$ *satisfies either*

(i) for $r_0<r_1$, W *decreases and*

$$\int_0^t W(s)s^{1/r-1/r_1-1}(1+|\log s^{\sigma}|)^{\delta}ds\leq W(t)t^{1/r-1/r_1}(1+|\log t^{\sigma}|)^{\delta}$$

or

(ii) for $r_0>r_1$, W *increases and*

$$\int_t^\infty W(s)s^{1/r-1/r_1-1}(1+|\log s^{\sigma}|)^{\delta}ds\leq W(t)t^{1/r-1/r_1}(1+|\log t^{\sigma}|)^{\delta},$$

then there exists a constant $M>0$, *such that, for* $0<q<\infty$,

$$\int_0^\infty W(t)[(Tf)^*(t)t^{1/r}(1+|\log t^{\sigma}|)^{\delta}]^q\frac{dt}{t}\leq M\int_0^\infty W(t^{1/\sigma})[f^*(t)t^{1/p}(1+|\log t|)^{\delta}]^q\frac{dt}{t}.$$

Proof: We consider only the case $r_0<r_1$, the case $r_1<r_0$ is similar, except that one applies Theorems 3.10 and 3.12.

Let $X_0=L^{p_0}$, $X_1=L^{p_1}$, $Y_0=L^{r_0}$, $Y_1=L^{r_1}$ and $X=L^{pq}(Log)^{\delta}$. By Lemma 4.7, $\phi_X(t)\approx t^{1/p}(1+|\log t|)^{\delta}$, and we also have $\xi(t)=\phi_{Y_0}(t)/\phi_{Y_1}(t)=t^{1/r_0-1/r_1}$, $\eta(t)=\phi_{X_0}(t)/\phi_{X_1}(t)=t^{1/p_0-1/p_1}$, $\psi(t)=t^{\sigma}$ and $\phi_Y(t)=\phi_X(\psi(t))\phi_{Y_0}(t)/\phi_{X_0}(\psi(t))$ $\approx t^{1/r}(1+|\log t^{\sigma}|)^{\delta}$. In order to apply Theorems 3.9 and 3.11 with $C(x)=x^q$, we must verify

(a) $\int_0^t s^{1/p_0-1/p-1}(1+|\log s|)^{-\delta}ds \leq M t^{1/p_0-1/p}(1+|\log t|)^{-\delta}$

(b) $\int_t^{\infty} s^{1/p_1-1/p-1}(1+|\log s|)^{-\delta}ds \leq M t^{1/p_1-1/p}(1+|\log t|)^{-\delta}$

and

(c) $\int_t^{\infty} s^{1/r-1/r_0-1}(1+|\log s^{\sigma}|)^{\delta}ds \leq M t^{1/r-1/r_0}(1+|\log t^{\sigma}|)^{\delta}$.

An identical argument of Lemma 4.7 proves (a) and for (b) we note that the integral tends to infinity as t tends to 0. Hence, by L'Hospital's rule,

$$\lim_{t\to 0} t^{1/p-1/p_1}(1+|\log t|)^{-\delta}\int_t^{\infty} s^{1/p_1-1/p-1}(1+|\log s|)^{\delta}ds$$
$$= 1/(1/p-1/p_1),$$

from which (b) follows. The inequality (c) is proved similarly, which completes the proof of the theorem.

Recall now that if C is a Young's function, then

$$L_C = \{f: \int_{\mathfrak{m}} C\left(\frac{|f(x)|}{k}\right)dx\leq 1 \text{ , for some } 0<k<\infty\}.$$

Theorem 4.9: Let C_0, C_1 and C be Orlicz functions, and suppose

T is a sublinear operator of types (L_{C_0}, L_{C_0}) *and* (L_{C_1}, L_{C_1}). *If* $\phi_{L_{C_0}}$, $\phi_{L_{C_1}}$, *W and* ϕ_{L_C} *satisfy the conditions of Theorem 3.9, and* $C(a/b)=\frac{C(a)}{C(b)}$, *then*

$$\int_0^\infty W(t)C[(Tf)^*(t)]dt \leq M\int_0^\infty W(t)C(f^*(t))dt.$$

Proof: Since $C(\phi_{L_C}(t)(Tf)^*(t))=C(\frac{(Tf)^*(t)}{C^{-1}(1/t)})=C((Tf)^*(t))t$, and similarly $C(\phi_{L_C}(t)f^*(t))=f^*(t)t$, the result follows.

