CONTRIBUTIONS TO REVERSE LOGISTICS WITH GAME THEORETIC APPLICATIONS

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Abstract

The last two decades witnessed an increasing emphasis on reverse logistics (RL) which is both economically attractive and ecologically beneficial. Our thesis attempts to investigate a few research problems in RL and explore the application of game theoretic models in this field.

In Chapter 1, we introduce SCM and RL, game theoretic applications in SCM and RL, and the organizational structure of this thesis.

In Chapter 2, we address a retailer's single-period inventory problem with high rate of resalable customer returns. We first develop a three-subperiod basic model with order quantity as the single decision variable and conduct concavity analysis. We then develop a general model in which the retailer determines both order quantity and two inventory thresholds as an easy-to-follow reference for inter-period inventory control. We use simulation for sensitivity analysis and investigate the timing effect of both customer demands and customer returns on the retailer's decision making.

In Chapter 3, we explore the application of game theoretic models with incomplete information in inventory management. Games with incomplete information may provide a more realistic modeling framework. We hope that our exposition may help researchers interested in applying game theoretic models and computing the equilibriums in their specific problems in SCM and RL.

In Chapter 4 we consider a remanufacturing competition problem between an original equipment manufacturer (OEM) and a pure remanufacturer (REM) with the OEM's incomplete information on the REM's unit cost. We apply the type-III model in Chapter 3 for formulation and derive the closed-form Bayesian Nash equilibrium with the OEM's priority of accessing available shells. We use sensitivity analysis, both analytically and numerically, to investigate the effect of such incomplete information on both competitors' decision making.

In Chapter 5 we summarize our thesis and provide a general direction for future research on game theoretic applications in RL.

Dedication

I would like to dedicate this Doctoral thesis to Hai Wang, my beloved husband. I could not have completed this six-and-half-year journey without his love, patience, acceptance, counsole, encouragement, and support. The tremendous sacrifices he has made for me, ever since we met, have made this mission-impossible become true. More than half of this precious PhD degree belongs to him.

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Chapter 1

Introduction

This chapter is composed of three sections. Section 1.1 gives a brief introduction of supply chain management (SCM) and reverse logistics (RL). Section 1.2 presents the applications of game theoretical models in SCM and the potential for further application in RL. Section 1.3 explains the organizational structure of this thesis. Our introduction indicates that RL has been drawing growing attention of both academic researchers and industrial practitioners. We also show that game theoretic models, which have found wide applications in SCM, have potential to be applied in RL.

1.1 Reverse Logistics

Supply chain management has been a dominant topic in operations research / management science / operations management (OR/MS/OM) for about two decades. After being first introduced in the mid of 1980s, the terms Supply Chain (SC) and Supply Chain Management have been used in a very wide range in both academic circles and industrial community (Nikbakhsh [67]). As Chopra and Meindl [15] defined,

"*A supply chain consists of all stages involved, directly or indirectly, in fulfilling a customer request. The supply chain not only includes the manufacturer and suppliers, but also transporters, warehouses, retailers and customer themselves.*"

SCM, accordingly, involves all kinds of processes that a firm may use to control the SC related behaviors so that its predefined goals can be achieved (Nikbakhsh, [67]). The traditional domain of SC and SCM considers all kinds of "forward" handling or processing until a product is delivered to customers. Recovery or recycling of used products and materials, which is both economically competitive and environmentally beneficial, requires a material flow from the end customer back to manufacturers. Waste paper, used bottles and containers, and organic waste are materials collected for various reusing purposes in our daily life. The number of recycling stations for batteries and printer cartridges and even used cell phones keeps increasing. Professional brokers collect used items ranging widely from used auto tires, out-of-dated computers, second-hand textbooks, to name a few. All these phenomena are links of a "backward" material flow.

Reverse logistics is concerned with the management of material flow in the opposite direction to the traditional supply chain. The definition of RL varied over time. Viewed narrowly, as Fleischmann *et al.* [31] summarized, RL "*includes the logistics activities from used products no longer required by the user to products again usable in a market.*" Probably the first formal definition of RL was proposed by the Council of Logistics Management in 1992 which stresses the recovery aspects of RL:

"*... the term often used to refer to the role of logistics in recycling, waste disposal, and management of hazardous materials; a broader perspective includes all relating to logistics activities carried out in source reduction, recycling, substitution, reuse of materials and disposal.*" (Stock, [85])

A more comprehensive definition which emphasizes the goal and processes involved in RL was given by Rogers and Tibben-Lemke [73] in 1999:

"*RL is the process of planning, implementing, and controlling the efficient, cost-effective flow of raw materials, in-process inventory, finished goods and related information from the point of consumption to the point of origin for the purpose of recapturing value or proper disposal*".

A broader definition given more recently (2011) by the Reverse Logistics Association refers to RL as:

"*all activity associated with a product/service after the point of sale, the ultimate goal to optimize or make more efficient aftermarket activity, thus saving money and environmental resources.*" (Bernon and Rossi, [6])

Research on RL can be traced back to as early as the 60s and 70s. Publications in that age concerned more about solid waste recycling and its environmental implications (Pokharel and Mutha [72]). Papers about RL strategic models started to appear in the 80s. It is about two decades ago when RL started to emerge as an OR/MS/OM field (Stock, [85]).

The content of research covered under the field of RL is very extensive. In an earlier stage, Fleischmann *et al.* [31] reviewed the RL quantitative models from three perspectives — distribution planning, inventory management, and production planning — and pointed out how to integrate the backward flow of returned products and materials into traditional SCM models is the critical feature of RL research. Carter and Ellram [13] reviewed the RL literature from transportation, packaging, purchasing, and environmental aspects. More recently, Rubio *et al.* [75] did a review focusing on recovery production management and production planning, end-of-life products distribution, and inventory management. Guide and Van Wassenhove [39] did a review focusing on profitable value recovery from returned products. Pokharel and Mutha [72] reviewed the RL literature from an input-and-output perspective and divided RL research topics into four subgroups, namely, RL inputs, RL structure, RL processes, and RL outputs. Their review shows that deterministic models compose a major stream in RL research while the application of stochastic models is very limited. In spite of the appearance of a number of reviews for RL related literature, the large volume of RL research still calls for a systematic synthesis into a broad-based conceptual framework. Bernon and Rossi [6] presented a conceptual framework for managing retail RL operations. Lambert *et al.* [50] proposed a reverse logistics decisions conceptual framework in which a large variety of practical situations are considered. These two, to the best of our knowledge, are the most recent efforts that has been made to fill in this research gap.

We observe that, along with RL, there exist a few well-known OR/MS/OM terms which overlap with RL in various magnitude. They are closed-loop supply chain (CLSC), green supply chain management (GrSCM), and sustainable supply chain management (SSCM). Here we would like to briefly state the relations between RL and these terms.

- CLSC stresses one essential figure of SCM. That is to close the loop of a supply chain by integrating waste materials into logistic management decisions (Dyckhoff *et al.* [24]). Accordingly, researchers in CLSC take RL as an element of SCM for the reason that the backward material flows do not surpass the generalized concept of supply chain management (Dekker *et al.* [21]);

- GrSCM stresses on the environmental beneficial efforts made with SCM. As Srivastava [84] summarized, the scope of GrSCM includes the importance of GrSCM, green design, and green operations. RL, along with remanufacturing and waste management, belongs to the category of green operation;

- SSCM emphasizes the sustainable development of SCM and, as a matter of fact, has been used to a large extent as synonyms or near-synonyms to GrSCM. The concept of sustainable development — originally proposed by World Commission on Environment and Development in 1980s — includes three major dimensions — economic prosperity, social justice, and environmental quality (Nikbakhsh, [67]). John Elkington [25], in his book "*Cannibals with forks*", pointed out that these three dimensions are also the "triple bottom line" of the 21st century business featured with rapidly evolving capitalist economies. Correspondingly, SSCM, as its name implies, considers economic, social and environmental issues in supply chain management.

Based on the above, to sort these terms in the order of their research scope, we would like to put them as:

$$RL \subset CSCM \subset GrSCM \subset SSCM,$$

where "$A \subset B$" means A is a subset of B.

1.1.1 Customer Returns

Customer returns compose one of the upstream links along the RL backward material flow. Most available research on customer returns focus on return policies. Return policies allow consumers to return purchased products for refund under certain conditions. Comprehensive surveys on return policies can be found in David *et al.* [19] and Rogers and Tibben-Lemke [73]. Earlier studies reveal that return policies can benefit customer relationships by improving customer satisfaction and retention (Pokharel and Mutha [72]). Along with the e-business revolution in recent years, competition in retailing industry is more intensive. More and more retailers are adopting return policies as an effective competitive weapon. Meanwhile, the prevalence of return policies has also given rise to a remarkable growth in customer returns and this has brought new challenges to retailers' inventory management, a crucial managerial aspect in both RL and the retailing industry. Research efforts assessing customer returns in inventory management started to appear in early 2000s, although the volume so far is still limited.

For a retailer's strategic decision making, the remarkable growth in customer returns requires a deliberate consideration on how to involve the return flows into their inventory management. More specifically, to comprehend the timing and patterns of both customer demands and customer returns and to understand their effects on inventory management are crucial aspects for a retailer to consider. These concerns motivate us to do the research work presented in Chapter 2 of this thesis.

1.1.2 Remanufacturing

Remanufacturing is another essential aspect in RL, although, in some literature, remanufacturing is positioned in parallel with RL under the field of green operations (Srivastava [84]). Meanwhile, the definitions and statements regarding RL, which have been introduced earlier in this chapter, all imply that remanufacturing can be understood as a research field within the scope of RL. We take this point of view throughout our study.

As defined by Majumder and Groenevelt [58], remanufacturing is *the process of disassembling used items, inspecting and repairing/reworking the components, and using these in new product manufacture*. Having been around for over half a century, remanufacturing is generally considered an effective and efficient way to handle returned but reusable products and to fulfill multi-dimensional purposes, i.e., alternative raw material, cost saving, waste deduction and value generation. The remanufacturing industry is also called an "invisible industry". Its scope is wide and diverse, but the research available on this topic is sparse (Parkinson and Thompson [68]). The first comprehensive overview of research for remanufacturing was completed by Bras and McIntosh [7]. More recently, Parkinson and Thompson [68] examined the terminology surrounding remanufacturing and clarified the definitions of related terms such as reusing, recycling, refurbishing, and reconditioning.

Uncertainty is involved in several aspects in remanufacturing. The quantity, quality, and timing of returned products and the demand of remanufactured products are the major ones which have been considered in the literature of remanufacturing. Recent years have witnessed growing remanufacturing competition between original equipment manufacturers and remanufacturers. We notice that these remanufacturing competitors may not always know each other's feature information such as production cost. It would be interesting to examine how the uncertainty of such feature information may impact the decision making of remanufacturing competitors. This thought motivates us to do the research work presented in Chapter 4.

1.2 Game Theoretic Applications in SCM and RL

Game theory (GT) is an effective tool for the analysis of situations involving conflict, competition, and cooperation. The development of game theory dates back to early 1940s and its applications are found in diverse areas including economics, political science, management-labour arbitration, philosophy, warfare, auctions, etc. Applications of game theory in management science peaked in the 1950s but waned during the 1960s and 1970s. Shubik [83] gave an early survey of game theory in management science. Feichtinger and Jorgensen [26] gave a specific review on differential game models in management science. The real proliferation of game theoretic applications in MS/OR happened in the late 1980s, following the revolution of SCM in early 1980s. Leng and Parlar [54] provided a review on game theoretic applications in SCM. Nagarajan and Greys Soŝić [65] did a survey on some applications of cooperative game theory in SCM. Fiestras-Janeiro *et al.* [30] did a review of the applications of cooperative game theory in the management of centralized inventory systems. To the best of our knowledge, there have not been any review or survey on the applications of game theoretic models in the field of RL. Our thesis focuses on the applications of non-cooperative game models in SCM and RL.

Among the applications of game theoretic models in SCM, most consider games of complete information. That is, all players are assumed to know each player's objective function as common knowledge. This assumption is usually thought to be stringent and unrealistic. In many situations, a player's payoff function may not be known by all players. For example, a local pure remanufacturer (REM) is about to enter the market of a remanufactured product and to compete with an original equipment manufacturer (OEM). The OEM gets informed about the forthcoming entrance of the REM. But as the competitor, the OEM may be unsure about the REM's unit production cost. For this kind of situations, games with incomplete information rather than games with complete information are preferable modeling tools.

Currently, applications of incomplete information games in SCM focus on information sharing and incentive mechanism design for contracting problems. Applications in other SCM facets such as production planning and inventory management are limited. Applications of dynamic games with incomplete information are even more limited. In the domain of RL, more competitive or cooperative business behaviors such as the competition in remanufacturing mentioned above are arising and deserve further investigation. Such an investigation will be valuable for both industry practitioners and academic researchers who are involved in RL and wish to make improvements both economically and ecologically. Hence, RL has opened up a wider sphere for the applications of game theoretic models. In order to help researchers in our field to get a better understanding of how to construct games with incomplete information and to compute the corresponding equilibrium, we are motivated to provide in Chapter 3 a detailed elucidation of game models with incomplete information and their solution concepts with simplified applications in stochastic inventory management.

1.3 Organization

We present the organization of this thesis in the following.

Chapter 1 (this chapter) briefly introduces the crucial concepts covered in this thesis.

Chapter 2 studies a retailer's single-period inventory management problem with resalable returns. We divide a single period into three subperiods to examine the timing effects of portions of demand and customer returns on the retailer's decision making on inventory management. In the basic model, the retailer chooses an order quantity to optimize his total expected profit. We do concavity analysis for this model and conclude the sufficient conditions for the strict concavity of the retailer's total expected profit. In the general model, the retailer makes decision for both the order quantity and two inventory thresholds for the last two subperiods. The two thresholds will be used as the retailer's inventory reference when it is found better to return extra inventory to the supplier. We do simulation for the general model and study the timing effects of portions of demand and customer returns on the retailer's inventory policy and profit performance. This chapter is, to the best of our knowledge, the first investigation of the timing effects of customer returns on inventory policy making. Another two contributions we make are: (1) both customer returns and return-to-supplier are considered in a retailer's inventory management, and (2) a feasible and easy-to-follow inventory policy is provided for a retailer who faces high rates of customer return and, simultaneously, has a return-to-supplier option.

Chapter 3 presents a simplified exposition of applications of incomplete information games into inventory management. After presenting a brief review of the static and dynamic games under complete information, we illustrate the application of these two games in inventory management by using a single-period stochastic inventory problem with two competing retailers. Next illustrated is the Bayesian Nash equilibrium and perfect Bayesian equilibrium solution concepts for the static and dynamic games under incomplete information with two competing retailers. The expository nature of this chapter give MS/OR researchers an easy-to-follow access to the applications of incomplete information games to more specific areas in SCM and RL.

Chapter 4 studies the remanufacturing competition problem between an OEM and a REM with incomplete information. We use linear demand functions for both the OEM and the REM. We first study the OEM-REM remanufacturing competition problem with complete information. We prove the existence and uniqueness of the Nash equilibrium to this problem and obtain the closed-form NE solution. We then assume that the OEM is uncertain about the REM's unit remanufacturing cost but has the priority of accessing available shells and apply the type-III model in Chapter 3 to formulate this remanufacturing competition problem with incomplete information. We apply Bayesian Nash equilibrium as the solution concept, prove the existence of the BNE solution and obtain the closed-form BNE solution to this model. We then conduct sensitivity analysis and analyze the effect of such uncertainty on the OEM-REM remanufacturing competition.

Finally Chapter 5 summarizes our results in this thesis and identifies a general direction for future research.

Chapter 2

The Newsvendor Problem with Resalable Returns

2.1 Introduction

The retailing business world is experiencing a significant increase in customer returns (Mostard and Teunter [62]). This phenomenon occurs for several reasons.

The first reason is government related. Legislation systems in many countries now define it as customers' *legal right* to return a purchased product within a certain time frame. It is also regulated as the retailers' obligation to refund the customer, either fully or partially, if the condition of the returned product is acceptable.

Secondly, the increasingly fierce competition among retailers promotes customer returns. On the one hand, a return policy is regarded as a signal of high quality which can be used to stimulate market demand (Xiao *et al.* [95]). It is also well known as an effective weapon that retailers can use to attract consumers. Therefore more retailers are making proactive offering of return policies to strengthen their competitive position in the marketplace. Threatened with the growing commercial competition from these proactive return-policy takers, some conservative retailers are forced to apply return policies to retain their competitiveness. On the other hand, peer competition keeps challenging the retailers' leniency on their return policy. A customer can return a purchased product with no obligation to provide any excuse or explanation for his/her decision, although an optional explanation of the reason is helpful for the purpose of improving the retailer's business. The return time frame is getting extended as well. An overtime return is no longer unacceptable for many retailers, although it is common that for an overtime return the customer gets refunded either partially or fully in the form of store credit. In particular, the leniency of return policies in Europe has already exceeded the government mandated regulation (Mostard and Teunter [62]).

The consumer is the third aspect. Modern managerial theory states that the easier a purchased product can be returned the more likely it will be returned (Davis *et al.*

[19]). The growing leniency grants customers substantial flexibility in making a return decision and reduces their cost to reverse a bad decision. Although a lenient return policy gives more liberalization to consumers, it simultaneously exposes a retailer to consumers' abuse of their leniency (Ketzenberg and Zuidwijk [47]). In addition, the increase of product variety, which is commonly considered as a fruit of economic development, raises the uncertainty in the consumer's decision making (Xiao *et al.* [95]). That is, the consumer is more uncertain about whether, or to which extent, the purchased item meets his/her need or desire. Intuitively speaking, the more uncertain a customer is about the purchased product, the more likely the product ends up with being returned.

Another reason for increasing customer returns lies in "mail sales". Mail sales, originally in the form of catalogues, hold a considerable share in the total sales of commercial businesses. Along with the prevalence of the Internet and the boom of E-commerce since early 21th century, more and more retailers are seeking online business opportunities. The Internet has replaced catalogues and become the dominant form of mail sales. Hence, the share of mail sales has been continuingly increasing in the last two decades.

Mail sales are distinguished from traditional retail sales in several aspects. Firstly, rather than checking the physical body of the product, a consumer using mail sales views the product image provided in catalogues or on the Internet. This exposes the consumer to a higher risk that the product may be unsatisfactory. Secondly, a consumer using mail sales can easily place an order with a remote retailer with no need for travelling. Purchasing on the Internet or via fax machine can even save a trip that is only a walking distance.

It is observed that another significant difference lies in customer return rates which tend to be much higher in mail sales than in traditional retail sales. Mostard and Teunter [62] attributed this difference to the two features of mail sales we have mentioned above. As revealed in a survey, the majority of consumers acknowledge that return policies play an important role to encourage them to purchase from remote retailers they are unfamiliar with (Ketzenberg and Zuidwijk [47]). This kind of easy though curt purchasing behavior is another source of high volumes of customer returns. Customer return rates for some mail sales can be as high as 35%. In some extreme cases, it can even reach a peak of 70%. Returns of seasonal products are even more likely to be problematic (Vlachos and Dekker [89]).

The notable increase of customer returns has been challenging to the retail business world. For companies having high volumes of customer returns, how to handle these returns has become one of the most important management problems. There are multiple options for handling this issue. For returns that can be resold directly, retailers may put them *on sale* with a reduced price, organize a clearance sale event, or deliver them to an outlet store. Items that need to be recovered are normally sent for remanufacturing or refurbishing.

For seasonal products having a limited sale period, the better and faster the return products are recovered, the higher profit they can generate. When a product is recovered to be resalable as a brand new product, we say it is in *as-good-as-new* condition and call it as a *resalable return*. Returned products in mail sales, compared with traditional sales, are easier to be recovered to an *as-good-as-new* condition. For some seasonal products, such as textbooks, a large portion of returns are seldom used, so they can be directly put back into inventory in an as-good-as-new condition. Hence, customer returns actually benefit a retailer as a complementary supply channel that can often merge with the primary one. Meanwhile, the retailer is compelled to take the high-volume resalable returns into inventory policy making — an essential management problem in retail business. For a strategic decision maker, it implies more than to attach substantial importance to customer returns in the domain of consciousness. It requires a deliberate consideration on how to involve the return flows into inventory management. More specifically, to comprehend the timing and patterns of both customer demands and customer returns and to understand their effects on inventory dynamics are crucial for the retailer's decision making. These concerns motivate the work in this chapter.

This chapter considers a single-period single-order newsvendor problem with resalable returns faced by a mail sales retailer. We assume that the returned items can be recovered, easily and quickly, to an as-good-as-new condition and can be used to satisfy an arriving customer at the same price as a brand new product. We divide the single period into three subperiods and assume that a product sold at an earlier subperiod can be returned and recovered as a resalable product in later subperiod(s). We first develop *a basic model* in which the retailer maximizes the total expected profit with order quantity as the unique decision variable. We do concavity analysis for the basic model and provide the sufficient conditions for the concavity of the total expected profit. Then we develop *a general model* in which the retailer determines, besides the order quantity at the beginning of the entire period, the inventory thresholds for the beginning of each following subperiods. At the beginning of each subperiod (except the first one), the retailer will determine whether to return a certain amount of products to the supplier by referring to the corresponding threshold. By doing sensitivity analysis for this general model, we observe the timing effect of portions of customer demand and customer returns on the retailer's inventory decision making.

This chapter is organized as follows. Section 2.2 gives a literature review. In Section 2.3, we define a basic model and do concavity analysis for this model. In Section 2.4, we extend to a general model which involves two possible intermediate returns to the supplier and do simulation for sensitivity analysis. Section 2.5 gives several interesting managerial insights and Section 2.6 concludes.

2.2 Literature Review

The tremendous increase in customer returns is drawing the attention of OR/OM researchers. Inventory management is one of the research topics that are closely related. To fit the scope of our chapter, our literature review in this chapter focuses on single-period inventory models dealing with customer returns to retailers and retailer returns to suppliers.

To the best of our knowledge, the earliest paper dealing with inventory management with customer returns to a retailer was completed by Vlachos and Dekker [89] who studied a single period inventory problem based on two important assumptions: 1) a fixed percentage of sold products will be returned and, 2) all products are to be resold at most once. Six options for handling customer returns are considered, each corresponding to a specific setting of return reusing (no or partial reuse), return recovery (partial or full recovery), and recovery cost (with or without fixed recovery cost). The paper provides closed form analytical expressions for the optimal order quantity of each option and presents guidelines for selecting a proper return option.

Mostard and Teunter [62] argued that the two assumptions that Vlachos and Dekker [89] held are restrictive and unrealistic and moreover, ignore the net demand variability. In this sense, the optimal order quantities derived in [89] are sub-optimal. To tackle this problem, Mostard and Teunter [62] analyzed a newsboy problem with resalable returns in a case study where the above two assumptions are relaxed. They derived a simple closed-form formula that determines the optimal order quantity Q^*. By comparing Q^* with the order quantity that is used by the company in practice and an order quantity approximated by the formula Vlecho and Dekker provided in [89], they concluded that applying Q^* improves the company's profit.

The paper written by Ketzenberg and Zuidwijk [47] was probably the most recent one to study customer returns. They modelled a retailer's single selling season by splitting it into two periods and assume that 1) the returns in the first period can be recovered and sold in the second one and; 2) the returns in the second period can only be salvaged. The customers they consider are assumed sensitive to both price and return policy. Besides making decision for the optimal ordering quantity, they address issues about optimal pricing and return policy making as well. As far as we know, this is the only published paper that considers customer returns in multiple periods within a single selling season.

Compared with the literature on newsvendor problems with return options for customers (to retailers), the literature on newsvendor problems with various return options for retailers (to suppliers or manufacturers) is more abundant. Earlier papers are extensions regarding additional *replenishment or re-ordering* options. For the classical newsvendor problem to be applicable, the demand distribution is an essential piece of information. However it is often the case that the information available for the retailer to estimate the demand distribution is insufficient. As an alternative, Scarf [79] proposed a "distribution-free" model which requires merely the mean and standard deviation of the demand distribution to obtain the optimal order quantity that maximizes the minimum profit. Pasternack [70] considered the pricing decision of a producer of perishable product and investigated the effect of several return policies (with a retailer) on the producer's pricing decision. Gallego and Moon [33] extended

Scarf's work to a recourse case in which the retailer, after observing the demand, will meet extra demand by placing an emergency order with a higher unit purchasing cost. This is probably the first extension of the newsvendor problem that involves a second ordering opportunity for the retailer. Khouja [48] further extended this recourse model by, in case of shortage, allowing partial extra demand be lost and the remainder be satisfied with an emergency supply option. Khouja [48] considered two maximization objective functions: one is the expected profit and the other is the probability of reaching a desired profit which has been justified as an applicable managerial target. Lau and Lau [51] did an exploratory study of the newsvendor problem with mid-period replenishment and developed a semi-analytical solution procedure. Buchanan and Abad [8] studied the inventory control problem in a periodic review system where containers are returned by consumers to the manufacturer for reuse. They viewed the returns in a given period as a stochastic function of the number of containers out in the field and used dynamic programming to derive the optimal inventory control policy for the system. Ma and Meng [57] made the extension by offering the retailer a second ordering opportunity in case that the first order quantity Q_1 is insufficient to meet the customer demand. The paper derives a simple formula to determine the optimal values for the first and second order quantities which jointly maximize the retailer's expected profit. The computational experiments also show a substantial increase in profit as the benefit of allowing an emergency order option.

Kabak [45] considered a partial return for the retailer. This is probably the first newsvendor model dealing with the retailer's *return* option. Kodama [49] extended Kabak's model by considering both partial returns in case of surplus and additional orders under shortage, under general demand.

As supply chain management has been a dominant topic in OR/OM, we have seen studies of retailer returns in the context of supply chain management. Lee [52] investigated the coordination related issues among a supplier, a retailer and a discount sale outlet (DSO). In Lee's model [52], the single period for the newsvendor's regular sale is followed by a markdown period. The supplier is in charge of designing the return policy, either with or without cooperation with the retailer-DSO alliance. The retailer has two return options, either to return the surplus to the supplier per return policy or to leave them to the discount sale outlet. Their numerical results demonstrate offering a return option to retailers as an effective tool for the supplier to conquer a common problem in a decentralized supply chain — the double marginalization. Probably the latest extension of the newsvendor problem with return policy was attributed to Lee and Rhee [53]. In the context of a decentralized supply chain, Lee and Rhee [53] proposed three coordination contracts, each including a return policy and a benefit transfer scheme. The supplier is assumed to use these coordination contracts to motivate the retailer to cooperate so that the outcome of the decentralized system can reach the same level as that of an integrated supply chain.

In this chapter, we make a further extension for the newsvendor problem with resalable customer returns. We take both customer returns and the retailer's return option into consideration. We divide the single period into three subperiods so as to investigate the timing effect of both portions of demand and customer returns. The contribution of this chapter is twofold. We believe this is the first investigation of the timing effect of portions of demand and customer returns on the retailer's inventory management. We also provide a feasible and easy-to-follow inventory policy for the retailer facing high-volume customer return rates and/or return-to-supplier options.

2.3 The Basic Model

In this section, we consider a basic model for the newsvendor problem with resalable returns. More specifically, we consider a single period inventory system for a single seasonal product, in which a significant fraction of customer purchases will be returned to the retailer. We start with assumptions and notations.

2.3.1 Assumptions and Notations

We first assume that the retailer, whose target is to maximize the total expected profit, will determine a single order quantity which is to arrive at the beginning of the period. Considering that mail-sale retailing business is normally characterized with long lead times and short selling seasons (Mostard and Teunter [62]), we think it is reasonable to have a single order.

We assume the customer return rate to be significantly high, hence the formula provided in the classical newsboy problem is not sufficient to estimate the optimal order quantity. We need to deal with the customer returns more carefully. We assume that the returned items can be easily processed to an as-good-as-new condition and be ready to satisfy new arriving customer orders. Moreover, we assume that when a customer returns a purchased item to the retailer, it would be the customer's responsibility to pay the packing and shipping fee. Hence, when the retailer receives a return-from-customer, he will refund exactly the whole purchase price s to the customer. Meanwhile, we assume a customer return to incur a certain cost for administration and processing so that it can be resold at the original sale price s. We denote this unit cost as b and assume the value of b be significantly smaller than the value of unit sale price s.

In Table 2.1, we list the notations that we use for the basic model. The operation of this single-period inventory system with customer returns is depicted in Figure 2.1.

2.3.2 The Model

In the original newsvendor problem, the retailer determines an order size to satisfy the stochastic demand in a single period and the order will be available at the beginning of the period (Axsäter [3]). Similarly, the retailer in our resalable return problem will

Symbol	**Description**
b	Unit handling cost for a return from customer;
c	Unit purchase cost;
p	Unit penalty cost for shortage;
s	Unit sale price;
v	Unit salvage value at the end of the period;
X	Total demand for the entire period;
$f(x)$	Probability density function of the total demand of the entire period;
$F(x)$	Cumulative density function of the total demand of the entire period, $F(x) = \int_0^x f(t)\,dt$;
T_i	Subperiod i in which a proportion (α_i) of the total demand occurs;
α_i	Portion of demand that will occur in subperiod T_i $(i = 1, 2, 3)$; $0 \leq \alpha_i \leq 1$, $\alpha_1 + \alpha_2 + \alpha_3 = 1$;
β_{ij}	Portion of products sold at T_i that are returned and resalable in T_j, $(i = 1, 2;\ j = 2, 3;\ i < j; \sum_j \beta_{ij} \leq 1)$;
I_i	Initial inventory level for subperiod T_i;
S_i	Sales in subperiod T_i, $i = 1, 2, 3$;
R_j	Items that were sold in previous subperiod(s) but have been returned and become resalable in subperiod T_j, $R_j = \sum_{i \leq j} \beta_{ij} S_i$.

Table 2.1: Summary of notations for the newsvendor problem with resalable returns.

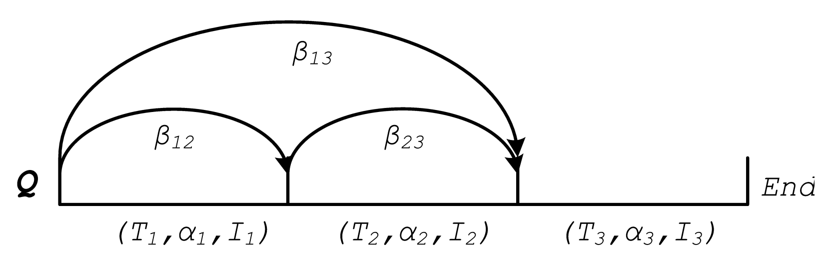

Figure 2.1: The flow diagram of the basic model for the newsboy problem with resalable returns.

make the decision on the order size Q and the order will be available at the beginning of the period.

As mentioned in Section 2.1, we are interested in investigating the timing effect of both customer demand and customer returns. To facilitate the investigation, we divide the single period into three subperiods, T_1, T_2, and T_3. Correspondingly, portion α_i of the total period demand X would occur in subperiod T_i, $(i = 1, 2, 3)$, i.e., for each α_i we have

$$0 \leq \alpha_i \leq 1 \text{ and } \sum_{i=1}^{3} \alpha_i = 1.$$

If a demand can't be satisfied at its occurrence, it will be considered as a shortage. Alternatively speaking, any demand that can't be satisfied instantly is regarded as a shortage.

Also we assume that among the items that are sold in subperiod T_i, portion β_{ij} will be returned and recovered to an as-good-as-new condition for resale at subperiod T_j. We understand that, even an easy and simple recovery process still takes certain time. Hence, we assume that an item that is sold in subperiod T_i, if it is returned, can only be ready for resale at a later subperiod T_j, where j should be greater than but not equal to i. For example, an item sold in subperiod T_1, if being returned, can only be used to satisfy the demand in either T_2 or T_3. Hence, for β_{ij} we have

$$\sum_{j=i+1}^{3} \beta_{ij} \leq 1, \ (i = 1, 2; \ j = 2, 3; \ j > i)$$

The retailer is to determine an optimal order quantity Q which maximizes the total expected profit $E(\Pi)$. We need to introduce a few intermediate variables to obtain the correct expression of the total expected profit. They are, for each subperiod T_i, inventory level I_i, sale S_i, and customer returns R_i, $(i = 1, 2, 3)$. The inventory level I_i at the beginning of subperiod T_i is determined by the selling and returning behavior in previous subperiods. The sale S_i is determined by the available inventory level I_i and subperiod demand $\alpha_i x$. The customer return R_i is determined by sales in previous subperiods. We provide in detail the expression of each intermediate variable in the following.

For the first subperiod T_1. At the beginning of the first subperiod, neither any sale nor any return has occurred yet. So we have customer returns $R_1 = 0$ and the inventory level I_1 equal to the initial order size Q. The subperiod sale would be the minimum of inventory level I_1 and customer demand $\alpha_1 x$, i.e., $S_1 = \min(\alpha_1 x, I_1) = \min(\alpha_1 x, Q)$.

