EQUIVARIANT PRINCIPAL BUNDLES OVER THE 2-SPHERE

By

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To my father Ömer

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Abstract

Isotropy representations provide powerful tools for understanding the classification of equivariant principal bundles over the 2-sphere. We consider a Γ-equivariant principal G-bundle over S^2 with structural group G a compact connected Lie group, and $\Gamma \subset SO(3)$ a finite group acting linearly on S^2. Let X be a topological space and Γ be a group acting on X. An isotropy subgroup is defined by $\Gamma_x = \{\gamma \in \Gamma | \gamma x = x\}$. Assume X is a Γ-space and A is the orbit space of X. Let $\varphi : A \to X$ be a continuous map with $\pi \circ \varphi = 1_A$. An isotropy groupoid is defined by $\mathfrak{I} = \{(\gamma, a) \in \Gamma \times A | \ \gamma \in \Gamma_{\varphi(a)}\}$. An isotropy representation of $\mathfrak{I}$ is a continuous map $\iota : \mathfrak{I} \to G$ such that the restriction map $\mathfrak{I}_a \to G$ is a group homomorphism. Γ- equivariant principal G-bundles are studied in two steps;

1) the restriction of an equivariant bundle to the Γ equivariant 1-skeleton $X \subset S^2$ where $\mathfrak{I}$ is isotropy representation of X over singular set of the Γ-sets in S^2

2) the underlying G-bundle ξ over S^2 determined by $c(\xi) \in \pi_2(BG)$.

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Chapter 1

Introduction

Let Γ and G be a Lie groups. A principal (Γ, G)-bundle ξ over B is a locally-trivial Γ-equivariant principal G-bundle $p : E \to B$ such that E and B are left Γ-spaces. We denote the bundle $\xi{=}(E, B, p, G, \Gamma)$. The projection map is Γ-equivariant and $\gamma(e \cdot g) = (\gamma e) \cdot g$ where $\gamma \in \Gamma$ and $g \in G$ acting on $e \in E$ by the principal action. Equivariant principal bundles and their natural generalizations were studied by T. E. Stewart [12] T. tom Dieck [13], R. Lashof [6], P. May [7], G. Segal [9]. These authors study equivariant principal bundles by homotopy-theoretic methods. There exists a classifying space $B(\Gamma, G)$ for principal (Γ, G)-bundles [13], so the classification of equivariant bundles in particular cases can be approached by studying the Γ-equivariant homotopy type of $B(\Gamma, G)$. If the structural group G of the bundle is abelian, the main result of Lashof states that equivariant bundles over a Γ-space B are classified by ordinary homotopy classes of maps $[E\Gamma \times_\Gamma X, BG]$ [9].

Another approach is given by Hambleton and Hausmann [5] for classifying equivariant principal bundles. They used the local invariants arising from isotropy representations at singular points of (B, Γ). For each Γ-fixed point $b_0 \in B$ there exists an isotropy representation. It means that we obtain a homomorphism $\alpha_{b_0} : \Gamma_{b_0} \to G$ defined by the formula

$$\gamma \cdot e_0 = e_0 \cdot \alpha_{b_0}(\gamma)$$

where $e_0 \in p^{-1}(b_0)$. Denote the collection of isotropy representations of ξ by $Rep_\Gamma^G(\mathfrak{I})$.

The homomorphism α is independent of the choice of e_0 up to conjugation in G.

In this thesis, Γ-equivariant principal G bundles over S^2 will be classified. First of all, let $X \subset S^2$ be a Γ-equivariant 1-*skeleton* in a Γ-CW complex decomposition for S^2. The classification of Γ-equivariant principal G-bundle over X will be determined by using the paper of Hambleton and Hausmann since X is a split-Γ space [5] , So we will use a different method to classify Γ-equivariant principal G-bundles over S^2 since S^2 is not a split-Γ space. Since $X \subset S^2$ is a Γ-equivariant CW- subcomplex, then

$$X \xrightarrow{i} S^2 \xrightarrow{j} S^2 \cup CX \xrightarrow{k} \Sigma(X) \xrightarrow{m} \Sigma(S^2) \xrightarrow{\Sigma i} \Sigma(S^2 \cup CX) \to \cdots$$

is a cofibration sequence where CX= cone on $X = (X \times [0,1])/(a,0) \sim pt$ and $\Sigma(S^2)$= the suspension of S^2.

If we take Γ-equivariant homotopy classes of maps into the space $Y = B(\Gamma, G)$, then the following sequence

$$[\Sigma(S^2), Y] \xrightarrow{m^*} [\Sigma(X), Y] \xrightarrow{k^*} [S^2 \cup CX, Y] \xrightarrow{j^*} [S^2, Y] \xrightarrow{i^*} [X, Y]$$

is a exact sequence of abelian groups since $B(\Gamma, G) = Y = \Omega Z$ is a loop space [3]. We will determine $[S^2, Y]$ by $m^* = \Sigma i^*$ and i^* where Σi^* is the suspension map of i^*.

Secondly, we will use the classification of principal G-bundles over the 2-sphere by using Steenrod‘s book [11, p,96]. Let S^1 be a great 1-sphere on S^2 and V_1, V_2 be the closed hemispheres of S^2 and bounded by 1-sphere parallel to S^1. Then $V_1 \cup V_2$ cover S^2 and $V_1 \cap V_2$ is equatorial band containing S^1. Let x_0 be a reference point on S^1 is said to be in normal form if its coordinate neighborhoods are V_1, V_2 and $g_{12}(x_0) = 1_G$ in structural group. Any bundle ξ over S^2 is strictly equivalent to a bundle in normal form. Hence there exist bundle maps

$$\phi_i' : V_i \times G \to \xi_i$$

so we have the following maps:

$$g_{ij} : V_i \cap V_j \to G.$$

Then ϕ'_1, ϕ'_2 are coordinate functions of a bundle ξ' strictly equivalent to ξ. If $g'_{12}(x) = a$, we alter ξ' to strictly equivalent bundle by setting $\lambda_1(x) = e = 1_G$ for $x \in V_1$ and $\lambda_2(x) = a$ for $x \in V_2$. The resulting bundle is in normal form.

A bundle ξ is in normal form, the map $T = g_{12}|S^1$ which maps S^1 into G is called the characteristic map of ξ. Now, since $T(x_0) = e$, and we regard T as a map

$$(S^1, x_0) \to (G, e).$$

Lemma 1.0.1. *[11, p,97] Any map $T : (S^1, x_0) \to (G, e)$ is the characteristic map of some bundle over S^2 in normal form.*

Finally, we will introduce the Theorem from [11, p,99] .

Theorem 1.0.2. *The equivalance classes of bundles over S^2 with group G are in $1-1$ correspondence with equivalance classes of elements of $\pi_1(G)$ under the operations of $\pi_0(G)$. Such a correspondence is provided by $\xi \to \chi(\alpha)$ where α is a generator of $\pi_2(S^2)$ and $\chi : \pi_2(S^2) \to \pi_1(G)$ is a characteristic homomorphism of ξ.*

We conclude that we can determine bundles over S^2 by a characteristic classes $c(\xi) \in [S^2, BG] = \pi_2(BG) = \pi_1(G)$. Our main result is following theorem.

Theorem 1.0.3. *Let G be a compact connected Lie group, $\Gamma \subset SO(3)$ be a finite subgroup acting linearly on S^2. A Γ-equivariant principal G-bundle over (S^2, Γ) is classified by $Rep^G_\Gamma(\mathcal{I})$ and $c(\xi) \in \pi_2(BG)$.*

Corollary 1.0.4. *If $Rep^G_\Gamma(\xi_1) \cong Rep^G_\Gamma(\xi_2)$ then $c(\xi_1) \equiv c(\xi_2)$ $mod|\Gamma|$.*

Chapter 2

Introductory Material

2.1 Basic Definitions

In this chapter, we will give some basic definitions and theorems from tom Dieck's book [13] which are useful for later chapters to classify equivariant principal bundles over S^2. Firstly, we shall define a *group action* since it is the one of the main objects of this research. In algebraic topology, groups generally act on topological spaces (more specifically manifolds).

Definition (Lie Group)**.** *Let G be a topological group and a finite-dimensional smooth manifold. Then G is called a* ***Lie group****, if the map μ: $G \times G \to G$, $\mu\,(x, y) = x^{-1}y$ is smooth.*

Definition (Group Action)**.** *Let G be a topological group and X be a topological G-space. We define a* ***(left) group action*** *to be a continuous map ϕ: $G \times X \to X$ such that*

(i) $\phi(g,(\phi(h,x)))= \phi(gh,x)$ *for* $g,h \in G$, $x \in X$

(ii) $\phi(e,x) = x$ *for* $x \in X$ *for* $e \in G$ *the identity element.*

The left group action ϕ(g,x) is generally denoted by g·x. Under the same conditions, a right group action can be defined. Now, we will consider the group action as a left group action. If there exists a topological group acting on a topological space, we consider the isotropy subgroup (stabilizer group) of each point in X and the orbit space of the space X under the group action. An action is called *transitive* if for every $x_1, x_2 \in X$ there exists a group element g such that $gx_1 = x_2$ and the action is called *free* if for every $x \in X$ the only identity element of G fixes x.

Definition (Isotropy Subgroup)**.** *Let X be a set and G be a group acting on X. For each $x \in X$, the **isotropy subgroup** of x is denoted by $G_x = \{g \in G \mid g{\cdot}x{=}x\}$. It is obvious that the G_x is a subgroup of G.*

Definition (Orbit Space)**.** *Let X be a set and G be a group action on X. $Gx := \{gx \mid \text{for all } g \in G\}$ is the **orbit** of x. The set of orbits induce an equivalence relation. The **orbit space** of X is denoted by $G\backslash X$. Commonly, we will use X/G for the orbit space if the action is defined before as a right action.*

For every subgroup H of G acting on X, the fixed points set of H will be denoted by X^H.

$$Fix(X,H) = X^H = \{x \in X \mid h \cdot x = x \quad for\ all\ h \in H, \ \}.$$

Definition (Equivariant Map)**.** *Let $\rho : X \to Y$ be a map between G-sets. The map ρ is called equivariant if we have $g \cdot \rho(x) = \rho(g \cdot x)$ for every $g \in G$ and every $x \in X$.*

Two G-spaces are G-equivalent if there exists an equivariant homeomorphism between them. As an example, there is an equivariant homeomorphism

$$\begin{array}{lcl} \alpha : G/G_x & \to & G \cdot x \\ gG_x & \to & g \cdot x \end{array}$$

is equivariant, and G/G_x and $G \cdot x$ are G-equivalent spaces. We say that an orbit which is equivalent to G/H is of type G/H.

Definition. *Let X and Y be sets and G act on X as a right space and act on Y as a left space. We define the equivalence relation*

$$(x, y) \sim (x \cdot g^{-1}, g \cdot y) \quad \forall g \in G.$$

The set of equivalence classes of this relation is $X \times_G Y$.

Definition. *Let* $P \subset X$ *be an orbit of type* G/H. *A* ***tube*** *about* P *is an equivariant homeomorphism into an open neighborhood of* P

$$\phi : G \times_H A \to X$$

where A *is a space on which* G *acts.*

For $x \in S \subset M$, a manifold, such that $G_x S = S$, S is a **slice** at x if the map

$$\begin{array}{ccc} G \times_{G_x} S & \to & M \\ (g, s) & \to & g \cdot s \end{array}$$

is a tube about $G \cdot x$.

Equivalently, We say that a slice at x is defined as a subspace S of X with the following properties:

(i) S is closed in $G \cdot S$

(ii) $G \cdot S$ is an open neighborhood of $G \cdot x$

(iii) $G_x \cdot S = S$

(iv) If $(g \cdot S) \cap S$ is non-empty, then g is an element of G_x

The following theorem says that there exists a slice under certain conditions. It was proven by Montgomery-Yang[1957] and Mostow[1957]

Theorem. *[2, p,40] Let* G *be a compact Lie group and* X *a* G*-space which is completely regular. Then there is a slice at each point and a tube around each orbit.*

Now, the *Riemann-Hurwitz Formula* is a useful formula especially in algebraic geometry. In this context, we will give the related formula for group actions. One can find a detailed proof in [13, p,30] for a general case.