Observe that if $L_C(W)=\{f: \inf\{k: \int_0^\infty W(t)C(\frac{f^*(t)}{k})dt\leq 1\} \equiv \|f\|_{L_C(W)}\}$, then the last inequality may be written as

$$\|Tf\|_{L_C(W)} \leq M\|f\|_{L_C(W)}.$$

Corollary 4.10: *Let T be a sublinear operator, simultaneously of weak types (1,1) and (p,p),* $1<p<\infty$. *Let C be a Young's function, such that* $C(t)/t^p$ *decreases,* $C(t)/t$ *increases and*

$$\int_0^t C(s)\frac{ds}{s^2} \leq MC(t)/t.$$

Then

$$\int_0^\infty W(t)C((Tf)^*(t))dt \leq M\int_0^\infty W(t)C(f^*(t))dt$$

where

$$\int_0^t W(s)s^{-1/p-1}\frac{ds}{C^{-1}(1/s)} \leq M\frac{W(t)t^{-1/p}}{C^{-1}(1/t)}.$$

Proof: This is Theorem 4.9 with $C_0(t)=t$ and $C_1(t)=t^p$.

§4.3: Integral operator inequalities.

In this section, we give four examples of integral operator inequalities.

The Hilbert transform H on $[-\pi,\pi]$ is of weak type (1,1) and of type (p,p) for any $1<p<\infty$. Thus, we obtain the following weighted inequality for this transform:

Corollary 4.11: *For $1<p<\infty$, $0<q<\infty$, and $W\geq 0$ satisfying (i) of Theorem 4.8,*

$$\int_0^\infty W(t)[(Hf)^*(t)t^{1/p}(1+|\log t|)^\delta]^q \frac{dt}{t} \leq M\int_0^\infty W(t)[t^{1/p}f^*(t)(1+|\log t|)^\delta]^q \frac{dt}{t}.$$

Let $0<\alpha<1$ and let

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n(1-\alpha)}}\,dy.$$

This is the Riesz fractional integral of f of order α.

Lemma 4.12 ([76], Lemma 4.5): Let E be a measurable subset of $\mathbb{R}^n$, then $t^{1/p-\alpha}(I_\alpha \chi_E)^{**}(t) \leq M|E|^{1/p}$, $1\leq p\leq\infty$. Thus I_α maps $L(p,1)$ continuously into $L(q,\infty)$ for $0<1/q=1/p-\alpha<1$, $p>1$.

Corollary 4.13: *Suppose $p>1$, $0<1/q=1/p-\alpha<1$ and $0<s<\infty$. Then under the conditions of Theorem 4.8,*

$$\int_0^\infty W(t)[(I_\alpha f)^*(t)t^{1/q}(1+|\log t^\sigma|)^\delta]^s \frac{dt}{t}$$

$$\leq M\int_0^\infty W(t^{1/\sigma})[f^*(t)t^{1/p}(1+|\log t|)^\delta]^s \frac{dt}{t}.$$

For our next example, let f be defined on $(-\infty,\infty)$. The Fourier transform, $Tf=\hat{f}$, is known to be of type (2,2) and $(1,\infty)$. Using these facts, we obtain:

Corollary 4.14: *For $q=p/(p-1)$, $1<p<2$, and W satisfying (i) of Theorem 4.8, we have for $0<s<\infty$,*

$$\int_0^\infty W(t)[(\hat{f})^*(t)t^{1/q}(1+|\log t|)^\delta]^s \frac{dt}{t} \leq M\int_0^\infty W(1/t)[f^*(t)t^{1/p}(1+|\log t|)^\delta]^s \frac{dt}{t}.$$

As a final example, let f be a real valued, measurable function on $[0,\infty)$. Define the Hardy-Littlewood maximal function m(f) by

$$m(f)(x) = \sup_\varepsilon \frac{1}{x-\varepsilon}\int_\varepsilon^x f(t)dt, \quad 0<\varepsilon<x.$$

If the operator T is defined by $T(f)=m(f)$, then T is linear, and clearly, if $f\in L^\infty$, then so is m(f). T is also of weak type (1,1). Thus, we have the following:

Corollary 4.15: *Under the conditions of Theorem 4.8, (i), it follows that for any $0<q<\infty$,*

$$\int_0^\infty W(t)[(m(f))^*(t)t^{1/p}(1+|\log t^\sigma|)^\delta]^q \frac{dt}{t} \leq M\int_0^\infty W(t)[f^*(t)t^{1/p}(1+|\log t|)^\delta]^q \frac{dt}{t}.$$

§4.4: Sobolev spaces.

In this section, we sketch the recent abstract generalizations of Heinig[21] and provide an application.