For the second subperiod T_2. At the beginning of the second subperiod, the resalable customer returns collected from the first subperiod are $R_2 = \beta_{12} S_1 = \beta_{12} \min(Q, \alpha_1 x)$. Hence the inventory level I_2 would be the sum of the brand new items left over from the first subperiod, which is $Q - \min(\alpha_1 x, I_1)$, and the resalable customer returns R_2, i.e., $I_2 = Q - \min(\alpha_1 x, I_1) + R_2$. The demand in the second subperiod is $\alpha_2 x$, so the sale in the second subperiod would be $S_2 = \min(\alpha_2 x, I_2)$.

For the third subperiod T_3. In a similar way, for the third subperiod, the beginning inventory level I_3 is also composed of two parts: the leftover from the second subperiod, which is $I_2 - \min(I_2, \alpha_2 x)$, and the returns that are collected from the first two subperiods but become resalable at the beginning of the third subperiod, which is $R_3 = \beta_{13} \min(Q, \alpha_1 x) + \beta_{23} \min(I_2, \alpha_2 x)$. That means $I_3 = I_2 - \min(I_2, \alpha_2 x) + R_3$. Also the demand in the third subperiod is $\alpha_3 x$, so the sale would be $S_3 = \min(\alpha_3 x, I_3)$.

To summarize, the intermediate variables for each subperiod T_i are:

$$
\begin{aligned}
T_1 &: \begin{cases} R_1 = 0; \\ I_1 = Q; \\ S_1 = \min(\alpha_1 x, Q); \end{cases} \\
T_2 &: \begin{cases} R_2 = \beta_{12} \min(Q, \alpha_1 x); \\ I_2 = Q - (1 - \beta_{12}) \min(Q, \alpha_1 x); \\ S_2 = \min(\alpha_2 x, I_2); \end{cases} \\
T_3 &: \begin{cases} R_3 = \beta_{13} \min(Q, \alpha_1 x) + \beta_{23} \min(I_2, \alpha_2 x); \\ I_3 = I_2 - (1 - \beta_{23}) \min(I_2, \alpha_2 x) + \beta_{13} \min(Q, \alpha_1 x); \\ S_3 = \min(\alpha_3 x, I_3). \end{cases}
\end{aligned}
\tag{2.1}
$$

Given unit sale price s, unit purchase cost c, unit shortage penalty cost p, unit return processing cost b and unit salvage value v at the end of the period, we can obtain the profit function π_i for each subperiod T_i $(i = 1, 2, 3)$ as:

$$
\pi_1 = \begin{cases} s\alpha_1 x - cQ, & \text{if } \alpha_1 x \le Q; \\ sQ - p(\alpha_1 x - Q) - cQ, & \text{if } \alpha_1 x \ge Q; \end{cases} \tag{2.2}
$$

$$
\pi_2 = \begin{cases} s\alpha_2 x - (s + b) R_2, & \text{if } \alpha_2 x \le I_2; \\ sI_2 - p(\alpha_2 x - I_2) - (s + b) R_2, & \text{if } \alpha_2 x \ge I_2; \end{cases} \tag{2.3}
$$

$$
\pi_3 = \begin{cases} s\alpha_3 x + v(I_3 - \alpha_3 x) - (s + b) R_3, & \text{if } \alpha_3 x \le I_3; \\ sI_3 - p(\alpha_3 x - I_3) - (s + b) R_3, & \text{if } \alpha_3 x \ge I_3. \end{cases} \tag{2.4}
$$

The expected profit of each subperiod is simply denoted as J_i for T_i, $i = 1, 2, 3$. It follows that, in this basic model, the expected total profit, which we denote as J, is the sum of the expected profit J_i of each subperiod T_i, i.e.,

$$
J = J_1 + J_2 + J_3 = E(\Pi_1) + E(\Pi_2) + E(\Pi_3). \tag{2.5}
$$

We will provide in detail the expressions of expected subperiod profits J_1, J_2, and J_3

in the following concavity analysis.

2.3.3 Concavity Analysis

After we introduce the basic model and obtain the expected total profit in (2.5), it is important to investigate the concavity feature of this model. We are interested to see whether the expected total profit is concave in regard to the decision variable, the order quantity Q.

As explained in 2.3.1, when the retailer receives a customer return, it incurs an administrative and processing fee b and the retailer would refund the total price of the item s to the customer. In general practice, the value of b is significantly smaller than the value of the whole sale price s. Hence, we first do the concavity analysis with $b = 0$.

When $b = 0$

To investigate the concavity of the total expected profit, we first present the concavity feature of each J_i, $(i = 1, 2, 3)$, in Lemma 2.1, 2.2, and 2.3, respectively.

Lemma 2.1 *For the first subperiod T_1, the first derivative of the expected profit dJ_1/dQ is positive at $Q = 0$ (the left end point) and negative at $Q = \infty$ (the right end point). The second derivative of the expected profit d^2J_1/dQ^2 is negative.*

Lemma 2.2 *For the second subperiod T_2, the first derivative of the expected profit dJ_2/dQ is positive at $Q = 0$ (the left end point) and zero at $Q = \infty$ (the right end point).*

Lemma 2.3 *For the third subperiod T_3, the first derivative of the expected profit dJ_3/dQ is positive at $Q = 0$ (the left end point) and negative at $Q = \infty$ (the right end point) if $v/s < \beta_{13}$ and $\alpha_1\beta_{12} \geq \alpha_2$.*

The proofs of the above three Lemmas are provided in Appendix A. Based on Lemma 2.1, 2.2, and 2.3, we derive the concavity feature of the total expected profit as in Lemma 2.4.

Lemma 2.4 *Regarding the total expected profit J, its first derivative dJ/dQ is positive at $Q = 0$ and negative at $Q = \infty$; its second derivative d^2J/dQ^2 is negative if $\alpha_1\beta_{12} \geq \alpha_2$.*

The proof of Lemma 2.4 is presented in Appendix A as well. Given the concavity feature of the total expected profit as stated in Lemma 2.4, we have the following theorem:

Theorem 2.1 *The total expect profit of the basic model is maximized at a unique optimal order quantity if* $\alpha_1\beta_{12} \geq \alpha_2$.

Furthermore, we derive the sufficient conditions for the strict concavity of the total expected profit J with $b = 0$ and summarize in Table 2.2, where θ_k, $(k = 1, \ldots, 5)$ are intermediate parameters which can be found in Appendix A, page 113.

Case	Case Conditions	Sufficient Conditions for Strict Concavity
1	$\theta_1 \leq 0,\ \theta_2 \leq 0$	$\beta_{23}-\beta_{12}\beta_{23}-\beta_{13}-\beta_{12} < 0$
2	$\theta_1 \leq 0,\ \theta_2 \geq 0,\ \theta_3 \geq 0$	$(v-s)(\beta_{23}-\beta_{12}\beta_{23}-\beta_{13}) - p\beta_{12} < 0$
3	$\theta_1 \leq 0,\ \theta_2 \geq 0,\ \theta_3 \leq 0$	$\beta_{23}-\beta_{12}\beta_{23}-\beta_{13}-\beta_{12} < 0$
4	$\theta_1 \geq 0,\ \theta_5 \geq 0$	Concave
5	$\theta_1 \geq 0,\ \theta_5 \leq 0,\ \theta_1+\theta_2 \leq 0$	Concave
6	$\theta_1 \geq 0,\ \theta_5 \leq 0,\ \theta_1+\theta_2 \geq 0$	Concave

Table 2.2: The sufficient conditions for the strict concavity of the total expected profit when $b = 0$.

As Table 2.2 shows, when the first subperiod customer return is not enough to satisfy the second subperiod demand, i.e., $\theta_1 \leq 0$ or $\alpha_1\beta_{12} \leq \alpha_2$, the strict concavity of the total expected profit depends on return ratios β_{ij} and monetary parameters v, s, and p. When the first subperiod return exceeds the second subperiod demand, i.e., $\theta_1 \geq 0$ or $\alpha_1\beta_{12} \geq \alpha_2$, the total expected profit is strictly concave with respect to the order quantity Q. We understand that it may be simpler to group Cases 4, 5, and 6 into one with condition $\theta_1 \geq 0$, but we still decide to list them all there in order to keep consistency with the proofs in Appendix A.

When $b > 0$

We further investigate the concavity feature of the basic model with $b > 0$, by following the same procedure as with $b = 0$. We derive the expressions of the total expected profit and the first and second derivatives for each of the six cases and obtain the sufficient conditions for the concavity of the total expected profit as shown in Table 2.3.

Table 2.3 shows that the sufficient conditions for strict concavity of the total expected profit is more complicated with $b > 0$ than the conditions with $b = 0$ in Table 2.2. If we set $b = 0$ in Table 2.3, all the six sufficient conditions become identical with the counterparts in Table 2.2. It implies the consistency between Table 2.2 and Table 2.3. Furthermore, we note that even when $b > 0$, most cases in Table 2.3 are identical with Table 2.2 if only $b < p$. For example, if $b < p$, then $b - p < 0$ and $p - b > 0$. It follows that the sufficient conditions for Case 1 in Table 2.3, which are $b - p < 0$ and $(\beta_{23} - \beta_{12}\beta_{23} - \beta_{13} - \beta_{12}) < 0$, will be simplified into $\beta_{23} - \beta_{12}\beta_{23} - \beta_{13} - \beta_{12} < 0$, which is exactly the sufficient condition for Case 1 in Table 2.2. This fact applies to Case 3, 4, 5, and 6 as well. Case 2 is the only exception, because having Case 2

Case	Case Conditions	Sufficient Conditions for Strict Concavity
1	$\theta_1 \leq 0,\ \theta_2 \leq 0$	$b-p<0$ and $\beta_{23}-\beta_{12}\beta_{23}-\beta_{13}-\beta_{12}<0$
2	$\theta_1 \leq 0,\ \theta_2 \geq 0,\ \theta_3 \geq 0$	$b-p<0$ and $(v-s-b)\left(\beta_{23}-\beta_{12}\beta_{23}-\beta_{13}\right)-(p-b)\beta_{12}<0$
3	$\theta_1 \leq 0,\ \theta_2 \geq 0,\ \theta_3 \leq 0$	$b-p<0$ and $\beta_{23}-\beta_{12}\beta_{23}-\beta_{13}-\beta_{12}<0$
4	$\theta_1 \geq 0,\ \theta_5 \geq 0$	$b-p<0$
5	$\theta_1 \geq 0,\ \theta_5 \leq 0,\ \theta_1+\theta_2 \leq 0$	$b-p<0$
6	$\theta_1 \geq 0,\ \theta_5 \leq 0,\ \theta_1+\theta_2 \geq 0$	$b-p<0$

Table 2.3: The sufficient conditions for the strict concavity of the total expected profit when $b > 0$.

in Table 2.2 identical with Case 2 in 2.3 requires $\beta_{23} - \beta_{12}\beta_{23} - \beta_{13} - \beta_{12} < 0$ as an additional condition. Therefore, we conclude that the customer return processing fee b does affect the concavity feature of the total expected profit. But when the value of b is small enough, or, more specifically, when the value of b is smaller than the shortage penalty cost p, its effect can be ignored and Theorem 2.1 still holds if only $\beta_{23} - \beta_{12}\beta_{23} - \beta_{13} - \beta_{12} < 0$.

2.3.4 Numerical Examples

We now present two numerical examples to illustrate the concavity feature of the basic model with $b = 0$. In both examples, we assume that the entire period demand follows an exponential distribution with pdf $f(x) = \lambda e^{-\lambda x}$, rate $\lambda = 1/200$, and mean value $E(X) = 200$. Also we set monetary parameters $c = 25$, $s = 55$, $p = 30$, and $v = 7$.

In the first example, we assume $\alpha = [1/3, 1/3, 1/3]$ and $\beta = [0, 1/4, 1/4; 0, 0, 1/2; 0, 0, 0]$. We then have $\theta_1 = \alpha_1\beta_{12} - \alpha_2 = (1/3)(1/4) - 1/3 = -1/4 < 0$ and $\theta_2 = \alpha_1\beta_{13} + \alpha_2\beta_{23} - \alpha_3 = (1/3)(1/4) + (1/3)(1/2) - (1/3) = -1/6 < 0$. This indicates that this example belongs to Case 1. As Table 2.2 shows, the sufficient condition for the strict concavity of Case 1 is $\beta_{23} - \beta_{12}\beta_{23} - \beta_{13} - \beta_{12} < 0$. Here we have it satisfied since $\beta_{23} - \beta_{12}\beta_{23} - \beta_{13} - \beta_{12} = (1/2) - (1/4)(1/2) - (1/4) - (1/4) = -1/8 < 0$. So the total expected profit in this example should be strictly concave in respect of Q. The optimal order quantity $Q^* = 204.88$ maximizes the total expected profit at 2953.81. The curve of the total expected profit for this example is shown in Figure 2.2.

In the second example, we assume $\alpha = [1/2, 1/4, 1/4]$ and $\beta = [0, 2/3, 1/3; 0, 0, 2/3; 0, 0, 0]$. We then have $\theta_1 = 1/12 > 0$ and $\theta_5 = \alpha_2(\beta_{13} + \beta_{12}\beta_{23}) - \alpha_3\beta_{12} = (1/4)(1/3 + (2/3)(2/3)) - (1/4)(2/3) = 1/9 > 0$. This means that this example belongs to Case 4 whose total expected profit should be strictly concave in respect of Q with no further sufficient condition. As Figure 2.3 shows, the optimal order quantity $Q^* = 110.10$

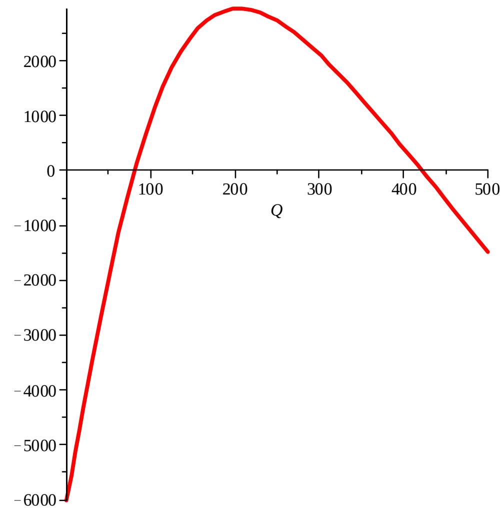

Figure 2.2: The total expected profit of a Case-1 example with $E(X) = 200$.

maximizes the total expected profit at -1571.38.

Remark 2.1 ***The "negative-optimal-profit" phenomenon*** *Note that, in Figure 2.3, we have the optimal total expected profit being negative for any $Q > 0$. It is not surprising if we expect a high demand to occur in the first subperiod as well as a high return rate from the sales in this subperiod. In addition, this "negative-optimal-profit" phenomenon may seem more irrational in the eyes of a retailer, for he/she may ask why not just discontinue the business if continuing it would bring him/her negative profit regardless of the order size. In fact, the presence of the shortage penalty cost p confirms the validity of this "negative-optimal-profit" phenomenon. With the presence of the shortage penalty cost p, what the retailer loses in case of a shortage is not only the sale revenue (i.e., the sale price s) but also the shortage penalty cost p. When the value of p is high, the retailer would naturally choose to continue the business even it is not profitable, which is merely because he/she would lose more by choosing to discontinue the business.*

It would be more convincing if we could provide a numerical example whose total expected profit is a non-concave curve. However, this is quite a challenging task. We've derived the sufficient conditions for the strict concavity of the total expected profit with $b = 0$ in Table 2.2. They are sufficient but not necessary conditions. It means that when the sufficient condition of any one of the six cases is not satisfied, we can not yet guarantee the non-concavity of the total expected profit. For example, if we have $\alpha = [1/3, 1/3, 1/3]$ and $\beta = [0, 1/100, 1/100; 0, 0, 49/50; 0, 0, 0]$. This belongs to Case 1 and the correspondent sufficient condition is unsatisfied since $\beta_{23} - \beta_{12}\beta_{23} - \beta_{13} - \beta_{12} = 4751/5000 > 0$. However, its total expected profit curve turns out concave

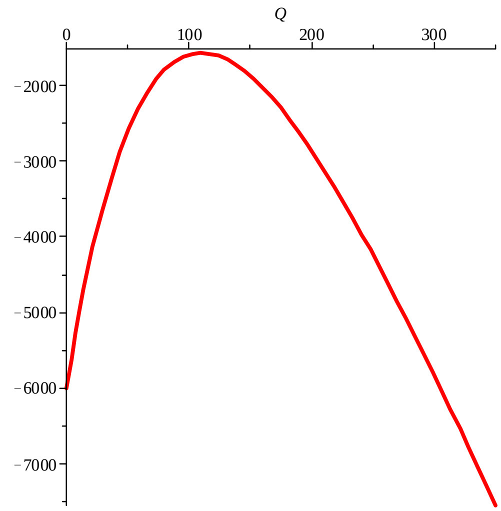

Figure 2.3: The total expected profit of a Case-4 numerical example with $E(X) = 200$.

too, as shown in Figure 2.4. This numerical example reveals a fact that the *IFF* (i.e., if and only if, or, sufficient and necessary) conditions for the concavity of the total expected profit should be less stringent than the correspondent sufficient conditions we've derived. This fact can also be observed without too much difficulty in the derivations in the proof of Lemma 2.4 provided in Appendix A.

2.4 The General Model

In the basic model we have introduced above, the unique decision variable is the order quantity Q. It means that the original inventory level Q is the only thing "controllable" by the retailer. Once this is determined and the order of that size is placed accordingly, the retailer will passively receive the order before the period starts, and then perform regular sale operations until the period ends. In some sense the retailer, right after placing the order, loses control over the inventory. They would have no more chance to adjust the inventory to improve monetary performance. Facing the stochastic demand in a real business, a retailer is very likely to be more active by considering other options than merely choosing the order quantity Q. They may either place an emergency order to satisfy a foreseen high demand or sign a buy-back contract beforehand to evade certain risk. We have mentioned some recent research works on these options in Section 2.2.

We believe a retailer would be more willing to manage the inventory in a dynamic way and it would be more practical if a retailer has a chance to monitor the inventory level within the single period and take proper action in case of need. For example, if a single period can be viewed as a series of successive subperiods, it is reasonable

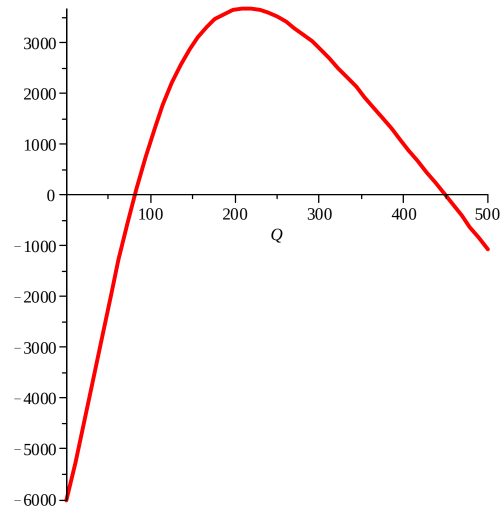

Figure 2.4: The total expected profit of a Case-1 numerical example with $E(X) = 200$ but the sufficient condition unsatisfied.

for the retailer to place an extra order when inventory on hand is insufficient to meet the demand in forthcoming subperiods. This is called *emergency order* and has been extensively studied in previous literature. Similarly, when inventory on hand is found more than the forecasted demand in the forthcoming subperiods, it would also be natural for a retailer to cut off certain inventory before the sale season ends. This option will be more attractive when it is available with a payoff better than a buy-back contract by which, in general, the supplier will buy back the surplus after the season ends. We will consider such an option in our general model. We propose an easy-to-follow inventory policy and expect that the retailer would benefit by following this policy. Meanwhile we will use this general model to investigate the timing effect of portions of demand and customer returns on the retailer's total expected profit, too.

2.4.1 Assumptions and Notations

Based on the above considerations, we propose a general model for the single-period inventory problem with stochastic demand and resalable returns. As in the basic model, we separate the entire period into three subperiods in this general model. Here, the retailer would not only determine the order quantity Q, but also the inventory thresholds Y_2 and Y_3 for the beginning of the second and third subperiods, respectively. The retailer will still receive the order of size Q at the beginning of the entire period. After that, he will examine the inventory level I_i, $(i = 2, 3)$ at the beginning of the second and the third subperiods, respectively. If the inventory level I_i is higher than the threshold Y_i, $(i = 2, 3)$, the retailer will return the extra units, which equals I_i-

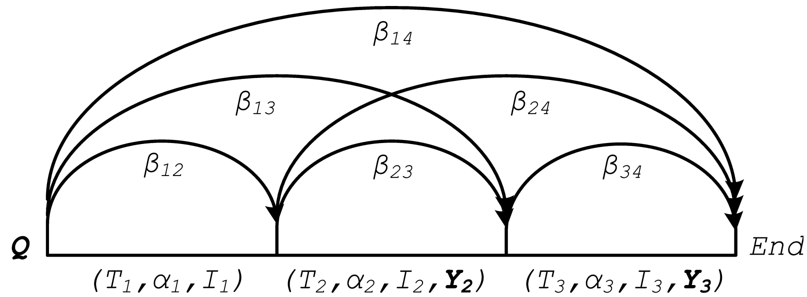

Figure 2.5: The flow diagram of the general model for the newsboy problem with resalable return.

Y_i, to the supplier with a unit price of v_i. It would be better to have the retailer order the shortage amount, which equals $I_i - Y_i$, when the inventory level I_i is lower than the threshold Y_i, $(i = 2, 3)$. Since this option has been well studied by previous literature, and also for the sake of simplicity, we ignore this option in this general model (i.e., the retailer will only take action in case of extra inventory).

In addition, in this general model we assume that the occurrence of customer returns will be extended until the entire period ends. This means that the retailer will receive customer returns not only at the beginning of each subperiod (except the first one), but also at the end of the entire period. For example, an item sold in the first subperiod may be returned and recovered to be resalable at the beginning of the second or third subperiods, or the end of entire period. In the same way, an item sold in the second subperiod may be returned and recovered to be resalable at the beginning of the third subperiod or the end of the entire period. Finally an item sold in the third subperiod may be returned and recovered to be resalable at the end of entire period. Figure 2.5 shows the flow diagram of this general model.

For each subperiod, the inventory level I_i, resalable return from customers R_i, return-to-supplier u_i, subperiod sale S_i, and profit $\hat{\pi}_i$ can be obtained as following.

For the first subperiod T_1, we have:

$$T_1 : \begin{cases} I_1 = Q, \\ S_1 = \min\left(Q, \alpha_1 x\right), \\ \hat{\pi}_1 = sS_1 - p\max\left(\alpha_1 x - S_1, 0\right) - cQ. \end{cases}$$

For the second subperiod T_2:

$$T_2 : \begin{cases} R_2 = \beta_{12} S_1, \\ I_2 = Q - S_1 + R_2, \\ u_2 = \max\left(I_2 - Y_2, 0\right), \\ S_2 = \min\left(Y_2, \alpha_2 x\right), \\ \hat{\pi}_2 = sS_2 + v_2 u_2 - \left(s + b\right) R_2 - p\max\left(\alpha_2 x - S_2, 0\right). \end{cases}$$

For the third subperiod T_3:

$$T_3 : \begin{cases} R_3 = \beta_{13} S_1 + \beta_{23} S_2, \\ I_3 = I_2 - u_2 - S_2 + R_3, \\ u_3 = \max\left(I_3 - Y_3, 0\right), \\ S_3 = \min\left(Y_3, \alpha_3 x\right), \\ \hat{\pi}_3 = sS_2 + v_3 u_3 - (s+b) R_3 - p \max\left(\alpha_3 x - S_3, 0\right). \end{cases}$$

For the end of entire period, we can model this as the fourth subperiod with no portion of demand. So we have:

$$T_4 : \begin{cases} R_4 = \beta_{14} S_1 + \beta_{24} S_2 + \beta_{34} S_3, \\ I_4 = I_3 - u_3 - S_3 + R_4, \\ u_4 = \max\left(I_4, 0\right), \\ \hat{\pi}_4 = v_4 u_4 - (s+b) R_4. \end{cases}$$

Correspondingly, the retailer's total expected profit in this general model, which we denote as $\hat{J}$, would be the sum of all subperiod expected profits $\hat{J}_k$, i.e.,

$$\hat{J} = \sum_{k=1}^{4} \hat{J}_k = \sum_{k=1}^{4} E(\hat{\Pi}_k).$$

Note that, in order to differentiate from the notations in the Basic Model, here we use $\hat{\Pi}_k$ to denote the profit of subperiod k, $k = 1, 2, \ldots, 4$.

2.4.2 Simulation and Sensitivity Analysis

Here we have three decision variables, Q, Y_2 and Y_3. It is difficult to do concavity analysis in an analytic approach, so we choose to run simulation and do sensitivity analysis accordingly.

Demand distribution

The demand X is the unique random variable so it requires an input probability distribution for simulation. Here we choose to use Erlang distribution whereby the probability density function $f(x)$ is in the form of

$$f(x) = \lambda^k x^{k-1} e^{-\lambda x} / (k-1)!.$$

We specifically set shape parameter $k = 6$. The reasons for choosing an Erlang distribution with $k = 6$ as the input probability distribution are twofold. Firstly, the support of an Erlang distribution is $[0, \infty)$. It captures the non-negativity of customer demand. Secondly, given a fixed expected value of X, as the value of k increases, the curve of the probability density function (PDF or pdf) of an Erlang

Confidence Level	$t_{N-1,1-\alpha/2}$	CI of μ
90%	1.645	$[196.08, 202.16]$
95%	1.961	$[195.50, 202.74]$
99%	2.578	$[194.36, 203.88]$

Table 2.4: Confidence intervals of the mean of demand $\mu = 200$.

distribution is more similar to that of a Normal distribution, the one commonly used as the distribution of random customer demand (See Figure 2.6).

Setting the shape parameter $k = 6$ and mean of demand $\mu = E(X) = 200$, we obtain the rate parameter $\lambda = k/\mu = 3/100$ and the variance $Var(X) = k/\lambda^2 = 6666.67$.

We run the simulation in Maple 12. We first use a Maple command X := Random-Variable(Erlang(b,a)) to generate a random variable X of Erlang distribution with b $= E(X)/k$ and a $= k$ and then command Sample(X,N) to generate a set of N random samples of X, i.e., $X_1, X_2, \cdots, X_N$. The values of b, a and N are set as b $= 200/6\,(= \mu/k)$, a $= 6\,(= k)$, and N $= 2000$, respectively. In this way, it yields sample mean $\bar{X}(N)$ and sample variance $S^2(N)$ as

$$\bar{X}(N) = \sum_{i=1}^{N} X_i\,(N-1) \Big/ N = 199.12$$

and

$$S^2(N) = \sum_{i=1}^{N} \left[X_i - \bar{X}(N)\right]^2 \Big/ (N-1) = 6818.73.$$

The approximate confidence intervals of the mean of demand μ, given different confidence levels, are shown in Table 2.4.

Table 2.4 indicates that with the 2000 random samples we generate, we have exactly 90% confidence level that the mean of demand μ will be located in the interval of $[196.08, 202.16]$, 95% confidence level in $[195.50, 202.74]$, and 99% confidence level in $[194.36, 203.88]$. Hence, 2000 random samples is sufficient for the simulation with the mean of demand $\mu = 200$. Also we use the same set of random sample to run the simulation for all cases.

Original parameters setting and the optimal solution

We initialize the value of parameters as following:

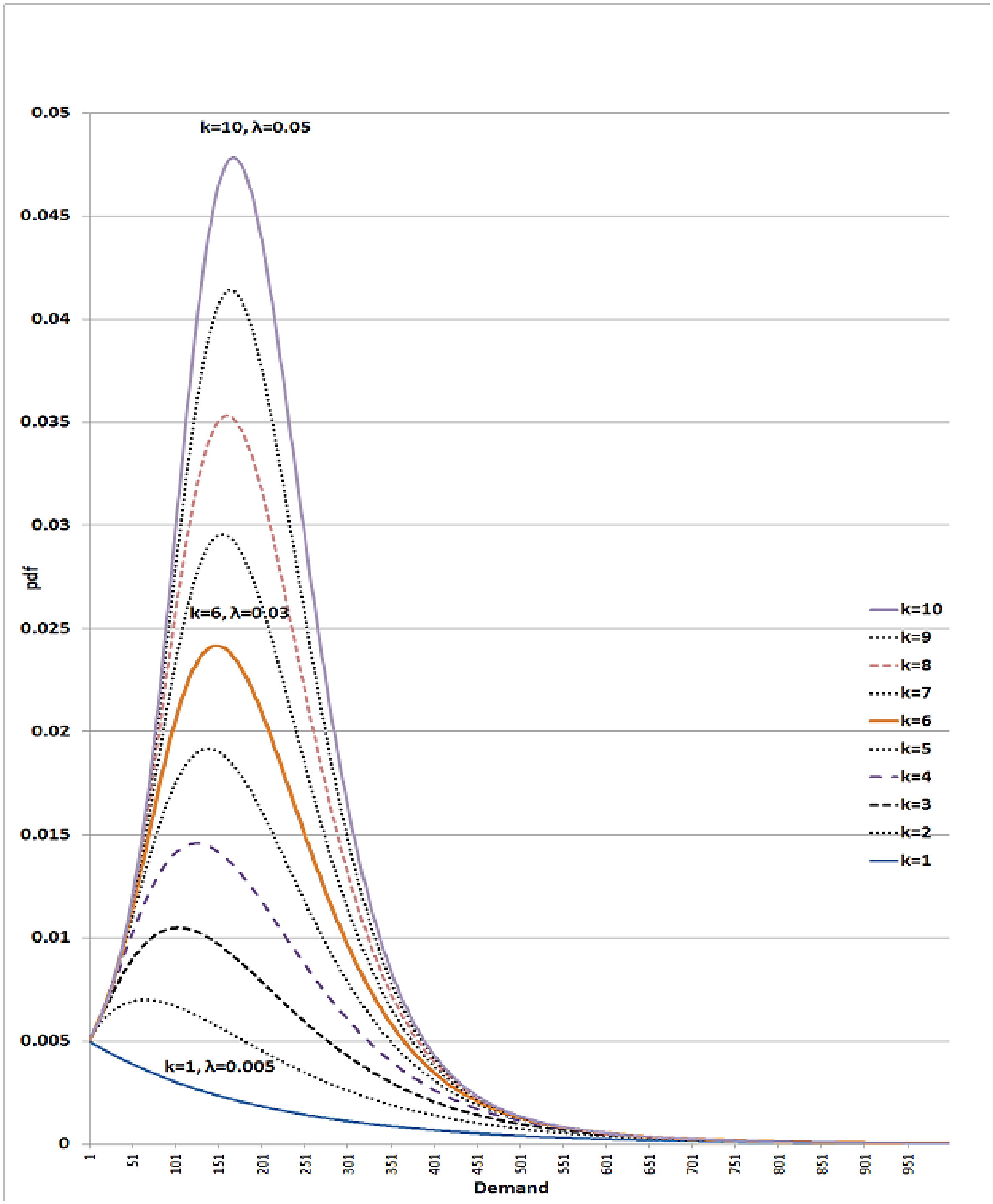

Figure 2.6: Probability density functions of Erlang distributions with $E(X) = 200$.

- Unit prices and costs:

$$\begin{aligned}
\text{Unit purchase cost} &: c = 25, \\
\text{Unit sale price} &: s = 55, \\
\text{Unit shortage penalty} &: p = 30, \\
\text{Unit return-from-customer processing cost} &: b = 3, \\
\text{Unit return-to-supplier prices} &: \mathbf{v} = [0, 13, 10, 7];
\end{aligned}$$

- Subperiod portions of demand:

  We assume that in the original setting, the period demand is evenly allocated in the three subperiods, i.e.:

$$\alpha = [\alpha_1, \alpha_2, \alpha_3] = [1/3, 1/3, 1/3]\,;$$

- Customer return rates:

In the original setting, we assume that half of customer purchases in each subperiod will be returned, and the returns will be evenly distributed among the relevant subperiods. The corresponding mathematical depiction is:

$$\sum_{j=i+1}^{4} \beta_{ij} = \frac{1}{2}, (i = 1, 2, 3) \text{ or } \beta = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \\ \beta_{31} & \beta_{32} & \beta_{33} & \beta_{34} \\ \beta_{41} & \beta_{42} & \beta_{43} & \beta_{44} \end{bmatrix} = \begin{bmatrix} 0 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Given each setting of parameters, we run two groups of simulation, one group of rough simulation and one group of fine simulation. We start with the group of rough simulation. For the group of rough simulation, we choose a large range of possible values for each decision variable. Normally we have 21 possible values within the range of 100 for Q, 9 possible values within the range of 80 for Y_2, and 7 possible values within the range of 60 for Y_3. We run the simulation and observe the concavity performance of the total expected profit per each decision variable, given the other two decision variables as parameters. Then, in case of need, we adjust the range of each decision variable so that, for each of them given the other two as parameters, the total expected profit shows strictly concave with the local maximum visible within the chosen range. We choose the set of decision variable values that yield the maximum expected profit as the "near-optimal solution". Then we use this "near-optimal solution" to run the group of fine simulation, that is to re-run the simulation by narrowing down the range of each decision variable's possible values. Normally we have 9 possible values within the range of 8 for Q, Y_2 and Y_3 respectively. We run the simulation with these possible values and choose the set of decision values that gives the maximum expected profit as the optimal solution for the set of parameters we have set in the beginning.