A simplicial G-complex is **regular** if given elements $g_1,\ g_2,\ \ldots\ ,\ g_n$ of the G and two simplices $(v_1,\ v_2,\ \ldots\ ,\ v_n)$ and $(g_1\cdot v_1,\ g_2\cdot v_2,\ \ldots\ ,\ g_n\cdot v_n)$, there is an element $g\in G$ such that $gv_i = g_iv_i$ for all i. Not all simplicial complexes are regular but we can make a regular simplicial complex by using barycentric subdivision [15] . The **singular set** of X consists of points x whose isotropy subgroup different than identity and it will be denoted by $Sing(X, G) = \{\ x \in X \mid G_x \neq 1\ \}$

Theorem (Riemann-Hurwitz Formula)**.** *Let Γ be a finite group. Let X be a compact connected, oriented surface and $\Gamma \times X \to X$ be an effective orientation preserving group action with the orbit space $X/\Gamma \cong A$. Then*

$$\chi(X) = |\Gamma|\chi(A) - \sum_{x\epsilon X}(|\Gamma_x| - 1)$$

Here, it is not necessary to prove this theorem since we will use an analogous result for the graph.

Proposition 2.1.1. *Let X be a compact connected regular graph, Γ be a finite group and $\Gamma \times X \to X$ be an group action with the orbit space $X/\Gamma \cong A$. Then*

$$\chi(X) = |\Gamma|\chi(A) - \sum_{v_i\epsilon X}(|\Gamma_{v_i}| - 1)$$

Proof. Let the orbit space A be a graph with v vertices and e edges and Γ be a group with the order n and the singular set be $Sing(X, \Gamma) = \{v_1, v_2, ..., v_k\}$. We relate the space X with v' vertices and e' edges to the orbit space A. Since no edges are fixed by the group Γ, we can say that the number of edges of the space X is ne. if the group Γ acts freely on the X, then the number of vertices of X would be nv. On the other hand, we know that there exist some singular points with fewer pre-images. We should subtract them for the correct number of vertices of X. For v_i we counted it $|\Gamma_{v_i}|$ times. We should do this for all fixed points, then

$$v' - e' = n(v - e) - \sum_{v_i\epsilon X}(|\Gamma_{v_i}| - 1)$$

$$\chi(X) = |\Gamma|\chi(A) - \sum_{v_i \epsilon X}(|\Gamma_{v_i}| - 1)$$

□

There are various maps between two topological spaces. We would like to classify them under some restrictions. A continuous function is a special function which satify the continuity definition. If we have any two continuous functions, we wonder how one can compare them or relate them.

Definition. *Let $f, g : X \to Y$ be two continuous map. f and g are* ***homotopic*** *if and only if there exists a continuous map $H : X \times I \to Y$ such that, for $x \in X$,*

$$H(x, 0) = f(x)$$

$$H(x, 1) = g(x).$$

if two maps are are homotopic, it is denoted by

$$f \simeq g.$$

One can show that this is an equivalance relation on the set of all continuous maps from X to Y. Therefore, equivalence classes are called *homotopy classes* of maps.

2.2 Fiber Bundles

Now, a fiber bundle will be introduced with Steenrod definitions [11, p,6]. Actually, a manifold is locally homeomorphic to R^n. The fiber bundle idea is the same as a manifold structure but the fiber bundle is locally homeomorphic to cross product of open set and fiber. Let E denote a bundle space, B denote a base space, and F denote a fiber. Let $p : E \to B$ be a projection map if arbitrary open set U is taken from B then the inverse image of U under the projection map is homeomorphic to $U \times F$.

Definition (Fiber Bundle)**.** *A fiber bundle ξ consists of a topological space E called the bundle space, a topological space B called base space, a continuous map*

$$p : E \to B$$

of E onto B called projection and a space F called the fiber. The fiber over the point x of B is defined by F_x=$p^{-1}(x)$ and F_x is homeomorphic to F. Therefore, for each x $\in B$, there is a neighborhood U of x and a homeomorphism

$$\varphi : U \times F \to p^{-1}(U)$$

such that

$$p\varphi(x', y) = x'$$

where $x' \in B$, $y \in F$.

The fiber bundle can be illustrated as below, where the fiber is denoted by F, the bundle space is denoted by E, and the base space by B.

$$\begin{array}{ccc} F & \to & E \\ & & \downarrow_p \\ & & B \end{array}$$

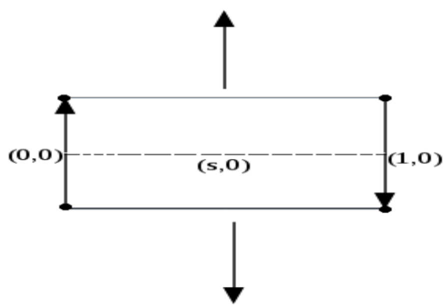
The first example is **product bundle** E, sometimes called trivial bundle, where the bundle space is $B \times F$, $p(x, y) = x$, $x \in B$, $y \in F$.

. The second example is Mobius line bundle μ defined to be a one-dimensional real vector bundle over the circle given as follows. Let $E = ([0, 1] \times R)/\sim$ where $(0, t) \sim (1, -t)$ and C be the middle circle $C = \{(s, 0) \in E\}$. Then μ is the line bundle defined by the projection

$$p : E \to C \qquad where \;\; (s, t) \to (s, 0)$$

If we glue together the line $x = 0$ and the line $x = 1$ with the relation $\sim$, we will get

Mobius line bundle with the bundle space E. It can be seen the figure below the base space C is the middle circle and the fiber of each point $(s,t) \in E$ is homeomorphic to the real line R.



Definition (Coordinate Bundle). *[11, p,7] A coordinate bundle ξ consists of a topological space E called the bundle space, a topological space B called base space, a continuous map*

$$p : E \to B$$

of E onto B called projection and a space F called the fiber, an effective topological transformation group G of F called the group of the bundle, a family of $\{U_j\}$ of open sets covering B with an index set J, the U_j's are called coordinate neighborhoods, and for each j in J, a homomorphism

$$\varphi_j : U_j \times F \to p^{-1}(U_j)$$

called the coordinate function.

The coordinate functions are required to satify the following conditions:

$$p\varphi_j(x,y) = x \qquad for\ x \in U_j,\ y \in F$$

if the map $\varphi_{j,x} : F \to p^{-1}(x)$ is defined by $\varphi_{j,x}(y) = \varphi_j(x,y)$, then, for each pair i, j in J, and each $x \in U_i \cap U_j$, the homeomorphism

$$\varphi_{j,x}^{-1}\varphi_{i,x} : F \to F$$

coincides with the operation of an element of G and for each pair $i, j \in J$, the map

$$g_{ji} : U_i \cap U_j \to G$$

defined by $g_{ji}(x) = \varphi_{j,x}^{-1}\varphi_{i,x}$ is continuous. One can check that the functions g_{ji}'s satisfy the axioms of a group (closure, associativity, an identity element, an inverse element, cocycle condition).

Definition (Vector Bundle)**.** *Let ξ be a fiber bundle with the fiber F. The fiber bundle ξ is called a* ***real vector bundle*** *if the fiber is $\mathbb{R}^n$ and the transformation group G is the general linear group $GL_n(\mathbb{R})$.*

Here, if the group is $GL_n(\mathbb{C})$ and acting on E and $F = \mathbb{C}$ then the fiber bundle is called a complex vector bundle.

Definition (Principal Fiber Bundle)**.** *Let ξ be a fiber bundle with the fiber F. The fiber bundle ξ is called a* ***principal fiber bundle*** *if the fiber is homeomorphic to the transformation group G and the transformation group G acts freely on F and $F = G$.*

Definition. *Let ξ =($p : E \to B$) and ξ' =($p' : E' \to B'$) be two principal G-bundles and let $u : E \to E'$ and $f : B \to B'$ be two maps such that $p'u = fp$.*

$$\begin{array}{ccc} E & \xrightarrow{u} & E' \\ \downarrow p & & \downarrow p' \\ B & \xrightarrow{f} & B' \end{array}$$

A pair (u, f) is called a principal bundle map (or just a principal map) if $u : E \to E'$ is G-equivariant in the sense that

$$u(eg) = u(e)g$$

for all $e \in E$ and $g \in G$.

Definition (Associated Bundle)**.** *Let ξ=($p : E \to B$) be a fiber bundle over a topological space X with the structural group G and let G be a left action on F and let G*

be a right action on E. Then ξ_p (=$E \times_G F \to B$) is called an associated bundle of ξ with fiber F.

Definition. *Let ξ and ν be two fiber bundles with the same base space B and the same fiber F . Then they are called equivalent if their associated bundles are equivalent.*

One can generalize this definition for different structural groups i.e. ξ =($p : E \to B$) is a G-bundle, ξ' =($p' : E' \to B'$) is a G'-bundle, and $\phi : G \to G'$ is a group homeomorphism. We shall require

$$u(eg) = u(e)\phi(g)$$

for all $e \in E$ and $g \in G$.

Theorem 2.2.1. *Let (u, f) be a principal map between a pair of principal G-bundles ξ =(p: $E \to B$) and ξ' =(p': $E' \to B'$). If $B' = B$,*

$$\begin{array}{ccc} E & \rightarrow & E' \\ \downarrow_p & & \downarrow_{p'} \\ B & \xrightarrow{f} & B \end{array}$$

then E and E' are equivalent to each other.

Let ξ be a fiber bundle with fiber F and let $f : X \to B$ be a smooth function.

$$\begin{array}{ccc} & & E \\ & & \downarrow_p \\ X & \xrightarrow{f} & B \end{array}$$

Then there is a fiber bundle $f^*(\xi) = \{\ (x, e) \in X \times E \mid p(e){=}f(x)\}$ over X with the fiber F, in the sense that the following diagram commutes;

$$\begin{array}{ccc} f^*(E) & \xrightarrow{u} & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & B \end{array}.$$

It can be seen that if $f,\ g:\ X \to B$ are two homotopic maps, then $f^*(E)$ and $g^*(E)$ are isomorphic bundles over the space X [11, p,20]. We denote by $Bun^G(X)$ the isomorphism classes of principal G-bundles over X. Any G- bundle ξ over X generates a map

$$\begin{array}{rcl} \alpha_\xi : [X,B] & \stackrel{u}{\to} & Bun^G(X) \\ f & \mapsto & f^*(\xi) \end{array}$$

where $[X,B]$ denotes the set of homotopy classes of maps from X into B.

Generally, this map is neither one-to-one nor onto. We will study on principal G-bundles and we call universal bundle that makes this map as one-to-one and onto.

Theorem 2.2.2. *[11] Let ξ =($p : E \to B$) be a principal G-bundle. There exists a classifying space BG and principal bundle UG ($= G \to EG \to BG$). Then there exists a function $f : B \to BG$, unique up to homotopy, such that $f^*(EG) = E$.*

2.3 Equivariant Bundles

Now, we consider a principal G-bundle together with an automorphism group Γ of bundle maps. Let Γ be a compact Lie group, G a topological group and

$$\alpha : \Gamma \to Aut(G)$$

a homomorphism from Γ into the automorphism group of G. Let $\gamma \in \Gamma$, $\alpha(\gamma) \in Aut(G)$ can be denoted by α_γ and the map

$$\begin{array}{rcl} \Gamma \times G & \to & G \\ (\gamma, g) & \mapsto & \alpha_\gamma(g) \end{array}$$

has to be continuous.

Definition. *A (Γ, α, G)-bundle consists of a locally trivial principal G-bundle $p : E \to B$ composed with left Γ-actions on E and B such that the following holds:*

(i) The projection map p is a Γ-equivariant map.

(ii) For $\gamma \in \Gamma$, and $e \in E$, we have the relation $\gamma(eg) = (\gamma e) \cdot \alpha_\gamma(g)$.

If the α is trivial homomorphism, then Γ acts as a group of a principal bundle automorphisms and it is just written as a (Γ, G)-bundle or Γ-equivariant principal G-bundle.