Definition 4.16: *Let U_i, V_i, $i=0,1$, be rearrangement invariant spaces,* *and* $U_i(t)$, $V_i(t)$ *their respective fundamental functions. Define*

$$\eta(t) = U_0(t)/U_1(t) \qquad \xi(t) = V_0(t)/V_1(t)$$

$$\psi(t) = \eta^{-1}(\xi(t)) \qquad V(t) = U(\psi(t))V_0(t)/U_0(\psi(t)).$$

<u>*Definition 4.17*</u>*: Let* U_i, V_i*,* $i=0,1$*, be rearrangement invariant spaces and* (X_0,Y_0)*,* (X_1,Y_1) *two pairs of Banach spaces. A sublinear operator* T *is of generalized weak type* $(U_0,V_0;U_1,V_1)$ *with respect to the couples* (X_0,Y_0)*,* (X_1,Y_1) *if for every* $t>0$

$$K_Y(t,Tf)/t \leq A\left\{\frac{1}{V_0(t)}\int_0^{\psi(t)} U_0(s)(K_X(s;f)/s)\,\frac{ds}{s} + \frac{1}{V_1(t)}\int_{\psi(t)}^{\infty} U_1(s)(K_X(s;f)/s)\,\frac{ds}{s}\right\},$$

provided the right side exists. Here, K *denotes the* K*-functional defined in the introduction.*

If U_i, V_i, $i=0,1$, are the spaces L^{p_i} and L^{q_i} respectively, this definition reduces to that of [15].

The interpolation theorem corresponding to Theorem 3.9 is, in this context, the following:

<u>*Theorem 4.18 [21]*</u>*: Let* T *be of generalized weak type* $(U_0,V_0;U_1,V_1)$ *with respect to the couples* (X_0,Y_0)*,* (X_1,Y_1) *and suppose* ξ*,* η*, and* ψ *are as in Definition 4.16, with* $U(t)/U_1(t)$ *increasing and* $U(t)/U_0(t)$ *decreasing. Suppose*

$$\int_0^t \frac{U_0(s)}{U(s)}\,\frac{ds}{s} \leq \theta_0 U_0(t)/U(t)$$

$$\int_t^{\infty} \frac{U_1(s)}{U(s)}\,\frac{ds}{s} \leq \theta_1 U_1(t)/U(t)$$

$$\int_t^{\infty} \frac{V(s)}{V_0(s)}\,\frac{ds}{s} \leq \theta_2 V(t)/V_0(t)$$

and

$$\int_0^t W(s)\,\frac{V(s)}{V_1(s)}\,\frac{ds}{s} \leq \Theta_3 W(t)V(t)/V_1(t)$$

hold, where W is a non-negative non-increasing function on $(0,\infty)$. *If C is as in Theorem 3.9, then*

$$\int_0^\infty W(t)C[V(t)K_Y(t;Tf)/t]\,\frac{dt}{t} \leq M\int_0^\infty W(\psi^{-1}(t))C[U(t)K_X(t;f)/t]\,\frac{dt}{t}\,.$$

For further results, see [21].

The following example is a generalization of the Hausdorff-Young inequality:

Let $N>0$ be an integer. The Sobolev spaces $W_p^N(\mathbb{R}^n)$ are defined to be the spaces of functions $f\in L^p(\mathbb{R}^n)$ such that $\frac{\partial^\alpha f}{\partial x^\alpha}$, $|\alpha|=\alpha_1+\ldots+\alpha_n\leq N$, exists in the sense of distributions and are in $L^p(\mathbb{R}^n)$, $1\leq p\leq\infty$. On $W_p^N(\mathbb{R}^n)$, we define a semi-norm

$$|f|_{p,N} = \max_{|\alpha|=N}\left\|\frac{\partial^\alpha f}{\partial x^\alpha}\right\|_p, \qquad 1\leq p\leq\infty$$

The K-functional, $K_p^N(t;f) \equiv K(t;f,L^p(\mathbb{R}^n),W_p^N(\mathbb{R}^n))$, is defined by

$$K_p^N(t;f) = \inf_{f=g+h}\left\{\|g\|_p + |h|_{p,N}\right\}.$$

The Hausdorff-Young inequality is then given by the following theorem:

<u>Theorem 4.19[21]</u>: Suppose that W is a non-negative, non-decreasing function on $(0,\infty)$, *such that,*

$$\int_t^\infty W(s)s^{((Np/nq')+(p/p'-2))}\,ds \leq MW(t)t^{(Np/nq')+(p/p'-1)},$$

where $1<p<2$, $N\geq n$, $1<q<N/[N-n((1/p)-(1/p'))]$. *If C is as in Theorem 3.9, then*

$$\int_0^\infty W(t)C[t^{(Np/nq')+p/p'-1}K(t;\hat{f},L^{p'},L^\infty)]\,\frac{dt}{t}$$

$$\leq M\int_0^\infty W(t^{-n/Np})C[t^{1/q-1}K_p^N(t;f)]\,\frac{dt}{t}.$$

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