Confidence Level	$t_{N-1,1-\alpha/2}$	CI of P^*
90%	1.645	$[299.98, 381.74]$
95%	1.961	$[292.13, 389.59]$
99%	2.578	$[276.79, 404.92]$

Table 2.5: Confidence intervals of the total expected profit P^*.

Given the above parameters setting, we run the simulation for the original setting and get the optimal solution as:

$$\begin{aligned}\text{Order quantity :}&Q^* = 191,\\ \text{Inventory threshold levels :}&Y_2^* = 121,\\ &Y_3^* = 75,\\ \text{Total expected profit :}&P^* = 340.86.\end{aligned}$$

The total expected profit P^* is the output variable of the simulation, so its statistics features need to be examined, too. We find its sample mean and variance are 340.86 and 1.235008213×10^6 respectively and its approximate confidence intervals at each confident level are summarized in Table 2.5.

As Table 2.5 shows, we have 90% confidence level that the total expected profit P^* will be located in the interval of $[299.98, 381.74]$, 95% confidence level in $[292.13, 389.59]$, and 99% confidence level in $[276.79, 404.92]$. The final total expected profit $P^* = 340.86$ drops within all these three confidence intervals. This is statistically satisfying, too.

Sensitivity analysis (SA)

Having the simulation results for the original setting as the benchmark, we will do sensitivity analysis for this general model as following:

SA for monetary parameters $c, s, p, b, \mathbf{v}$ Firstly, we do sensitivity analysis for the monetary parameters (i.e., unit purchase cost c, sale price s, shortage penalty cost p, customer return processing cost b, and return-to-supplier price vector $\mathbf{v}$). We want to see these monetary parameters' effect on the optimal value of decision variables Q^*, Y_2^*, and Y_3^* and the total expected profit P^*. Considering that a retailer would rather withdraw from the business when the profit is expected to be negative, we use "$\rightarrow 0$" to make it zero for any negative profit in our numerical examples.

- Purchase cost c

As shown in Table 2.6, as the unit purchase cost c increases, the optimal values of all the three decision variables Q^*, Y_2^*, and Y_3^* decrease. The total expected profit P^* decreases too. This is intuitive and consistent with practical business: when unit purchase cost is higher, a retailer will place a smaller order, keep a lower inventory level, and expect a lower profit.

Case	c	Q^*	Y_2^*	Y_3^*	P^*	
C1	15	257	155	94	2519.40	
C2	20	216	136	84	1352.33	
Original	25	191	121	75	340.86	
C3	30	173	111	69	−566.59	$\rightarrow 0$
C4	35	159	102	64	−1396.31	$\rightarrow 0$

Table 2.6: The unit purchase cost c vs Q^*, Y_2^*, Y_3^* and P^*.

- Sale price s

Table 2.7 shows that as the unit sale price s increases the optimal values of all the three decision variables Q^*, Y_2^*, and Y_3^* increase too. This also increases the total expected profit P^* and is intuitive and consistent with practical business. Facing a higher unit sale price s, a retailer would place a larger order, keep a higher inventory level and expect a higher profit.

Case	s	Q^*	Y_2^*	Y_3^*	P^*	
S1	35	178	113	70	−1466.60	$\rightarrow 0$
S2	45	183	116	72	−568.71	$\rightarrow 0$
Original	55	191	121	75	340.86	
S3	65	196	124	77	1259.36	
S4	75	203	129	80	2185.63	

Table 2.7: The unit sale price s vs Q^*, Y_2^*, Y_3^* and P^*.

- Shortage penalty p

Table 2.8 indicates that, as the unit shortage penalty cost p increases, the optimal values of all the three decision variables Q^*, Y_2^*, and Y_3^* increase while the total expected profit P^* decreases. This is consistent with the intuition and business behavior too. A retailer who knows that he will get a heavier shortage penalty would place a larger order and keep a higher inventory to prevent shortage occurrence. Meanwhile, he may expect a lower profit as the larger order may cause unnecessary surplus at the end of the entire period.

- Customer return processing cost b

Case	p	Q^*	Y_2^*	Y_3^*	P^*
p1	5	161	102	63	765.10
p2	10	181	115	71	521.63
Original	20	191	121	75	340.86
p3	30	203	129	80	195.75
p4	50	209	133	83	75.96

Table 2.8: The unit shortage penalty p vs Q^*, Y_2^*, Y_3^* and P^*.

As shown in Table 2.9, as the unit customer return processing cost b increases, the optimal values of all the three decision variables Q^*, Y_2^*, and Y_3^* decrease while the expected total profit Q^* decreases too. The effect of unit customer return processing cost b is similar to the effect of unit purchase cost c but with a weaker magnitude. It is intuitive as well when customer return processing cost b is higher, a retailer would order less and keep a lower inventory level to avoid a high volume of customer returns.

Case	b	Q^*	Y_2^*	Y_3^*	P^*	
b1	0	193	122	76	615.46	
b2	1	191	121	75	523.77	
Original	3	191	121	75	340.86	
b3	5	191	121	75	157.96	
b4	7	188	119	74	−24.35	$\rightarrow 0$

Table 2.9: The unit return-from-customer processing cost b vs Q^*, Y_2^*, Y_3^* and P^*.

- Return-to-supplier prices $\mathbf{v}$

Table 2.10 shows that as unit return-to-supplier prices $\mathbf{v}$ increase, the optimal values of all the three decision variables Q^*, Y_2^*, and Y_3^* increase and the total expected profit Q^* increases too. The effect of unit return-to-supplier prices $\mathbf{v}$ is similar to that of the sale price s, which is consistent with intuition. When return-to-supplier prices $\mathbf{v}$ are higher, a retailer would place a larger order, keep a higher inventory level and expect a higher total profit.

SA for portions of demand α

Now we do sensitivity analysis for the portions of demand. In the original setting, the entire period demand is evenly allocated to the three subperiods (i.e., we set $\alpha = [\alpha_1, \alpha_2, \alpha_3] = [1/3, 1/3, 1/3]$). We will investigate the timing effect of portions of demand on Q^*, Y_2^* and Y_3^*, and P^*. We run simulation for twelve cases of portions of demand and show the simulation results in four tables. Each table includes three new cases that are in one pattern of portions of demand but with different magnitudes. We will do sensitivity analysis for each pattern separately.

Case	v	Q^*	Y_2^*	Y_3^*	P^*	
v1	$[0, 9, 6, 3]$	178	114	71	-35.13	$\rightarrow 0$
v2	$[0, 11, 8, 5]$	184	117	73	147.58	
Original	$[0, 13, 10, 7]$	191	121	75	340.86	
v3	$[0, 15, 12, 9]$	198	125	77	545.87	
v4	$[0, 17, 14, 11]$	207	130	80	765.61	

Table 2.10: The unit return-to-supplier **v** vs Q^*, Y_2^*, Y_3^* and P^*.

- Cases A_1-A_3

In these three cases, the major portion of demand occurs in the first subperiod (i.e., $\alpha_1 = 0.4$, 0.6, and 0.8 in respective). The remaining demand is evenly distributed in the other two subperiods. The simulation results are shown in Table 2.11.

Case	$[\alpha_1, \alpha_2, \alpha_3]$	Q^*	Y_2^*	Y_3^*	P^*
Original	$[1/3, 1/3, 1/3]$	191	121	75	340.86
A_1	$[.4, .3, .3]$	191	106	68	410.46
A_2	$[.6, .2, .2]$	191	59	47	586.91
A_3	$[.8, .1, .1]$	198	33	34	485.70

Table 2.11: The simulation results for portions of demand - cases A_1-A_3.

Table 2.11 shows that, as α_1 increases, the order quantity Q^* increases mildly (i.e., the retailer requires more inventory in the beginning of the entire period). Meanwhile, Y_2^* and Y_3^* obviously decrease, for the decreasing portions of demand in the last two subperiods require less inventory. A higher expected total profit P^* is another effect of having major portion of demand in the first subperiod. We think it is reasonable. On the one hand, a higher portion of demand in the first subperiod results a higher volume of customer returns for the last two subperiods. We understand that the earlier a customer returns, the more opportunity it will have to be resold before the entire period ends. On the other hand, the earlier an item is returned from a customer, the higher unit price it will take if it is returned to the supplier between two successive subperiods. However, we note that the expected total profit P^* drops as α_1 increases from 0.6 to 0.8. A possible explanation is that the further increase of demand in the first subperiod comes along with the shrinking demand in the last two subperiods. This makes the increased customer returns from the first subperiod be a burden rather than a benefit to the last two subperiods as the customer returns end up being returned to the supplier rather than being resold. As a result of that, the expected total profit P^* reduces.

- Cases A_4-A_6

In these three cases, the major portion of demand occurs in the second subperiod ($\alpha_2 = 0.4, 0.6$, and 0.8) and the demand in the other two subperiods is evenly distributed.

Case	$[\alpha_1, \alpha_2, \alpha_3]$	Q^*	Y_2^*	Y_3^*	P^*
Original	[1/3, 1/3, 1/3]	191	121	75	340.86
A_4	[.3, .4, .3]	192	128	68	346.48
A_5	[.2, .6, .2]	196	152	47	331.61
A_6	[.1, .8, .1]	204	184	34	5.54

Table 2.12: The simulation results for portions of demand - cases A_4-A_6.

Table 2.12 shows the simulation results for cases A_4-A_6. We compare these three cases pairwisely with cases A_1-A_3 (i.e., compare the case with $\alpha_2 = 0.4$ in Table 2.12 with the case with $\alpha_1 = 0.4$ in Table 2.11, and so on). We found that the order quantity Q^* here is slightly higher, the inventory level threshold Y_2^* is much higher, and Y_3^* is at about the same level. It seems obvious that a higher Y_2^* is necessary to satisfy the higher demand in the second subperiod. Since there is no difference in the third subperiod demand, it is reasonable to have Y_3^* remain at the same level. Furthermore, the customer returns from the major portion of demand can only be resold or returned to supplier in the third subperiod, so there is smaller chance to be resold. However, the expected total profit P^* is found much lower here. The possible reason, as we understand, is that with the major portion of demand occuring later than in cases A_1-A_3, fewer customer returns are available for being resold and, if a product is returned to the supplier, the price is lower too.

- Cases A_7-A_9

In these three cases, the majority of demand occurs in the last subperiod ($\alpha_3 = 0.4, 0.6$,and 0.8) and the demand in the first two subperiods is evenly distributed.

Case	$[\alpha_1, \alpha_2, \alpha_3]$	Q^*	Y_2^*	Y_3^*	P^*	
Original	[1/3, 1/3, 1/3]	191	121	75	340.86	
A_7	[.3, .3, .4]	192	130	89	253.62	
A_8	[.2, .2, .6]	198	158	131	-62.24	$\rightarrow 0$
A_9	[.1, .1, .8]	177	157	144	-564.50	$\rightarrow 0$

Table 2.13: The simulation results for portions of demand - cases A_7-A_9.

Similarly, we compare these three cases pairwisely with the first two case groups A_1-A_3 and A_4-A_6, and find that the inventory level threshold Y_3^* here is much higher. This is necessary to meet the major portion of demand in the last subperiod. The values of Y_2^* is slightly higher than the ones for case group A_4-A_6 (with case A_9 as an exception). With major portion of demand being delayed, fewer customer returns have chance to be resold. Hence, a higher inventory threshold level Y_2^* is set as a compensation. However, both Q^* and Y_2^* in case A_9 are lower than in case A_6. This looks abnormal at first glance, as lower Q^* and Y_2^* will directly cause shortages in each subperiod. However, compared with a higher Q^* and Y_2^* which would cause severe surplus at the end of the entire period, the lower Q^* and Y_2^* are regarded as the better choice.

- Cases A_{10}-A_{12}

In these three cases, portions of demand are assumed to be in a "V" shape, which means that the major portion of demand occurs in the first and last subperiods ($\alpha_1 = \alpha_3 = 0.35$, 0.4, and 0.45, respectively) and the demand in the second subperiod is low.

Case	$[\alpha_1, \alpha_2, \alpha_3]$	Q^*	Y_2^*	Y_3^*	P^*
Original	$[1/3, 1/3, 1/3]$	191	121	75	340.86
A_{10}	$[.35, .3, .35]$	191	118	79	335.76
A_{11}	$[.4, .2, .4]$	190	108	89	312.83
A_{12}	$[.45, .1, .45]$	191	99	99	280.38

Table 2.14: The simulation results for portions of demand - cases A_{10}-A_{12}.

The simulation results show that as the "V" valley goes deeper (i.e., both α_1 and α_3 increases while α_2 decreases further), Y_2^* decreases moderately but Y_3^* increases. These are reasonable since the total demand in the last two subperiods decreases moderately but the demand for the last subperiod increases. However, it is interesting to see that the order quantity Q^* maintains roughly at the same level as in the original case. It implies that there is a minor effect of "V"-shaped portions of demand on the order quantity. At the same time, the expected total profit P^* decreases. We understand this as the result of increased customer returns from the higher demand in the third subperiod which causes a larger surplus at the end of the entire period.

We note that the optimal expected profit in some cases above turns out with negative value. They are: cases C4 and C5 in Table 2.6, cases S1 and S2 in Table 2.7, case b4 in Table 2.9, and cases A_8 and A_9 in Table 2.13. We have confirmed the validity of such "negative-optimal-profit" phenomenon in Remark 2.1, Section 2.3.4.

SA on customer return ratio vector β In our general model, the customer return ratio β is in the form of a matrix

$$\beta = \begin{bmatrix} 0 & \beta_{12} & \beta_{13} & \beta_{14} \\ 0 & 0 & \beta_{23} & \beta_{24} \\ 0 & 0 & 0 & \beta_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is originally set as $\sum_{j=i+1}^{4} \beta_{ij} = 1/2$ $(i = 1, 2, 3)$ and for each $i = 1, 2, 3$, β_{ij} shares the same value over j. With this setting, we assume that half of the items sold in each subperiod will be returned in following subperiods. For sensitivity analysis, we designed another five cases which can be classified into three groups.

- Cases B_1 & B_2

The first group includes two cases: B_1 & B_2. Their simulation results are displayed in Table 2.15.

Case	β		$\begin{bmatrix} Q^* \\ Y_2^* \\ Y_3^* \\ P^* \end{bmatrix}$
Original	$\sum_{j=i+1}^{4} \beta_{ij} = 1/2$ $(i = 1, 2, 3)$ β_{ij} $(i = 1, 2, 3)$ evenly distributed over j	$\begin{bmatrix} 0 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 191 \\ 121 \\ 75 \\ 340.86 \end{bmatrix}$
B_1	$\sum_{j=i+1}^{4} \beta_{ij} = \frac{1}{4}$ $(i = 1, 2, 3)$	$\begin{bmatrix} 0 & 1/12 & 1/12 & 1/12 \\ 0 & 0 & 1/8 & 1/8 \\ 0 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 227 \\ 146 \\ 80 \\ 2213.44 \end{bmatrix}$
B_2	$\sum_{j=i+1}^{4} \beta_{ij} = \frac{3}{4}$ $(i = 1, 2, 3)$	$\begin{bmatrix} 0 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 3/8 & 3/8 \\ 0 & 0 & 0 & 3/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 157 \\ 98 \\ 70 \\ -1550.06 \rightarrow 0 \end{bmatrix}$

Table 2.15: The simulation results for customer return - cases B_1-B_2.

In Case B_1, the total return rate for sale in each subperiod is half of the rate in the original case. Q^*, $Y_{\mathbf{2}}^*$, and $Y_{\mathbf{3}}^*$ are all higher while the expected profit $P^* = 2213.44$ is significantly higher than $P^* = 340.86$ in the original case. On the contrary, as the total return rate is increased to 3/4 in Case B_2 (which is half more than in the original case), Q^*, Y_2^*, and Y_3^* are all lower while the expected profit P^* is much lower. It implies that as the total return rate for sale in each subperiod increases, the order quantity Q^* and the two inventory threshold levels Y_2^* and Y_3^* decrease while the total expected profit P^* decreases significantly.

- Case B_3

In case B_3, we set the return pattern as $\beta_{ij} = 1/4$. That means for sale in each subperiod, the return rates in each following subperiod are all set as 1/4. With this setting, the total return rates for the sales in each subperiod are not identical with

Case	β		$\begin{bmatrix} Q^* \\ Y_2^* \\ Y_3^* \\ P^* \end{bmatrix}$
Original	$\sum_{j=i+1}^{4} \beta_{ij} = 1/2$, $(i = 1, 2, 3)$ β_{ij} $(i = 1, 2, 3)$ evenly distributed over j	$\begin{bmatrix} 0 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 191 \\ 121 \\ 75 \\ 340.86 \end{bmatrix}$
B_3	$\beta_{ij} = \frac{1}{4}$, $(i = 1, 2, 3)$ $(j = i + 1, ..., 4)$	$\begin{bmatrix} 0 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 187 \\ 121 \\ 79 \\ 479.47 \end{bmatrix}$

Table 2.16: The simulation results for customer return - case B_3.

each other. Instead, the sales in an earlier subperiod incur a higher total return rate, i.e.,

$$\sum_{j=2}^{4} \beta_{1j} = 3/4, \ \sum_{j=3}^{4} \beta_{2j} = 1/2, \text{ and } \sum_{j=4}^{4} \beta_{3j} = 1/4.$$

Here the order quantity Q^* and the inventory threshold levels Y_2^* and Y_3^* are approximately the same as in the original case. The change of the return pattern in this way doesn't have much of an effect on these decision variables. However, the total expected profit P^* is much higher. We understand that, with customer return in this pattern, the total volume of customer returns is lower than in the original setting. Hence the valid sale (i.e., the demand that is satisfied but not returned) is higher and the total expected profit P^* is also higher.

- Cases B_4 & B_5

In Case B_4, we set $\beta_{ij} > \beta_{i(j+1)}$, $(i = 1, 2, 3, \ i < j < 4)$. It represents the scenario in which customers prefer to return products earlier. Compared with the original case, the order quantity Q^* and the inventory threshold level Y_2^* are lower but Y_3^* is slightly higher. Meanwhile, the expected total profit P^* is significantly higher. In Case B_5, we set $\beta_{ij} < \beta_{i(j+1)}$, $(i = 1, 2, 3, \ i < j < 4)$. It represents the scenario in which customers prefer to return products later. Compared with the original case, the order quantity Q^* and the inventory threshold level Y_2^* are higher but Y_3^* is slightly lower. The expected total profit P^* drops substantially. The simulation results for this case group indicate that if customers prefer to return items at an earlier time, the retailer should place a smaller order Q^*, keep a lower threshold level Y_2^* but a slightly higher threshold level Y_3^* and expect a significantly higher total profit. When customers prefer to return items at a later time, the retailer should place a larger order Q^*, keep a higher threshold level Y_2^* but a slightly lower threshold level Y_3^* and expect a significantly lower total profit.

Case	β		$\begin{bmatrix} Q^* \\ Y_2^* \\ Y_3^* \\ P^* \end{bmatrix}$
Original	$\sum_{j=i+1}^{4} \beta_{ij}= 1/2$, $(i = 1, 2, 3,\ i < j < 4)$ $\beta_{ij}\ (i = 1, 2, 3,\ i < j < 4)$ evenly distributed over j	$\begin{bmatrix} 0 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 191 \\ 121 \\ 75 \\ 340.86 \end{bmatrix}$
B_4	Customer prefer earlier returns; $\sum_{j=i+1}^{4} \beta_{ij}=\frac{1}{2}$, $(i = 1, 2, 3,\ i < j < 4)$; $\beta_{ij} > \beta_{i(j+1)}$ $(i = 1, 2, 3,\ i < j < 4)$.	$\begin{bmatrix} 0 & 1/4 & 1/6 & 1/12 \\ 0 & 0 & 1/3 & 1/6 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 180 \\ 116 \\ 76 \\ 598.14 \end{bmatrix}$
B_5	Customer prefer later returns; $\sum_{j=i+1}^{4} \beta_{ij}=\frac{1}{2}$, $(i = 1, 2, 3,\ i < j < 4)$; $\beta_{ij} < \beta_{i(j+1)}$ $(i = 1, 2, 3,\ i < j < 4)$.	$\begin{bmatrix} 0 & 1/12 & 1/6 & 1/4 \\ 0 & 0 & 1/6 & 1/3 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 202 \\ 126 \\ 74 \\ 81.15 \end{bmatrix}$

Table 2.17: The simulation results for customer return - cases B_4-B_5.

2.5 Managerial Insights

Based on the above sensitivity analysis, we observe some valuable managerial insights. Regarding the portions of demand $[\alpha_1, \alpha_2, \alpha_3]$, we observe that in general the earlier the major demand occurs, the lower the order quantity Q^* would be, although the reduction is not that significant. Meanwhile, the expected total profit P^* is higher and the inventory level thresholds Y_2^* and Y_3^* are lower for earlier large demand. This observation brings forward some interesting insights for a retailer facing a high volume of returns. On the one hand, they can estimate the order quantity Q^* and inventory level thresholds Y_2^* and Y_3^* according to the demand pattern if it is available. On the other hand, if there is opportunity to manipulate the demand pattern, it's better to move major demand to an earlier subperiod so that the expected total profit can be improved accordingly.

If the portions of demand are "V" shaped, the order quantity Q^* is about the same as in the original case, while the total expected profit P^* decreases as the "V" valley goes deeper. This indicates that if the portions of demand are estimated to be in a "V" shape, efforts to lessen the valley would benefit a retailer with a higher profit in case the cost of such an effort can be well balanced.

Regarding customer return pattern β, we find two interesting points. Firstly, as the total return rate for sale in each subperiod increases, the expected total profit P^* drops significantly. Hence, it would still be recommended to make effort to reduce customer returns. Secondly, since earlier customer returns result in a higher total expected profit, we believe it is better for the retailer to encourage customer returns, if they want to, as early as possible.

2.6 Conclusion and Future Work

2.6.1 Conclusion

We are in an age in which customer returns (protected by governments, intensified by the competitive retailing business and quick growing mail sales, and facilitated by the prevalence of the Internet) are growing fast. Retailers need strategic decision making tools more capable than the classic newsvendor problem. An effective and easy-to-follow inventory policy taking customer returns into consideration would be a valuable response to this need.

In this paper we analyze the single-period newsvendor problem with resalable returns in two models. In the basic model, we have the order quantity as the unique decision variable and investigate the concavity of the retailer's total expected profit in regards of this order quantity. In the general model, in addition to the order quantity, we have also two inventory thresholds as decision variables which can be easily understood and followed by a retailer if they would like to consider the return of extra inventory to the supplier during the single period. Moreover, with the help of simulation we are able to study the timing effect of the portions of demand and customer returns on the retailer's inventory policy and further provide some interesting managerial insights.

We believe that this is the first investigation of the timing effect of portions of demand and customer returns on a retailer's inventory policy. Also, we put both customer returns and return-to-supplier in consideration by providing a feasible and easy-to-follow inventory policy for retailers facing high customer return rates and/or return-to-supplier options.

2.6.2 Future Work

Both customer returns and return-to-supplier are involved in the general model. An underlying assumption we have made here is that the supplier would take all returns from the retailer unconditionally. In sensitivity analysis we have tried to reflect the supplier's choices by considering different scenarios of the return-to-supplier prices (see Section 2.4.2), we think it would be more practical to consider the supplier as an independent pricing decision maker. An appropriate game model can be applied to study the return-to-supplier behavior between the supplier and the retailer facing high volume of customer returns. Considering that high degree of uncertainty is the feature of customer returns, it would be valuable to develop a game model with the supplier having incomplete information as to the return from retailer. We will elaborate this type of game models in the next chapter.

Chapter 3

Games with Incomplete Information: A Simplified Exposition with Inventory Management Applications[1]

3.1 Introduction

Game theory studies multiple-person decision problems involving conflict, competition, and cooperation. Following the publication of von Neumann and Morgenstern's seminal book [90] in 1944, interest in potential applications of game theory reached a peak in the following decade. The fundamental solution concepts of game theory (e.g., the Nash equilibrium for non-cooperative games [66], and the Shapley value for cooperative games [82]) were developed in the 1950s and were used to analyze problems in diverse areas including economics, political science, management-labour arbitration, philosophy and warfare.

Early applications of game theory considered games of "complete information" where each player's payoff (objective) function is common knowledge for all players. However, the stringent (and unrealistic) assumption of complete information became a barrier to successful implementation of game theoretic ideas because in most competitive problems the players are not privy to each other's payoff functions. For example, two firms competing for the same market demand do not have complete information on each other's production cost functions. Similarly, in a sealed-bid auction, the bidders do not know each other's valuations. In the late 1960s Harsanyi [40] developed solution concepts for games with incomplete, i.e., asymmetric, information (also known as Bayesian games). In such games, players' payoff functions are no longer common knowledge; instead, at least one player is uncertain about another

[1] Accepted for publication in the International Journal of Production Economics, 133(2), 562-577, OCT 2011 (Wu and Parlar [94]).

player's payoff function. With Harsanyi's discovery of the new solution concepts for incomplete information games, interest in game theory was heightened in the last two decades and game theory once again became an important tool that can be used to analyze realistic problems of competitive situations.

Operations researchers were early users of game theory as can be seen in the operations research texts published in the 1950s and 1960s. The textbooks by Churchman, Ackoff and Arnoff [17], Sasieni, Yaspan and Friedman [76], Hillier and Lieberman [42] and Ackoff and Sasieni [1], published in 1956, 1959, 1967 and 1968, respectively, all include a chapter on competitive problems. All four texts cover zero-sum games and all, except Hillier and Lieberman [42], present a few examples of non-zero sum games involving bidding strategies. Shubik [83] reviewed early publications in this area. However, after the initial excitement generated by its potential applications, operations researchers' interest in game theory seemed to have waned during the 1970s and the 1980s. But the last two decades have witnessed a renewed interest by academics and practitioners in the management of supply chains and a new emphasis on the interactions among the decision makers ("players") constituting a supply chain. This has resulted in the proliferation of game theoretical publications in operations research/management science/operations management (OR/MS/OM) journals dealing with the use of game theory in the competitive and cooperative problems arising in supply chain management (SCM). For an excellent review of game theoretic applications in supply chain management we refer the reader to Cachon and Netessine [11]; see also a more recent review by Leng and Parlar [54].

In their respective reviews of game theory applications in SCM, both Cachon and Netessine [11] and Leng and Parlar [54] each cite more than 100 papers. It is interesting to note that a large majority of the reviewed papers make the simplified (and frequently unrealistic) assumption that all players know each other's objective functions with certainty. That is, they investigate problems dealing with games under *complete information*. Cachon and Netessine [11] briefly mention signalling, screening and Bayesian games where the games are played under *incomplete information*, i.e., at least one of the players does not know the other players' objective function. As examples of games played under incomplete information, they cite Cachon and Lariviere [10] who applied a signaling game to study a contracting problem with information sharing in a one-supplier, one-manufacturer supply chain, and Cachon and Lariviere [9] who studied a capacity allocation problem with information sharing issue between a supplier and several downstream retailers. Contract design problems also involve games of incomplete information; one of the earliest papers in this area is by Corbett [18] who applied the principal-agent theory to design an inventory contract in the context of (Q, r) model. In a more recent paper, Chu and Lee [16] studied an information sharing problem in a vertical supply chain with one vendor and one retailer and employed the perfect Bayesian equilibrium as the solution concept used in dynamic games played under incomplete information.

The reviews by Cachon and Netessine [11] and Leng and Parlar [54] reveal that there is a paucity of papers that deal with games played under incomplete information. However, in recent years publications have begun to appear that analyze games played under incomplete information. Since most realistic SCM problems involve competitive interactions with incomplete information, it would be useful to provide an exposition of such games with applications to a specific area in SCM, namely, inventory management. With this in mind, we write this chapter to present a simplified treatment of games with incomplete information with applications in stochastic inventory management.

We follow the same framework as in Gibbons [35] who has also considered static and dynamic *complete* and *incomplete* information games and their applications in economics. Gibbons' classification results in four categories: (i) Static games with complete information (for which the solution concept used is the Nash equilibrium), (ii) dynamic games with complete information (subgame perfection and Stackelberg equilibrium), (iii) static games with incomplete information (Bayesian Nash equilibrium), and (iv) dynamic game with incomplete information (perfect Bayesian Nash equilibrium). We start by briefly describing static and dynamic *complete* information games. This is followed by a more detailed exposition of static and dynamic *incomplete* information games. We first illustrate each of the four cases (which we call "Models") with a simple discrete game where each player has two moves. For each case, we then present a single-period stochastic inventory game with two competing newsvendors with the players' decision variables as continuous values. While the content in this chapter is expository in nature, it also contributes to the literature by presenting explicit methods for dealing with static and dynamic inventory games under incomplete information and computing the Bayesian Nash and perfect Bayesian equilibrium for such games.

In Section 3.2, we briefly review the well-studied games of complete information and discuss the solution concepts of Nash equilibrium (Model I, for static games) and subgame perfect equilibrium (Model II, for dynamic games). In Section 3.3, we present a discussion of a static game under incomplete information and discuss the solution concept of Bayesian Nash equilibrium. In Section 3.4 we discuss the case of dynamic games under incomplete information and use the solution concept of perfect Bayesian equilibrium to solve the game. Section 3.5 concludes the paper with a brief summary.

Since our inventory applications are concerned with games played by two newsvendors, we assign the male gender to the first newsvendor and the female gender to the second newsvendor in order to minimize the confusion that may arise when we refer to the players.

3.2 Games with Complete Information (Nash and Subgame Perfect Equilibria)

In this section we present a summary of games played under complete information by two players whose payoff functions are common knowledge; that is, known to both of them. For this class of simple games we first consider static games where the players choose their strategies simultaneously. We then consider a dynamic (two-stage) game where the players choose their strategies sequentially. For the static case, the solution concept is the Nash equilibrium which we compute using the best response analysis. For the dynamic case, the solution concept is subgame perfect equilibrium (SPE) which is computed using backward induction. We illustrate each solution concept by discussing two examples; one with discrete strategies and another with continuous strategies.

3.2.1 Model I: Static Games with Complete Information (Nash Equilibrium)

Consider two competing newsvendors (denoted by P1 and P2) who face random demand for their product. The newsvendors (also called "players") may lose customers to each other if their stock is not sufficient to meet the demand of their product. Thus, P1's expected profit J_1 is affected by his order quantity q_1 and also by P2's order quantity q_2. (Clearly, if P1 chooses a low value of q_1, this may result in shortages for him and thus P2 may benefit from this as some of P1's unsatisfied customers may switch to P2.) Similarly, P2's expected profit J_2 is affected by her order quantity q_2 and also by P1's order quantity q_1. Thus, we write $J_1(q_1, q_2)$ and $J_2(q_1, q_2)$ as the expected profits of players P1 and P2, respectively.

Discrete Strategies

Consider first a simple situation where the newsvendors' order quantities are limited to take only one of two possible values, say low or high. Thus, P1 chooses either L_1 (low) or H_1 (high), i.e., $q_1 \in \{L_1, H_1\}$, and P2 chooses either ℓ_2 (low) or h_2 (high), i.e., $q_2 \in \{\ell_2, h_2\}$. For each combination of order quantities, the newsvendors' expected profits are given in **strategic form** (or, **normal form**) in Table 3.1 as a pair of numbers (J_1, J_2). [In general, the (J_1, J_2) values represent the players' expected utilities (Luce and Raiffa [55], Straffin [86, Ch. 9], von Neumann and Morgenstern [90, Ch. 3]), but in this paper we assume that they are risk-neutral; thus (J_1, J_2) are taken as dollar values.] It is also possible to represent this game with discrete strategies using a game tree as in Figure 3.1. Note that both (a) and (b) in this figure are equivalent representations of the simultaneous game where the nodes connected by a dashed line constitute a player's **information set**. In Figure 3.1(a), when P2 makes a move, she is at a node in the information set indicated, but she does not know which node since in the simultaneous game P1 would not reveal his choice to P2. Figure 3.1(b) has essentially the same interpretation where the nodes connected by the dashed line constitute P1's information set. Game trees play a crucial role in identifying the equilibrium in dynamic games, but the normal form is more frequently used for analyzing static games.