A (Γ, α, G)-bundle $p : E \to B$ is called locally trivial if B admits an open covering $\mathfrak{U} = \{U_j | j \in J\}$ by Γ-sets U_j such that each restriction $p^{-1}U_j \to U_j$ admits a (Γ, α, G)-bundle map into local objects.

Definition. *If X is a G-space, an open cover $\{U_j\}_{j\in J}$ will be called an open G-cover if for each U_i, there exists a U_j such that $G.U_i = U_j$. An open G-cover will be called **numerable** if there is a subordinate partition of unity $\{U_{\lambda_i}\}_{\lambda_i \in J}$ such that each U_{λ_i} is G-invariant.*

If $\mathfrak{U}$ is numerable, then (Γ, α, G)-bundle is called a numerable bundle.

Theorem 2.3.1. *Existence of Universal Bundle*

(i) p is a numerable (Γ, α, G)-bundle.

(ii) Each numerable (Γ, α, G)-bundle admits a bundle-map into projection map p. Any two such bundle maps are homotopic to each other as (Γ, α, G)-bundle maps.

Theorem 2.3.2. *Let $p : E \to B$ be a numerable (Γ, α, G)-bundle and $f_t : B' \to B$ a Γ-homotopy. Then the induced bundles $f_0^* p$ and $f_1^* p$ are isomorphic as (Γ, α, G)-bundles.*

A (Γ, α, G)-bundle p for which the theorem 2.3.1 is true is called a **universal bundle.** From the Theorem 2.3.2, we say that $E(\Gamma, \alpha, G) \to B(\Gamma, \alpha, G)$ and any Γ-space X the homotopy set $[X,\ B(\Gamma, \alpha, G)]_\Gamma$ is canonically isomorphic to the set of isomorphism classes of numerable (Γ, α, G)-bundles over X which is denoted by $\mathfrak{B}(\Gamma, \alpha, G)(X)$. The space $B(\Gamma, \alpha, G)$ is called the **classifying space** for (Γ, α, G)-bundles. If α is trivial, we write $B(\Gamma, G)$, if Γ is trivial, it will be written as BG.

Consider the bundle $EG \to BG$ as (Γ, G)-bundle with trivial Γ- action. It is numerable and therefore has a classifying map $j : BG \to B(\Gamma, G)$. Let $J = id \times_\Gamma j$: $E\Gamma \times_\Gamma BG \to E\Gamma \times_\Gamma B(\Gamma, G)$. Notice that $E\Gamma \times_\Gamma BG = B\Gamma \times BG$.

Proposition 2.3.3. *The maps j and J are homotopy equivalences.*

Proof. $E(\Gamma, G) \to B(\Gamma, G)$ is a universal principal G- bundle. Then we know that j induces a universal bundle from another one, it must be a homotopy equivalence. We can think J fiberwisely then J must be a homotopy equivalence. □

Let $h^*(-)$ be an element of cohomology groups $H^*(-)$ which are defined on all spaces. Elements of $h^*(BG)$ are called universal characteristic classes for principal G- bundles. Given $x \in h^*(BG)$ and a classifying map $f : B \to BG$ for a principal bundle $p : E \to B$, then $f^*(x) \in h^*(B)$ is called a characteristic class for p of type x. [10] Similarly, we can use a type of equivariant cohomology theory defined for Γ-spaces. By the proposition 2.3.3, we can define simpler by using the homotopy equivalence $E\Gamma \times_\Gamma B(\Gamma, G) \cong BF \times BG$, one can define universal characteristic classes in $h^*(B\Gamma \times BG)$.

2.4 Equivariant CW-complexes

Cell complexes are constructed by iterated attaching of cells. To begin with, a **push out** is an useful object. A diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ \downarrow j & & \downarrow J \\ X & \xrightarrow{F} & Z \end{array}$$

of G spaces and G-maps is called push out if for each part of G-maps $f' : Y \to U$, $j' : X \to U$ with $f'f = j'j$, there exists a unique G-map $u : Z \to U$ with $uJ = f'$, $uF = j'$. If $A \subset X$ then j is a subcomplex and a closed embedding. Suppose X and Y are disjoint. We consider the following equivalence notation R on $X \cup Y$:

$$(z_1, z_2) \in R \Leftrightarrow z_1 \in A,\ z_2 \in f(A) \text{ and } f(z_1) = z_2.$$

The quotient space $(X \cup Y)/R = Z$ is denoted by $Y \cup_f X$. And the canonical maps turn out to be $F : X \to Y \cup_f X$ and $J : Y \to Y \cup_f X$.

Proposition 2.4.1. *J is a closed embedding. The morphism, $(F, f) : (X, A) \to (Y \cup_f X, Y)$ is a relative homomorphism.*

The closed inclusion $j : A \to X$ is called a **G-cofibration** if it has the **homotopy extension property** for all G-maps $f : X \to Y$ and for all G-homotopies $\varphi : A \times I \to Y$ with $\varphi(a, 0) = f(a)$ for $a \in A$. In other words, given f and φ , there must exist $\phi : X \times I \to Y$ such that $\phi|A \times I = \varphi$ and $\phi(x, 0) = f(x)$. Let $n \geq 0$ be an integer. Let A be a G-space. Given a family $(H_j | j \in J)$ of closed subgroup of H_j of G and G-maps

$$\varphi_j : G/H_j \times S^{n-1} \to A, \quad j \in J.$$

We consider push outs of G-spaces

$$\begin{array}{ccc} \coprod_{j \in J} G/H_j \times S^{n-1} & \to & E' \\ \bigcap & & \downarrow \\ \coprod_{j \in J} G/H_j \times D^n & \to & X \end{array}$$

We put $\varphi|G/H_j \times S^{n-1} = \varphi_j$ and $\phi|G/H_j \times D^n = \phi_j$. Now, X is obtained from A by attaching the family of equivariant n-cells $(G/H_j \times D^n | j \in J)$ of type $(G/H_j | j \in J)$. The map

$$(\phi_j, \varphi_j) : (G/H_j \times D^n, G/H_j \times S^{n-1}) \to (X, A)$$

is called the **characteristic map** of the corresponding n-cell, while φ_j is called the **attaching map.**

Chapter 3

Literature Review

T. E. Stewart has a paper on lifting the group action which is defined on base space B [12].

Definition 3.0.2. *Let $\xi = (E, F, B)$ be a fiber bundle. If $\overline{\alpha} : H \times B \to B$ is a group action of H on B he says that $\overline{\alpha}$ can be lifted in ξ to a group action on E such that the following diagram is commutative:*

$$\begin{array}{ccc} H \times E & \stackrel{\alpha}{\to} & E \\ \downarrow_{(1,p)} & & \downarrow_{p} \\ H \times B & \stackrel{\overline{\alpha}}{\to} & B \end{array} .$$

α will be said to be a lifting of $\overline{\alpha}$ in ξ.

Proposition 3.0.3. *If $\overline{\alpha}$ has bundle lifting in a principal bundle ξ over B then $\overline{\alpha}$ has a bundle lifting in every associated bundle.*

Proposition 3.0.4. *Let ξ be a bundle over B, and $\overline{\alpha}$, $\overline{\alpha'}$ actions of H on the spaces B and B' respectively. If $f : B' \to B$ is an equivariant map, i.e. $f(\overline{\alpha}(h, b')) = \overline{\alpha}(h, f(b'))$ and the action $\overline{\alpha}$ has a bundle lifting in ξ, then $\overline{\alpha'}$ has a bundle lifting in the induced bundle $f^*(\xi)$ over B'*

We can mention many other papers written on this topic. But rigorously, R. K. Lashof developed the theory of equivariant bundles. He defined equivariant bundles in [6]. Now, let $p : E \to X$ be a principal G-bundle. We call this bundle as a Γ-G bundle if E and X are Γ-spaces and p is Γ-equivariant map. (that is if we can lift X as a Γ-space to E via the map p) If two Γ-G bundles over X are G-equivalent via a Γ-equivariant map, then they are called Γ-G equivalent [11] and [6].

Lashof adds some remarks in [6, p,257]. $\xi =(p : E \to B)$ is a Γ-G-bundle if and only if the associated bundle of ξ (ξ_A) is a Γ-G-bundle. Furthermore, Two Γ-G-bundles with fiber F are Γ-G-bundle equivalent if and only if their associated principal Γ-G-bundles are Γ-G equivalent.

Theorem 2.1 gurantees that there exists a slice around $x \in X$ if X is a completely regular space. For every $x \in X$, there is a Γ_x- invariant subspace V_x, called Γ_x-slice, then we define a map ν such that

$$\nu : \Gamma \times_{\Gamma_x} V_x \to X, \qquad \nu(g, v) = gv$$

Definition. *A Γ-G bundle $p : E \to X$ with fiber F is called Γ-G locally trivial if there is an open cover $\{\Gamma V_\alpha\}_{\alpha \in I}$ of X, where V_α is an H_α slice, such that $E|\Gamma V_\alpha$ is Γ-G equivalent to a locally trivial bundle $\epsilon^{\rho_\alpha}(V_\alpha)$ for some homomorphism $\rho_\alpha : H_\alpha \to G$ under an identification $\Gamma \times_{H_\alpha} V_\alpha \to \Gamma V_\alpha$.*

In this definition, Lashof uses the open neighborhoods and coverings. In the same paper [6] he gave Bierstone's condition which is more related with group homomorphisms.

Definition (Bierstone's Condition)**.** *[1] For each $x \in X$ there is a Γ_x invariant neighborhood U_x such that $p^{-1}(U_x)$ is Γ_x-G is equivalent to $U_x \times F$ with the Γ_x action*

$$h(u, g) = (hu, \rho_x(h)y),$$

where $u \in U_x$, $h \in \Gamma_x$, $y \in F$ and $\rho_x : \Gamma_x \to G$ is an homomorphism.

Lashof proved in [6] that these two definitions are equivalent under some restricted conditions.

Lemma 3.0.5. *[6, p,259] A* $(\Gamma\text{-}G)$ *locally trivial bundle satisfies the Bierstone's condition.*

For the other direction of lemma we need other conditions for X. X is called a **completely regular space** if given any closed set F and any point x in $X - F$ then for every y in F there is a continuous function f from X to a real line $\mathbf{R}$ such that $f(x)$ is 0 and $f(y)$ is 1 in [15, p,50].

Lemma 3.0.6. *If X is a completely regular space, then a* $(\Gamma\text{-}G)$ *bundle over X which satisfies Bierstone's condition is* $\Gamma\text{-}G$ *locally trivial.*

We say that a $(\Gamma\text{-}G)$ bundle satisfies Bierstone's condition if and only if the associated principal bundle does.

Lemma 3.0.7. *Let a principal* $\Gamma\text{-}G$ *bundle* $p : P \to X$ *reduce to a* $\Pi\text{-}G$ *bundle,* Π *be a closed subgroup of* Γ *such that* Γ/Π *has local cross-section in* Γ*, if and only if the associated bundle* P/Π *with the fiber* Γ/Π *has an equivariant cross-section.*

Definition. *Let A,B be compact connected Lie group. If B is closed subgroup of A, A/B is called $A - B$ locally trivial if for each compact Lie group $C \subset A$, $\lambda : A \to A/B$ is an $C - B$ locally trivial bundle.*

$\Gamma\text{-}G$ locally trivial bundle property can be extended to its associated bundle and under some restriction we can reduce the $\Gamma\text{-}G$ locally trivial property.

Lemma 3.0.8. *Let $\rho : G \to H$ be a homomorphism. If $q : Q \to X$ is a* $\Gamma\text{-}G$ *locally trivial bundle, then associated* $\Gamma\text{-}H$ *bundle* $P = Q \times_G H$ *over X is locally* $\Gamma\text{-}H$ *locally trivial.*

We conclude that the following is converse lemma:

Lemma 3.0.9. *Let $p : P \to X$ be a principal* $\Gamma\text{-}G$ *locally trivial bundle and suppose p reduces to a* $\Gamma\text{-}K$ *bundle $q : Q \to X$, K a closed subgroup of G such that G/K is $G - K$ locally trivial. Then q is* $\Gamma\text{-}K$ *locally trivial bundle.*

Proposition 3.0.10 (Wasserman-Segal)**.** *Any* (Γ-L_n) *bundle over a completely regular space* X *is* Γ-L_n *locally trivial where* L_n *is a general linear group.*

Definition. *If* X *is a* G*-space, an open cover* $\{U_\alpha\}_{\alpha\in I}$ *will be called an open* G*-cover if for each* U_α*, there exists a* U_β *such that* $G.U_\alpha = U_\beta$ *. An open* G*-cover will be called numerable if there is a subordinate partition of unity* $\{U_\lambda\}_{\alpha\in I}$ *such that each* λ_α *is* G*-invariant.*

Now, it does not need to check partition of unity for all space, since in the same paper [6], Lashof introduces a lemma to guarantee a numerable refinement.

Lemma 3.0.11. *If* X *is paracompact* G*-space, then every open* G*-cover has a numerable refinement.*

Moreoever, If (Γ-G) bundle $p : P \to X$ will be called numarable if X has a trivializing numerable G-cover $\{GV_\lambda\}_{\alpha\in I}$.

Corollary 3.0.12. *Every* Γ-G *locally trivial bundle over a paracompact space* X *is numarable.*

Now, H is a closed subgroup of the Lie group G. Lashof gave proposition to compare Γ-H bundles and Γ-G bundles.

Proposition 3.0.13. *Let* H *be a closed subgroup of the Lie group* G*. If* G/H *is equivariantly contractible for a compact subgroup of* G*, then there is a bijection between equivalence classes of numarable* Γ-H *bundles and* Γ-G *bundles.*

Corollary 3.0.14. *There is a bijective corresponce between equivalence classes of numarable* Γ-L_n *and* Γ-O_n *bundles.*

Theorem 3.0.15. *A* Γ-G *bundle has the* ***equivariant covering homotopy property****(ECHP) if a numerable* Γ-G *bundle* E *over* $X \times I$ *is equivalent to* $E_0 \times I$ $(E_0 = E|(X \times (0)))$.

In the [14], Wassermann proved that(ECHP) is satisfed in the vector bundle case. And in the [1], Bierstone proved that it is satisfied in the differentiable G-bundle case. Finally, important theorem is given from Lashof at [6].

Theorem. *A numerable* Γ-G *bundle* E *over* $X \times I$ *is* (Γ-G) *bundle equivalent to* $(E|X) \times I$*, where by* $E|X$ *we mean* $E|(X \times (0))$.

Corollary 3.0.16. *Let* $p : E \to Y$ *be a numerable* Γ-G*and let* $f : X \times I \to Y$ *be an equivariant homotopy between the* G*-maps* $f_0, f_1 : X \to Y$*. Then* $f_0^*(E)$ *and* $f_1^*(E)$ *are* Γ-G *equivalent.*

Corollary 3.0.17. *A numerable* Γ-G *bundle satisfies the equivariant covering homotopy property.*

We discussed the classifying space. Now, in [6] Lashof and in [13] tom Dieck use the parallel definition induced from classifying spaces .

A **universal** Γ-G bundle is a numerable principal Γ-G bundle $p : E \to B$ such that for any G-space X, the equivalence classes of numerable Γ-G over X are bijective correspondence with $[X, B]_\Gamma$, the equivariant maps X into B; the correspondence being given by induced bundles. [13, p,59]

A **strongly universal** Γ-G bundle is a numerable principal Γ-G bundle $p : E \to B$ which satisfies: let $\pi : P \to X$ be a numerable principal Γ-G and let $X_0 \subset X$ be a closed invariant subspace with invariant halo W_0. If $\phi_0 P|W_0 \to E$ is a Γ-G bundle map, then there is a Γ-G bundle map $\phi : P \to E$ such that ϕ agrees with ϕ_0 on $P|X_0$.

Theorem 3.0.18. *A numerable principal* Γ-G *bundle* $\pi : P \to X$ *is strongly universal if and only if for each* $H \subset \Gamma$ *and homomorphism* $\rho : H \to G$*,* P *is contractible to a point under the action* $a \to ha\rho(h)^{-1}$*,* $a \in P$*,* $h \in H$*.*

There is a universal bundle for a given bundle space. Other method is given for equivariant case by using Steenrood's approach [11].

Lemma 3.0.19. *If a strongly universal* Γ-G *bundle exists, then every universal* Γ-G *bundle is strongly universal.*

Finally, Lashof prove the final theorem in [6]. Before that we will give some definitions and will define some sets which related by the theorem.

Let $H \subset \Gamma$ be a closed subgroup and $\rho : H \to G$ be a homomorphism. Suppose that G^ρ be the centralizer of $\rho(H)$ in G. That is as set notation

$$G^\rho = \{g \in G \mid \quad \rho(h) g \rho^{-1}(h) = g, \quad h \in H\}$$

Since G^ρ is a closed subgroup of G, we let BG^ρ denote its universal base space. Let R_H be a collection of homomorphism of H in G containing exactly one representative from each G-conjugacy class.

Theorem 3.0.20. *[6] Let $p : E \to B$ be a universal Γ-G bundle and $H \subset \Gamma$ a closed subgroup. Then B^H is the disjoint union of the BG^ρ where $\rho \in R_H$. If $K \subset H$, $B^H \subset B^K$ corresponce to the maps $BG^\rho \to BG^{\rho/K}$ induced by $G^\rho \subset G^{\rho/K}$.*

In [7], they generalize the equivariant principal Γ-G bundles and it is known that a numerable principal Γ-G bundle satisfies the equivariant bundle covering homotopy property.

Theorem 3.0.21. *A numerable principal Γ-G bundle $p : E \to B$ is universal if and only if E^H is contractible for all closed subgroup H of Γ such that $H \cap G = e$.*

Let $B(\Gamma, G)$ be the base space of a universal principal Γ-G bundle it is uniquely determined up to G- homotopy type.

Theorem 3.0.22. *For $H \subset \Gamma$, $A \subset \Gamma$*

$$B(\Gamma, G)^H = \coprod B(W_\Gamma A, G \cap N_\Gamma A)$$

such that $H \cap G = e$ where $N_\Gamma A$ is the normalizer of A in Γ and $W_\Gamma A = N_\Gamma A/A$. In particular, $B(\Gamma, G)^H$ is empty if there is no such A.

Now, we will mention the paper of Lashof , May and Segal [8]. For a space Γ-space X of the homotopy type of a Γ-CW complex, define $B(\Gamma, G)(X)$ to be the set of equivalence classes of principal (Γ, G)-bundles over X. For the space Y of the homotopy type of a CW-complex, define $B(G)(Y)$ to be the set of equivalence classes of principal G- bundles over Y. Let $X_\Gamma = E\Gamma \times_\Gamma X$, where $E\Gamma$ is a contractible and Γ-free Γ-CW complex. Define a natural transformation

$$\Phi : B(\Gamma, G)(X) \to B(G)(X_\Gamma)$$

by sending a (Γ, G)-bundle $p : D \to X$ to the G-bundle $p_\Gamma : D_\Gamma \to X_\Gamma$.

Theorem 3.0.23. *If G is abelian, then Φ is an isomorphism.*

Proof. A choice of base point in $E\Gamma$ determines an injection $i : X \to X_\Gamma$ and we write a natural transformation

$$i^* : B(G)(X_\Gamma) \to B(G)(X).$$

$\pi : E\Gamma \times X \to X_\Gamma$ is a quotient map and $\epsilon : E\Gamma \times X \to X$ is the projection, then $i \circ \epsilon \simeq \pi$ and thus i^* agrees with the composite

$$B(G)(X_\Gamma) \xrightarrow{\pi^*} B(G)(E\Gamma \times X) \xrightarrow{\epsilon^{*-1}} B(G)(X).$$

The composition is

$$i^*\Phi : B(\Gamma, G)(X) \to B(G)(X)$$

from (Γ, G)-bundles to G-bundles. Its image consists of those G-bundles over X which admits a structure of (Γ, G)-bundles. In other words, an action of Γ on X lifts appropriately to the total space.

Moreover, we show this result by different method. Let $p : E \to X$ be a Γ-equivariant principal G-bundle with $\gamma(e \cdot g) = (\gamma e) \cdot g$ where $\gamma \in \Gamma$, $e \in E$ and $g \in G$. And the isotropy representation at $x \in X$ is the homomorphism $\alpha_x : \Gamma_x \to G$ defined by the formula $\gamma \cdot \tilde{x} = \tilde{x} \cdot \alpha_x(\gamma)$ where $\tilde{x} \in p^{-1}(x)$. We know that $i^*\Phi$ is injective up to homotopy equivalance. Surjectivity can be shown by the following. $\gamma(\tilde{x} \cdot g) = (\gamma\tilde{x}) \cdot g = \tilde{x} \cdot \alpha_x(\gamma) \cdot g$ since G is abelian we can write $= \tilde{x} \cdot g\alpha_x(\gamma)$ then it is surjective. □

Now, for Γ-spaces X and X', let $M(X, X')$ denote the function Γ-space of continuous map $X \to X'$ with Γ acting by conjugation. Define $B(\Gamma, G) = M(pt, B(\Gamma, G))$

maps from point to classifying space and recall that Γ-map $f : D \to E$ is said to be a **weak Γ-equivalence** if its fixed point map $f^H : D^H \to E^H$ is an ordinary weak equivalence for each closed subgroup H of G. By the Γ-Whitehead theorem

$$f_* : [X, D]_\Gamma \to [X, E]_\Gamma$$

is then a bijection for any Γ-space X of the homotopy type of a Γ-CW complex.

Lemma 3.0.24. $\xi_* : M(E\Gamma, BG) \to M(E\Gamma, B(\Gamma, G))$ *is a weak* Γ*-equivalence.*

Theorem 3.0.25. *[9]* $\epsilon^* : B(\Gamma, G) \to M(E\Gamma, B(\Gamma, G))$ *is a weak-equivalence when* G *is abelian.*

Proposition 3.0.26. $B : Hom(\Gamma, G) \to [B\Gamma, BG]$ *is an isomorphism when* G *is abelian.*

Now, we will classify Γ-equivariant principal G-bundles over 2-sphere. We approach the problem as in the paper of Hambleton and Hausmann [5] and tom Dieck [13]. Hambleton and Hausmann studied on local invariants. This paper will be used for the classification equivariant principal G-bundles over 1-skeletons which are subset of 2-sphere. For the 2-sphere, we will classify by Chern classes. Therefore, we need two invariants for the classification of equivariant principal G-bundles over 2-sphere.

Chapter 4

Classifying Equivariant Principal G-Bundles

4.1 Split Equivariant Principal Bundles

We will introduce the definition of a *split* Γ-*space* since the theorem [5, Theorem 3.2] basically shows that if X is *split* Γ-*space* on S^2, the equivalance classes of split Γ-equivariant G-bundles over X are in bijection with $Rep^G(\mathfrak{I})$.

Definition (Split-Γ Space)**.** *Let A be a topological space and Γ be a topological group. A triple (X,π, φ) is called a split Γ-space over A if;*

(i) *X is a Γ-space,*

(ii) *$\pi : X \to A$ is a continuous surjective map such that for each $a \in A$, the preimage $\pi^1(a)$ is a single orbit, and*

(iii) *φ: $A \to X$ is a continuous section of π, i.e. $\pi \circ \varphi = 1_A$*

In this definition, the maps π and φ may be omitted from the notation and we change of a *split* Γ-*space* over A. In (ii), the map π induces $\bar{\pi}$: $\Gamma \setminus X \to A$ which is a homeomorphism since φ provides its continuous inverse. We defined split Γ-space. Now, we will introduce the isotropy groupoid for given split-Γ space. Then we will define the isotropy representation of an isotropy groupoid. Finally, we will give the theorem which was proven by Hambleton and Hausmann [5, p,134].

A (Γ, A)-groupoid is a subspace $\mathfrak{I} \subset \Gamma \times A$ such that, for each $a \in A$, the space $\mathfrak{I}_a = \mathfrak{I} \cap (\Gamma_a \times \{a\})$ is the form $\tilde{\mathfrak{I}} \times \{a\}$, where $\tilde{\mathfrak{I}_a}$ is a closed subgroup of Γ.