P1\P2	ℓ_2		h_2
L_1	$(3,1)$	$\longrightarrow$	$(6,2^*)$
	$\downarrow$		$\downarrow$
H_1	$(\mathbf{5}^*,\mathbf{4}^*)$	$\longleftarrow$	$(7^*,3)$

Table 3.1: Payoff table for the two newsvendors' expected profits for the static game where the players make their moves simultaneously. Here, each player has two strategies.

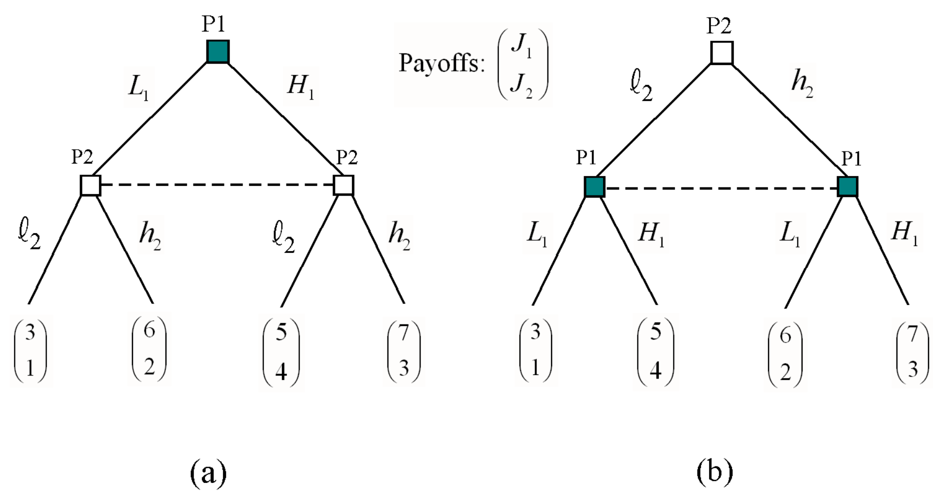

Figure 3.1: Two equivalent game tree representations of the static game between two players P1 and P2. The nodes connected by a dashed line represent the information set of a player.

In static games of complete information which are represented in normal form, each player has exactly the same number of actions (i.e., moves) as the number of strategies. In this example, both P1 and P2 have two actions/strategies to choose from, thus the normal form Table 3.1 simply consists of two rows (for P1) and two columns (for P2). When games become sequential and/or they involve incomplete information, one needs to distinguish between 'actions' and 'strategies.' We will have more to say about this distinction in subsequent sections.

How should the newsvendors choose their order quantities (i.e., determine their moves) recognizing that each newsvendor's expected profit depends on both players' decisions? To answer this question we determine the best response of each player to the other's decision. If P1 chooses L_1, then P2 should choose h_2 as this choice gives her an expected profit of 2 which is higher than 1 if she had chosen ℓ_2. (We indicate this with an asterisk * placed next to 2.) Similarly, P2's best response to H_1 is ℓ_2 indicated by 4^*. What are P1's best responses to P2's moves? If P2 chooses ℓ_2, it is best for P1 to choose H_1 and receive 5^*, and if h_2 is chosen, it is still best for P1 to choose H_1 and receive 7^*. The $H_1\ell_2$ cell in the second row and first column of the table is significant as this gives the **Nash equilibrium** for this problem. The directions of the arrows in Table 3.1 indicate that any movement away from the equilibrium will not last long and the players will eventually settle at the equilibrium solution of $H_1\ell_2$ with payoffs $(J_1, J_2) = (5, 4)$.

The Nash equilibrium is a "trap" in the sense that the players would have no incentive to deviate away from it. For example, if P1 moves to L_1, then P2 would play h_2 resulting in a reduction of P1's payoff from 5 to 3. But if P2 plays h_2, then P1 would prefer H_1 in which case P2 would choose ℓ_2 thus ending up in the "trap" again.

We formalize the above discussion with the following definition: **Nash equilibrium** for a two-player non-cooperative game is a pair (q_1^N, q_2^N) with the property that,

$$J_1(q_1^N, q_2^N) \geq J_1(q_1, q_2^N) \text{ for all } q_1, \tag{3.1}$$

$$J_2(q_1^N, q_2^N) \geq J_2(q_1^N, q_2) \text{ for all } q_2. \tag{3.2}$$

Remark 3.1 ***Computing the equilibria in the extensive form*** *We can also quickly identify the Nash equilibrium by determining whether a given strategy is part of an equilibrium* without *first computing the normal form of the game. This method becomes important in identifying the equilibria in games of incomplete information. To illustrate this method, assume that P1 chooses L_1. From Figure 3.1(a), we see that Player P2's best response to this choice is h_2 because 2 is better than 1. But if P1 realizes that P2 will choose h_2, we again see from Figure 3.1(a) that now P1 would play H_1 because 7 is better than 6; thus the given strategy L_1 cannot be part of an equilibrium. Now assume that P1 chooses H_1 to which P2's best response would be ℓ_2 because 4 is better than 3. But then P1 would have no incentive to move away from H_1 because 5 is better than 3, thus resulting in the equilibrium $H_1\ell_2$ with the payoffs (5,4).* ♦

Continuous Strategies

When the expected profit functions $J_i(q_1, q_2)$ are continuous in the strategies, the best response functions and the Nash equilibrium are determined as follows: Given P2's strategy q_2, P1 must maximize his expected profit; thus P1 finds his best response by maximizing $J_1(q_1, q_2)$ for a given q_2. That is, P1 must solve $\max_{q_1} J_1(q_1, q_2)$ for each possible value of q_2 and obtain $R_1(q_2)$ as his best response. Similarly, P2 solves $\max_{q_2} J_2(q_1, q_2)$ for each possible value of q_1 and obtain $R_2(q_1)$ as her best response. Provided that the payoff functions $J_i(q_1, q_2)$ are continuously differentiable in their argument q_i and concave for every q_j, $i \neq j$, $i, j = 1, 2$, the best response function is found from $\partial J_i / \partial q_i \equiv I_i(q_1, q_2) = 0$. It then follows that a Nash equilibrium (if it exists) is found as a solution of the system of two equations

$$I_1(q_1, q_2) \equiv \frac{\partial J_1}{\partial q_1} = 0, \quad \text{and} \quad I_2(q_1, q_2) \equiv \frac{\partial J_2}{\partial q_2} = 0.$$

To illustrate these results, consider a simplified version of the competitive newsvendor model discussed in Parlar [69]. As in the discrete strategy example discussed above, the newsvendors face random demands X and Y with respective densities $f(x)$ and $h(y)$ and if one newsvendor runs out of stock, some of the unsatisfied customers may switch to the other newsvendor if he/she has any units available. For simplicity of exposition in this paper, we assume that both the salvage value and the penalty costs are zero. With these assumptions, the expected profit function of the first newsvendor (P1) is given in Parlar [69] as,

$$\begin{aligned} J_1(q_1, q_2) = s_1 \int_0^{q_1} x f(x)\ dx + s_1 \int_{q_1}^{\infty} q_1 f(x)\ dx + s_1 \int_0^{q_1} \int_{q_2}^{B} b(y - q_2) h(y) f(x)\ dy\ dx \\ + s_1 \int_0^{q_1} \int_B^{\infty} (q_1 - x) h(y) f(x)\ dy\ dx - c_1 q_1, \end{aligned} \tag{3.3}$$

where s_1 is the unit sales revenue, c_1 is the unit purchase cost and $B \equiv (q_1 - x)/b + q_2$, with b as the fraction of P2's demand that will switch to P1's product when P2 is sold out. The second newsvendor's expected profit is obtained similarly as,

$$\begin{aligned} J_2(q_1, q_2) = s_2 \int_0^{q_2} y h(y)\ dy + s_2 \int_{q_2}^{\infty} q_2 h(y)\ dy + s_2 \int_0^{q_2} \int_{q_1}^{A} a(x - q_1) f(x) h(y)\ dx\ dy \\ + s_2 \int_0^{q_2} \int_A^{\infty} (q_2 - y) f(x) h(y)\ dx\ dy - c_2 q_2, \end{aligned} \tag{3.4}$$

where s_2 and c_2 are the unit sales revenue and unit purchase cost, respectively, and $A \equiv (q_2 - x)/a + q_1$, with a as the fraction of P1's demand that will switch to P2's product when P1 is sold out.

Parlar [69] has shown that,

$$\frac{\partial J_1}{\partial q_1} \equiv I_1(q_1,q_2) = s_1 \int_{q_1}^{\infty} f(x)\ dx + s_1 \int_0^{q_1} \int_B^{\infty} h(y) f(x)\ dy\ dx - c_1, \tag{3.5}$$

$$\frac{\partial J_2}{\partial q_2} \equiv I_2(q_1,q_2) = s_2 \int_{q_2}^{\infty} h(y)\ dy + s_2 \int_0^{q_2} \int_A^{\infty} f(x) h(y)\ dx\ dy - c_2, \tag{3.6}$$

and that $\partial I_1/\partial q_1 = \partial^2 J_1/\partial q_1^2 < 0$, indicating the strict concavity of J_1 for each q_2. [Using parallel arguments, it is also possible to show that $\partial^2 J_2/\partial q_2^2 < 0$, i.e., J_2 is strictly concave for each q_1.] Employing these results, Parlar [69] proves the uniqueness of the Nash equilibrium for this problem.

To illustrate the above results, let us assume that demand densities are exponential, i.e., $f(x) = \lambda e^{-\lambda x}$ and $h(y) = \mu e^{-\mu y}$ with respective parameters $[\lambda, \mu] = \left[\frac{1}{30}, \frac{1}{20}\right]$, and means $E(X) = 30$ and $E(Y) = 20$. The other parameters are given as $[a, b \mid s_1, s_2 \mid c_1, c_2] = [0.9, 0.9 \mid 15, 9 \mid 8, 5]$. With these data values, we find the newsvendors' expected profits as,

$$J_1(q_1,q_2) = 450(1-e^{-q_1/30}) + 270e^{-q_2/20} + 405e^{-(q_1/18+q_2/20)} - 675e^{-(q_1/30+q_2/20)} - 8q_1,$$

$$J_2(q_1,q_2) = 180(1-e^{-q_2/20}) + 243e^{-q_1/30} - \frac{6561}{7}e^{-(q_1/30+q_2/27)} + \frac{4860}{7}e^{-(q_1/30+q_2/20)} - 5q_2.$$

The three-dimensional surfaces corresponding to $J_1(q_1,q_2)$ and $J_2(q_1,q_2)$ are displayed in Figure 3.2. Note that if P1 could choose both q_1 and q_2 at will, he would solve the optimization problem $\max_{q_1,q_2} J_1(q_1,q_2)$ and obtain $(q_1,q_2) = (37.21, 0)$ and $J_1 = 148.11$. Naturally, in a competitive setting P1 has no control over P2's order quantity and thus this solution would not be possible. Similarly, if P2 could choose P1's order quantity, then she would solve $\max_{q_1,q_2} J_2(q_1,q_2)$ and obtain $(q_1,q_2) = (0, 35.21)$ and $J_2 = 80.98$, but this solution would also not be possible since P2 has no control over P1's decisions.

To determine the Nash equilibrium for this problem where P1 chooses his strategy q_1 and P2 chooses her strategy q_2 we proceed as follows: Differentiating the expected profit functions we have

$$\frac{\partial J_1}{\partial q_1} \equiv I_1(q_1,q_2) = 15e^{-q_1/30} - \frac{45}{2}e^{-(q_1/18+q_2/20)} + \frac{45}{2}e^{-(q_1/30+q_2/20)} - 8, \tag{3.7}$$

$$\frac{\partial J_2}{\partial q_2} \equiv I_2(q_1,q_2) = 9e^{-q_2/20} + \frac{243}{7}e^{-(q_1/30+q_2/27)} - \frac{243}{7}e^{-(q_1/30+q_2/20)} - 5. \tag{3.8}$$

The best response function $R_1(q_2)$ for P1 can be extracted (in this case, numerically) by solving $I_1(q_1,q_2) = 0$ for each value of q_2. Similarly, the best response function $R_2(q_1)$ for P2 can be extracted (again, numerically) by solving $I_2(q_1,q_2) = 0$ for each value of q_1. [See Figure 3.3 for the curves representing the relations $I_1(q_1,q_2) = 0$ and $I_2(q_1,q_2) = 0$.] Thus, to compute the equilibrium we solve the system of

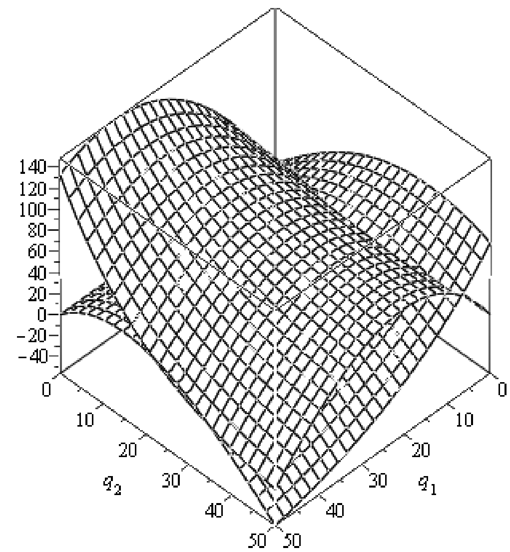

Figure 3.2: Three-dimensional surfaces corresponding to the expected profits $J_1(q_1, q_2)$ [upper surface] and $J_2(q_1, q_2)$ [lower surface] of the two newsvendors. The global maximum of $J_1(q_1, q_2)$ is at $(q_1, q_2) = (37.21, 0)$ with $J_1 = 148.11$, and the global maximum of $J_2(q_1, q_2)$ is at $(q_1, q_2) = (0, 35.21)$ with $J_2 = 80.98$.

two equations $I_1(q_1, q_2) = 0$ and $I_2(q_1, q_2) = 0$ for the two unknowns. This gives $(q_1^N, q_2^N) = (25.38, 19.55)$ as the unique Nash equilibrium with $J_1(q_1^N, q_2^N) = 83.63$ and $J_2(q_1^N, q_2^N) = 35.91$. In this competitive scenario each player receives a lower expected profit than what they would have obtained if they could have chosen both decision variables freely.

3.2.2 Model II: Dynamic Games with Complete Information (Subgame Perfect Equilibrium)

We now consider a dynamic (two-stage) version of the game discussed above. Whereas in Section 3.2.1 the players were choosing their strategies simultaneously, now the decisions are made sequentially (perhaps because one of the players can act quickly and make his decision before the other one). Retaining the assumption of complete information, we now examine the resulting complications arising from the sequential nature of the game.

Discrete Strategies

Let us return to the same problem we discussed in Section 3.2.1 but now suppose that P1 acts *before* P2 in choosing his order quantity strategy; thus in this version of the game P1 becomes the “leader” and P2 becomes the “follower.” Will this result

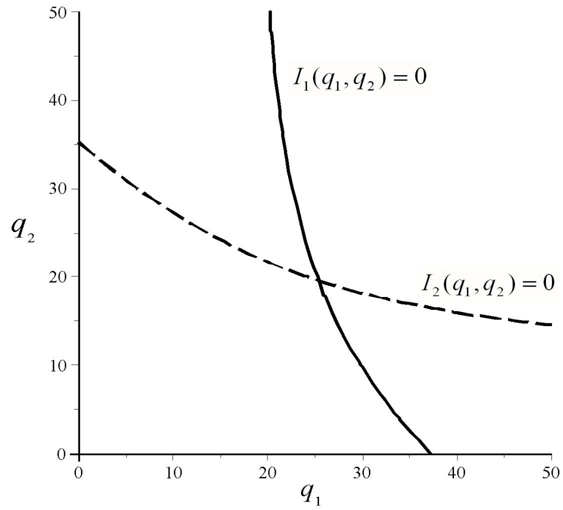

Figure 3.3: Best response functions of the two newsvendors are represented by the implicit relations $I_1(q_1, q_2) = 0$ and $I_2(q_1, q_2) = 0$. The Nash equilibrium in this case is found by solving for the point of intersection of these two nonlinear equations and is calculated as $(q_1^N, q_2^N) = (25.38, 19.55)$.

in a "first mover advantage" for P1? To answer this question, we first consider the **extensive form** representation of the game given in Figure 3.4 where P1 moves first and P2 moves next. Depending on the combination of moves made by the players, the payoffs obtained are indicated at the endpoints of the game tree. In dynamic games with sequential decisions, a **subgame** is defined as that part of the game tree that starts at a particular node of the original game. For example, one of the subgames in Figure 3.4 starts at node indicated by <2a>; and another subgame starts at <2b>. The complete game itself which starts at node <1> is also considered a "subgame."

Now that the structure of the game has changed and the choices are made sequentially, P2 no longer has ℓ_2 and h_2 as the only strategies (which was the case in the static game of Section 3.2.1). In dynamic games the follower's moves are conditional on the leader's moves and thus the follower must have a complete plan of action specified for all the possibilities that she may face. This means that P2 now has a total of four strategies available given in Table 3.2:

For ease of reference, we use the "`mapsto`" notation $\mapsto$ to denote P2's strategies as a function of P1's moves. For example, P2's first strategy is denoted by $(L_1, H_1) \mapsto \ell_2\ell_2$ which indicates that P2 will always choose ℓ_2 regardless of what P1 does; and P2's third strategy is denoted by $(L_1, H_1) \mapsto h_2\ell_2$ which indicates that P2 will choose h_2 if P1 chooses L_1, and she will choose ℓ_2 if P1 chooses H_1. Thus, a **strategy** for a player is a complete plan to play the game. (Note that if P1 had three moves and P2 had two, then P2 would have a total of $2^3 = 8$ strategies. In general, with m moves for P1 and n moves for P2, the former has m strategies and the latter has n^m strategies. See,

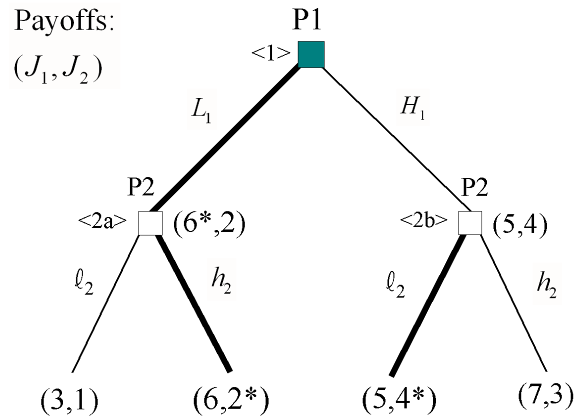

Figure 3.4: Extensive form of the dynamic game with two players where the subgame perfect equilibrium is found as $(L_1, h_2\ell_2)$.

		P2's strategies			
		#1	#2	#3	#4
P1's moves	L_1	ℓ_2	ℓ_2	h_2	h_2
	H_1	ℓ_2	h_2	ℓ_2	h_2
Notation		$(L_1, H_1) \mapsto \ell_2\ell_2$	$(L_1, H_1) \mapsto \ell_2 h_2$	$(L_1, H_1) \mapsto h_2\ell_2$	$(L_1, H_1) \mapsto h_2 h_2$

Table 3.2: Strategies for player P2 as a function of player P1's moves.

Peters [71, Ch. 4.2] and Webb [92, Ch. 2.2] for a good discussion of the enumeration of strategies in games with finite number of moves for each player.)

Given the two strategies (moves) available to P1 and four strategies available to P2, we can establish the normal formal of this dynamic game and attempt to identify the equilibrium. The normal form is given in Table 3.3 where the first and the last columns are exactly the same as the first and second columns of the normal form matrix for the simultaneous game discussed in Section 3.2.1 (Table 3.1). Note, for example, that the strategy combination $(L_1, \ell_2\ell_2)$ results in payoff vector $(3, 1)$ since P1's choice of L_1 is followed by P2's choice of ℓ_2. Similarly, the strategy combination $(H_1, h_2\ell_2)$ results in payoff vector $(5, 4)$ since P1's choice of H_1 is followed by P2's choice of ℓ_2.

It is easy to see that the direction of arrows in Table 3.3 show that this game has two Nash equilibria; one being $(H_1, \ell_2\ell_2)$ with payoffs $(\mathbf{5}^*, \mathbf{4}^*)$; and the other $(L_1, h_2\ell_2)$ with payoffs $(\mathbf{6}^*, \mathbf{2}^*)$. Which equilibrium is the one that will/should be used in this dynamic game? Before we answer this question, we point out that we can determine which strategy will be part of an equilibrium without computing the normal form.

P1\P2	$(L_1, H_1) \mapsto \ell_2\ell_2$		$(L_1, H_1) \mapsto \ell_2 h_2$		$(L_1, H_1) \mapsto h_2\ell_2$		$(L_1, H_1) \mapsto h_2 h_2$
L_1	$(3, 1)$	$\leftrightarrow$	$(3, 1)$	$\rightarrow$	$(\mathbf{6}^*, \mathbf{2}^*)$	$\leftrightarrow$	$(6, 2^*)$
	$\downarrow$		$\downarrow$		$\uparrow$		$\downarrow$
H_1	$(\mathbf{5}^*, \mathbf{4}^*)$	$\leftarrow$	$(7^*, 3)$	$\rightarrow$	$(5, 4^*)$	$\leftarrow$	$(7^*, 3)$

Table 3.3: Payoff table for the two newsvendors' expected profits for the dynamic game. In this table P1 is the "leader" and P2 is the "follower" where the former has two strategies, but the latter has four strategies.

Remark 3.2 ***Computing the equilibria in the extensive form*** *We observe here that one could again quickly determine whether a strategy is part of a Nash equilibrium without first computing the normal form of the game—as was done in Section 3.2.1. When we consider L_1 we note that P2's best responses to L_1 are both $h_2\ell_2$ and h_2h_2. Faced with $h_2\ell_2$, P1 will have no incentive to deviate to H_1 (since that would result in receiving 5 rather than 6), but faced with h_2h_2, P1 will deviate to H_1. Thus, $(L_1, h_2\ell_2)$ must be an equilibrium. Similarly, if P1 chooses H_1, then P2's best responses are $\ell_2\ell_2$ and $h_2\ell_2$, but P1 will not deviate from H_1 when faced $\ell_2\ell_2$, but will deviate when faced with $h_2\ell_2$. Thus, $(H_1, \ell_2\ell_2)$ is also a Nash equilibrium.* ♦

Returning to the question of which equilibrium is the one that will/should be used in this dynamic game, we refer to Figure 3.4 and observe that if the game ever arrived at node <2a>, it is optimal for P2 to choose h_2 with a payoff of $(6, 2^*)$, which is better for her than $(3, 1)$; and if it arrived at <2b> it is optimal for P2 to choose ℓ_2 with a payoff of $(5, 4^*)$, which is better than (7,3). That is, the best response $R_2(q_1)$ for P2 is given as,

$$R_2(q_1) = \begin{cases} h_2, & \text{if } q_1 = L_1, \\ \ell_2, & \text{if } q_1 = H_1. \end{cases} \tag{3.9}$$

Moving back to node <1>, P1's problem is now to maximize $J_1(q_1, R_2(q_1))$ where P2's best response $R_2(q_1)$ is given in (3.9). Thus, P1 makes his choice by comparing $(6, 2)$ [if he chooses L_1] and $(5, 4)$ [if he chooses H_1], resulting in the optimal choice of L_1. This method of solving the game is known as **rollback** (Dixit, Skeath and Reily [23, Ch. 3]) or **backward induction** (Gibbons [35, Ch. 2]) which uses the same principle as dynamic programming (Bellman [5]). The rollback equilibrium is shown as thick lines in Figure 3.4 and is denoted by $(L_1, h_2\ell_2)$ since it is optimal for P1 to choose L_1 at <1>, and it is optimal for P2 to choose h_2 if P1 chooses L_1, and ℓ_2 if P1 (ever) chooses H_1.

Note that at each subgame the rollback principle produces a choice that is optimal for that player resulting in the equilibrium for the game. In games with finite trees and **perfect information** (where the players know the result of all previous moves, as the ones considered in this chapter) the equilibrium found by the rollback principle is also called *subgame-perfect equilibrium* (SPE). More formally, a **subgame-perfect equilibrium** is defined as a combination of strategies for both players that result in a Nash equilibrium in every subgame which specify moves that are best responses to each other (as we saw above); see, Dixit and Skeath [23, Ch. 3], Gibbons [35, Ch. 2], and Selten [81] (where the concept was first introduced).

In summary, the subgame-perfect equilibrium in our example is $(L_1, h_2\ell_2)$ which can also be written as $(q_1^*, R_2(q_1))$ where q_1^* is P1's optimal choice at node <1>. It is important to note that if P1 chooses his optimal strategy L_1, then P2's optimal move is to choose h_2, and thus the *equilibrium path* arising from SPE is $L_1 \to h_2$. Thus, even though ℓ_2 is a part of the SPE, P2 will never choose it *unless* P1 makes a non-optimal decision at node <1> and chooses H_1 at that node. In summary, the SPE $(L_1, h_2\ell_2)$ is the backward induction *equilibrium*, whereas (L_1, h_2) is the backward induction *outcome* that is on the equilibrium path. If both players choose their strategies optimally, then the resulting sequence of decisions $L_1 \to h_2$ that are on the equilibrium path is known as *Stackelberg solution* (von Stackelberg [91]) which we denote by $(q_1^S, R_2(q_1^S))$. Thus, the (rational) choice of the players leading to the equilibrium path eliminates the other Nash equilibrium, i.e., $(H_1, \ell_2\ell_2)$.

Continuous Strategies

Let us now return to the newsvendor problem with continuous strategies discussed in Section 3.2.1 and assume that P1 is the first mover (the "leader") and P2 is the second mover (the "follower") with respective objective functions $J_1(q_1, q_2)$ and $J_2(q_1, q_2)$ given in (3.3) and (3.4). For this game, the best response function $R_2(q_1)$ for P2 is obtained by maximizing $J_2(q_1, q_2)$ for each q_1, or equivalently, solving $\partial J_2/\partial q_2 = I_2(q_1, q_2) = 0$ for each value of q_1; that is,

$$R_2(q_1) = \arg\max_{q_2 \geq 0} \; J_2(q_1, q_2) = \left\{ q_2 : \frac{\partial J_2}{\partial q_2} = I_2(q_1, q_2) = 0 \right\}.$$

Thus, P1 must choose his order quantity that maximizes $J_1(q_1, q_2)$ subject to the constraint $I_2(q_1, q_2) = 0$. For this game, the **Stackelberg solution** for P1 is obtained by solving the optimization problem

$$\max_{q_1 \geq 0} \; J_1(q_1, q_2) \text{ subject to } I_2(q_1, q_2) = 0,$$

where $J_1(q_1, q_2)$ and $I_2(q_1, q_2) = 0$ are given in (3.3) and (3.8), respectively, and depicted in the three-dimensional Figure 3.5. The same problem is shown in two dimensions by projecting the contours of $J_1(q_1, q_2)$ onto the (q_1, q_2) plane and by choosing the highest-valued contour that is tangent to P2's best response curve $I_2(q_1, q_2) = 0$ as in Figure 3.6. Solving the optimization problem with the same data as in Section 3.2.1 gives the Stackelberg solution as $(q_1^S, q_2^S) = (28.38, 18.60)$ and $(J_1(q_1^S, q_2^S), J_2(q_1^S, q_2^S)) = (84.35, 33.94)$. This result indicates that P1 has obtained a first-mover *advantage* compared to the Nash solution since $J_1(q_1^S, q_2^S) > J_1(q_1^N, q_2^N)$, which has resulted in a second-mover *disadvantage* for P2 since $J_2(q_1^S, q_2^S) < J_2(q_1^N, q_2^N)$.

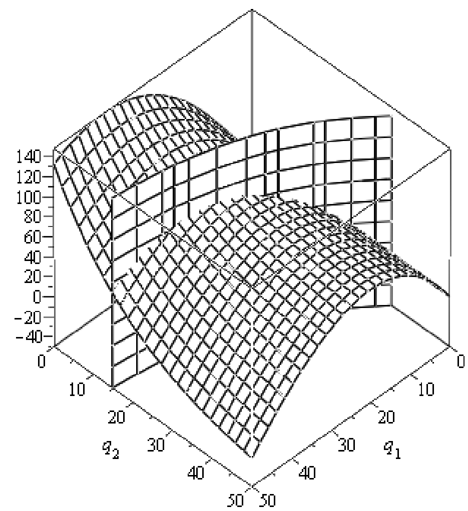

Figure 3.5: The Stackelberg equilibrium is obtained by maximizing P1's objective function $J_1(q_1, q_2)$ [upper surface] subject to the constraint $I_2(q_1, q_2) = 0$.

3.3 Model III: Static Games with Incomplete Information (Bayesian Nash Equilibrium)

The games of complete information described in Models I and II above had the common feature that both players were informed about each other's payoff functions. In a game of **incomplete information** players may not know the payoff function of some other player, or they may not know what actions are available to other player(s). For example, even though P2 would know her own purchase cost c_2 with certainty, she may only know that P1's purchase cost is c_{1L} (low) with probability θ_1, or c_{1H} (high) with probability $1 - \theta_1$. Since P1 knows his cost (which is either c_{1L} or c_{1H}), he has superior information, i.e., information structure is asymmetric in favour of P1.

Using Harsanyi's approach [40] elucidated in a three-part essay, we solve the resulting **Bayesian game** by assuming that every player can be of several possible **types**

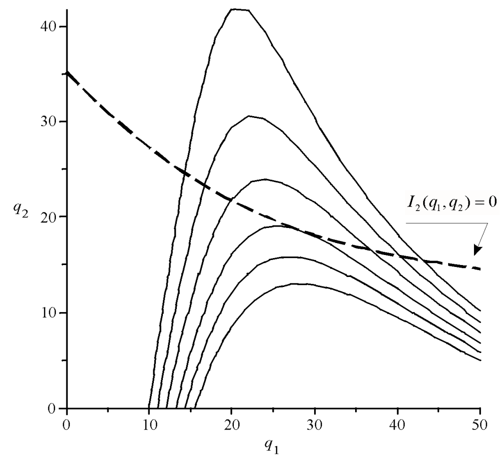

Figure 3.6: The Stackelberg equilibrium is obtained by solving the following problem: max $J_1(q_1, q_2)$ subject to $I_2(q_1, q_2) = 0$. This figure superimposes the contours of P1's objective $J_1(q_1, q_2)$ with P2's best response $I_2(q_1, q_2) = 0$. In the equilibrium we have $(q_1^S, q_2^S) = (28.38, 18.60)$.

where a type summarizes all relevant information about a player such as the payoffs and possible moves. (See Myerson [64] for an illuminating commentary on Harsanyi's three papers.) In such a game, **Harsanyi transformation** [40] introduces an artificial player, called Nature, which chooses a particular type for all players (according to some joint probability distribution) and reveals to player Pi his type t_i. Thus, some players cannot observe the move of Nature regarding the actual type of the other player(s), but they know the joint probability distribution from which Nature selects the types. This means that the players face a game with imperfect information (because of the uncertainty about the move of the "player" Nature), and hence the incompleteness of information about payoffs is transformed into uncertainty about the move of Nature.

For example, if P1's purchase cost is either c_{1L} or c_{1H}, then this player will have two types $T_1 = \{c_{1L}, c_{1H}\}$, and if P2's purchase cost is c_2, then P2 will have one type $T_2 = \{c_2\}$, only. It is assumed that each player knows his/her type, and given this, each player can compute his/her beliefs on the types of other players. That is, for $t_1 \in T_1$ and $t_2 \in T_2$, the **beliefs** $p_1(t_2 \mid t_1)$ and $p_2(t_1 \mid t_2)$ are computed by the conditional probability formula (i.e., the Bayes' rule) as,

$$p_1(t_2 \mid t_1) = \frac{p(t_2, t_1)}{p(t_1)}, \quad \text{and} \quad p_2(t_1 \mid t_2) = \frac{p(t_1, t_2)}{p(t_2)},$$

where the joint probability $p(t_1, t_2)$ is assumed common knowledge.

Harsanyi's approach [40] assumes that the strategy for player Pi is a function

$\sigma_i(t_i)$ for each type $t_i \in T_i$ which specifies a feasible action. For example, if P1 has two types, and he only has two moves $\{L_1, H_1\}$, then he has four possible strategies. These are listed in Table 3.4 where L_1L_1 indicates that P1 will choose L_1 when his type is either c_{1L} or c_{1H} (that is, he always chooses L_1); L_1H_1 indicates that P1 will choose L_1 if his type is c_{1L}, will choose H_1 if his type is c_{1H}, etc.

		P1's type t_1		
		c_{1L}	c_{1H}	Notation
	#1	L_1	L_1	$(c_{1L}, c_{1H}) \mapsto L_1L_1$
P1's	#2	L_1	H_1	$(c_{1L}, c_{1H}) \mapsto L_1H_1$
strategies	#3	H_1	L_1	$(c_{1L}, c_{1H}) \mapsto H_1L_1$
	#4	H_1	H_1	$(c_{1L}, c_{1H}) \mapsto H_1H_1$

Table 3.4: Strategies for player P1 as a function of his type.