An isotropy groupoid $\mathfrak{I}$ is called *weakly locally maximal* if each point $a \in A$ admits a neighbourhood U such that $\mathfrak{I}_u$ is a subgroup of $\mathfrak{I}_a$ for all $u \in U$. A space X is called *locally compact* if it is Hausdorff and every point of X admits a compact neighbourhood. The following proposition says that given Γ, A and isotropy groupoid, we have a split Γ-space.

Proposition 4.1.1. *Let Γ be a compact topological group and A be a locally compact space. Let $\mathfrak{I}$ be a weakly locally maximal isotropy groupoid. Then the following properties hold.*

(i) There is a split Γ-space $(Y_{\mathfrak{I}}, \Pi, \phi)$ over A with isotropy groupoid $\mathfrak{I}$; the space $Y_{\mathfrak{I}}$ is locally compact.

(ii) Let (X, π, φ) and (X', π', φ') be two split Γ-spaces over A with isotropy groupoid $\mathfrak{I}$. Suppose that X and X' are locally compact. Then there is a unique Γ-equivariant homeomorphism $F : X \to X'$ such that $\varphi' = F \circ \varphi$.

Definition (Isotropy Groupoid)**.** *Let (X,π,φ) be a split Γ-space over the space A with relative topology. The **Isotropy groupoid** of X is denoted by*

$$\mathfrak{I}(X) := \{(\lambda, a) \in \Gamma \times A | \lambda \in \Gamma_{\varphi(a)}\}$$

Definition (Split Bundle)**.** *Let (X,π,φ) be a split Γ-space over A with isotropy groupoid $\mathfrak{I}$ and let η:$(E \to X)$ be a principal G-bundle over X. Then, the bundle η is called a **split bundle** if $\varphi^*\eta$ is trivial.*

A topological space is called a *contractible space* if it is homotopic to a point. For instance, a one-dimensional disk D^1 is a contractible space but a 1-sphere is not a contractible space. Moreover, it is known that every compact spaces is a paracompact space. All spaces which we use in this paper are compact spaces, then they are paracompact. Two split Γ-equivariant principal G-bundles over (X,π,φ) are *isomorphic* if they are isomorphic just as Γ-equivariant principal G-bundles over X. The set of isomorphism classes of Γ-equivariant split G-principal bundles over (X, π, φ) simply by $SBun_{\Gamma}^{G}(X)$. It is still a subset of Bun_{Γ}^{G}.

Any Γ-equivariant principal G-bundle is split bundle if the orbit space A is a contractible, paracompact space, since the contractible spaces are homotopy equivalent to a point. If the orbit space A is homotopic to a point, all pull-back bundles over the space X are trivial, then all equivariant principal bundles are *split*. This is very useful for classifying the equivariant principal bundles over X. For the finite subgroups of $SO(3)$ acting linearly on S^2, we will get different orbit spaces. Except for the orbit space of dihedral subgroup case, all orbit spaces of subgroups of $SO(3)$ are contractible and paracompact. Hence, the theorem proven by Hambleton and Hausmann [5] will be used.

Definition (Isotropy Representation)**.** *Let $\mathfrak{I}$ be a (Γ,A)-groupoid and G be a topological group. A continuous representation of $\mathfrak{I}$ is a continuous map*

$$\iota : \mathfrak{I} \to G$$

such that the restriction of ι to each point $a \in A$ is a group homomorphism from $\mathfrak{I}_a$ to G. It can be denoted by ι_a: $\mathfrak{I}_a \to G$.

A continuous representation of $\iota : \mathfrak{I} \to G$ is called **locally maximal** if for each point $a \in A$, there exists a neighborhood U such that $\mathfrak{I}_u$ is subgroup of $\mathfrak{I}_a$ for all $u \in U$ together with a continuous map g: $U \to G$ such that $\alpha_u\ (\gamma) = g(u)\ \alpha_a\ (\gamma)\ g(u)^{-1}$ for all $u \in U$ and $\gamma \in \mathfrak{I}_u$. Moreover, a (Γ, A)-groupoid $\mathfrak{I}$ such that $\mathfrak{I}_a$ is a compact Lie group for all $a \in A$ is called *proper.* We denote by $Rep^G(\mathfrak{I})$ the set of conjugacy classes of locally maximal continuous representations of $\mathfrak{I}$.

Let A be a CW-complex filtered by its skeleta $A^{(n)}$. We denote by $\Omega = \Omega(A)$ the set of cells of A, and by $d(e)$ the dimension of a cell $e \in \Omega$ and $\Omega_n = \{\ e \in \Omega \mid d(e) = n\ \}$. For $a \in A$, we denote by $e(a) \in min\{d(e)|a \in e\}\ \Omega$

Definition. *Let Γ be a topological group and A be a CW-complex. The isotropy groupoid $\mathfrak{I}(X)$ is called cellular if it is locally maximal and if $\tilde{\mathfrak{I}}_{\mathfrak{a}} = \tilde{\mathfrak{I}}_{\mathfrak{b}}$ when $e(a) = e(b)$.*

Let (X,π,φ) be a split Γ-space over the space A with isotropy groupoid $\mathfrak{I}$. Let $\eta : (E \to X)$ be a split Γ-equivariant G-principal bundle over the space X.

$$\begin{array}{ccc} \varphi^*(\eta) & \to & E \\ \downarrow & {}^{\tilde{\varphi}}\nearrow & \downarrow \\ A & \xrightarrow{\varphi} & X \end{array}$$

Since $\varphi^*(\eta)$=$(A \times G)$ is trivial, there exists a continuous lifting $\tilde{\varphi^*}(\eta) : A \to E$ of φ. The equation

$$\gamma\tilde{\varphi}(a) = \tilde{\varphi}(a)\alpha_a(\gamma),$$

(valid for $a \in A$ and $\gamma \in \mathfrak{I}_{\mathfrak{a}}$,) determines a continuous representation $\alpha_{\eta,\tilde{\varphi}} : \mathfrak{I} \to G$. Hambleton and Hausmann [5] checked that α_η does not depend on the choices $\tilde{\varphi}$ and depends only on the Γ-equivariant isomorphism class of η. Then, classification theorem is the following.

Theorem (Classification Theorem of Hambleton and Hausmann)**.** *Let (X,π,φ) be a split-Γ space over the orbit space A with the isotropy groupoid $\mathfrak{I}$. Assume that A is locally compact, the group Γ is a compact Lie group and $\mathfrak{I}$ is locally maximal. Then for any compact Lie group G, the map*

$$\Phi : SBun_\Gamma^G \to Rep^G(\mathfrak{I})$$

is a bijection.

Now, S^2 is the CW-complex,let $X \subset S^2$ be Γ-equivariant 1-*skeleton* the theorem can be used to classify equivariant principal G-bundles over X. For the subgroups of the $SO(3)$ (except the dihedral group), the orbit spaces are contractible and para-compact. It follows that all equivariant bundles over X are the split bundles. For the dihedral group while we can use the same method we need to address the fact not all equivariant principal G bundles are split. In the theorem, the group is a compact Lie group. If the group G is abelian, then the non-split bundle space over the space X is isomorphic to the bundle space over the orbit space A since the projection maps are equivariant.

Proposition 4.1.2. *[5, p,137] Let (X,π,φ) be a split-Γ space over the orbit space A with the isotropy groupoid $\mathfrak{I}$, and Γ a compact Lie group. Suppose that $\mathfrak{I}$ is locally maximal and that A is a locally compact space. If the group G is a compact abelian group, then one has an isomorphism of abelian groups*

$$(\Phi, \varphi*) : Bun_{\Gamma}^{G}(X) \to Rep^{G}(\mathfrak{I}) \times Bun^{G}(A).$$

4.2 1-skeletons of S^2

After definitions and theorems, we will introduce the crucial theorem. For different subgroups of $SO(3)$, there exists a Γ-equivariant 1-*skeleton* which is a split Γ-space. Hence, if it is a split Γ-space, then equivariant principal G-bundles over the CW-complex will be classified by paper of Hambleton and Hausmann [5].

Theorem 1. *Let $\Gamma \subset SO(3)$ be a finite subgroup acting linearly on S^2. Then there exists a Γ-equivariant 1-*skeleton *$X \subset S^2$, which is a split Γ-space over $A = X/\Gamma$.*

Proof. We will prove this theorem by induction and directly. We need the induction method for the cyclic and the dihedral groups. For the other subgroups of the $SO(3)$, it will be proved by direct computation.

Finite Subgroups of $SO(3)$

(i) Cyclic Subgroups

(ii) Dihedral Subgroups

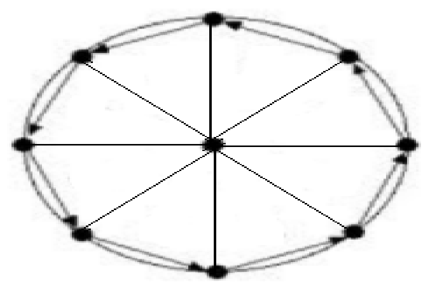
(iii) Tetrahedral Subgroup

(iv) Octahedral Subgroup

(v) Icosahedral Subgroup

4.2.1 Cyclic Subgroups

Let $\mathfrak{C}_\mathfrak{n}$ be a *1-skeleton* over S^2 such that it is composed of 2 vertices and n edges , $\mathbb{C}_n$ be a cyclic group containing n elements and acting on $\mathfrak{C}_\mathfrak{n}$ and E_n be the orbit space of $\mathfrak{C}_\mathfrak{n}$ under the group action of $\mathbb{C}_n$. We will look the CW- complex $\mathfrak{C}_\mathfrak{n}$ illustrated below as a projection from the north pole to the xy-plane.



The CW-complex $\mathfrak{C}_\mathfrak{n}$ is a split-$\mathbb{C}_n$ space It is proven by induction that E_n and $\mathfrak{C}_\mathfrak{n}$ satisfy the *proposition of Riemann-Hurwitz Formula*

$$\chi(\mathfrak{C}_\mathfrak{n}) = n\chi(E_n) - \sum_{p \epsilon E_n} (|\mathbb{C}_{np}| - 1)$$

and by definition of Euler Characteristic

$$\chi(\mathfrak{C}_\mathfrak{n}) = 2 - n \qquad \chi(E_n) = 1$$

the claim is that

$$\chi(\mathfrak{C}_{\mathfrak{n}+1}) = \chi(\mathfrak{C}_\mathfrak{n}) - 1 \qquad \chi(E_{n+1}) = \chi(E_n)$$

After we add one edge to the $\mathfrak{C}_\mathfrak{n}$, the new CW-complex become the $\mathfrak{C}_{\mathfrak{n}+1}$. Furthermore, we will subtract one from Euler characteristic since vertices are fixed in the north and south poles. On the other hand, the orbit space E_n does not change since the orbit space E_{n+1} is actually composed of one edge and two vertices.

if $n = 2$, $\Gamma = \mathbb{C}_2$, then $\chi(\mathfrak{C}_2) = 0$, $\chi(E_2) = 1$ and

$$\chi(\zeta_n) = 2 - n$$

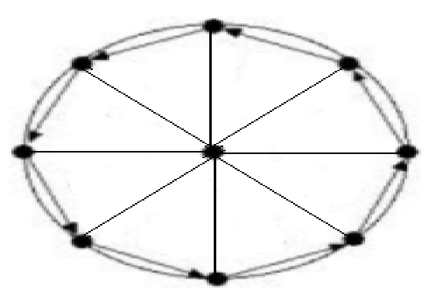
$\chi(\mathfrak{C}_{n+1}) = 1 - n$ since the CW-complex $\mathfrak{C}_{n+1}$ has 2 vertices and $n+1$ edges. Therefore, $\chi(\mathfrak{C}_{n+1}) = \chi(\mathfrak{C}_n) - 1$, $\mathfrak{C}_{n+1}$ and E_{n+1} satisfies the *proposition of Riemann-Hurwitz Formula.*

Then we can conclude that there is a continuous section from orbit space to $\mathfrak{C}_n$. It can be said that the isotropy groupoid of the CW-complex $\mathfrak{C}_n$ is $\mathfrak{I}$ cellular since it is defined by $\mathfrak{I}_0$=$\mathfrak{I}_1$=$\mathbb{C}_n$ and $\mathfrak{I}_{01}$ =id and $\mathfrak{I}_a$ is constant on the interior of each cell [5, p.141].

4.2.2 Dihedral Subgroups

Let $\mathfrak{D}_n$ be a *1-skeleton* over S^2 such that it is composed of $2n + 2$ vertices, $6n$ edges since the CW-complex $\mathfrak{D}_n$ is actually composed of n-gon (n vertices and n edges) stated at equator of a sphere.

Let D_{2n} be a dihedral group of order $2n$ acting on the CW-complex $\mathfrak{D}_n$ and we should add $n + 2$ vertices and $5n$ edges on the CW-complex $\mathfrak{D}_n$ to provide regular CW-complex where n is the number of vertices of polygon. Let $\mathcal{D}_n$ be the orbit space of $\mathfrak{D}_n$ under the group action of D_{2n} .



The CW-complex $\mathfrak{D}_n$ is a split-D_{2n} space, it is proven by induction since the orbit space $\mathcal{D}_n$ and the CW-complex $\mathfrak{D}_n$ satisfy the *Riemann-Hurwitz Formula.*

$$\chi(\mathfrak{D}_n) = 2n\chi(\mathcal{D}_n) - \sum_{x\epsilon\mathfrak{D}_n}(|D_{2np}| - 1)$$

$$\chi(\mathfrak{D}_n) = (2n+2) - 6n = 2 - 4n \quad \chi(\mathcal{D}_n) = 0$$

the claim is that

$$\chi(\mathfrak{D}_{n+1}) = \chi(\mathfrak{D}_n) - 4 \quad \textit{and} \quad \chi(\mathcal{D}_{n+1}) = \chi(\mathcal{D}_n)$$

After we add one vertex to the CW-complex $\mathfrak{D}_n$, we will subtract four from the Euler characteristic of $\mathfrak{D}_n$ to construct $\mathfrak{D}_{n+1}$. Indeed, one vertex is not sufficient since it is necessary to construct a regular CW-complex. Therefore, it needs 2 extra vertices different than the first vertex which is added first. Otherwise it would not be a regular CW-complex. On the other hand, the orbit space does not change after adding vertices and edges since the orbit space is actually composed of 3 edges and 3 vertices as well. If $n = 2$, then $\Gamma = D_4$, the CW-complex $\mathfrak{D}_2$ is 1-skeleton on S^2 such that it has square stating on equator of S^2. Let the vertices of the square be named 1,2,3,4.

$$\Gamma = D_4 = \{(), (1234), (1423), (1432), (14)(23), (12)(34), (13), (24)\}$$

then

$$\chi(\mathfrak{D}_2) = 6 \qquad \chi(\mathcal{D}_2) = 0$$

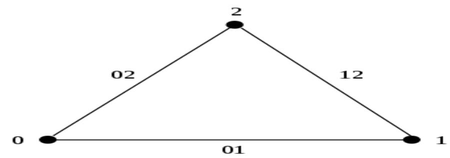
and

$$\chi(\mathfrak{D}_n) = 2 - 4n$$

$\chi(\mathfrak{D}_{n+1}) = -2 - 4n$ since $\mathfrak{D}_{n+1}$ has $2n + 4$ vertices and $6n + 6$ edges. Therefore,

$$\chi(\mathfrak{D}_{n+1}) = \chi(\mathfrak{D}_n) - 4,$$

$\mathfrak{D}_{n+1}$ and $\mathcal{D}_{n+1}$ satisfy the *Riemann-Hurwitz Formula.*

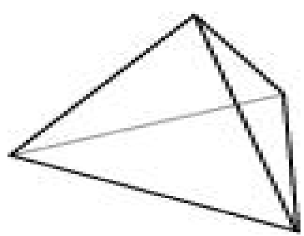


Then we can find a continuous section from orbit space to base space. We can say that the isotropy groupoid of $\mathfrak{D}_\mathfrak{n}$ is $\mathfrak{I}$ since it is defined by $\mathfrak{I}_0$=$\mathfrak{I}_1$=$\mathbb{C}_2$, $\mathfrak{I}_2$=$\mathbb{C}_n$and $\mathfrak{I}_{01}$ = $\mathfrak{I}_{12}$=$\mathfrak{I}_{02}$=id.

4.2.3 Tetrahedral Subgroup

Let $\mathfrak{T}$ be a tetrahedron and A_4 be a tetrahedral group of order 12 acting on $\mathfrak{T}$ and let T be an orbit space of $\mathfrak{T}$ under the group action of A_4.

$A_4 = \{(), (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243)\}$



Firstly, a tetrahedron has 4 vertices, and vertices of $\mathfrak{T}$ are named by 1,2,3,4. We will add extra vertices and edges to provide a regular CW-complex. It is necesary to put 6 vertices in the middle of edges and 4 vertices in the middle of the faces. Finally, $\mathfrak{T}$ has 14 vertices and 24 edges.

$$\chi(\mathfrak{T}) = -10$$

The tetrahedron has 4 vertex-rotation of order 3 (i.e. $(123), (142), (143), (243)$) and 3 edge-rotation order of 2 (i.e.$(12)(34), (13)(24), (14)(23)$). Therefore, the orbit space T is composed of 2 vertices and 1 edge. The tetrahedron is split-A_4 : space we can prove this by the *Riemann-Hurwitz Formula*.

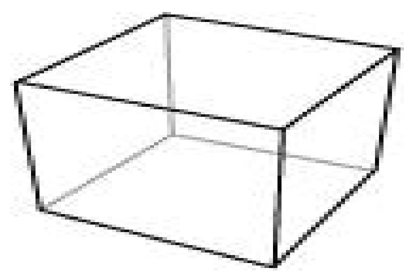
$$-10 = 12\chi(T) - (4(2+2) + 3(1+1))$$

$$\chi(T) = 1$$

Then we can find a continuous section from *orbitspace* to $\mathfrak{T}$. We can say that the isotropy groupoid of $\mathfrak{T}$ is $\mathfrak{I}$ cellular since it is defined by $\mathfrak{I}_0$=$\mathbb{C}_2$ $\mathfrak{I}_1$=$\mathbb{C}_3$ and $\mathfrak{I}_{01}$ =id .

4.2.4 Octahedral Subgroup

Let $\mathcal{C}$ be a cube and $\mathfrak{O}$ be an octahedral group of order 24 and acting on the cube $\mathcal{C}$ and O be the orbit space of $\mathcal{C}$ under the group action of $\mathfrak{O}$. The octahedral group $\mathfrak{O}$ can be considered as a subgroup of S_8.



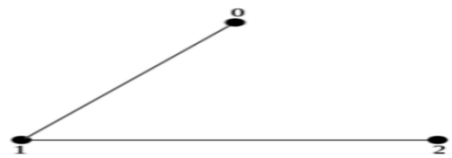
Firstly, the cube $\mathcal{C}$ has 8 vertices and 12 edges. We must add extra vertices and edges to satisfy the regularity of CW-complex. It is necessary to add 12 vertices in the middle of edges and 6 vertices in the middle of faces. Finally, the cube $\mathcal{C}$ has 26 vertices and 48 edges.

$$\chi(\mathcal{C}) = -22$$

The cube $\mathcal{C}$ has 4 vertex-rotation of order 3, 6 edge-rotation of order 2, and 3 face-rotation of order 4. Therefore, the orbit space O is composed of 3 vertices and 2 edges. The cube $\mathcal{C}$ is a split-$\mathfrak{O}$ space since it satisfies the *Riemann-Hurwitz Formula.*

$$-22 = 24\chi(O) - (3(3+3) + 4(2+2) + 6(1+1))$$

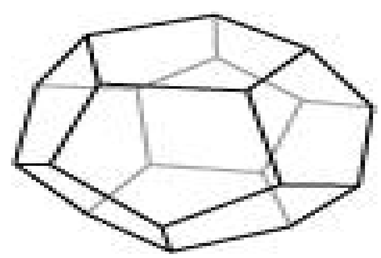
$$\chi(O) = 1$$



Then we can construct a continuous section from orbit space to $\mathcal{C}$. We can say that the isotropy groupoid of the cube $\mathcal{C}$ is $\mathfrak{I}$ cellular since it is defined by $\mathfrak{I}_0$=$\mathbb{C}_2$, $\mathfrak{I}_1$=$\mathbb{C}_3$, $\mathfrak{I}_2$=$\mathbb{C}_4$, $\mathfrak{I}_{01}$ =id and $\mathfrak{I}_{12}$ =id .

4.2.5 Icosahedral Subgroup

Let $\mathcal{I}$ be a Icosahedron and H be an icosahedral group of order 60 acting on $\mathcal{I}$ and let I be an orbit space of $\mathcal{I}$ under the group action of H. The icosahedral group H can be considered as a subgroup of S_{12}.



Firstly, the icosahedron $\mathcal{I}$ has 12 vertices and 30 edges. (20 faces which is useful for adding extra vertices) We will add extra vertices and edges to provide regular CW-complex. It is necessary to put 30 vertices in the middle of edges and 20 vertices in the middle of faces. Finally, the dodecahedron $\mathcal{I}$ has 62 vertices and 120 edges.

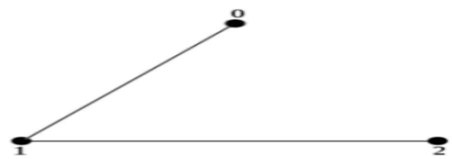
$$\chi(\mathcal{I}) = -58$$

The icosahedron $\mathcal{I}$ has 6 vertex-rotation of order 5, 15 edge-rotation of order 2, and 10 face-rotation of order 3. Then, the orbit space I is composed of 3 vertices and 2 edges. Therefore, the icosahedron $\mathcal{I}$ is a split-H space: we show this by the *Riemann-Hurwitz*

Formula.

$$-22 = 60\chi(I) - (6(4+4) + 15(2+2) + 10(1+1))$$

$$\chi(I) = 1$$

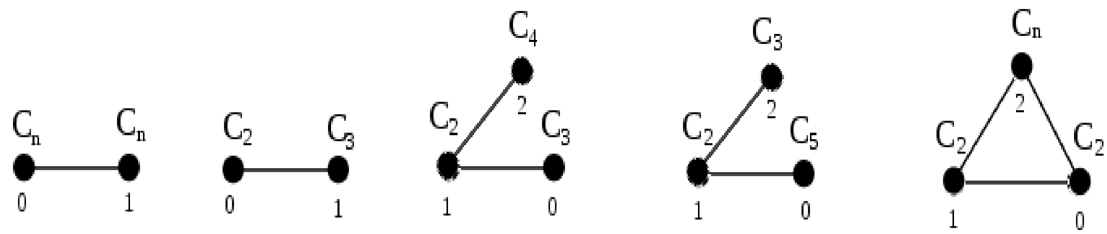


Then we can construct a continuous section from orbit space to $\mathcal{I}$. It is said that the isotropy groupoid of the dodecahedron $\mathcal{I}$ is $\mathfrak{I}$ cellular since it is defined by $\mathfrak{I}_0$=$\mathbb{C}_2$, $\mathfrak{I}_1$=$\mathbb{C}_3$, $\mathfrak{I}_2$=$\mathbb{C}_5$, $\mathfrak{I}_{01}$ =id and $\mathfrak{I}_{12}$ =id . $\square$

Now, we know that for each finite subgroup of $SO(3)$ we have found a *1-skeleton* CW-complex on S^2 such that it *splits* over A. In [5], Hambleton and Hausmann proved that there is a bijection between split Γ-bundle space of X and isotropy representation of the groupoid if the space X splits under some restrictions.