Harsanyi [40] proposed modelling static games with incomplete information by including Nature as an imaginary player. In the above game Nature moves first and determines P1's type which is c_{1L} with probability θ_1 and c_{1H} with probability $1-\theta_1$. P1 knows his type (i.e., his payoffs), but P2 knows only that she is facing an opponent (P1) whose purchase cost is either c_{1L} or c_{1H} with probabilities θ_1 and $1-\theta_1$, respectively. One interpretation of this game is that P1 and P2 are randomly paired and the proportion of low cost P1s is θ_1. Since P2 has only one type, the conditional probabilities for P1 are simply $p_1(t_2 \mid c_{1L}) = 1$ and $p_1(t_2 \mid c_{1H}) = 1$. However, since P1 has two types, P2's conditional probabilities are $p_2(c_{1L} \mid t_2) = p_2(c_{1L}) = \theta_1$, and $p_2(c_{1H} \mid t_2) = p_2(c_{1H}) = 1-\theta_1$. It is important to note that even though P1 knows his type, the Bayesian equilibrium solution must still provide a complete plan of action for both players, i.e., in the discrete version of the game, P1 must consider his four strategies $\{L_1L_1, L_1H_1, H_1L_1, H_1H_1\}$ shown in Table 3.4, and P2 must consider her two strategies (moves) $\{\ell_2, h_2\}$.

When the players adopt the strategy profile $(\sigma_1(t_1), \sigma_2(t_2))$ we define the conditional expected payoffs for player Pi of type t_i as

$$\hat{J}_1(\sigma_1(t_1), \sigma_2(t_2), t_1) = \sum_{t_2 \in T_2} J_1(\sigma_1(t_1), \sigma_2(t_2); t_1, t_2) p_1(t_2 \mid t_1), \quad \text{for all } t_1 \in T_1, \tag{3.10}$$

$$\hat{J}_2(\sigma_1(t_1), \sigma_2(t_2), t_2) = \sum_{t_1 \in T_1} J_2(\sigma_1(t_1), \sigma_2(t_2); t_1, t_2) p_2(t_1 \mid t_2), \quad \text{for all } t_2 \in T_2, \tag{3.11}$$

where $J_i(\sigma_1(t_1), \sigma_2(t_2); t_1, t_2)$ is player Pi's payoff when this player's type t_i adapts the strategy $\sigma_i(t_i)$ for $i = 1, 2$. With this definition, a strategy profile $(\sigma_1^*(t_1), \sigma_2^*(t_2))$ is a (pure) **Bayesian Nash equilibrium** of a static Bayesian game if for each player Pi, every type $t_i \in T_i$ of player Pi, and every alternative strategy $\sigma_i'(t_i)$ of player Pi, we have

$$\hat{J}_1(\sigma_1^*(t_1), \sigma_2^*(t_2), t_1) \geq \hat{J}_1(\sigma_1'(t_1), \sigma_2^*(t_2); t_1), \tag{3.12}$$
$$\hat{J}_2(\sigma_1^*(t_1), \sigma_2^*(t_2), t_2) \geq \hat{J}_2(\sigma_1^*(t_1), \sigma_2'(t_2); t_2). \tag{3.13}$$

Similar to the definition of Nash equilibrium given in (3.1)–(3.2), this definition states that whatever a player's type is, this player's strategy is a best response to the strategies of the other player.

3.3.1 Discrete Strategies

To illustrate the above discussion of the Bayesian Nash equilibrium, let us consider a discrete strategy problem where each player has two possible moves (i.e., P1 orders low L_1 or high H_1, and P2 orders low ℓ_2 or high h_2). But now P1 knows that his type (purchase cost) is either c_{1L} (low) or c_{1H} (high), but P2 only knows the probability distribution of P1's type, that is, that Pr(P1's type is c_{1L}) $= \theta_1$ and Pr(P1's type is c_{1H}) $= 1 - \theta_1$. P2 has only one type, c_2, and both players know this.

The game trees for this game played under incomplete information are represented in Figure 3.7. Player P1 knows exactly which game is being played, but P2 knows that the game on the left will be played with probability θ_1 and the game on the right with probability $1 - \theta_1$. One can interpret this game of incomplete information by saying that as far as P2 is concerned P1 has two "personalities" and P2 faces the type 1 personality (purchase cost c_{1L}) with probability θ_1 and type 2 personality (purchase cost c_{1H}) with probability $1 - \theta_1$.

The Harsanyi transformation involves converting the game of incomplete information to a game of complete but imperfect information. The game tree for the transformed version of the problem is given in Figure 3.8 where Nature moves first and determines P1's type, i.e., low or high purchase cost. (We will use the same payoff values we have in Figure 3.8 when we consider the dynamic version of this problem with incomplete information in Section 3.4.) Once Nature makes her choice, P1 knows his type but this information is not revealed to P2. With the inclusion of Nature as one of the players, the game becomes one of complete information since the players know all the payoffs on the extensive form; the game also becomes one of imperfect information because P2 will not be aware of what Nature has chosen initially.

We now represent this extensive game of complete but imperfect information to an equivalent strategic form game and determine the Bayesian Nash equilibrium. First, recall that, as indicated in Table 3.4, P1 has a total of four strategies as he must have a complete plan of action following Nature's choice. On the other hand, since P2 has one type only (purchase cost of c_2), this player still has two strategies, one for each possible move ℓ_2 and h_2. The (expected) payoffs for all possible combinations of all strategy pairs are given in Table 3.5. Consider, for example, the strategy combination (L_1L_1, h_2), that is, regardless of Nature's choice, P1 always orders low (L_1L_1), and P2

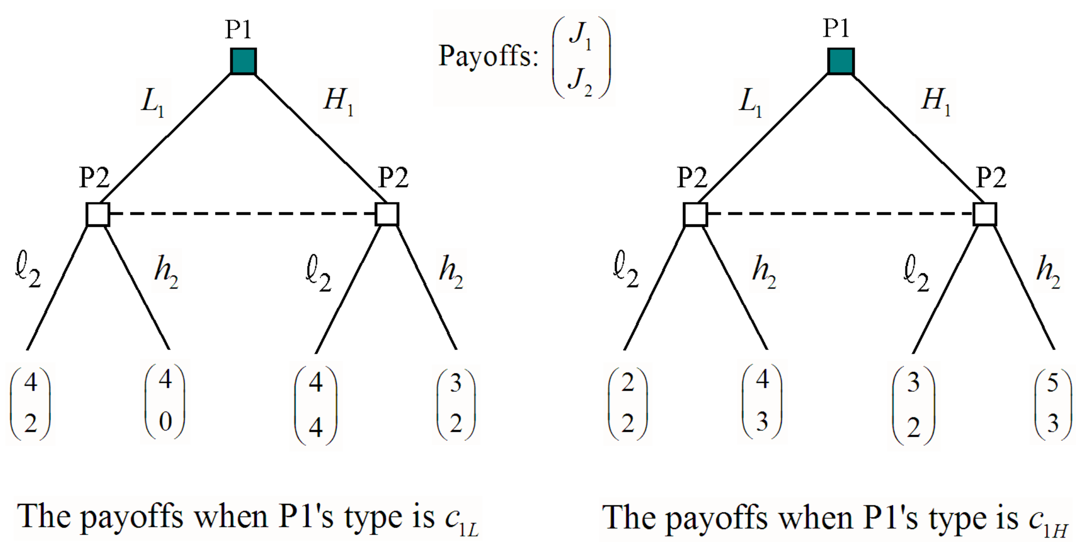

Figure 3.7: The incomplete information game where P1 knows which game is being played, i.e., either the one on the left where his type is c_{1L}, or the one on the right where his type is c_{1H}. P2 only knows the probability distribution of P1's type, i.e., $(\theta_1, 1-\theta_1)$.

orders high (h_2). Now, referring to Figure 3.8, if P1's cost is c_{1L}, then the expected conditional payoff for P1 is computed as

$$\hat{J}_1(L_1L_1, h_2; c_{1L}) = J_1(L_1L_1, h_2; c_{1L}, c_2)p_1(c_2 \mid c_{1L}) = 4 \cdot 1 = 4,$$

since P2 has only one type, i.e., c_2. Similarly, If P1's cost is c_{1H}, then the expected conditional payoff is,

$$\hat{J}_1(L_1L_1, h_2; c_{1H}) = J_1(L_1L_1, h_2; c_{1H}, c_2)p_1(c_2 \mid c_{1H}) = 4 \cdot 1 = 4.$$

To compute P2's expected payoff, we recall that $p_2(c_{1L} \mid c_2) = \frac{1}{2}$ and $p_2(c_{1H} \mid c_2) = \frac{1}{2}$ which gives,

$$\begin{aligned}\hat{J}_2(L_1L_1, h_2; c_2) &= J_2(L_1L_1, h_2; c_{1L}, c_2)p_2(c_{1L} \mid c_2) + J_2(L_1L_1, h_2; c_{1H}, c_2)p_2(c_{1H} \mid c_2)\\ &= 0 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} = 1.5.\end{aligned}$$

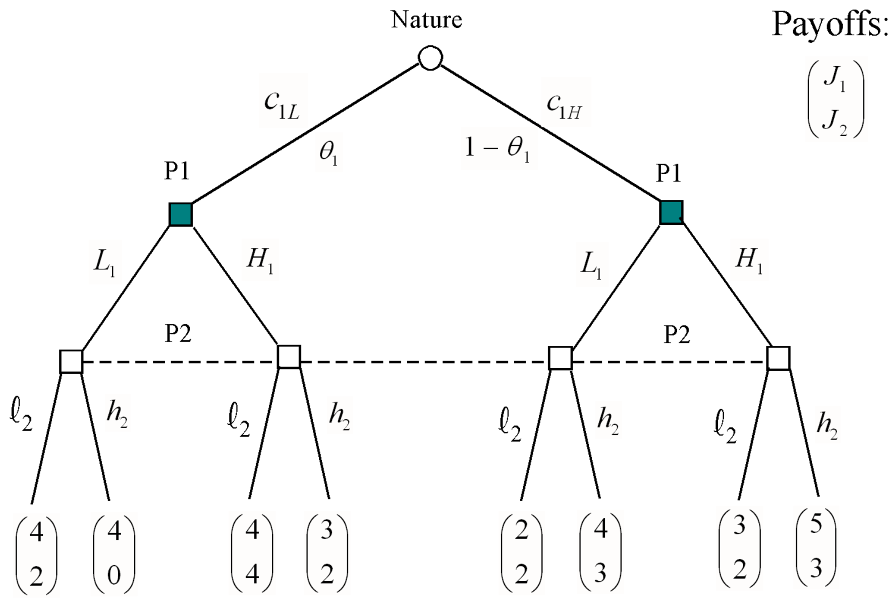

Figure 3.8: The incomplete information game in Figure 3.7 becomes an equivalent complete but imperfect information game through Harsanyi transformation.

The other expected conditional payoffs in Table 3.5 are calculated in a similar manner.

P1\P2	ℓ_2	h_2
$(c_{1L}, c_{1H}) \mapsto L_1L_1$	$(4^*, 2), 2^*$	$(4^*, 4), 1.5$
$(c_{1L}, c_{1H}) \mapsto L_1H_1$	$(\mathbf{4^*}, \mathbf{3^*}), \mathbf{2^*}$	$(4^*, 5^*), 1.5$
$(c_{1L}, c_{1H}) \mapsto H_1L_1$	$(4^*, 2), 3^*$	$(3, 4), 2.5$
$(c_{1L}, c_{1H}) \mapsto H_1H_1$	$(\mathbf{4^*}, \mathbf{3^*}), \mathbf{3^*}$	$(3, 5^*), 2.5$

Table 3.5: Payoff table obtained after Harsanyi transformation of the original payoffs in the discrete strategy, static incomplete information game.

To determine the Bayesian Nash equilibrium for this game, we use the familiar approach and identify the best responses for P2 for each possible strategy of P1. For example, if P1 uses H_1L_1, it is best for P2 to use ℓ_2 since this gives her an expected payoff of 3^* (rather than 2.5 if h_2 is used). Similarly, P1's best responses are found by identifying the strategy that gives him the highest expected payoff, given P2's strategy. For example, if P2 chooses ℓ_2, then the best response for P1 is either L_1H_1 or H_1H_1. Thus, the pure strategy equilibria in this game are found as (L_1H_1, ℓ_2) and

(H_1H_1, ℓ_2) with payoffs of (4,3) to P1 and 2 to P2 in the first case; and $(4, 3)$ for P1 and 3 for P2 in the second case. In this problem with multiple equilibria, (L_1H_1, ℓ_2) may be the one that is implemented if P1 wants to see P2 receive the least amount. On the other hand, if negotiations are possible and if P2 can motivate P1 to move to H_1H_1 with the potential promise of a side-payment to P1, then (H_1H_1, ℓ_2) could be the equilibrium of the game. (See Schelling [80, pp. 54–58, 89–118] for an explanation of the concept of a focal point in games with multiple Nash equilibria.)

The above procedure illustrates Harsanyi's insight [40] which transforms an incomplete information game (as shown in Figure 3.7) to a complete but imperfect information game (as shown in Figure 3.8). While the game in Figure 3.7 involves P2's uncertainties about P1's purchase cost—and hence the incompleteness of information—the strategic form game of Table 3.5 does not involve any uncertainty because it is subsumed in the expected payoff calculations for P2. However, the game in Figure 3.8 is of *imperfect* information variety since P1 and P2 make simultaneous decisions after Nature moves and P2 does not know which move Nature has made when she makes her decision.

Remark 3.3 ***Computing the equilibria in the extensive form*** *As in previous models, it is easy to determine whether a given strategy is part of an equilibrium without computing the payoffs in the normal form of Table 3.5. First, consider L_1L_1 for which P2's best response is ℓ_2. But, if P2 plays ℓ_2, then P1's type c_{1H} can improve by deviating to either L_1H_1 or H_1H_1, thus L_1L_1 cannot be part of an equilibrium. Using the same reasoning it can be shown that H_1L_1 cannot be part of an equilibrium, either. However, if P1 chooses L_1H_1, then P2's choice of ℓ_2 does not offer any motivation for P1 to move to a different strategy, thus L_1H_1 is part of an equilibrium. Similar arguments lead to the conclusion that H_1H_1 must also be part of an equilibrium.*

3.3.2 Continuous Strategies

Let us return to the competitive newsvendor problem discussed in Section 3.2.1 but assume now that at least one of the players is unsure about the actual purchase cost of the other player.

Player P1 has two types and player P2 has one type

Assume here that P2 is not sure about P1's purchase cost which is either c_{1L} or c_{1H} with probabilities θ_1 and $1 - \theta_1$. Given this uncertainty that P2 faces, what are the best strategies for P1 and P2, i.e., what should be the Bayesian Nash equilibrium order quantities for each newsvendor?

Since P1's cost (types) may be c_{1L} or c_{1H}, this newsvendor's strategy set is $[0, \infty) \times [0, \infty)$ with moves (q_{1L}, q_{1H}). Similarly, P2 has only one cost c_2, thus her strategy set is $[0, \infty)$ with the move q_2. Now, referring to the conditional expected payoff expressions in (3.10)–(3.11), we write the first newsvendor's objective function, conditional on his cost as,

$$\hat{J}_{1L}(\sigma_1(t_1), \sigma_2(t_2)) = J_{1L}(q_{1L}, q_2)p_1(c_2 \mid c_{1L}) = J_{1L}(q_{1L}, q_2),$$
$$\hat{J}_{1H}(\sigma_1(t_1), \sigma_2(t_2)) = J_{1H}(q_{1H}, q_2)p_1(c_2 \mid c_{1H}) = J_{1H}(q_{1H}, q_2),$$

where $J_{1L}(q_{1L}, q_2)$ and $J_{1H}(q_{1H}, q_2)$ are P1's expected profits with purchase cost c_{1L} and c_{1H}, respectively, obtained from (3.3), and $p_1(c_2 \mid c_{1L}) = p_1(c_2 \mid c_{1H}) = 1$. The second newsvendor's objective function must take into account the uncertainty she faces and thus, we have,

$$\begin{aligned}\hat{J}_2(\sigma_1(t_1), \sigma_2(t_2)) &= J_2(q_{1L}, q_2)p_2(c_{1L} \mid c_2) + J_2(q_{1H}, q_2)p_2(c_{1H} \mid c_2) \\ &= \theta_1 J_2(q_{1L}, q_2) + (1-\theta_1)J_2(q_{1H}, q_2),\end{aligned}$$

where $J_2(q_{1L}, q_2)$ and $J_2(q_{1H}, q_2)$ are P2's expected profits when P1 chooses q_{1L} and q_{1H}, respectively, obtained from (3.4). In our numerical calculations we will set $\theta_1 = \frac{1}{2}$, as before.

The first newsvendor P1 chooses his Bayesian Nash equilibrium strategies to maximize $\hat{J}_{1L}(\sigma_1(t_1), \sigma_2(t_2))$ when his cost is c_{1L}, and to maximize $\hat{J}_{1H}(\sigma_1(t_1), \sigma_2(t_2))$ when his cost is c_{1H}. The other newsvendor P2 chooses q_2 to maximize $\hat{J}_2(\sigma_1(t_1), \sigma_2(t_2))$. The first-order conditions for the equilibrium are determined from (3.12)–(3.13) and are found by solving a system of *three* nonlinear equations in three unknowns q_{1L}, q_{1H} and q_2:

$$\frac{\partial}{\partial q_{1L}}\hat{J}_{1L}(\sigma_1(t_1), \sigma_2(t_2)) = \frac{\partial}{\partial q_{1L}}J_{1L}(q_{1L}, q_2) = 0, \tag{3.14}$$
$$\frac{\partial}{\partial q_{1H}}\hat{J}_{1H}(\sigma_1(t_1), \sigma_2(t_2)) = \frac{\partial}{\partial q_{1H}}J_{1H}(q_{1H}, q_2) = 0, \tag{3.15}$$
$$\frac{\partial}{\partial q_2}\hat{J}_2(\sigma_1(t_1), \sigma_2(t_2)) = \frac{\partial}{\partial q_2}[\theta_1 J_2(q_{1L}, q_2) + (1-\theta_1)J_2(q_{1H}, q_2)] = 0. \tag{3.16}$$

The explicit expressions for $J_{1L}(q_{1L}, q_2)$ and $J_{1H}(q_{1H}, q_2)$ are found from (3.3); the expression for $J_2(\cdot, q_2)$ follows from (3.4). The first partial derivatives of these functions are also available from (3.5) and (3.6) as $\partial J_{1L}/\partial q_{1L} = I_{1L}(q_{1L}, q_2)$, and $\partial J_{1H}/\partial q_{1H} = I_{1H}(q_{1H}, q_2)$, and $\partial J_2/\partial q_2 = I_2(\cdot, q_2)$, respectively. Thus, the Bayesian Nash equilibrium conditions of (3.14)–(3.16) simplify to,

$$\begin{aligned} I_{1L}(q_{1L}, q_2) &= 0, \\ I_{1H}(q_{1H}, q_2) &= 0, \\ \theta_1 I_2(q_{1L}, q_2) + (1-\theta_1)I_2(q_{1H}, q_2) &= 0. \end{aligned} \tag{3.17}$$

Let us now return to the continuous strategy example discussed in Section 3.2.1 where the demand densities were exponential with means $E(X) = 30$ and $E(Y) = 20$. As before, we set the other parameters as $[a, b \mid s_1, s_2 \mid c_2] = [0.9, 0.9 \mid 15, 9 \mid 5]$, but now since the unit purchase cost of the first newsvendor could be low or high, we let $[c_{1L}, c_{1H}] = [6, 10]$. With these values the first order conditions given in (3.17) reduce to

$$\begin{aligned} I_{1L}(q_{1L}, q_2) &= 15e^{-q_{1L}/30} - \frac{45}{2}e^{-(q_{1L}/18+q_2/20)} + \frac{45}{2}e^{-(q_{1L}/30+q_2/20)} - 6, \\ I_{1H}(q_{1H}, q_2) &= 15e^{-q_{1H}/30} - \frac{45}{2}e^{-(q_{1H}/18+q_2/20)} + \frac{45}{2}e^{-(q_{1H}/30+q_2/20)} - 10, \\ \theta_1 I_2(q_{1L}, q_2) + (1-\theta_1)I_2(q_{1H}, q_2) &= 9e^{-q_2/20} + \frac{243}{14}e^{-(q_{1L}/30+q_2/27)} - \frac{243}{14}e^{-(q_{1L}/30+q_2/20)} \\ &\quad + \frac{243}{14}e^{-(q_{1H}/30+q_2/27)} - \frac{243}{14}e^{-(q_{1H}/30+q_2/20)} - 5. \end{aligned}$$

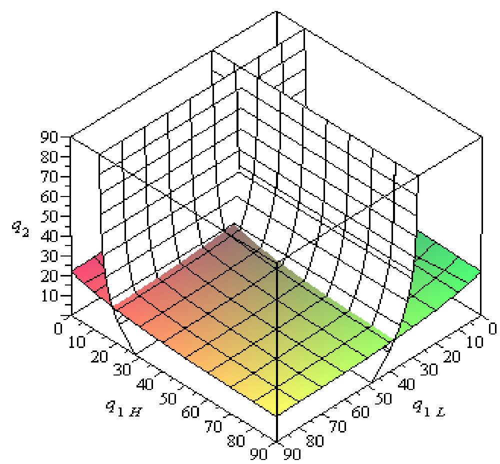

Figure 3.9: The Bayesian Nash equilibrium solution of Model III with two players (newsvendors) where player 1 has two types (c_{1L}, c_{1H}), and player 2 has one type (c_2). The solution is the intersection of three surfaces where the vertical surfaces correspond to $I_1(q_{1L}, q_2) = 0$ and $I_1(q_{1H}, q_2) = 0$, and the horizontal surface corresponds to $\theta_1 I_2(q_{1L}, q_2) + (1-\theta_1)I_2(q_{1H}, q_2) = 0$.

We plot the implicit surfaces for these first order conditions in Figure 3.9 and note that they intersect at a unique point. Solving this system of three nonlinear equations, we find $[\sigma_1^*(t_1), \sigma_2^*(t_2)] = [(q_{1L}, q_{1H}), q_2] = [(35.46, 17.02), 19.75]$. Comparing this to the result obtained in Section 3.2.1 where we had found $(q_1^N, q_2^N) = (25.38, 19.56)$ as the Nash equilibrium, the Bayesian Nash equilibrium result shows that the first newsvendor should order a higher quantity than before if he has a lower purchase cost $(c_{1L} = 6)$, and a lower quantity than before if he has a higher cost $(c_{1H} = 10)$. The Bayesian Nash order quantity for the second newsvendor is only slightly higher than its Nash counterpart. Substituting these values in the respective objective functions, we find the Bayesian Nash payoffs as,

$$[\hat{J}_{1L}(q_{1L}, q_2), \hat{J}_{1H}(q_{1H}, q_2)] = [143.74, 41.23] \quad \text{and} \quad \hat{J}_2(q_{1L}, q_{1H}, q_2) = 36.30.$$

This result indicates a substantial increase in expected payoff for P1 when his cost is low, as should be expected. The second newsvendor also expects a slightly higher expected payoff.

Players P1 and P2 both have two types

As a further extension, let us now assume that not only P2 is uncertain about P1's purchase cost, but also P1 is uncertain about P2's purchase cost. That is, as before, P1 has two types (c_{1L}, c_{1H}) for which P2 holds the distribution $(\theta_1, 1-\theta_1)$, but now P2 also has two types (c_{2L}, c_{2H}) for which P1 holds the distribution $(\theta_2, 1-\theta_2)$. In other words, the conditional probabilities for P1 and P2, respectively, are,

$$p_1(c_{2L} \mid c_{1L}) = p_1(c_{2L} \mid c_{1H}) = \theta_2, \quad \text{and} \quad p_1(c_{2H} \mid c_{1L}) = p_1(c_{2H} \mid c_{1H}) = 1-\theta_2,$$

$$p_2(c_{1L} \mid c_{2L}) = p_2(c_{1L} \mid c_{2H}) = \theta_1, \quad \text{and} \quad p_2(c_{1H} \mid c_{2L}) = p_2(c_{1H} \mid c_{2H}) = 1-\theta_1.$$

How does one determine the Bayesian Nash equilibrium for this problem with two-sided incomplete information? To answer this question, we find the conditional expected payoffs for each player from (3.10) and (3.11) as,

$$\begin{aligned}
\hat{J}_{1L}(\sigma_1(t_1), \sigma_2(t_2)) &= \sum_{t_2 \in T_2} J_{1L}(q_{1L}, \sigma_2(t_2); t_2) p_1(t_2 \mid c_{1L}) \\
&= \theta_2 J_{1L}(q_{1L}, q_{2L}) + (1-\theta_2) J_{1L}(q_{1L}, q_{2H}), \\
\hat{J}_{1H}(\sigma_1(t_1), \sigma_2(t_2)) &= \sum_{t_2 \in T_2} J_{1H}(q_{1H}, \sigma_2(t_2); t_2) p_1(t_2 \mid c_{1H}) \\
&= \theta_2 J_{1H}(q_{1H}, q_{2L}) + (1-\theta_2) J_{1H}(q_{1H}, q_{2H}), \\
\hat{J}_{2L}(\sigma_1(t_1), \sigma_2(t_2)) &= \sum_{t_1 \in T_1} J_{2L}(\sigma_1(t_1), q_{2L}; t_1) p_2(t_1 \mid c_{2L}) \\
&= \theta_1 J_{2L}(q_{1L}, q_{2L}) + (1-\theta_1) J_{2L}(q_{1H}, q_{2L}), \\
\hat{J}_{2H}(\sigma_1(t_1), \sigma_2(t_2)) &= \sum_{t_1 \in T_1} J_{2H}(\sigma_1(t_1), q_{2H}; t_1) p_2(t_1 \mid c_{2H}) \\
&= \theta_1 J_{2H}(q_{1L}, q_{2H}) + (1-\theta_1) J_{2H}(q_{1H}, q_{2H}).
\end{aligned}$$

Since each newsvendor's strategy set is $[0,\infty) \times [0,\infty)$ with moves (q_{1L}, q_{1H}) for P1 and (q_{2L}, q_{2H}) for P2, we follow the standard steps of partially differentiating the expected payoffs and obtain

$$\frac{\partial}{\partial q_{1L}}\hat{J}_{1L}(\sigma_1(t_1),\sigma_2(t_2)) = \theta_2 I_{1L}(q_{1L},q_{2L}) + (1-\theta_2)I_{1L}(q_{1L},q_{2H}) = 0,$$
$$\frac{\partial}{\partial q_{1H}}\hat{J}_{1H}(\sigma_1(t_1),\sigma_2(t_2)) = \theta_2 I_{1H}(q_{1H},q_{2L}) + (1-\theta_2)I_{1H}(q_{1H},q_{2H}) = 0,$$
$$\frac{\partial}{\partial q_{2L}}\hat{J}_{2L}(\sigma_1(t_1),\sigma_2(t_2)) = \theta_1 I_{2L}(q_{1L},q_{2L}) + (1-\theta_1)I_{2L}(q_{1H},q_{2L}) = 0,$$
$$\frac{\partial}{\partial q_{2H}}\hat{J}_{2H}(\sigma_1(t_1),\sigma_2(t_2)) = \theta_1 I_{2H}(q_{1L},q_{2H}) + (1-\theta_1)I_{2H}(q_{1H},q_{2H}) = 0.$$

The resulting system of four nonlinear equations in the four unknowns $(q_{1L},q_{1H}; q_{2L},q_{2H})$ can be solved to determine the Bayesian Nash equilibrium for this game where incomplete information is two-sided. As an example, consider the problem discussed in Section 3.3.2 with exponential demand densities having means $E(X)=30$ and $E(Y)=20$. As before, the other parameters are $[a,b \mid s_1,s_2] = [0.9, 0.9 \mid 15, 9]$ and $[c_{1L},c_{1H}] = [6,10]$, but now since the unit purchase cost of the second newsvendor P2 could be low or high, we let $[c_{2L},c_{2H}] = [3,5]$. Thus, in this case P2 is in a better position than before as her cost could be even as low as 3, and hence we would expect P2 to have a higher expected profit than before.

Solving the resulting system of four nonlinear equations given above with $(\theta_1,\theta_2) = (0.5, 0.5)$, we find the Bayesian Nash equilibrium as

$$[\sigma_1^*(t_1),\sigma_2^*(t_2)] = [(q_{1L},q_{1H}),(q_{2L},q_{2H})] = [(33.00, 15.35), (36.80, 20.47)].$$

The expected profits for each newsvendor, given their type are computed as

$$[J_{1L}(q_{1L},q_{2L}), J_{1L}(q_{1L},q_{2H}), J_{1H}(q_{1H},q_{2L}), J_{1H}(q_{1H},q_{2H})] = [119.68, 141.75, 32.75, 40.36]$$
$$[J_{2L}(q_{1L},q_{2L}), J_{2L}(q_{1L},q_{2H}), J_{2H}(q_{1H},q_{2L}), J_{2H}(q_{1H},q_{2H})] = [78.75, 108.98, 30.71, 44.93].$$

These results imply that

$$[\hat{J}_{1L}(q_{1L},q_{2L},q_{2H}), \hat{J}_{1H}(q_{1H},q_{2L},q_{2H})] = (130.71, 36.56),$$
$$[\hat{J}_{2L}(q_{1L},q_{2L},q_{1H}), \hat{J}_{2L}(q_{1L},q_{2H},q_{1H})] = (93.86, 37.82).$$

Since in this case P2's purchase can be lower than before, she can compete better resulting in an increase in her expected profits. Faced with a lower-cost competitor, P1 fares worse and his expected profits decrease.

3.4 Model IV: Dynamic Games with Incomplete Information (Perfect Bayesian Equilibrium)

Let us recall that in Model II we considered a *dynamic* game with complete information and in Model III we considered a static game with *incomplete information*. For Model II we used subgame perfect equilibrium as the solution concept; for Model III the solution concept was Bayesian Nash equilibrium. In this section we consider a more general model (Model IV) of a dynamic game with incomplete information which combines the two important features of Model II and Model III. The solution concept that is relevant for Model IV is known as the **perfect Bayesian equilibrium** which is a combination of strategies and specification of beliefs with the conditions of, (i) **belief consistency** (the beliefs are consistent with the strategies under consideration, i.e., they satisfy Bayesian updating), and (ii) **sequential rationality (**the players choose their strategies optimally given their beliefs); see Gibbons [35, Ch. 4]. As in previous sections, our objective is to identify pure strategy perfect Bayesian equilibria, if they exist. When such equilibria do not exist, it is still possible to determine the mixed strategy equilibria (see, Carmichael [12, Ch. 7], Dixit and Skeath [22, Ch. 9] and Montet and Serra [61, pp. 179–183]), but in this paper we limit our discussion to pure strategy equilibria.

3.4.1 Discrete Strategies

We illustrate the calculation of the perfect Bayesian equilibrium (PBE) by considering an important class of dynamic games with incomplete information known as a *signalling game*. In the context of our two-player inventory game, the signalling game starts with a chance move (by Nature) that determines the type $t_i \in T = \{c_{1L}, c_{1H}\}$ of P1 who is informed of this outcome and hence of his type. In such a game we assume as in Model II that player P1 is the leader and P2 is the follower. P1 moves first by sending a signal/message $m_j \in M = \{L_1, H_1\}$ and announcing his choice of a low order quantity $q_{1L} = L_1$, or a high order quantity $q_{1H} = H_1$. The second player P2, who does not know P1's type when Nature reveals it, observes P1's message m_j, and then chooses an action $a_k \in A = \{\ell_2, h_2\}$ which determines her order quantity. The sequence, (i) chance move, (ii) P1's signal/message, and (iii) P2's decision, determine the players' payoff as $J_1(t_i, m_j, a_k(m_j))$ and $J_2(t_i, m_j, a_k(m_j))$.

What distinguishes a signalling game in Model IV is the belief structure player P2 must evaluate once she observes P1's signal/message. As in Gibbons [35, Ch. 4], let the conditional probability $\mu(t_i \mid m_j)$ be player P2's updated belief that P1 is of type t_i given that P1 sends the message m_j. (A **belief** of a player is defined as a probability distribution over the nodes of an information set in the extensive game.) For a given strategy chosen by P1 [for example, "H_1 if c_{1L}, and L_1 if c_{1H}", i.e., in our notation, $(c_{1L}, c_{1H}) \mapsto H_1L_1$], player P2 updates her belief at each information set using the Bayes' rule [74, p. 54],

$$\mu(t_i \mid m_j) = \frac{\Pr(m_j \mid t_i)\Pr(t_i)}{\sum_{t_i'} \Pr(m_j \mid t_i')\Pr(t_i')}.$$

where we use t_i' to denote all possible types of P1.