4.3 Classification of Principal G-bundles Over 1-Skeletons of S^2

Although we know the general theory about the classification of equivariant principal bundles from [5] and [4]. The computation of $Rep^G(\mathcal{I})$ is generally hard. On the other hand, we can compute $Rep^G(\mathcal{I})$ since the isotropy groupoid of X and the orbit spaces of X under $\Gamma \subset SO(3)$ are simpler than the other spaces. In the proof of theorem 1, we chose special 1-skeletons for the different subgroups of $SO(3)$. After that, the orbit spaces are different than each other. The common point of the cyclic group, the tetrahedral group, the icosahedral group, the dodecahedral group is that their orbit spaces are homeomorphic to [-1,1]. Hence, their orbit spaces are contractible and paracompact. Every Γ-equivariant principal-G bundle is a *split*-Γ *space* where the orbit space A is a contractible and paracompact space. For the other case, the orbit space A is a triangle (in other words A is a graph) then there is a bijection between split bundle space of the CW-complex X and the isotropy representation of groupoid $\mathfrak{I}$. In this case, there exist some non-split bundle over the space X. If the group G is abelian, then there is an isomorphism between the bundle spaces of X and $Rep^G(\mathcal{I}) \times Bun^G(A)$. The orbit spaces of the different CW-complexes follow from the figure. We can say that all the orbit spaces are paracompact since all the orbit spaces are compact in our problem.



However, we shall separate two cases since one of the orbit space is not contractible.

(i) The orbit space A is contractible.

(ii) The orbit space A is not contractible.

4.3.1 Contractible Case

All of the equivariant bundles over the space X are split bundles, since the orbit space of the space X (where the group is one of the cyclic group, the tetrahedral group, the octahedral group, and icosahedal group) is contractible and paracompact. Therefore, we will use the classification theorem which is proven by Hambleton and Hausmann [5]. Since, the possible specific spaces ($*$,π,φ) is split over its orbit spaces. All the possible orbit spaces are locally compact, all the possible isotropy groupoids are locally maximal and the group Γ is a compact Lie group. Thus

$$\Phi : SBun^G_\Gamma \to Rep^G(\mathfrak{I})$$

is a bijection for all case where the orbit space is contractible.

4.3.2 Non-contractible Case

The bijection given is available for the dihedral case since the CW-complex $\mathfrak{D}_n$ is split over the orbit space $\mathcal{D}_n$ as well. But we cannot guarantee that all equivariant principal G bundles are split over the space X. (i.e. there may exist some non-split bundles over the space X). For the non-split bundles over the space X, we should restrict our conditions. If we choose abelian G, then we can use the proposition of the theorem. The space ($\mathfrak{D}_n$,π,φ) is a split-D_{2n} space over $\mathcal{D}_n$ with the isotropy groupoid $\mathfrak{I}$, and $\mathcal{D}_n$ is locally compact and $\mathfrak{I}$ is locally maximal. Then for any abelian Lie group G , one has an isomorphism of

$$(\Phi, \varphi*) : Bun_{\Gamma}^{G}(X) \to Rep^{G}(\mathfrak{I}) \times Bun^{G}(A).$$

Since the orbit space $\mathcal{D}_n$ is a triangle, the orbit space $\mathcal{D}_n$ is homeomorphic to circle S^1. We know the bundles over the circle S^1, it induce map

$$\overline{(\Phi, \varphi*)} : Bun_{\Gamma}^{G}(X) \xrightarrow{\approx} Rep^{G}(\mathfrak{I}) \times Bun^{G}(S^1).$$

If the group G is a connected compact Lie group, it follows

$$[S^1, BG] \cong \pi_1(BG) \cong \pi_0(G) \cong 0$$

If the group G is not connected but still a compact Lie group, it follows

$$[X, S^1] \cong \pi_1(BG) \cong \pi_0(G).$$

Now, we will define some new concepts to can calculate $Rep^{G}(\mathfrak{I})$ easier.

4.4 Calculation of $Rep^{G}(\mathfrak{I})$

Let $\iota : \mathfrak{I} \to G$ be an isotropy representation and let $\mathfrak{I}$ be a (Γ,A)-groupoid. The isotropy representation ι is called cellular if $\iota_a = \iota_b$ when $e(a) = e(b)$. For each $e \in$

$\Omega(A)$, this defines Hom$(\mathfrak{I}_e, G)$ with face compatibility conditions $\iota_e = \iota_f \mid \mathfrak{I}_e$ whenever $f \leq e$. The set of conjugacy classes of cellular representations of $\mathfrak{I}$ into G is denoted by Rep^G_{cell} $(\mathfrak{I})$. For a cellular representation $\iota : \mathfrak{I} \to G$ and a cell e of A, we can associate its conjugacy class $[\iota_e] \in \overline{Hom}$ $(\mathfrak{I},G)$. It follows that there is a map β

$$\beta : Rep^G_{cell}(\mathfrak{I}) \to \prod_{e \in \Omega(A)} \overline{Hom}(\mathfrak{I}_e, G).$$

If we pick an element b_e from this product, it must satisfy the face compatibility condition. Now, we will define

$$\overline{Rep}^G_{cell}(\mathfrak{I}) = \{(b_e) \in \prod_{e \in \Omega(A)} \overline{Hom}(\mathfrak{I}_e, G) | \quad b_e = b_f|\mathfrak{I}_e \ \ for \ \ f \leq e\}$$

and we can replace β as a map $\beta :$ $Rep^G_{cell}(\mathfrak{I}) \to \overline{Rep}^G_{cell}(\mathfrak{I})$. Then the following diagram is commutative,

$$\begin{tikzcd} Rep^G_{cell}(\mathfrak{I}) \arrow[rr, "\tau"] \arrow[dr, "\beta"'] & & Rep^G(\mathfrak{I}) \arrow[dl, "\upsilon"] \\ & \overline{Rep}^G_{cell}(\mathfrak{I}) & \end{tikzcd}$$

when $\mathfrak{I}$ is proper $(\Gamma$,$A)$-groupoid.

Theorem 4.4.1. *Let $X \subset S^2$ $A = X/\Gamma$ Let $\Gamma \subset SO(3)$ be a finite subgroup and G be a topological group. Let $\mathfrak{I}$ be a $(\Gamma$,$A)$-groupoid, where A is an orbit space. Then β : $Rep^G_{cell}(\mathfrak{I}) \to \overline{Rep}^G_{cell}(\mathfrak{I})$ is surjective.*

Proof. We gave before the possible finite subgroups of $SO(3)$. The orbit spaces of subgroups are either a tree or not a tree (for the dihedral case the orbit space is triangle). One can find the detailed proof in [4] when the orbit space is a tree. Now, we only prove the dihedral case. Let $b \in \overline{Rep}^G_{cell}(\mathfrak{I})$ and let v be a vertex of A. Choose ι_v *in* $Hom(\mathfrak{I}_{\mathfrak{v}}, G)$ representing b_v. For an edge e between v and v' we define $\iota_e \in Hom(\mathfrak{I}_{\mathfrak{e}}, G)$ by $\iota_e = \iota_v \mid \mathfrak{I}(e)$. Since t $b \in \overline{Rep}^G_{cell}(\mathfrak{I})$, we can choose ι'_v *in* $Hom(\mathfrak{I}'_{\mathfrak{v}}, G)$ where $\iota'_{v|} = \iota_e$. Therefore, we can construct a cellular representation ι^1 over the tree $A(v, 1)$ of the points of distance ≤ 1 from v. By this method, we can construct ι^2 over

$A(v, 2)$. Now, when one defines to $A(v, 3)$, we choose the points of distance < 3 from v. Now, this defines $\iota \in Rep^G_{cell}(\mathfrak{I})$ with $\beta(\iota) = b$. $\square$

Proposition 4.4.2. *Let $\mathfrak{I}$ is a (Γ,A)-groupoid, where $\Gamma \subset SO(3)$ and the orbit space A is a graph. Let G be a path-connected topological group. Then $\upsilon : Rep^G(\mathfrak{I}) \to \overline{Rep}^G_{cell}(\mathfrak{I})$ is surjective.*

Proof. One can use the classification theorem and the paper of Hambleton and Hausmann [5, p,138]. $\square$

Theorem 4.4.3. *Let $\mathfrak{I}$ be a proper (Γ,A)-groupoid with $\Gamma \subset SO(3)$ a finite topological group and let A be an orbit space. Let G be a compact connected Lie group. Then $\tau : Rep^G_{cell}(\mathfrak{I}) \to Rep^G(\mathfrak{I})$ is bijective.*

Proof. Let $\tau(\alpha) = \tau(\alpha')$ with the two cellular representations. One can see $\alpha = \alpha'$ by taking the conjugate of the one of them. For detailed proof one can look [4, p.251] . τ is surjective. If we choose $a \in Rep^G(\mathfrak{I})$ then in our case, it will be cellular representation since isotropy group is different than identity only vertices it is identity in edges. $\square$

We use the theorem in the paper of Hambleton and Hausmann [4, p.252] [theorem b]. Let κ_1,$\kappa_2 : \mathfrak{I} \to G$ be two isotropy representations such that $\kappa_1 = \kappa_2 \mid_{\mathfrak{I}_e} =: \kappa$. Then $\tau^{-1}(\kappa_1,\kappa_2)$ is in bijection with the set of double cosets $\pi_0(Z_{\kappa_1})\backslash\pi_0(Z_\kappa)/\pi_0(Z_{\kappa_2})$.

4.5 Classification of equivariant bundles on the 1-skeleton

Now, we know the relation between $Bun^G_\Gamma(X)$ and $Rep^G(\mathfrak{I})$ and we can calculate $Rep^G(\mathfrak{I})$. Hence, we classify the equivariant principal G-bundles over X where $X \subset S^2$ is the 1-*skeleton* of a regular Γ structure for S^2. We shall consider $\Gamma = C_n$, D_{2n}, A_4, $\mathfrak{O}$, H.

Theorem 1. *Let* $X = \mathfrak{C}_\mathfrak{n}$ *be a* Γ*-equivariant* 1-skeleton *over* S^2 *with* 2 *vertices and* n *edges ,* C_n *be a cyclic group of order* n *and acting on* $\mathfrak{C}_\mathfrak{n}$ *and* E_n *be the orbit space of* $\mathfrak{C}_\mathfrak{n}$ *under the group action of* C_n *with isotropy groupoid* $\mathfrak{I}_1$*. Then, there is a bijection*

$$Bun^G_{C_n}(\mathfrak{C}_\mathfrak{n}) \to Rep^G(\mathfrak{I}_1)$$

and

$$Rep^G(\mathfrak{I}_1) \cong \overline{Rep}^G(\mathfrak{I}_1) \cong \overline{Hom}(C_n, G) \times \overline{Hom}(C_n, G).$$

Proof. $\mathfrak{C}_\mathfrak{n}$ is a split C_n-space, all equivariant bundles are split bundles. □

Theorem 2. *Let* $X = \mathfrak{D}_n$ *be a* 1-skeleton *over* S^2 *with* $2n+2$ *vertices,* $6n$ *edges and let* D_{2n} *be a dihedral group of order* $2n$ *acting on the CW-complex* $\mathfrak{D}_n$ *and let* $\mathcal{D}_n$ *be an orbit space with isotropy groupoid* $\mathfrak{I}_5$*. If* G *is abelian and connected, Then there is a bijection*

$$Bun^G_{D_{2n}}(\mathfrak{D}_n) \to Rep^G(\mathfrak{I}_2)$$

and

$$Rep^G(\mathfrak{I}_2) \cong \overline{Rep}^G(\mathfrak{I}_2) \cong \overline{Hom}(C_2, G) \times \overline{Hom}(C_2, G) \times \overline{Hom}(C_n, G).$$

Proof. $\mathfrak{D}_n$ is split D_{2n}-space and

$$Bun^G_\Gamma(X) \to Rep^G(\mathfrak{I}) \times Bun^G(A)$$

since the group G is a connected compact Lie group, it follows

$$[S^1, X] \cong \pi_1(BG) \cong \pi_0(G) \cong 0.$$

□

Theorem 3. *Let* $X = \mathfrak{T}$ *be a tetrahedron and* A_4 *be the tetrahedral group of order 12 acting on* $\mathfrak{T}$ *and let* T *be an orbit space of* $\mathfrak{T}$ *under the group action of* A_4 *with isotropy groupoid* $\mathfrak{I}_3$*. Then, there is a bijection*

$$Bun^G_{A_4}(\mathfrak{T}) \to Rep^G(\mathfrak{I}_3)$$

and

$$Rep^G(\mathfrak{I}_3) \cong \overline{Rep}^G(\mathfrak{I}_3) \cong \overline{Hom}(C_2, G) \times \overline{Hom}(C_3, G).$$

Proof. $\mathfrak{T}$ is a split A_4-space, all equivariant bundles are split bundle. □

Theorem 4. *Let $X = \mathcal{C}$ be a cube and $\mathfrak{O}$ be the octahedral group of order 24 and acting on the cube $\mathcal{C}$ and O be the orbit space of $\mathcal{C}$ under the group action of $\mathfrak{O}$ with isotropy groupoid $\mathfrak{I}_4$. Then, there is a bijection*

$$Bun^G_{\mathfrak{O}}(\mathcal{C}) \to Rep^G(\mathfrak{I}_4)$$

and

$$Rep^G(\mathfrak{I}_4) \cong \overline{Rep}^G(\mathfrak{I}_4) \cong \overline{Hom}(C_2, G) \times \overline{Hom}(C_3, G) \times \overline{Hom}(C_4, G).$$

Proof. $\mathcal{C}$ is a split $\mathfrak{O}$-space, all equivariant bundles are split bundle. □

Theorem 5. *Let $X = \mathcal{I}$ be a icosahedron and H be an icosahedral group of order 60 acting on $\mathcal{I}$ and let I be an orbit space of $\mathcal{I}$ under the group action of H with isotropy groupoid $\mathfrak{I}_5$. Then, there is a bijection*

$$Bun^G_H(\mathcal{I}) \to Rep^G(\mathfrak{I}_5)$$

and

$$Rep^G(\mathfrak{I}_5) \cong \overline{Rep}^G(\mathfrak{I}_5) \cong \overline{Hom}(C_3, G) \times \overline{Hom}(C_4, G) \times \overline{Hom}(C_5, G).$$

Proof. $\mathcal{I}$ is a split H-space, all equivariant bundles are split bundle. □

4.6 Classification of Γ-G bundles over S^2

We classified Γ-G bundles over the Γ-equivariant 1-skeleton S^2. If $X \subset S^2$ be a Γ-equivariant 1-skeleton, then Γ-equivariant principal G-bundles over X are classified by the isotropy representation since X is split-Γ space. However, for S^2, a different technique is used to determine Γ-equivariant principal G-bundles over S^2 since S^2 is not a split-Γ space. Now, let $X \subset S^2$ be Γ-equivariant 1-skeleton, then

$$X \xrightarrow{i} S^2 \xrightarrow{j} S^2 \cup CX \xrightarrow{k} \Sigma(X) \xrightarrow{m} \Sigma(S^2) \xrightarrow{l} \Sigma(S^2 \cup CX) \to \cdots \tag{4.1}$$

is the cofibration sequence of Γ-equivariant CW-complexes where CX= cone on $X = (X \times [0,1])/(a,0) \sim pt$ and $\Sigma(S^2)$= suspension of $S^2 = (X \times [-1,1])/(a,-1) \sim pt,\ (a,1) \sim pt$

The Γ-fixed set of homotopy classes of maps into space $B(\Gamma, G)$,then the following sequence

$$[\Sigma(S^2), Y] \xrightarrow{m^*} [\Sigma(X), Y] \xrightarrow{k^*} [S^2 \cup CX, Y] \xrightarrow{j^*} [S^2, Y] \xrightarrow{i^*} [X, Y] \tag{4.2}$$

is the exact sequence of abelian groups provided that $B(\Gamma, G) = Y = \Omega Z$ is a loop space which is defined by Costenable and Waner at [3]. Now, $[S^2, Y]$ is determined by j^* and i^*. Here, $S^2 \cup CX \simeq \bigvee S^2$ (induced from 2-cells) and $\Sigma(X) \simeq \bigvee S^2$ (induced from 1-cells) then the exact sequence at 4.2 will be the following

$$[\bigvee S^2, Y] \xrightarrow{k^*} [\bigvee_{2-cells} S^2, Y] \xrightarrow{j^*} [S^2, Y].$$

(4.3)
$$[\bigvee S^2, Y]_{\Gamma \neq D_{2n}} \simeq \bigoplus_{1-\chi(X)} \pi_2(BG) \; and \; \Gamma \; acts \; on \; product \; = I \otimes \pi_1(G) \; as \; a \; \Gamma - module$$

$$[\bigvee S^2, Y] \xrightarrow{k^*} [\bigvee_{2-cells} S^2, Y] \xrightarrow{j^*} [S^2, Y]$$

counting Γ-orbits of 2-cells

(4.4)
$$[\bigvee S^2, Y]_{\Gamma = D_{2n}} \simeq \bigoplus_{1-\chi(X)} \pi_2(BG) \; and \; \Gamma \; acts \; on \; product \; = (I \oplus \mathbb{Z}\Gamma) \otimes \pi_1(G) \; as \; a \; \Gamma - module$$

(4.5)
$$[\bigvee_{2-cells} S^2, Y]_\Gamma \simeq \bigoplus_{\mathcal{N}} \pi_2(BG) \; and \; \Gamma \; acts \; on \; product \; = \mathbb{Z}\Gamma \otimes \pi_1(G) \; as \; a \; \Gamma - module$$

provided that the ideal $I = \mathbb{Z}\{(\gamma - 1)|\gamma \in \Gamma\} \subset \mathbb{Z}\Gamma$.

The number of copies of $\pi_2(BG)$ for 4.4 will be calculated by counting rotations and order of groups.

For cyclic case, we have 1-skeleton containing 2 vertices and n edges. Then if one edge collapse to a point, then other edges will become S^1-circles, then we have $(n-1)$ S^1-circles. These circles will become sphere if we get suspension of X.

The same idea for the dihedral group, since there is an orbit with $2n$ elements after an edge collapse the point, there will be $(2n-1)$ S^1-circles. The other $2n$ orbits have 2 elements, after one edge collapse a point, there will be $(2n)$ S^1-circles. Therefore there are $(4n-1)$ circles for the dihedral case.

For the tetrahedral group, there are 4-vertex rotations with order 3, after one edge collapse to a point for each rotation, there will be 8 circles. There are 3- edge rotations order with 2. There will be 3-circles. Totally, we have 11 circles for the tetrahedral group.

For octahedral group, there are 4-vertex rotations with order 3, after one edge collapse to a point for each rotation, there will be 8 circles. There are 6- edge rotations order with 2. There will be 6-circles.There are 3 face rotations order with 4. After collapsing, there will be 9-circles. Totally, we have 23 circles for the octahedral group. Therefore, we use same idea for the icosahedral group , we have 59 circles for the icosahedral group. Briefly, we say that there are $1 - \chi(X)$ where $X \subset S^2$ is Γ-equivariant 1-skeleton.

The number of copy of $\pi_2(BG)$ for 4.5 is denoted by $\mathcal{N}$ and depends on the number of orbits and the order of group Γ. The only case for the dihedral group, there are two orbits and the other finite subgroup of $SO(3)$ have a single orbit.

Group	1-$\chi(X)$	$\mathcal{N}$
Cyclic group	$n-1$	n
Dihedral group	$4n-1$	$4n$
Tetrahedral group	11	12
Dodecahedral group	23	24
Icosahedral group	59	60

Since k^* is an injective map, the following map holds;

$$0 \to I \otimes \pi_1(G) \xrightarrow{k^*} \mathbb{Z}\Gamma \otimes \pi_1 G \xrightarrow{j^*} \mathbb{Z} \otimes \pi_1(G) \qquad if \quad \Gamma \neq D_{2n}$$

or

$$0 \to (I \oplus \mathbb{Z}\Gamma) \otimes \pi_1(G) \xrightarrow{k^*} (\mathbb{Z}\Gamma \oplus \mathbb{Z}\Gamma) \otimes \pi_1 G \xrightarrow{j^*} \mathbb{Z} \otimes \pi_1(G) \qquad if \quad \Gamma = D_{2n}.$$

$$Coker(k^*) \simeq \mathbb{Z} \otimes \pi_1(G) \simeq [S^2, B(\Gamma, G)]_\Gamma$$

$$[\Sigma(X), Y] \xrightarrow{k^*} [S^2 \cup CX, Y] \xrightarrow{j^*} [S^2, Y] \xrightarrow{i^*} [X, Y]$$

Now, let Z and Y be two Γ-space and $f : Z \to Y$ be continuous. Define $f^\gamma(z) = \gamma^{-1} f(\gamma z)$ $f \to f^\gamma$ gives an action of Γ on $[Z, Y]$. Then $f = f^\gamma \leftrightarrow \gamma f(z) = f(\gamma z) \leftrightarrow$ f is a Γ-map. Therefore, we shall say the following $[Z, Y]_\Gamma = Fix(\Gamma, [Z, Y])$.

$Fix(\Gamma, I) = \{x \in I | \gamma x = x\} = 0$ and we will determine $Fix(\Gamma, \mathbb{Z}\Gamma)$. Let $t \in \Gamma$ be a generator. For the cyclic group, Γ is acting on $\mathbb{Z}\Gamma = \mathbb{Z} \oplus \mathbb{Z}t \oplus \cdots \oplus \mathbb{Z}t^{n-1}$ where $|\Gamma| = n$. The fixed set of $\mathbb{Z}\Gamma$ will be determined by $(a_0 + a_1 t + \cdots + a_{n-1} t^{n-1})\gamma = (a_0 + a_1 t + \cdots + a_{n-1} t^{n-1})$ $\forall \gamma \in \Gamma$ implies that fixed elements are $\mathbb{Z}(1 + t + \cdots + t^{n-1})$. For other subgroups, we have more generators but the idea is the same. Then,

$Fix(\Gamma, \mathbb{Z}\Gamma) = \mathbb{Z}(1 \sum_{\gamma\in\Gamma} \gamma)$. Therefore,

$$[\Sigma X, Y]_\Gamma = Fix(\Gamma, I \otimes \pi_2 Y) = 0 \tag{4.6}$$

and

$$[S^2 \cup C(X), Y]_\Gamma = Fix(\Gamma, \mathbb{Z}\Gamma \otimes \pi_2 Y) = \pi_2 Y. \tag{4.7}$$

since in the sequence 4.2 , we have $[S^2, Y] = \pi_2 Y$

Theorem 4.6.1. *Let G be a compact Lie group, $\Gamma \subset SO(3)$ finite and ξ be a principal G-bundle over S^2. Γ-equivariant principal G-bundle over (S^2, Γ) is classified by $Rep^G_\Gamma(\xi)$ and $c(\xi) \in \pi_2(BG)$.*

Proof. Let $[\nu]$ and $[\xi] \in [S^2, Y]_\Gamma$. $[S^2, Y]_\Gamma \xrightarrow{i^*} [X, Y]_\Gamma$ and $[X, Y]_\Gamma \cong Rep^G_\Gamma(\mathfrak{I})$ If $Rep^G_\Gamma(\nu) \ncong Rep^G_\Gamma(\xi)$ then one concludes that they are not equivalent to each other. if $Rep^G_\Gamma(\nu) \cong Rep^G_\Gamma(\xi)$ then

$$[\Sigma X, Y]_\Gamma \to [S^2 \cup C(X), Y]_\Gamma \to [S^2, Y]_\Gamma$$

if we let Γ drop then we have $[S^2 \cup C(X), Y] = \pi_1(G)$ and $[S^2, Y] = \pi_1(G)$. Therefore the map k^* multiply each element of $\pi_1(G) by |\Gamma|. Now$, by 4.6 and 4.7, we have

$$0 \to \pi_1(G) \xrightarrow{|\Gamma|} [S^2, Y]$$

then they will be determined by first Chern class. □

Corollary 4.6.2. *If $Rep^G_\Gamma(\xi_1) \cong Rep^G_\Gamma(\xi_2)$ then $c(\xi_1) \equiv c(\xi_2)$ $mod|\Gamma|$.*

After this theorem, we can classify the equivariant principal bundles over 2-sphere. Later studies will be focusing on how we can generalize this theorem for S^n by these ideas.

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