The procedure to find the PBE is as follows: Given her updated beliefs $\mu(t_i \mid m_j)$, P2 must then apply **sequential rationality** to determine her best response by solving the optimization problem $\max_{a_k \in A} \sum_{t_i \in T} J_2(t_i, m_j, a_k)\mu(t_i \mid m_j)$. In view of P2's response, P1 has to check to see if he has any incentive to deviate from the strategy assigned to him by solving the problem $\max_{m_j \in M} J_1(t_i, m_j, a_k(m_j))$. If P1 has no incentive to deviate, then P1's strategy and P2's best response and her updated beliefs constitute a PBE. Otherwise, we have to assign a different strategy to P1 and continue the above process.

When only pure strategies are considered, it is sufficient to examine the equivalent *normal* (strategic) form of the dynamic game with incomplete information and determine the Nash equilibria from the normal form. If multiple Nash equilibria exist, then those that do not qualify as perfect Bayesian equilibrium can be eliminated using the above procedure.

Strategic Form

We will illustrate this process by considering the example in Figure 3.10 where the numbers in brackets such as $[\alpha]$ and $[\beta]$ at P2's decision nodes correspond to that player's updated beliefs about P1's type. As the figure illustrates, P1 learns the choice of Nature, i.e., that his cost is either low (c_{1L}) or high (c_{1H}), but P2 does not know this. In the strategic form of this game, P1 has the strategy set $\{L_1L_1, L_1H_1, H_1L_1, H_1H_1\}$ as in Model III where the first letter corresponds to the action when P1 is of type c_{1L}, and the second letter when P1 is of type c_{1H}. Similarly, for P2 the strategy set is $\{\ell_2\ell_2, \ell_2h_2, h_2\ell_2, h_2h_2\}$ as in Model II where the first letter corresponds to P2's action if P1 chooses L_1, and the second letter refers to P2's action if P1 chooses H_1.

The 4×4 strategic form corresponding to the (expected) payoffs of the players is given in Table 3.6. To illustrate the calculations in the table, consider P1's strategy L_1L_1 and P2's strategy h_2h_2. Here, P1 always chooses L_1 regardless of his type, and P2 always chooses h_2 regardless of what P1 does. These lead us to the terminal payoffs (4,0) and (4,3) indicating that P1 gets (4,4), and P2 gets an expected payoff of $0 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} = 1.5$. Similarly, consider H_1H_1 vs. $h_2\ell_2$. In this case, we are led to the payoffs (4,4) and (3,2), thus P1 gets (4,3) and P2 gets $4 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = 3$.

The pure best responses in Table 3.6 are indicated by an asterisk showing that the game has three Nash equilibria, i.e., (L_1H_1, ℓ_2h_2), $(H_1L_1, h_2\ell_2)$ and $(H_1H_1, \ell_2\ell_2)$. We now examine these Nash equilibria separately to determine whether or not they also qualify as perfect Bayesian equilibria.

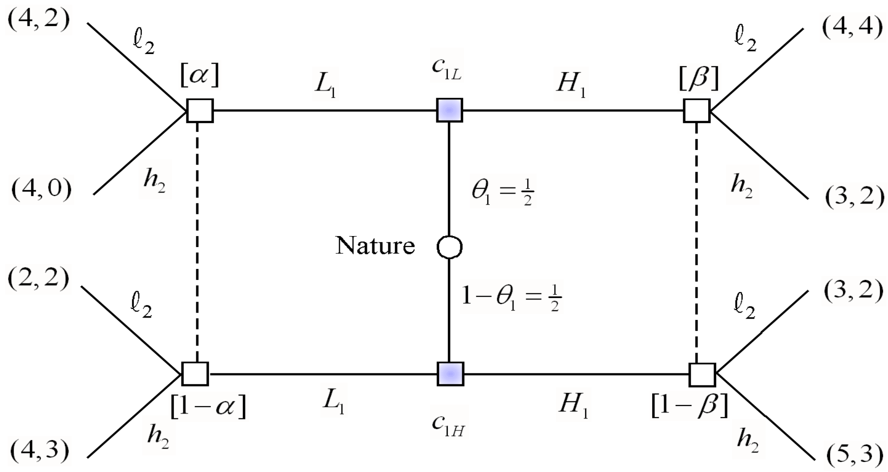

Figure 3.10: Extensive form of the dynamic game with incomplete information where each player has two moves (Model IV).

P1\P2	$(L_1, H_1) \mapsto \ell_2\ell_2$	$(L_1, H_1) \mapsto \ell_2 h_2$	$(L_1, H_1) \mapsto h_2\ell_2$	$(L_1, H_1) \mapsto h_2 h_2$
$(c_{1L}, c_{1H}) \mapsto L_1L_1$	$(4^*, 2), 2^*$	$(4^*, 2), 2^*$	$(4^*, 4^*), 1.5$	$(4^*, 4), 1.5$
$(c_{1L}, c_{1H}) \mapsto L_1H_1$	$(4^*, 3^*), 2$	$(4^*, 5^*), 2.5^*$	$(4^*, 3), 1$	$(4^*, 5^*), 1.5$
$(c_{1L}, c_{1H}) \mapsto H_1L_1$	$(4^*, 2), 3$	$(3, 2), 2$	$(4^*, 4^*), 3.5^*$	$(3, 4), 2.5$
$(c_{1L}, c_{1H}) \mapsto H_1H_1$	$(4^*, 3^*), 3^*$	$(3, 5^*), 2.5$	$(4, 3), 3^*$	$(3, 5^*), 2.5$

Table 3.6: Payoff table for the dynamic game with incomplete information where P1 and P2 have four pure strategies.

Equilibrium #1 (L_1H_1, ℓ_2h_2) In this combination we have $(c_{1L}, c_{1H}) \mapsto L_1H_1$ and $(L_1, H_1) \mapsto \ell_2h_2$ and the strategies are represented in Figure 3.11 where the updated beliefs α and β are calculated as follows. With P1's strategy being $(c_{1L}, c_{1H}) \mapsto L_1H_1$, i.e., P1 of type c_{1L} will choose L_1 and P1 of type c_{1H} will choose H_1, P2 will know with certainty that P1 is of type c_{1L} if she observes L_1. In the same way, P2 will know for sure that P1 is of type c_{1H} if she observes H_1. Hence, regarding α — P2's conditional probability that P1 is of type c_{1L} given that P1 chose L_1, we have, from Bayes' theorem;

$$\alpha = \Pr(c_{1L} \mid L_1) = \frac{\Pr(L_1|c_{1L})\Pr(c_{1L})}{\Pr(L_1|c_{1L})\Pr(c_{1L}) + \Pr(L_1|c_{1H})\Pr(c_{1H})} = \frac{1 \cdot 0.5}{1 \cdot 0.5 + 0 \cdot 0.5} = 1.$$

Thus, $(\alpha, 1-\alpha) = (1, 0)$. Similarly, $1-\beta$ is P2's conditional probability that P1 is of type c_{1H} given that P1 chose H_1, so we have,

$$1-\beta = \Pr(c_{1H}|H_1) = \frac{\Pr(H_1|c_{1H})\Pr(c_{1H})}{\Pr(H_1|c_{1H})\Pr(c_{1H}) + \Pr(H_1|c_{1L})\Pr(c_{1L})} = \frac{1 \cdot 0.5}{1 \cdot 0.5 + 0 \cdot 0.5} = 1.$$

Thus, $(\beta, 1-\beta) = (0, 1)$. These calculations imply that $[(L_1H_1, \ell_2h_2); \alpha = 1, \beta = 0]$ is a perfect Bayesian equilibrium. This is known as a **separating** strategy where P1's type c_{1L} plays L_1, and P1's type c_{1H} plays H_1.

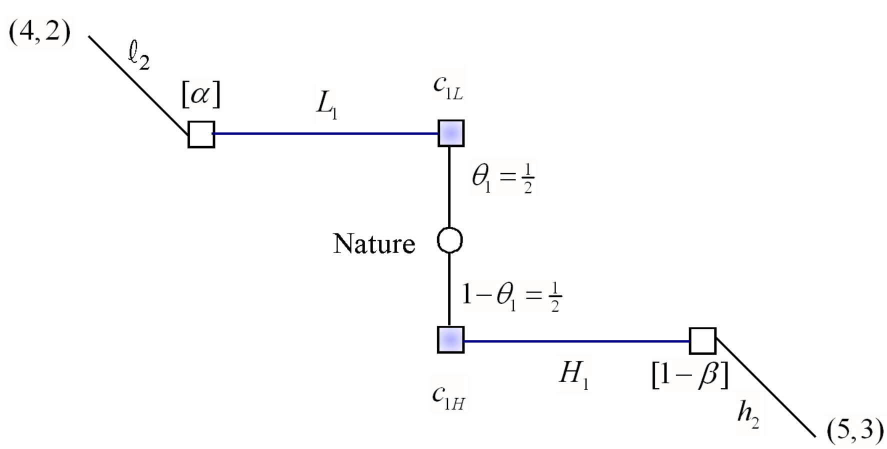

Figure 3.11: Calculating the beliefs and testing whether L_1H_1 and ℓ_2h_2 constitute a PBE (Model IV).

Equilibrium #2 $(H_1L_1, h_2\ell_2)$ For this combination, $(c_{1L}, c_{1H}) \mapsto H_1L_1$ and $(L_1, H_1) \mapsto h_2\ell_2$. Using similar arguments;

$$\beta = \Pr(c_{1L}|H_1) = \frac{\Pr(H_1|c_{1L})\Pr(c_{1L})}{\Pr(H_1|c_{1L})\Pr(c_{1L}) + \Pr(H_1|c_{1H})\Pr(c_{1H})} = \frac{1 \cdot 0.5}{1 \cdot 0.5 + 0 \cdot 0.5} = 1,$$

and,

$$1 - \alpha = \Pr(c_{1H}|L_1) = \frac{\Pr(L_1|c_{1H})\Pr(c_{1H})}{\Pr(L_1|c_{1H})\Pr(c_{1H}) + \Pr(L_1|c_{1L})\Pr(c_{1L})} = \frac{1 \cdot 0.5}{1 \cdot 0.5 + 0 \cdot 0.5} = 1.$$

Thus, $(\alpha, 1-\alpha) = (0, 1)$ and $(\beta, 1-\beta) = (1, 0)$. This implies that $[(H_1L_1, h_2\ell_2); \alpha = 0, \beta = 1]$ is a perfect Bayesian equilibrium. This is also known as a **separating** strategy where P1's type c_{1L} plays H_1, and P1's type c_{1H} plays L_1, i.e., the two types of P1 play different strategies from each other.

Equilibrium #3 $(H_1H_1, \ell_2\ell_2)$ For this case, $(c_{1L}, c_{1H}) \mapsto H_1H_1$ and $(L_1, H_1) \mapsto \ell_2\ell_2$. Since P1 always plays H_1, the left-hand-side of the game tree will be off the equilibrium path. To calculate P2's belief that P1 is of type c_{1L} given that P2 observed H_1, we have

$$\beta = \Pr(c_{1L}|H_1) = \frac{\Pr(H_1|c_{1L})\Pr(c_{1L})}{\Pr(H_1|c_{1L})\Pr(c_{1L}) + \Pr(H_1|c_{1H})\Pr(c_{1H})} = \frac{1 \cdot 0.5}{1 \cdot 0.5 + 1 \cdot 0.5} = \frac{1}{2},$$

so that $(\beta, 1-\beta) = (\frac{1}{2}, \frac{1}{2})$. Now, the beliefs $(\alpha, 1-\alpha)$ are not restricted, but they should be such that if P1 ever plays L_1, then this strategy combination results in P2 choosing ℓ_2. Thus, we must find a condition on α that will satisfy this requirement. This implies that the expected payoff to P2 from choosing ℓ_2 must be at least as much from choosing h_2. Thus,

$$E(\text{P2 chooses } \ell_2) = 2\alpha + 2(1-\alpha) = 2$$
$$E(\text{P2 chooses } h_2) = 0 \cdot \alpha + 3(1-\alpha) = 3 - 3\alpha,$$

and so P2 chooses ℓ_2 iff $2 \geq 3 - 3\alpha$, or $\alpha \geq \frac{1}{3}$. These arguments show that $[(H_1H_1, \ell_2\ell_2); \alpha \geq \frac{1}{3}, \beta = \frac{1}{2}]$ is a perfect Bayesian equilibrium. This is known as a **pooling** strategy where P1's both types c_{1L} and c_{1H} play the same strategy H_1.

Computing the equilibria in the extensive form

Of course, it is not necessary to use the strategic form to determine the PBEs in such games. We can start with any strategy for P1 and find the corresponding best response for P2. If P1 responds to P2's best response using his initially chosen strategy, then the combination of the two players' strategies considered would be an equilibrium.

For example, if we start with $(c_{1L}, c_{1H}) \mapsto L_1L_1$ for P1 we can find P2's best response to this strategy. This would imply $(\alpha, 1-\alpha) = (\frac{1}{2}, \frac{1}{2})$ for P2. At the left information set of Figure 3.10, P2's optimal action is ℓ_2 since on that set $E(\text{P2 chooses } \ell_2) = 2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = 2$ and $E(\text{P2 chooses } h_2) = 0 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} = 1.5$. Now that we have determined that $L_1 \mapsto \ell_2$ for P2, we need to find P2's best response to P1's choice of H_1. At the right information set, P2's beliefs $(\beta, 1-\beta)$ are not restricted. On this set, P2's optimal action is ℓ_2 if $\beta \geq \frac{1}{3}$, and h_2 if $\beta \leq \frac{1}{3}$ since $E(\text{P2 chooses } \ell_2) = 4\beta + 2(1-\beta) = 2\beta + 2$, and $E(\text{P2 chooses } h_2) = 2\beta + 3(1-\beta) = 3 - \beta$. Now, if P2 chooses ℓ_2 after H_1, then type c_{1H} of P1 can improve his payoff by choosing H_1 instead of L_1 (and thus obtain a payoff of 3 rather than 2). Thus, $(L_1L_1, \ell_2\ell_2)$ cannot be an equilibrium. Similarly, if P2 chooses h_2 after H_1, then type c_{1H} of P1 can improve by choosing H_1 instead of L_1 (and thus obtain a payoff of 5 rather than 4). Thus, (L_1L_1, ℓ_2h_2) cannot be an equilibrium, either. These arguments imply that strategy L_1L_1 is not part of any equilibrium in this game.

Let us now consider $(c_{1L}, c_{1H}) \mapsto L_1H_1$ with the beliefs for P2 as $(\alpha, \beta) = (1, 0)$. Then the extensive form indicates that P2's best response is $L_1H_1 \mapsto \ell_2h_2$. Now, given ℓ_2h_2 , P1's type c_{1L} would have no incentive to deviate to H_1 since $3 < 4$. [Note that, if P1 deviates to H_1, then P2's strategy of $L_1H_1 \mapsto \ell_2h_2$ calls for her to use h_2 resulting in payoffs of $(3, 2)$.] Similarly, P1's type c_{1H} would also have no incentive to deviate to L_1 since $2 < 5$. [In this case, if P1 deviates to L_1, then P2 would choose ℓ_2 with payoffs $(2, 2)$.] Thus, $[(L_1H_1, \ell_2h_2); \alpha = 1, \beta = 0]$ is a separating PBE as noted in the discussion of Equilibrium #1 above.

By similar arguments, one can also show (without constructing the complete strategic form) that $[(H_1L_1, h_2\ell_2); \alpha = 0, \beta = 1]$ and $[(H_1H_1, \ell_2\ell_2); \alpha \geq \frac{1}{3}, \beta = \frac{1}{2}]$ are also separating and pooling PBEs, respectively, as noted in the discussion of Equilibria #2 and #3 above.

We should note again that the above analysis applies when the game has pure strategy equilibria. When this is not the case, one can still compute the mixed strategy equilibria for the strategic form of the game, but the calculations become more involved. We refer the reader to the treatment in Carmichael [12, Ch. 7], Dixit and Skeath [22, Ch. 9] and Montet and Serra [61, pp. 179–183] for examples of computing mixed strategy PBE in signalling games.

Intuitive Criterion

In the above PBE analysis, the only requirement on a player's belief is the consistency requirement. Intuitive criterion is another important restriction on player's belief which can be used to exclude unreasonable or implausible equilibria (Cho and Kreps [14]). Briefly speaking, **intuitive criterion** requires that, in any information set, the uninformed player should assign probability zero to the type of the informed player that could never possibly gain (if compared with the equilibrium payoff) by playing the action leading to this information set.

Let us apply this criterion to the equilibrium $[(H_1H_1, \ell_2\ell_2); \alpha \geq \frac{1}{3}, \beta = 0.5]$ in the above game. Referring to Figure 3.10, we can see that P1's type c_{1L} does not have a reason to deviate from H_1 to L_1 since he will not gain from such a deviation. However, P1's type c_{1H} may have a reason to deviate since the maximum payoff from L_1 is 4 which is higher than 3 obtained from staying with H_1. Hence, we have $\alpha = \Pr(c_{1L} \mid L_1) = 0$. This contradicts the equilibrium beliefs ($\alpha \geq \frac{1}{3}, \beta = 0.5$). With intuitive criterion, $[(H_1H_1, \ell_2\ell_2); \alpha \geq \frac{1}{3}, \beta = 0.5]$ is thus no longer a PBE.

The intuitive criterion does not eliminate the PBE $[(L_1H_1, \ell_2h_2), \alpha = 1, \beta = 0]$ for the following reasons: (i) P1's type c_{1L} would not be motivated to move to H_1 implying that we still have $\beta = 0$, (ii) P1's type c_{1H} would also not be motivated to deviate to L_1, thus $\alpha = 1$ as before. Using similar reasoning it can also be shown that the PBE $[(H_1L_1, h_2l_2), \alpha = 0, \beta = 1]$ also survives the intuitive criterion.

3.4.2 Continuous Strategies

In this section we have so far assumed that the players can use discrete strategies where each newsvendor is limited to choosing one of two possible order quantities. It would be desirable to analyze dynamic inventory problems with incomplete information where the order quantities can be any nonnegative quantity and compute the PBE for the resulting game. We now consider such a case where it is assumed that player P2 can choose her order quantity $q_2 \in [0, \infty)$, but the first player P1 still has two possible order quantities L_1 and H_1 as his signals. We assume, as before, that the leader P1's purchase cost is either c_{1L} or c_{1H} with probabilities θ_1 and $1 - \theta_1$ and each type can choose one of two discrete order levels $\{L_1, H_1\}$. Given such a strategy setting, what is the perfect Bayesian Nash equilibrium for the dynamic game with incomplete information with the extensive form given in Figure 3.12?

As we illustrated in the discrete strategy example, the procedure to identify a perfect Bayesian Nash equilibrium involves three steps: (i) Construct the extensive form for the game, (ii) transform the extensive form into a strategic form table to identify pure Nash equilibria, if there exists one, (iii) determine perfect Bayesian Nash equilibrium from the Nash equilibria identified in the second step by examining two rules, *belief consistency* and *sequential rationality*. However, for this problem with continuous order quantities for P2, this approach fails since it is impossible to construct a strategic form with infinitely many columns as P2's strategy set is the Cartesian product $[0, \infty) \times [0, \infty)$ which consists of infinitely many elements. Thus, for this problem, we use the best response analysis to determine the PBE.

To illustrate our approach, we solve the problem with the same parameter values we used in previous examples, i.e., $[a, b \mid s_1, s_2 \mid (c_{1L}, c_{1H}), c_2] = [0.9, 0.9 \mid 15, 9 \mid (6, 10), 5]$ with expected demands $E(X) = 30$ and $E(Y) = 20$. We then consider four strategies of P1 separately and attempt to determine if any of these strategies is part of a PBE. We assume in our example that P1 can choose a low level of order quantity $L_1 = 20$, or a high level of order quantity $H_1 = 40$, and of course P2's order quantity $q_2 \in [0, \infty)$.

First consider $(c_{1L}, c_{1H}) \mapsto L_1L_1$ with updated beliefs $(\alpha, 1 - \alpha) = (0.5, 0.5)$ from

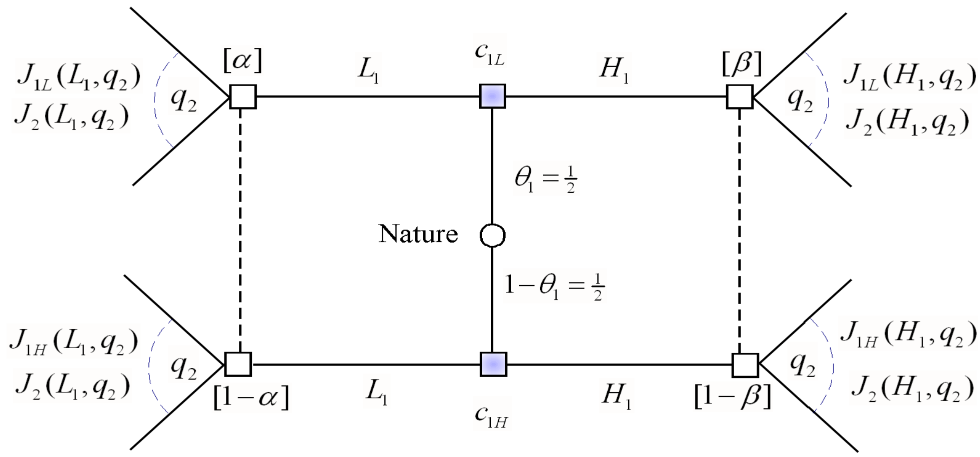

Figure 3.12: Extensive form of the dynamic game with incomplete information where P1 has two moves and P2 has infinitely many moves (Model IV).

Figure 3.12. Given that P1 will always order $L_1 = 20$ regardless of his type, we determine P2's optimal order quantity by solving the optimization problem $\max_{q_2} J_2(20, q_2)$ where J_2 is defined as in (3.4). This gives $q_2'' = \arg\max J_2(20, q_2) = 21.68$ as P2's best response with expected payoff $J_2(20, 21.68) = 40.46$. To make P2's strategy complete, we need to identify P2's response to an observation of H_1, although there is zero probability for it to occur if P1 chooses L_1L_1. To determine how P2 would respond if she observes P1 choosing $H_1 = 40$, we solve $\max_{q_2} J_2(40, q_2)$ and obtain $q_2' = \arg\max J_2(40, q_2) = 15.95$. Thus, in response to P1's strategy $(c_{1L}, c_{1H}) \mapsto L_1L_1$, player P2's best strategy is $(L_1, H_1) \mapsto q_2''q_2'$ with the corresponding payoffs of

$$\hat{J}_{1L}(L_1L_1, q_2''q_2') = J_{1L}(L_1, q_2'') = 118.16 \quad \text{and} \quad \hat{J}_{1H}(L_1L_1, q_2''q_2') = J_{1H}(L_1, q_2'') = 38.16$$

for P1 where $J_1$is defined as in (3.3), and expected profit of

$$\hat{J}_2(L_1L_1, q_2''q_2') = \theta_1 J_2(L_1, q_2'') + (1-\theta_1)J_2(L_1, q_2'') = \frac{1}{2}40.46 + \frac{1}{2}40.46 = 40.46$$

for P2 where J_2 is defined as in (3.4).

Now we check to see if P1 has motivation to switch from L_1L_1 with the hope of improving his payoff. If P1's type c_{1L} switches from L_1 to $H_1 = 40$, he will receive a payoff $J_{1L}(H_1, q_2') = 152.63$ which is higher than $J_{1L}(L_1, q_2'') = 118.16$. So, P1's type

c_{1L} would prefer to switch and thus $(L_1L_1, q_2''q_2')$ cannot be a PBE, i.e., L_1L_1 is not part of an equilibrium.

Second, consider P1's strategy $(c_{1L}, c_{1H}) \mapsto L_1H_1$ with updated beliefs $(\alpha, \beta) = (1, 0)$ from Figure 3.12. In this case, if P2 observes L_1, then P1 must be of type c_{1L}; and if P2 observes H_1, then P1 must be of type c_{1H}. To determine P2's optimal order quantity when she observes $L_1 = 20$, we solve, as before, the optimization problem $\max_{q_2} J_2(20, q_2)$ and obtain $q_2'' = \arg\max J_2(20, q_2) = 21.68$. To determine P2's optimal order quantity when she observes $H_1 = 40$, we solve $\max_{q_2} J_2(40, q_2)$ and obtain $q_2' = \arg\max J_2(40, q_2) = 15.95$. Thus, the best response of P2 to P1's strategy of $(c_{1L}, c_{1H}) \mapsto L_1H_1$ is $q_2''q_2'$ with the payoffs

$$\hat{J}_{1L}(L_1H_1, q_2''q_2') = J_{1L}(L_1, q_2'') = 118.16 \text{ and } \hat{J}_{1H}(L_1H_1, q_2''q_2') = J_{1H}(H_1, q_2') = -7.37,$$

and,

$$\hat{J}_2(L_1H_1, q_2''q_2') = \theta_1 J_2(L_1, q_2'') + (1-\theta_1) J_2(H_1, q_2'') = \frac{1}{2}40.46 + \frac{1}{2}28.81 = 34.63.$$

Now we check to see if P1 has motivation to switch from L_1H_1 with the hope of improving his payoff. If type c_{1L} switches from L_1 to H_1, he will receive a payoff of $J_{1L}(H_1, q_2') = 152.63$ which is higher than $J_{1L}(L_1, q_2'') = 118.16$. Thus, P1 would switch to H_1 implying that L_1H_1 cannot be part of an equilibrium.

Third, let's examine P1's strategy $(c_{1L}, c_{1H}) \mapsto H_1L_1$ to see if it is part of an equilibrium. In this case, P2's updated beliefs are $(\alpha, \beta) = (0, 1)$ and as in the previous case, P1's type can be easily identified after observing his order quantity. We already know that $q_2'' = \arg\max J_2(20, q_2) = 21.68$, and $q_2' = \arg\max J_2(40, q_2) = 15.95$, so that P2's best response to P1's $(c_{1L}, c_{1H}) \mapsto H_1L_1$ is still $q_2''q_2'$, with payoffs

$$\hat{J}_{1L}(H_1L_1, q_2''q_2') = J_{1L}(H_1, q_2') = 152.63 \text{ and } \hat{J}_{1H}(H_1L_1, q_2''q_2') = J_{1H}(L_1, q_2'') = 38.16,$$

and

$$\hat{J}_2(H_1L_1, q_2''q_2') = \theta_1 J_2(H_1, q_2') + (1-\theta_1) J_2(L_1, q_2'') = \frac{1}{2}28.81 + \frac{1}{2}40.46 = 34.63.$$

Does P1 have any motivation to move away from H_1L_1 to a different strategy? To check this, we note that if type c_{1L} switches to L_1, his payoff will be $J_{1L}(L_1, q_2'') = 118.16$, an amount lower than $J_{1L}(H_1, q_2') = 152.63$. Thus, type c_{1L} has no motivation to switch. Similarly, if type c_{1H} switches to H_1, his payoff will be $J_{1H}(H_1, q_2') = -7.37$ which is lower than $J_{1H}(L_1, q_2'') = 38.16$. So, this type wouldn't switch either. We therefore conclude that $[(H_1L_1, q_2''q_2'); \alpha = 1, \beta = 0]$ is a PBE.

Finally, P1's strategy $(c_{1L}, c_{1H}) \mapsto H_1H_1$ with updated beliefs $(\beta, 1-\beta) = (0.5, 0.5)$ results in $q_2' = \arg\max J_2(40, q_2) = 15.95$. To complete P2's strategy, we need to identify P2's response to an observation of L_1 even though there is zero probability for it to occur if P1 takes H_1H_1. As before, we find $q_2'' = \arg\max J_2(20, q_2) = 21.68$, and

$\hat{J}_{1L}\left(H_1L_1, q_2''q_2'\right) = J_{1L}\left(H_1, q_2'\right) = 152.63$ and $\hat{J}_{1H}\left(H_1H_1, q_2''q_2'\right) = J_{1H}\left(H_1, q_2''\right) = -7.37,$

and

$$\hat{J}_2(H_1H_1, q_2''q_2') = \theta_1 J_2(H_1, q_2') + (1-\theta_1)J_2(H_1, q_2') = \frac{1}{2}28.81 + \frac{1}{2}28.81 = 28.81.$$

If P1's type c_{1L} switches from H_1 to L_1, he receives $J_{1L}(L_1, q_2'') = 118.16$ which is less than $J_{1L}(H_1, q_2') = 152.63$, thus he will not switch. However, if P1's type c_{1H} switches from H_1 to L_1, his payoff will be $J_{1H}(H_1, q_2'') = 38.16$ which is better than -7.37, thus he would switch implying that H_1H_1 cannot be part of an equilibrium, either.

3.5 Conclusion

In this paper we consider static and dynamic inventory games played under complete and incomplete information between two players, i.e., newsvendors, facing random but substitutable demands. For each of the resulting four cases ("Models") we first consider a simple discrete game where each player has two moves. We then present a continuous strategy version of each game where at least one of the newsvendors can choose their order quantities from a continuous strategy set. Nash and subgame-perfect equilibria are the solution concepts used to solve the games with complete information. The more realistic game problems with incomplete information are solved using the Bayesian Nash and perfect Bayesian equilibria.

Even though the analysis of games with *complete* information has dominated the recent literature in supply chain management modeling, there have been very few papers dealing with non-cooperative games under *incomplete* information. Our chapter presents explicit methods for dealing with static and dynamic inventory games under incomplete information and computing the Bayesian Nash and perfect Bayesian equilibrium for such games with special emphasis on inventory management problems. Since games of incomplete information provide a more realistic modeling framework, we believe that there will be more applications of such games. It is our hope that researchers who are interested in using incomplete information games in their papers may find our exposition helpful in constructing their models and computing the equilibria in their specific problems.

Chapter 4

Competition in Remanufacturing with Incomplete Information

4.1 Introduction

In this chapter we study a remanufacturing competition problem between an original equipment manufacturer (OEM) and a pure remanufacturer (REM) where the OEM has uncertainty about the REM's unit remanufacturing cost. In this section we first introduce some basic facts about remanufacturing and then explain reasons for the development of remanufacturing industries. We then state the competition in remanufacturing and further focus our attention on the uncertainties existing in remanufacturing competition.

4.1.1 Remanufacturing

The remanufacturing process normally includes disassembling used products, inspecting and repairing the disassembled components and reusing those components for new manufacturing (Majumder and Groenevelt [58]). The fraction of used parts in a remanufactured item varies from product to product. In general, a product can be categorized as a remanufactured one if its essential components are taken from disassembled used products (Majumder and Groenevelt [58]). A used product can also be remanufactured by having the worn-out components be replaced with new ones without being disassembled (Debo *et al.* [20] and Thierry *et al.* [88]). Quality of remanufactured products can vary dramatically. Remanufactured products can be sold as perfect substitutes for brand new products if they are produced using the same manufacturing process and quality control systems. They can also be made as lower-end products with inferior quality. Remanufactured products are commonly seen in industry sectors such as automotive, toner cartridges, compressors, electrical equipment, tires, and so on (Majumder and Groenevelt [58]).

Remanufacturing as a production strategy is both environmentally friendly and economically profitable. The development of remanufacturing industries is expedited by consumers, governments, and manufacturers from diverse areas.

Consumers

Environmental concerns among consumers are getting stronger. People are willing to get involved in efforts to improve the ecological situation. Consumers give support by abiding by governmental recycling regulations and original manufacturers' specification for disposal of products such as batteries, printer cartridges, to name a few. More and more consumers are giving a higher priority to the purchase of products from manufacturers with a positive social image (Ferrer and Swaminathan [28]). More firms are thus encouraged to implement manufacturing processes for environmental-friendly products (Majumder and Groenevelt [58]). Moreover, entry-level customers are attracted by reputable brands but often find it hard to afford the price of a brand new product (Ferrer and Swaminathan [29]). The low-end market they compose attracts more investments into remanufacturing activities.

Governments

More and more governments endeavor to seek macroeconomic strategies that bring benefits both economically and ecologically. They tighten landfill regulations to mandate that firms must take back used products for recycling or remanufacturing (Majumder and Groenevelt [58]). Much attention has been given to recycling in which the structure of used products are destroyed and only constituent materials are collected and processed for re-use. In remanufacturing, however, used products are retained in component level and reused for the same purposes (e.g., refillable drink bottles and printer cartridges) or secondary purposes (e.g., reuse of automotive tires as anticollision cushions in a harbor or for race car tracks). Hence, remanufacturing is arguably said to bring a more dramatic reduction in environmental impact than recycling (Bras and McIntosh [7]).

Governments encourage remanufacturing as it makes macroeconomic impact in several ways. Firstly, it creates sales opportunity (Debo *et al.* [20]). As of 1996, the remanufacturing industry in US had the same sales figure (US $53 billion) as the American steel industry (Lund [56]). Secondly, remanufacturing satisfies the demand for raw material by using components from used products. As a result of that, it helps to conserve resources by reducing the consumption of raw materials. Thirdly, it creates job opportunities. In the US there are 73,000 remanufacturing companies and the total direct employment of these firms is 480,000, which is identical with the employment of the consumer durables industry and twice that of the steel industry (Lund [56] and Webster and Mitra [93]). Ferrer and Ayres [34] observed that, since remanufacturing reduces the consumption of raw material, it reduces the related suppliers' labor demand. However, disassembly, as an essential link of remanufacturing, is well known for its inherent labor-intensiveness and obstruction of being automated.

Therefore the labor demand from the remanufacturing disassembly is larger than the portion cannibalized from the original manufacturing. In addition, remanufacturing reduces industrial waste disposal and the requirement for landfill (Ayres *et al.* [4] and Ferrer and Ayres [34]). Finally, besides the rigid legislation (such as take-back laws), public information campaigns and financial incentives are less heavy-handed alternatives that governments may take to promote remanufacturing activity (Mitra and Webster [60]).

Manufacturers

Economical profitability is the major factor driving manufacturers to advocate remanufacturing (Ginsberg [36] and Webster and Mitra [93]). Debo *et al.* [20] argued that the profitability of a remanufactured product is a fundamental question to ask. However, the verification of the profitability of a remanufactured product is a complicated task. It is generally acknowledged that remanufactured products play an important role in increasing manufacturers' profit as the result of reducing production costs. This includes savings in labor costs, materials, energy costs, and disposal costs (McConocha and Speh [59]). For example, remanufactured sensors are 40% less expensive than brand new ones, and purchasing remanufactured sensors for monitoring patients' pulses saved $30,000 a year for Dartmouth Hitchcock (Mitra and Webster [60]). Ford avoided disposal of more than 67,700 pounds of toner cartridges and saved $180,000 in disposal costs (Mitra and Webster [60]).

A manufacturer needs to deal with the trade-off between retreatability and production costs when making decisions relevant to remanufacturing, such as production technology and product sale price (Debo *et al.* [20]). The choice of production technology influences the residual value that can be retrieved from used products. On the one hand, the manufacturing cost of a retreatable (or remanufacturable) product is typically higher than that of a single-use product. On the other hand, it is also true that the manufacturing cost of a remanufactured product is lower than the originally produced item. It is estimated that the production cost of a remanufactured product is 40-65% less than that of a new product (Mitra and Webster [60]). Sales price plays a decisive role in the profitability of a remanufactured product. The price of a remanufactured product is typically 30-40% lower than the price of a brand-new product. When the remanufactured product is priced identically with the originally produced item, the benefits of remanufacturing become even greater (Ferrer and Swaminathan [28]). The disposable camera is a good example of remanufactured products that have the same price but lower cost than brand new products (Ferrer and Swaminathan [28]). Xerox, as a well known manufacturer of photocopy equipment, has implemented an extensive program of recycling and remanufacturing for their production of photocopiers and printer toner cartridges (Majumder and Groenevelt [58]).

Brand effect stimulates manufacturers to use remanufacturing as a strategy to develop new market share among low-end consumers (Ferrer and Swaminathan [28] and McConocha and Speh [59]). Kodak's family of single-use cameras was once a success in this aspect, although this over-one-hundred-years-old giant of photographic film products has filed bankruptcy protection recently. Furthermore, Ferrer and Swaminathan [29] argued that a well-designed product line with remanufactured and new products merged together may increase market share while sustaining high profit margins. Also remanufacturing helps firms improve their competitiveness via shorter production lead times (McConocha and Speh [59]).

4.1.2 Competitions in Remanufacturing

Used products are widely distributed among consumers. Before being used for remanufacturing, they need to be retrieved from the market. The retrieval process occurs in the reverse logistics chain, which is also the link to where competition occurs (Majumder and Groenevelt [58]). The group of agencies in charge of gathering, classifying and delivering used products back to original manufacturers is growing fast. The agencies may also undertake some simple remanufacturing operations such as disassembly and cleaning. Meanwhile, agencies may also supply to pure remanufacturers who are in need of the used items, too. The original manufacturers are often referred to as OEMs. They normally play the leading role in industry competition. Compared with OEMs, remanufacturers are often smaller-sized firms using different production systems and technology. Remanufacturing of a product often starts here, because the smaller size is an advantage to catch business opportunities in a timely fashion. In some industries like automobile and diesel engines, OEMs cooperate with smaller firms by contracting with them the remanufacturing of their own product (Lund [56]). When OEMs build up their own production line for remanufacturing, they switch the role to compete with local remanufacturers for the supply of used products. Meanwhile, competition in remanufacturing may occur in sales as well which is more straightforward to understand (Majumder and Groenevelt [58]). Atasu and Sarvary [2], in their investigation on the profitability of a remanufacturing system, emphasized that, under competition, remanufacturing as an effective marketing strategy for manufacturers to protect market share via pricing discrimination.

OEMs have invested in product design and marketing of their products. Therefore they wish to capture the benefits of remanufacturing which can be achieved by controlling the reverse logistics chain (Majumder and Groenevelt [58]). However, to keep complete control over the reverse logistic chain has been very challenging for them. In order to encourage consumers to return the used remanufacturable products, some OEMs give certain price discounts, such as a prebate program in toner cartridge industry, to the customers who agree to either destroy or return the product back to the OEM. Some apply specific technologies such as an encrypted counter in a cartridge to restrict those smaller remanufacturing firms to access used products. The remanufacturing firms, on the other hand, are protesting such monopolistic behaviors so that their supply of reusable shells can be guaranteed to a certain degree.

"Shell" and "core" are two terms frequently used in remanufacturing. Sometimes they are used as synonyms to each other. As far as we understand, there is slight difference between these two terms. A "shell" generally refers to a returned product, part of which or the whole of which can be used for remanufacturing, while a "core" normally refers to the reusable part of a returned product.

4.1.3 Uncertainties in Remanufacturing Competition

Another fact in remanufacturing and remanufacturing competition is that uncertainty is involved in several aspects. Guide *et al.* [37] enumerated several complicating features related to the uncertainty in remanufacturing, the timing and volume of product returns, yield estimation, balancing demand with core returns, and managing reverse logistics, etc. Also commonly known are the facts that the volume of returned products accessible to either OEMs or pure remanufacturers cannot be guaranteed; the quality of returned products varies from piece to piece; and the demand for remanufactured products is uncertain as well.

We observe that uncertainty may also lie in the competition between OEMs and local remanufacturers. A pure remanufacturer, in order to be competitive to the OEMs, generally operates with a smaller size and a more flexible product transformation mechanism. Their production technology and remanufacturing cost are different than the OEM's as well. When a pure remanufacturer enters the market as a new competitor to the OEM, it is likely that the OEM does not understand the pure remanufacturer's production costs. It would be interesting to examine: what will be the effect of such uncertainty on the decision making of the OEM and the pure remanufacturer? Will the effect of such uncertainty differ, given different market preference settings and shell accessibility scenarios? We are to address these questions in this chapter.

Here is the organization of the rest of this chapter. In this Section 4.1 we have introduced the development of remanufacturing industries and the relevant competition behaviors and states that uncertainties exist in the competition levels for remanufacturing. Section 4.2 reviews the literature regarding remanufacturing with a focus on the literature as about remanufacturing competition behaviors. In Section 4.3 we formulate a simplified remanufacturing competition problem between an OEM and a REM as a static game with complete information. We obtain the Nash equilibrium (NE) solution to this game and proof the existence and uniqueness of the NE. In Section 4.4 we include the OEM's uncertainty about the REM's unit remanufacturing cost into their remanufacturing competition and extend the formulation in Section 4.3 to a static game with incomplete information, which shares the same structure as the type-III models in Section 3.3, Chapter 3. Then, for this type-III game model, we obtain the Bayesian Nash equilibrium solution when the OEM has priority to access the available shells. In Section 4.5, we conduct sensitivity analysis analytically, if possible, and numerically otherwise. Section 4.6 concludes this chapter and gives a plan for future research work.

4.2 Literature Review

Literature in the area of remanufacturing has not been pervasive (Bras and McIntosh [7]). Bras and McIntosh [7] and Guide *et al.* [38] provided two good reviews for early literature in this area. Generally speaking, the available remanufacturing related literature is mainly about operational issues arising in reverse logistics, production control, and inventory management (Fleischmann *et al.* [31]). Some literature studies remanufacturable product design problems (Debo *et al.* [20]). Another portion of the literature addresses the benefits and challenges, both economical and ecological, in remanufacturing (Guide *et al.* [37]). It is not long though since we started to see the appearance of research papers on competition in remanufacturing (Debo *et al.* [20]).

4.2.1 Competition of Remanufacturing

To the best of our knowledge, Majumder and Groenevelt [58] are probably the first authors investigating the effect of competition on remanufacturing. They formulated a two-period model between an OEM and a local remanufacturer (L). In their paper [58], the remanufactured products are assumed indistinguishable to the original products. In the first period, the OEM manufactures and sells newly-made products. In the second period, the OEM keeps manufacturing and both the OEM and L use shells returned from the first period for remanufacturing and therefore act as competitors. Both players determine, for each period, the manufacturing and remanufacturing quantities and the optimal prices. Considering a common phenomenon that no remanufacturing competitor can keep complete control over all returned shells, they develop four schemes to allocate returned shells between the OEM and L and, under each shell allocation scheme, prove the existence of a unique Nash equilibrium for the second-period competition. Their results show that the second-period competition drives the OEM to reduce the first-period manufacturing quantity and to increase the L's second-period remanufacturing cost. These two results, as a counteractive effect, actually suppress the second-period competition. It also reveals that, the L, by being involved in the second-period competition, actually helps the OEM to reduce their remanufacturing cost, since any factor making remanufacturing attractive compels the OEM to increase both their first-period manufacturing and second-period remanufacturing quantities.

Ferrer and Swaminathan [28] studied the competition problem between an OEM and an independent operator (IO). Their model framework is similar to Majumder and Groenevelt's model in [58], but they extended it to an infinite time horizon and characterized production quantities with self-selection (i.e., customers are ensured to show higher preference on OEM's products than on IO's products). Their managerial insights focus on the effect of competition on the OEM's pricing decisions. Their results show that when remanufacturing is profitable, the OEM would prefer to reduce first-period pricing so that additional units can be sold and the number of shells available for remanufacturing in future periods can increase. Meanwhile, it is also shown that increasing competition with the IO may drive the OEM to use up all available shells and sell remanufactured products at a lower price. In another paper which studies a single firm that makes new products in the first period and use returned shells for remanufacturing in future periods, Ferrer and Swaminathan [29] assumed the remanufactured products be distinguished from the new products with different prices and optimal remanufacturing quantities.

Jung and Hwang [44] applied a repeated game model to study the remanufacturing competition between an OEM and a remanufacturer in which the OEM is required to pay penalties for the shortage of the end-of-use products that are taken back for remanufacturing. They first obtained the optimal prices of new and remanufactured products and buy-back cost from the repeated game model and then extended to the case with the two players' cooperation involved. Their numerical results revealed that the competition raises return rate while the competition with cooperation increases the net profits.

Based on Majumder and Groenevelt's two-period model in [58], a few recent papers have studied the effects of different government policies on remanufacturing activities. Webster and Mitra [93] examined the impact of take-back laws within a manufacturer/remanufacturer competitive framework. Take-back laws generally regulate the collection/disposal costs as a mandatory responsibility for certain relevant firms to take. Mitra and Webster consider a manufacturer who does not engage in remanufacturing but is under the implementation of take-back laws. They considered two alternative implementations of take-back laws, collective take-back and individual take-back, and distinguished the two alternatives by setting different degree of the manufacturer's control on return products sold to a remanufacturer. Under a collective take-back law, the manufacturer has no control over returns products accessible to the remanufacturer. Under an individual take-back, the manufacturer is in full charge of collecting returns and has complete control over whether returns are recycled or sold to the remanufacturer. The authors find that collective take-back will increase both manufacturer and remanufacturer's profits but result in higher market prices. It is also revealed that the manufacturer, when being granted with the complete control over returns, often benefits from allowing the remanufacturer to enter the market, although this may force governmental policy-makers to deal with the manufacturer's monopolistic behavior.

Observing that governments use subsidies to promote remanufacturing activity, Mitra and Webster [60] examined the effects of alternative subsidy allocation schemes. They assume that the subsidy is proportional to remanufacturing volume paid solely to the remanufacturer, solely to the manufacturer, or shared among the firms, respectively. Their results show that a subsidy going to the OEM can benefit both the OEM and the remanufacturer. Comparatively, a subsidy going to the remanufacturer brings less profit for the OEM. It is also shown that a subsidy going to the OEM increases the total volume of remanufacturing. Another revelation is that subsidy sharing can motivate the OEM to design products more suitable for remanufacturing.

Generally speaking, OEMs prefer not to do remanufacturing as remanufactured products may hurt the higher-margin market of brand new products. However, they may change the strategy if they find their end-of-life products are appealing to remanufacturers. Ferguson and Toktay [27] first studied the competition between new and remanufactured products, both made by a monopolistic OEM. They identified the conditions under which the OEM can still be safe as a monopolist with no remanufacturing. They then analyzed the OEM's strategies when competitive threat from a remanufacturer is observed. Their results show that the OEM will consider remanufacturing or preemptive collection as effective strategies to deter a remanufacturer's entry even though he would not choose to do so as a monopolist.

As Majumder and Groenevelt [58] have mentioned, OEMs are making efforts to enhance the collection of returned products for remanufacturing. Some researchers have made further investigation of this aspect. Savaskan *et al.* [77] inspected how an OEM can choose a proper reverse channel structure to collect used products. They considered three channel options: the OEM may 1) collect directly from customers, 2) induce collection by providing incentive to existing retailers, or 3) subcontract the collection to an agency. Their research models each option as a decentralized decision-making system with the OEM as the Stackelberg leader and investigates the resulting effects on profitability and pricing strategy. It is also revealed that a desirable coordination can be achieved when a proper contracting mechanism is applied together with a good incentive scheme. Another insight tells that those agencies who are geographically closer to the manufacturer can collect used products for the OEM with better performance.

By modelling a direct product collection system and an indirect product collection system, Savaskan and Van Wassenhove [78] studied the interaction between a manufacturer's reverse channel design and two retailers' competitive pricing decisions in the forward channel. Both systems are analyzed in a decentralized closed-loop supply chain and a centrally coordinated supply chain. Another paper on pricing decision making between a manufacturer and a remanufacturer is due to Jia and Zhang [43] who managed to make the demand analysis simple by applying Bertrand game model.

Regarding remanufacturable product design, Debo *et al.* [20] solved the joint pricing and production technology selection problem in an infinite time horizon for a manufacturer introducing a remanufacturable product into a market consisted of miscellaneous consumers. They [20] determined the remanufacturability level and the optimal prices using a market model that reflects the consumers' perception on their remanufactured products (Debo *et al.* [20]). Different from Majumder and Groenevelt [58] who took the return fraction of products as exogenously given, Debo *et al.* [20] introduced the level of remanufacturability as a key variable that the OEM can determine.

Kaya [46] considered an OEM producing new products using brand new materials and remanufactured products with returns from the market. The amount of returns in that paper is assumed dependent on the incentive offered by the manufacturer and the OEM is to optimize the incentive as well as production quantities of both original and remanufactured products in a stochastic demand setting. They considered two business settings (centralized and decentralized) and three models for each setting (pure remanufacturing, manufacturing/remanufacturing with perfect substitution, and manufacturing/remanufacturing with partial substitution).

While most related literature consider competition between an OEM and one or more local remanufacturers, direct competition between OEMs is a topic less studied. Heese *et al.* [41] used a Stackelberg duopoly game model to analyze the profitability of remanufacturing under direct OEM competition and stated that remanufacturing can be a profitable strategy for the first-moving firm with proper cost structure and market share.

4.2.2 Uncertainties in Remanufacturing

Compared with the literature regarding remanufacturing competition, literature on uncertainties in remanufacturing is much less. Considering that the significant variation of core quality affects a remanufacturer's profitability, Teunter and Flapper [87] studied a remanufacturer's joint decision making on core acquisition and remanufacturing policy setting when he faces multiple core quality classes and the quality of each class is under uncertainty presented by multinomial quality distribution. They analyzed cases with deterministic demand and uncertain demand respectively. Results with deterministic demand show that ignoring quality uncertainty does cause inaccurate core acquisition. They derived newsboy-type solutions for the case with uncertain demand and the corresponding results showed that: 1) increasing demand variation leads to a larger number for core acquisition and a higher up-to-level for remanufacturing production, and 2) the value of using information on quality uncertainty is not quite significant in case of high demand uncertainty. Mukhopadhyay and Ma [63] applied stochastic analysis to study the joint procurement and production decisions in a hybrid manufacturing system, where both used and brand new parts can be used in production process, in the face of uncertainties in both quality of returned products as well as market demand. Galbreth and Blackburn [32] examined a remanufacturer's jointly acquisition and sorting policies making problem with both deterministic and uncertain demand in the presence of used product condition variability. They showed that an optimal acquisition and sorting policy exists and this policy is independent on production amount in case of linear acquisition costs.

4.2.3 Our Contributions

In summary, research efforts on remanufacturing, particularly on competition in remanufacturing, are not sufficient yet. Literature as to the uncertainty in remanufacturing is limited, too, and mainly focuses on the uncertainty in core quality and demand. Meanwhile, most available research work in this aspect uses stochastic analysis with continuous probability distributions to deal with uncertainty. To the best of our knowledge, there hasn't any literature dealing with remanufacturing competition with competitors' uncertainty on each other's feature information taken into consideration. Our major contributions in this chapter are two-fold. On the one hand, rather than focusing on the uncertainty of core quality and market demand, we manage to have the OEM's uncertainty as to the REM's remanufacturing unit cost considered in the context of remanufacturing competition. On the other hand, instead of using a continuous probability distribution, we apply a type-III game (see Section 3.3) for our formulation which models the uncertainty with a tri-vector $[\theta, u_{r_L}, u_{r_H}]$ by assuming the REM's unit cost be u_{r_L} with probability θ and u_{r_H} with probability $(1-\theta)$. We obtain the closed-form Bayesian Nash equilibrium solution for this static game with incomplete information which is tractable and easy for sensitivity analysis and examine the impact of such incomplete information on the OEM-REM remanufacturing competition.

4.3 Remanufacturing Competition with Complete Information

As a prelude to the illustration of our model for the remanufacturing competition problem with incomplete information, we first introduce the model for the remanufacturing competition problem with complete information. We consider an OEM and a REM who compete with each other on remanufacturing business. We know that regular production or manufacturing of brand new products is an OEM's major business. Since in this chapter we mainly focus on the effect of the incomplete information on remanufacturing competition, in our models with complete information and incomplete information, we consider the OEM's remanufacturing business and ignore his regular manufacturing business. We do the formulation based on Majumder and Groenevelt's model in [58]. Also we assign the male gender to the OEM and the female gender to the REM when we refer to them.

4.3.1 Notations

Table 4.1 lists the notations that we use in the remanufacturing competition problem with complete information. We use superscript "c" throughout this section (Section 4.3), in case of need, to label notations for the formulation with complete information.

4.3.2 Demand Functions

In order to achieve tractable analytic results, we adopt the linear demand functions in Majumder and Groenevelt [58] as well. The OEM and REM's demand functions

Symbol	Description
Parameters	
u_o	The OEM's unit remanufacturing cost;
u_r	The REM's unit remanufacturing cost;
(A_o, B_o, C_o)	The OEM's demand parameters;
(A_r, B_r, C_r)	The REM's demand parameters which are the same for all types;
$D_o(p_o, p_r)$	The OEM's demand function;
$D_r(p_o, p_r)$	The REM's demand function;
$D_o^m(p_r)$	$= \frac{1}{2}D_o(u_o, p_r)$, the OEM's median demand, given p_r;
$D_r^m(p_o)$	$= \frac{1}{2}D_r(p_o, u_r)$, the REM's median demand, given p_o;
S	The total number of available returned shells in the market;
S_o	The number of returned shells accessible to the OEM;
S_r	The number of returned shells accessible to the REM
Θ_o^c	The OEM's optimization problem;
Θ_r^c	The REM's optimization problem;
P_o or $P_o(p_r)$	The upper bound of the OEM's price, given p_r;
P_r or $P_r(p_o)$	The upper bound of the REM's price, given p_o;
Decision Variables	
q_o	The OEM's remanufacturing quantity;
q_r	The REM's remanufacturing quantity;
p_o	The OEM's unit sale price;
p_r	The REM's unit sale price;

Table 4.1: Summary of notations for the remanufacturing competition problem with complete information.

are in the form of

$$D_o(p_o, p_r) = A_o - B_o p_o + C_o p_r,$$
$$D_r(p_o, p_r) = A_r - B_r p_r + C_r p_o.$$

All demand parameters in A_o, B_o, C_o, A_r, B_r and C_r are assumed positive. Also both players should set their own prices so that their demand will not be negative, i.e., $D_o(p_o, p_r) \geq 0$ and $D_r(p_o, p_r) \geq 0$ for any valid value of p_o and p_r. Therefore, their price lower bounds are their respective unit remanufacturing cost (u_o for the OEM u_r for the REM) and their upper bounds are

$$P_o(p_r) = \arg_{p_o}[D_o(p_o, p_r) = 0] = (A_o + C_o p_r)/B_o,$$
$$P_r(p_o) = \arg_{p_r}[D_r(p_o, p_r) = 0] = (A_r + C_r p_o)/B_r.$$

Moreover, same as in Majumder and Groenevelt [58], we assume $B_o > C_r$ and $B_r > C_o$ so that the total demand

$$D(p_o, p_r) = (A_o + A_r) - (B_o - C_r) p_o - (B_r - C_o) p_r$$

is a decreasing function of both p_o and p_r.

4.3.3 The Optimization Problems

The two players' optimization problems in our model with complete information are similar to the ones in Majumder and Groenevelt [58], too.

The OEM's Optimization Problem

By applying our symbols and notations to the OEM's remanufacturing activity, we present his optimization problem with complete information as:

$$\begin{aligned} \Theta_o^c(p_o, q_o \mid p_r, S_o) : \max\ & \pi_o^c = q_o (p_o - u_o), & (4.1)\\ \text{s.t. } & q_o \leq S_o, & (4.2)\\ & q_o \leq D_o(p_o, p_r), & (4.3)\\ & q_o \geq 0,\ p_o \geq u_o. & (4.4) \end{aligned}$$

That is, given the OEM's accessible shells S_o and the REM's sale price p_r, the OEM is to maximize his profit by choosing his remanufacturing quantity q_o and sale price p_o. Constraint (4.2) guarantees that the remanufacturing quantity q_o won't exceed the number of accessible shells S_o; constraint (4.3) keeps the quantity q_o no more than his market demand under price p_o; constraint (4.4) is for the nonnegativity of the remanufacturing quantity q_o and the price p_o be no lower than the unit cost u_o. According to Lemma 1 in Majumder and Groenevelt [58], the OEM prefers to sell the quantity demanded if only he can make a positive profit. Hence, the OEM's optimization problem can be simplified to

$$\Theta_o^c(p_o, q_o \mid p_r, S_o) : \max\ \pi_o^c = [\min\ (S_o, D_o(p_o, p_r))] (p_o - u_o), \tag{4.5}$$

$\Theta_o^c(p_o, q_o \mid p_r, S_o)$ is a nonlinear optimization problem of sale price p_o. The closed-form optimal solution $(p_o^*, q_o^*)^c$, given the REM's price p_r and the OEM's accessible shells S_o, can be summarized as:

$$(p_o^*, q_o^*)^c = \begin{cases} \left(P_o(p_r) - \frac{S_o}{B_o}, S_o\right), & \text{if } S_o < D_o^m(p_r), \\ \left(\frac{1}{2}(u_o + P_o(p_r)), D_o^m(p_r)\right), & \text{if } S_o \geq D_o^m(p_r). \end{cases} \tag{4.6}$$

where $D_o^m(p_r)$ is the medium of the OEM's demand, given the REM's price p_r and it equals to half of the OEM's maximum demand $D_o(u_o, p_r)$ — the demand when the OEM sets his price p_o as his remanufacturing unit cost u_o. We illustrate in detail the optimal solution $(p_o^*, q_o^*)^c$ in (4.6) in Appendix B.1.1. In particular, when $S_o < D_o^m(p_r) = (A_o - B_o u_o + C_o p_r)/2$, we are supposed to have $p_o = P_o(p_r) - S_o/B_o = (A_o + C_o p_r)/B_o - S_o/B_o > (A_o + B_o u_o + C_o p_r)/2B_o = [u_o + (A_o + C_o p_r)/B_o]/2$ which is definitely greater than u_o since $(A_o + C_o p_r)/B_o = P_o(p_r)$ is the OEM's highest price given the REM's price p_r and it must be greater than his remanufacturing unit cost u_o.

The REM's Optimization Problem

The REM's optimization problem is symmetrically structured with the OEM's, i.e., it is in the form of,

$$\Theta_r^c(p_r, q_r \mid p_o, S_r) : \max \ \pi_r^c = q_r(p_r - u_r), \tag{4.7}$$

$$\text{s.t. } q_r \leq S_r, \tag{4.8}$$

$$q_r \leq D_r(p_o, p_r), \tag{4.9}$$

$$q_r \geq 0, \ p_r \geq u_r. \tag{4.10}$$

Given accessible shells S_r and the OEM's price p_o, the REM is to maximize her profit by determining production quantity q_r and sale price p_r,The constraints (4.8), (4.9), and (4.10) here are counter parts of the constraints (4.2), (4.3) and (4.4) in $\Theta_o^c(p_o, q_o \mid p_r, S_o)$. Similarly, the REM's optimization problem can be simplified to

$$\Theta_r^c(p_r, q_r \mid p_o, S_r) : \max \ \pi_r^c = [\min \ (S_r, D_r(p_o, p_r))](p_r - u_r), \tag{4.11}$$

and its optimal solution $(p_r^*, q_r^*)^c$, given S_r and p_o, is:

$$(p_r^*, q_r^*)^c = \begin{cases} \left(P_r(p_o) - \frac{S_r}{B_r}, S_r\right), & \text{if } S_r < D_r^m(p_o), \\ \left(\frac{1}{2}(u_r + P_r(p_o)), D_r^m(p_o)\right), & \text{if } S_r \geq D_r^m(p_o). \end{cases} \tag{4.12}$$

The optimal solution $(p_r^*, q_r^*)^c$ in (4.12) is actually a simplified version of the optimal solution that Majumder and Groenevelt presented in the Proposition 1 in [58].

4.3.4 Shell Accessibility Scenarios

In the above two optimization problems, the numbers of shells accessible to each player, S_o for the OEM and S_r for the REM, are all exogenously given parameters. We use S to denote the total number of available shells in the market. Both S_o and S_r should be portions of S and be dependent on each other. Similar to the first two shell allocation mechanisms in Majumder and Groenevelt [58], we consider two shell accessibility scenarios (SAS1 and SAS2) for the competition with complete information which we display in Table 4.2.

SAS	**Priority**	$\mathbf{S}_o$	$\mathbf{S}_r$
SAS1	OEM	S	$S - q_o$
SAS2	REM	$S - q_r$	S

Table 4.2: Shell accessibility scenarios for the remanufacturing competition with complete information.

In scenario 1 (SAS1), the OEM is assumed to be accessible to the total available shells (i.e., $S_o = S$) and the REM can only access to the shells that are left over by OEM (i.e. $S_r = S - q_o$). This scenario is applicable to the case when the OEM has a higher priority on shell accessibility. Similarly, scenario 2 (SAS2) assumes that the REM can access all available shells S (i.e., $S_r = S$) and the OEM is allowed to access to the left over (i.e., $S_o = S - q_r$). Similarly, this scenario is applicable to the case when the priority of shell accessibility is with the REM.

4.3.5 The NE Solutions

Having looked at the players' optimization problems $\Theta_o^c(p_o,q_o \mid p_r,S_o)$ in (4.5) and $\Theta_r^c(p_r,q_r \mid p_o,S_r)$ in (4.11) respectively, we need to identify the proper solution concept. We assume that the OEM and the REM make decisions independently and simultaneously. Therefore, having the total number of accessible shells S as an exogenously-given parameter, we are to solve $\Theta_o^c\,(p_o, q_o \mid p_r, S_o)$ and $\Theta_r^c\,(p_r, q_r \mid p_o, S_r)$ simultaneously under each shell accessibility scenario. That is, (Θ_o^c, Θ_r^c) constitutes a static game with complete information so the Nash Equilibrium (NE) applies. The following theorem applies to (Θ_o^c, Θ_r^c) under both shell accessibility scenarios and we provide the proof of the existence and uniqueness of the NE in Appendix B.1.2.

Theorem 4.1 *A unique pure strategy Nash Equilibrium exists for (Θ_o^c, Θ_r^c) with each shell accessibility scenario.*

Furthermore, we solve (Θ_o^c, Θ_r^c) to obtain the unique Nash Equilibrium in closed form. Based on the derivation process that is provided in Appendix B.1.3, we present $[p_o^*, q_o^*, p_r^*, q_r^*]_1^c$ — the NE for (Θ_o^c, Θ_r^c) under SAS1 — as:

$$[p_o^*, q_o^*, p_r^*, q_r^*]_1^c = \begin{cases} [p_o^*, q_o^*, p_r^*, q_r^*]_{11}^c, & \text{if } S \in [0, \bar{S}_{11}^c), \\ [p_o^*, q_o^*, p_r^*, q_r^*]_{12}^c, & \text{if } S \in [\bar{S}_{11}^c, \bar{S}_{12}^c), \\ [p_o^*, q_o^*, p_r^*, q_r^*]_{13}^c, & \text{if } S \in [\bar{S}_{12}^c, \infty). \end{cases} \tag{4.13}$$

We omit the derivation process for the NE in (Θ_o^c, Θ_r^c) under SAS2 for it duplicates the one under SAS1 in Appendix B.1.3 except that $S_r = S$ and $S_o = S - q_r$.

4.4 Remanufacturing Competition with Incomplete Information

After presenting the remanufacturing competition problem with complete information, we are now ready to formulate the remanufacturing competition problem with incomplete information with the OEM's uncertainty on the REM's unit cost taken into consideration.

4.4.1 Notations

Table 4.3 lists the notations we use in the remanufacturing competition problem with incomplete information. For the sake of consistency, we retain the notations in the competition problem with complete information as much as we can and only make changes out of necessity in the setting of incomplete information.

4.4.2 Incomplete Information

We are interested in the effect of incomplete information on the competition in remanufacturing. Specifically, we assume the uncertainty lie in a player's information on his/her competitor's production cost. We consider the case when the OEM is uncertain with the REM's unit remanufacturing cost u_r. To model this uncertainty, rather than assuming the REM's unit cost as a random variable following a continuous probability distribution, we choose to apply the type III game in Section 3.3, Chapter 3. To describe in detail, we assume that the OEM is still of a single type with unit remanufacturing cost u_o and this is common knowledge to both players. This is the same as our assumption in the setting with complete information. Meanwhile, we assume that there are two possible types of REM, namely, REM of type L and REM of type H. The type-L REM is featured with a lower production cost u_{r_L} and the type-H REM is featured with a higher production cost u_{r_H}. The REM is either of type L or of type H and is assumed to know her own type exactly, i.e., she knows her own unit remanufacturing cost. Then the OEM's incomplete information on the REM is reflected by our assumption that he knows his competitor be of type L with probability θ and of type H with probability $(1-\theta)$. We also assume that the REM knows the OEM's incomplete information about her types, i.e., the probabilities θ and $(1-\theta)$ are common knowledge to the REM as well. Briefly summarize, we use a tri-vector $[\theta, u_{r_L}, u_{r_H}]$ to present the OEM's incomplete information on the REM.

4.4.3 Demand Functions

With the above setting of incomplete information, we need to re-define each player's demand function. The REM, who is either of type L or of type H, knows her own production cost and the OEM's production cost clearly, so her demand functions, D_{r_L}

Symbol	Description
Parameters	
u_o	The OEM's unit remanufacturing cost;
u_{r_L} and u_{r_H}	The type-L and type-H REM's unit remanufacturing cost;
$\theta$and $1-\theta$	The probability that the REM is of type L and type H respectively;
(A_o, B_o, C_o)	The OEM's demand parameters;
(A_r, B_r, C_r)	The REM's demand parameters which are the same for both types;
$D_o\left(p_o, p_{r_L}, p_{r_H}\right)$	The OEM's demand function;
$D_{r_L}\left(p_o, p_{r_L}\right)$	The type-L REM's demand function;
$D_{r_H}\left(p_o, p_{r_H}\right)$	The type-H REM's demand function;
$D_o^m\left(p_{r_L}, p_{r_H}\right)$	$=\frac{1}{2}D_o(u_o, p_{r_L}, p_{r_H})$,
	the median of the OEM's demand, given $p_{r_L}$and p_{r_H};
$D_{r_L}^m\left(p_o\right)$	$=\frac{1}{2}D_{r_L}(p_o, u_{r_L})$,the median of the type-L REM's demand, given p_o;
$D_{r_H}^m\left(p_o\right)$	$=\frac{1}{2}D_{r_H}(p_o, u_{r_H})$,the median of the type-H REM's demand, given p_o;
S	The total number of available returned shells in the market;
S_o	The number of returned shells accessible to the OEM;
S_r	The number of returned shells accessible to the REM;
Θ_o	The OEM's optimization problem;
$\Theta_{r_L}$and Θ_{r_H}	The type-L and type H REM's optimization problem;
$P_o$or $P_o\left(p_{r_L}, p_{r_H}\right)$	The upper bound of the OEM's price p_o, given $p_{r_L}$and p_{r_H};
P_{r_L}or $P_{r_L}\left(p_o\right)$	The upper bound of the type-L REM's price p_{r_L}, given p_o;
P_{r_H}or $P_{r_H}\left(p_o\right)$	The upper bound of the type-H REM's price p_{r_H}, given p_o;
Decision Variables	
q_o	The OEM's remanufacturing quantity;
$q_{r_L}$and q_{r_H}	The type-L and type-H REM's remanufacturing quantity;
p_o	The OEM's unit sale price;
p_{r_L} and p_{r_H}	The type-L and type-H REM's unit sale prices;

Table 4.3: Summary of notations for the remanufacturing competition problem with incomplete information.

for the type-L REM and D_{r_H} for the type-H one, are in the form of

$$D_{r_L}(p_o, p_{r_L}) = A_r - B_r p_{r_L} + C_r p_o,$$
$$D_{r_H}(p_o, p_{r_H}) = A_r - B_r p_{r_H} + C_r p_o.$$

The demand parameters (A_r, B_r, C_r) here are the same as the ones in the formulation with complete information.

Regarding the OEM, his uncertainty on the REM's type is projected onto his uncertainty on the REM's sale price which is either p_{r_L} (if the REM is of type L) or p_{r_H} (if the REM is of type H). Given the OEM's sale price p_o and the REM's sale price, which is either p_{r_L} or p_{r_H}, the OEM's expected demand function D_o should be in the form of

$$D_o(p_o, p_{r_L}, p_{r_H}) = A_o - B_o p_o + C_o [\theta p_{r_L} + (1-\theta) p_{r_H}].$$

where his demand parameters (A_o, B_o, C_o) are identical with the ones in the formulation with complete information, too. All demands are assumed non-negative, i.e., $D_{r_L}(p_o, p_{r_L}) \geq 0$, $D_{r_H}(p_o, p_{r_H}) \geq 0$, and $D_o(p_o, p_{r_L}, p_{r_H}) \geq 0$. Correspondingly, their price lower bounds are the unit remanufacturing costs u_{r_L}, u_{r_H}, and u_o respectively and their price upper bounds are

$$P_{r_L}(p_o) = \arg_{p_{r_L}} [D_{r_L}(p_o, p_{r_L}) = 0] = [A_r + C_r p_o] / B_r,$$
$$P_{r_H}(p_o) = \arg_{p_{r_H}} [D_{r_H}(p_o, p_{r_H}) = 0] = [A_r + C_r p_o] / B_r,$$
$$P_o(p_{r_L}, p_{r_H}) = \arg_{p_o} [D_o(p_o, p_{r_L}, p_{r_H}) = 0] = [A_o + C_o(\theta p_{r_L} + (1-\theta) p_{r_H})] / B_o.$$

Note that $P_{r_L}(p_o) = P_{r_H}(p_o)$, i.e., the two types of REM share the same upper bound. Hence, we simply use $P_r(p_o)$ in the setting with incomplete information to denote both $P_{r_L}(p_o)$ and $P_{r_H}(p_o)$, i.e., $P_{r_L}(p_o) = P_{r_H}(p_o) \triangleq P_r(p_o)$. Moreover, we still assuming $B_o > C_r$ and $B_r > C_o$ so that the expected total demand

$$D(p_o, p_{r_L}, p_{r_H}) = (A_o + A_r) - (B_o - C_r) p_o - (B_r - C_o) [\theta p_{r_L} + (1-\theta) p_{r_H}]$$

be a decreasing function of all players' prices.

4.4.4 Shell Accessibility Scenarios

Same as in the setting with complete information, we assume the numbers of shells accessible to each player — S_o for the OEM and S_r for the REM of either type — are all exogenously given parameters. We continue to use S to denote the total number of shells available in the market and consider the shell accessibility scenario SAS1 which has the priority with the OEM. We notice that, the SAS1 in the setting of incomplete information is identical with the SAS1 in the setting of complete information, i.e., as shown in Table 4.2, we have $S_o = S$ and $S_r = S - q_o$. It implies that the existence of incomplete information has no effect on shell accessibility scenario when the priority is with the OEM.

4.4.5 The Optimization Problems

Now we are ready to introduce the two players' optimization problems in the existence of incomplete information as described in Section 4.4.2. Since the REM still has complete information (about herself and the OEM), her optimization problems are relatively easy to describe. Hence, we start with the REM's optimization problems.

The REM's Optimization Problems

Having accessible shells S_r and the OEM's price p_o as parameters, the REM, knowing her own type (type L or type H) with certainty, is to maximize her profit (π_{r_L} or π_{r_H}) by determining production quantity (q_{r_L} or q_{r_H}) and sale price (p_{r_L} or p_{r_H}) respectively. Her optimization problem, $\Theta_{r_L}(p_{r_L}, q_{r_L} \mid p_o, S_r)$ for type L and $\Theta_{r_H}(p_{r_H}, q_{r_H} \mid p_o, S_r)$ for type H, are:

$$
\begin{aligned}
\Theta_{r_L}\left(p_{r_L}, q_{r_L} \mid p_o, S_r\right) : \max \;\; & \pi_{r_L} = q_{r_L}\left(p_{r_L} - u_{r_L}\right), && (4.14)\\
\text{s.t. } & q_{r_L} \le S_r, && (4.15)\\
& q_{r_L} \le D_{r_L}\left(p_o, p_{r_L}\right), && (4.16)\\
& q_{r_L} \ge 0, \;\; p_{r_L} \ge u_{r_L}. && (4.17)
\end{aligned}
$$

$$
\begin{aligned}
\Theta_{r_H}\left(p_{r_H}, q_{r_H} \mid p_o, S_r\right) : \max \;\; & \pi_{r_H} = q_{r_H}\left(p_{r_H} - u_{r_H}\right), && (4.18)\\
\text{s.t. } & q_{r_H} \le S_r, && (4.19)\\
& q_{r_H} \le D_{r_H}\left(p_o, p_{r_H}\right), && (4.20)\\
& q_{r_H} \ge 0, \;\; p_{r_H} \ge u_{r_H}. && (4.21)
\end{aligned}
$$

Here, $\Theta_{r_L}(p_{r_L}, q_{r_L} \mid p_o, S_r)$ and $\Theta_{r_H}(p_{r_H}, q_{r_H} \mid p_o, S_r)$ are formulated identically with the optimization problem Θ_r^c in Section 4.3.3. Constraints (4.15) and (4.19), (4.16) and (4.20), and (4.17) and (4.21) are counterparts of constraints (4.8), (4.9), and (4.10), respectively. Similar to (4.11), we can re-write $\Theta_{r_L}(p_{r_L}, q_{r_L} \mid p_o, S_r)$ and $\Theta_{r_H}(p_{r_H}, q_{r_H} \mid p_o, S_r)$ as

$$
\begin{aligned}
\Theta_{r_L}\left(p_{r_L}, q_{r_L} \mid p_o, S_r\right) &: \max\left[\min\;\left(S_r, D_{r_L}\left(p_o, p_{r_L}\right)\right)\right]\left(p_{r_L} - u_{r_L}\right), && (4.22)\\
\Theta_{r_H}\left(p_{r_H}, q_{r_H} \mid p_o, S_r\right) &: \max\left[\min\;\left(S_r, D_{r_H}\left(p_o, p_{r_H}\right)\right)\right]\left(p_{r_H} - u_{r_H}\right), && (4.23)
\end{aligned}
$$

and their optimal solutions $\left(p^*_{r_T}, q^*_{r_T}\right), T \in \{L, H\}$, are:

$$\left(p^*_{r_T}, q^*_{r_T}\right)_{T\in\{L,H\}} = \begin{cases} \left(P_r\left(p_o\right) - \frac{S_r}{B_r}, S_r\right), & \text{if } S_r < D^m_{r_T}\left(p_o\right), \\ \left(\frac{1}{2}\left(u_{r_T} + P_r\left(p_o\right)\right), \frac{1}{2}D_{r_T}\left(p_o, u_{r_T}\right)\right), & \text{if } S_r \geq D^m_{r_T}\left(p_o\right). \end{cases} \tag{4.24}$$

Furthermore, we claim that the following results apply to the REM in the setting of incomplete information as defined in Section 4.4.2.

Lemma 4.1 *Given p_o and p^*_o, it is true that $D_{r_L}\left(p_o, u_{r_L}\right) > D_{r_H}\left(p_o, u_{r_H}\right)$ and $D_{r_L}(p^*_o, u_{r_L}) > D_{r_H}(p^*_o, u_{r_H})$.*

Lemma 4.2 *Given p_o and S_r, it is true that (a) $p^*_{r_L} \leq p^*_{r_H}$ and $q^*_{r_L} \geq q^*_{r_H}$; (b) $p^*_{r_L} = p^*_{r_H} = P_r - \frac{S_r}{B_r}$ and $q^*_{r_L} = q^*_{r_H} = S_r$ if only $S_r \leq D^m_{r_H}\left(p_o\right)$.*

Lemma 4.1 states that, given the OEM's price p_o, the demand of the type-L REM is always greater than the demand of the type-H REM. This statement holds true when it is given the OEM's optimal price p^*_o in Bayesian Nash Equilibrium which we will illustrate later. Lemma 4.2 says that, given the OEM's price p_o and accessible shells S_r, (a) the type-L REM would set price no higher than the type-H REM and it is the opposite as to their optimal quantities; (b) both type-L and type-H REM would share the same optimal price and optimal quantity if only the accessible shells is no more than the type-H REM's median demand. The proof of these two lemmas are provided in Appendix B.2.1 and B.2.2 separately.

The OEM's Optimization Problem

Knowing the REM be of type L or of type H with probabilities θ and $(1-\theta)$ respectively, the OEM would face $\Theta^c_o\left(p_o, q_o \mid p_{r_L}, S_o\right)$ with probability θ and $\Theta^c_o\left(p_o, q_o \mid p_{r_H}, S_o\right)$ with probability $(1-\theta)$. Hence, we present his optimization problem as

$$\Theta_o\left(p_o, q_o \mid p_{r_L}, p_{r_H}, \theta, S_o\right) : \theta\Theta^c_o\left(p_o, q_o \mid p_{r_L}, S_o\left(u_{r_L}\right)\right) + (1-\theta)\,\Theta^c_o\left(p_o, q_o \mid p_{r_H}, S_o\left(u_{r_H}\right)\right),$$

and transform it into

$$\max \pi_o = \{\theta\ [\min(S, D^c_o(p_o, p_{r_L}))] + (1-\theta)\,[\min\left(S, D^c_o(p_o, p_{r_H})\right)]\}(p_o - u_o), \tag{4.25}$$

or

$$\max \pi_o = \min(S, D_o(p_o, p_{r_L}, p_{r_H}))(p_o - u_o).$$

because under SAS1, we have $S_o\left(u_{r_L}\right) = S_o\left(u_{r_H}\right) = S$. We use D^c_o here to denote the OEM's demand if he know the REM's type with certainty. Accordingly, $\min(S, D^c_o\left(p_o, p_{r_L}\right))$ in (4.25) is the OEM's remanufacturing quantity at price p_o if the REM is of type L and $\min(S, D^c_o\left(p_o, p_{r_H}\right))$ is his remanufacturing quantity if he knows

Cases	p_{r_L} **vs.** p_{r_H}	S	$(p_o^*, q_o^*)_1$
1	$p_{r_L} = p_{r_H}$	$0 \leq S < D_o^m(\bar{p}_r) + D_{r_H}^m(\bar{p}_o)$	$(P_o(\bar{p}_r) - S/B_o, S)$
2	$p_{r_L} = p_{r_H}$	$D_o^m(\bar{p}_r) + D_{r_H}^m(\bar{p}_o) \leq S < \infty$	$(\bar{p}_o, D_o^m(\bar{p}_r))$
3	$p_{r_L} < p_{r_H}$	$0 \leq S < D_o^m(\bar{p}_r) + D_{r_H}^m(\bar{p}_o)$	Invalid case.
4	$p_{r_L} < p_{r_H}$	$D_o^m(\bar{p}_r) + D_{r_H}^m(\bar{p}_o) \leq S < \infty$	$(\bar{p}_o, D_o^m(\bar{p}_r))$

Table 4.4: The optimal solution to the OEM's optimization problem with incomplete information under SAS1.

that the REM is of type H. Note that, when $\theta = 0$, (4.25) becomes

$$\max \pi_o = \min (S, D_o^c(p_o, p_{r_H}))(p_o - u_o), \tag{4.26}$$

and, when $\theta = 1$, the maximization problem in (4.25) becomes

$$\max \pi_o = \min(S, D_o^c(p_o, p_{r_L}))(p_o - u_o). \tag{4.27}$$

We note that the optimization problems in (4.26) and (4.27) are identical with the OEM's optimization problem with complete information in (4.5). This is consistent with our intuition that the remanufacturing competition problem with complete information should be a special case of the remanufacturing competition problem with incomplete information.

We claim that the following lemma applies to the OEM in the formulation with incomplete information.

Lemma 4.3 *Given the REM's optimal solution* $\left(p_{r_L}^*, p_{r_H}^*\right)$*, it is true that* $D_o^c(p_o,p_{r_L}^*) \leq D_o^c(p_o,p_{r_H}^*)$ *and* $D_o^c(u_o,p_{r_L}^*) \leq D_o^c(u_o,p_{r_H}^*)$*.*

Lemma 4.3 states that, given the REM's optimal solution $\left(p_{r_L}^*, p_{r_H}^*\right)$, the OEM's demand under his price p_o if the REM is of type-H is no less than his demand when the REM is of type-L and this relation holds true as about the OEM's maximum demand. The proof to this lemma is provided in Appendix B.2.3. Then we provide further investigation on the OEM's optimization problem $\Theta_o(p_o, q_o \mid p_{r_L}, p_{r_H}, \theta, S_o)$ under SAS1 in Appendix B.2.4 and, given S and the REM's p_{r_L} and p_{r_H}, the OEM's optimal solution $(p_o^*, q_o^*)_1$ can be summarized in Table 4.4.

Note that for each case in Table 4.4, we have $\bar{p}_r = \theta p_{r_L} + (1-\theta)p_{r_H}$ and $\bar{p}_o = (u_o + P_o(\bar{p}_r))/2 = (u_o + [A_o + C_o(\theta p_{r_L} + (1-\theta)p_{r_H})]/B_o)/2$.

4.4.6 The BNE Solutions

Having looked at all players' optimization problems Θ_{r_L}, Θ_{r_H}, and Θ_o in (4.22), (4.23), and (4.25) respectively, we are to solve them simultaneously and this constitutes a static game with incomplete information so the Bayesian Nash equilibrium (BNE) applies. (More information about the static game with incomplete information and the Bayesian Nash equilibrium can be found in Section 3.3.) We will present the BNE in the form of $\left[p_o^*, q_o^*, , p_{r_L}^*, q_{r_L}^*, p_{r_H}^*, q_{r_H}^*\right]$. In addition, we will use subscript "1" to label the BNE under SAS1 and subscript "2" to label the BNE under SAS2 which will be worked out in the future:

Regarding the BNE, we claim that the following results applies to both SAS1 and SAS2.

Lemma 4.4 *In the BNE of* $(\Theta_o, (\Theta_{r_L}, \Theta_{r_H}))$*, it is true that* $p_{r_L}^* \leq p_{r_H}^*$ *and* $q_{r_L}^* \geq q_{r_H}^*$.

Lemma 4.4 states that, in the BNE of $(\Theta_o, (\Theta_{r_L}, \Theta_{r_H}))$, the type-L REM's price is no more than the type-H's price and the type-L's remanufacturing quantity is no less than the type-H's quantity. It is actually an application of Lemma 4.2 on the BNE. Moreover, we claim that the result below applies to the BNE with SAS1 and the proof is provided in B.2.5.

Theorem 4.2 *There exists a pure strategy Bayesian Nash equilibrium for* $(\Theta_o,\ (\Theta_{r_L},\ \Theta_{r_H})\ )$ *under SAS1.*

Then we solve $(\Theta_o, (\Theta_{r_L}, \Theta_{r_H}))$ to obtain the BNE in closed form. Based on the derivation process that is provided in Appendix B.2.6, we summarize the BNE for $(\Theta_o, (\Theta_{r_L}, \Theta_{r_H}))$ — the remanufacturing competition problem with incomplete information under SAS1 — as:

$$\left[p_o^*, q_o^*, p_{r_L}^*, q_{r_L}^*, p_{r_H}^*, q_{r_H}^*\right]_1 = \begin{cases} \left[p_o^*, q_o^*, p_{r_L}^*, q_{r_L}^*, p_{r_H}^*, q_{r_H}^*\right]_{11}, & \text{if } S \in [0, \bar{S}_{11}), \\ \left[p_o^*, q_o^*, p_{r_L}^*, q_{r_L}^*, p_{r_H}^*, q_{r_H}^*\right]_{12}, & \text{if } S \in [\bar{S}_{11}, \bar{S}_{12}), \\ \left[p_o^*, q_o^*, p_{r_L}^*, q_{r_L}^*, p_{r_H}^*, q_{r_H}^*\right]_{13}, & \text{if } S \in [\bar{S}_{12}, \bar{S}_{13}), \\ \left[p_o^*, q_o^*, p_{r_L}^*, q_{r_L}^*, p_{r_H}^*, q_{r_H}^*\right]_{14}, & \text{if } S \in [\bar{S}_{13}, \infty). \end{cases} \tag{4.28}$$

4.4.7 Consistency Between the BNE and NE

It is apparent that, when $\theta = 0$ or 1, the OEM is certain about the type of REM which is type H when $\theta = 0$ and type L when $\theta = 1$ respectively. Hence, we expect the BNE for $(\Theta_o, (\Theta_{r_L}, \Theta_{r_H}))$ and the NE for (Θ_o, Θ_r) are consistent with each other. In detail, we expect to see that the BNE with $\theta = 0$ would be identical with the NE with the REM being of type H and the BNE with $\theta = 1$ would be identical with the NE with the REM being of type L. In order to verify such consistency, we insert $\theta = 0$ and $\theta = 1$ into the BNE $[p_o^*, q_o^*, p_{r_L}^*, q_{r_L}^*, p_{r_H}^*, q_{r_H}^*]_1$ in (4.28) separately and compare the resulted BNEs with the NE $[p_o^*, q_o^*, p_r^*, q_r^*]_1^c$ in (4.13). The comparison, as presented in Table 4.5, shows that the closed-form BNEs with $\theta = 0$ and 1 are identical with the NE with the REM being of type H and type L respectively.

BNE with $\theta = 0$ (the REM is surely of type H)		NE with type-H REM
$[\bar{S}_{11}, \bar{S}_{12}]$	$=$	$[\bar{S}^c_{11}, \bar{S}^c_{12}]$
$\left[p^*_o, q^*_o, p^*_{r_H}, q^*_{r_H}\right]_{11}$	$=$	$[p^*_o, q^*_o, p^*_r, q^*_r]^c_{11}$
$\left[p^*_o, q^*_o, p^*_{r_H}, q^*_{r_H}\right]_{12}$	$=$	$[p^*_o, q^*_o, p^*_r, q^*_r]^c_{12}$
$\left[p^*_o, q^*_o, p^*_{r_H}, q^*_{r_H}\right]_{13}$	$=$	$[p^*_o, q^*_o, p^*_r, q^*_r]^c_{13}$
$\left[p^*_o, q^*_o, p^*_{r_H}, q^*_{r_H}\right]_{14}$	$=$	$[p^*_o, q^*_o, p^*_r, q^*_r]^c_{13}$
BNE with $\theta = 1$ (the REM is surely of type L)		NE with type-L REM
$[\bar{S}_{11}, \bar{S}_{13}]$	$=$	$[\bar{S}^c_{11}, \bar{S}^c_{12}]$
$\left[p^*_o, q^*_o, p^*_{r_L}, q^*_{r_L}\right]_{11}$	$=$	$[p^*_o, q^*_o, p^*_r, q^*_r]^c_{11}$
$\left[p^*_o, q^*_o, p^*_{r_L}, q^*_{r_L}\right]_{12}$	$=$	$[p^*_o, q^*_o, p^*_r, q^*_r]^c_{12}$
$\left[p^*_o, q^*_o, p^*_{r_L}, q^*_{r_L}\right]_{13}$	$=$	$[p^*_o, q^*_o, p^*_r, q^*_r]^c_{12}$
$\left[p^*_o, q^*_o, p^*_{r_L}, q^*_{r_L}\right]_{14}$	$=$	$[p^*_o, q^*_o, p^*_r, q^*_r]^c_{13}$

Table 4.5: The comparison between the BNE for $(\Theta_o, (\Theta_{r_L}, \Theta_{r_H}))$ with $\theta = 0$ and $\theta = 1$ respectively and the NE for (Θ_o, Θ_r) with type-H REM and type-L REM respectively.

4.5 Sensitivity Analysis (SA)

In the above, we have presented the closed-form NE solution for the remanufacturing competition problem with complete information under both SAS1 and SAS2 as well as the closed-form BNE solution for the remanufacturing competition problem with incomplete information under SAS1. In the following we will do sensitivity analysis in two parts — analytical results based on closed-form solutions and numerical study based on numerical examples.

4.5.1 Analytical Results

We will do this part of sensitivity analysis for the BNE thresholds, optimal prices, and optimal quantities.

BNE Thresholds $\left[\bar{S}_{11}, \bar{S}_{12}, \bar{S}_{13}\right]$

As a preparation for the analysis, we list the two players' optimal remanufacturing quantities q^*_o, $q^*_{r_L}$, and $q^*_{r_H}$ given different number of available shells S in Table 4.6. $\bar{S}_{11}$, $\bar{S}_{12}$ and $\bar{S}_{13}$ are the critical values of S for the OEM, type-H REM and type-L REM respectively, each of which flags the starting point when the relevant player's optimal quantity reaches his/her median demand.

Then we obtain the first derivative of $\bar{S}_{1i}$, $i \in \{1, 2, 3\}$ with respect to the competitors' remanufacturing costs u_o, u_{r_H}, u_{r_L} and the uncertainty factor θ and display the results in Table 4.7.

From Table 4.7, we have several observations:

$S \in$	$[0, \bar{S}_{11})$		$[\bar{S}_{11}, \bar{S}_{12})$		$[\bar{S}_{12}, \bar{S}_{13})$		$[\bar{S}_{13}, \infty)$
$q_o^* =$	S	$\Rightarrow$	D_o^m		D_o^m		D_o^m
$q_{r_H}^* =$	0		$S - D_o^m$	$\Rightarrow$	$D_{r_H}^m$		$D_{r_H}^m$
$q_{r_L}^* =$	0		$S - D_o^m$		$S - D_o^m$	$\Rightarrow$	$D_{r_L}^m$

Table 4.6: The optimal quantities q_o^*, $q_{r_L}^*$, $q_{r_H}^*$ vs the available shells S.

	$\bar{S}_{11}$	$\bar{S}_{12}$	$\bar{S}_{13}$
$\partial\bar{S}_{1*}/\partial u_o$	$-$	$-$	$-$
$\partial\bar{S}_{1*}/\partial u_{r_H}$	0	$-$	$+$
$\partial\bar{S}_{1*}/\partial u_{r_L}$	0	0	$-$
$\partial\bar{S}_{1*}/\partial\theta$	0	0	$-$

Table 4.7: Sensitivity Analysis of the BNE Thresolds S_{1i}, $i \in \{1, 2, 3, 4\}$.

1) $\bar{S}_{1i}/\partial u_o < 0$, $i \in \{1, 2, 3\}$

The first row shows that, when the OEM's remanufacturing unit cost increases, $\bar{S}_{1i}$, $i \in \{1, 2, 3\}$, decreases. We know from Table 4.6 that $\bar{S}_{11}$ flags the number of available shells with which 1): the OEM's optimal quantity q_o^* reaches his median demand D_o^m, and 2) the REM, regardless of her type, starts her remanufacturing production because $q_{r_H}^*$ and $q_{r_L}^*$ become nonzero after q_o^* reaches median demand D_o^m. Hence, a smaller $\bar{S}_{11}$ implies an earlier entrance of the REM into her remanufacturing production. Similarly, a smaller $\bar{S}_{12}$ and a smaller $\bar{S}_{13}$, which are also resulted from a larger u_o, imply that both type-H REM and type-L REM would reach their optimal quantities $q_{r_H}^*$ and $q_{r_L}^*$ at a smaller S value. Therefore, we claim that the OEM could consider to keep his unit remanufacturing cost u_o at a lower level as a competitive strategy to hold back or to slow down the REM's remanufacturing production.

2) As shown in the first column, only $\partial\bar{S}_{11}/\partial u_o < 0$ and the partial derivative of $\bar{S}_{11}$ w.r.t. the other three parameters are all zero. This indicates that $\bar{S}_{11}$ is more sensitive to the OEM's unit cost than to the REM's unit cost and is independent with the uncertainty factor θ. A further implication is that the OEM's uncertainty about the REM's unit cost has no effect on the REM's entrance of remanufacturing production.

3) $\partial\bar{S}_{12}/\partial u_{r_H} < 0$ and $\partial\bar{S}_{13}/\partial u_{r_L} < 0$

Mathematically, $\partial\bar{S}_{12}/\partial u_{r_H} < 0$ means that the higher the type-H REM's remanufacturing cost u_{r_H} is, the lower the value of $\bar{S}_{12}$ would be, and $\partial\bar{S}_{13}/\partial u_{r_L}$ can be interpreted in the similar way. We know that $\bar{S}_{12}$ is the critical value flagging the type-H REM's optimal quantity $q_{r_H}^*$ reaching her median demand $D_{r_H}^m$ and $\bar{S}_{13}$ flagging the type-L REM's $q_{r_L}^*$ reaching her median demand $D_{r_L}^m$. We understand that either a larger u_{r_H} or a larger u_{r_L} implies a less competitive REM. Being less competitive means that the REM of either type would end up with a smaller median demand, i.e., a smaller $\bar{S}_{12}$ for type-H REM or a smaller $\bar{S}_{13}$ for a type-L REM. Hence, it is reasonable to have $\partial\bar{S}_{12}/\partial u_{r_H} < 0$ and $\partial\bar{S}_{13}/\partial u_{r_L} < 0$.

4) $\partial\bar{S}_{13}/\partial u_{r_H} > 0$

It is the cell on the second row and third column. With $\bar{S}_{13}$ marking the type-L REM's reaching her median demand $D^m_{r_L}$ as her optimal quantity $q^*_{r_L}$, having $\partial\bar{S}_{13}/\partial u_{r_H} > 0$ seems counter-intuitive to us in the beginning, because, we have thought that $\bar{S}_{13}$ is irrelevant to u_{r_H}. In order to make it clear we look into further detail of it as

$$\partial\bar{S}_{13}/\partial u_{r_H} = \frac{B_r C_o C_r (1-\theta) + 2B_o B_r C_o (1-\theta)}{2(4B_o B_r - C_o C_r)}. \tag{4.29}$$

From (4.29) we see that, when $\theta = 1$, i.e., when the REM is of type L for sure, $\partial\bar{S}_{13}/\partial u_{r_H}$ becomes zero. This is rational, for, when the OEM knows the REM be of type L with certainty, it amounts to have the type-H REM excluded from this competition. Meanwhile, when $\theta < 1$, or in other words, if only there is any non-zero possibility that the REM is of type H, we still shave $\partial\bar{S}_{13}/\partial u_{r_H} > 0$. Then we realize that the proper explanation of this counter-intuitive phenomenon lies in our formulation. Remember that in our formulation we have only one OEM and one REM. The type-L REM and type-H REM are not two real-existing competitors. But rather, they should be properly understood as two parallel-existing features of the REM. Hence, any change of either feature is also a change of the REM. So it becomes reasonable to have a nonzero $\partial\bar{S}_{13}/\partial u_{r_H}$. Even so, we would still expect $\partial\bar{S}_{13}/\partial u_{r_H}$ to be negative rather than positive, for a higher u_{r_H} implies a less competitive REM and relevantly its value of $\bar{S}_{13}$ should be smaller. The only possible argument we can find to make it right is that, when the REM is considered less competitive with a higher u_{r_H}, it benefits the OEM with a higher median demand D^m_o and weaken the REM herself with a smaller median demand $D^m_{r_H}$. The resulted $\bar{S}_{13}$ would be larger if only the increase of D^m_o is greater than the reduction of $D^m_{r_H}$.

5) As the last row shows, $\partial\bar{S}_{11}/\partial\theta = \partial\bar{S}_{12}/\partial\theta = 0$ and $\partial\bar{S}_{13}/\partial\theta < 0$. It tells that the OEM's uncertainty on the REM's unit remanufacturing cost has no effect on the competition when the available shells are not sufficient to cover the median demand for both the OEM and the type-H REM. What the uncertainty factor θ affects is the amount of available shells at which the type-L REM's median demand could be satisfied. We know that, with $\partial\bar{S}_{13}/\partial\theta < 0$, a higher θ implies a smaller $\bar{S}_{13}$. At the same time, we know that a higher θ value means that the REM is more likely to be a type-L one. Although it is not quite clear to us yet why the type-L REM would end up with a smaller median demand when the OEM is more likely to believe she is a type-L one, we could still use this result to give a rough evaluation for the REM's payoffs.

BNE Prices $\left[p^*_o, p^*_{r_H}, p^*_{r_L}\right]$

In Table 4.8 we display the partial derivatives of the two players' optimal prices w.r.t. their remanufacturing costs and the uncertainty factor individually and make a few interesting observations:

$S \in$	$[0, \bar{S}_{11})$	$[\bar{S}_{11}, \bar{S}_{12})$	$[\bar{S}_{12}, \bar{S}_{13})$	$[\bar{S}_{13}, \infty)$
$\partial p_o^*/\partial u_o$	0	+	+	+
$\partial p_{r_H}^*/\partial u_o$	0	−	+	+
$\partial p_{r_L}^*/\partial u_o$	0	−	+	+
$\partial p_o^*/\partial u_{r_H}$	0	0	+	−
$\partial p_{r_H}^*/\partial u_{r_H}$	0	0	+	+
$\partial p_{r_L}^*/\partial u_{r_H}$	0	0	+	+
$\partial p_o^*/\partial u_{r_L}$	0	0	0	+
$\partial p_{r_H}^*/\partial u_{r_L}$	0	0	0	+
$\partial p_{r_L}^*/\partial u_{r_L}$	0	0	0	+
$\partial p_o^*/\partial \theta$	0	0	+/−	−
$\partial p_{r_H}^*/\partial \theta$	0	0	+/−	−
$\partial p_{r_L}^*/\partial \theta$	0	0	+/−	−

Table 4.8: Sensitivity Analysis of the BNE Prices $[p_o, p_{r_H}, p_{r_L}]$.

i) The OEM's optimal price p_o^* increases when his own unit cost u_o increases and it is the same to the optimal price of REM, regardless of her type, if only the available shells S can satisfy both the OEM and type-H REM's median demand;

ii) When the OEM's unit cost increases, the REM's optimal prices (both $p_{r_H}^*$ and $p_{r_L}^*$) decrease when the quantity of available shell S is not quite large and changes to increase when S is larger. Similarly, when the type-H REM's cost increases, the OEM's optimal price is not affected when S is small, starts to increase when S gets larger, and becomes to decrease when S becomes larger further;

iii) The type-H REM's unit cost starts to affect both players' optimal prices with an S value larger than the S value for the OEM's unit cost. Compared with the OEM and type-H REM's unit costs, the type-L REM's unit cost starts to make effect with an even larger S value;

iv) When the available shells are enough to satisfy all players' median demand, their optimal prices increase as the type-L REM's unit cost increases;

v) The uncertainty factor θ makes no impact on all players' prices when $S \leq \bar{S}_{12}$, i.e., the available shells are not enough to meet both the OEM and type-H REM's median demand. When $S \in [\bar{S}_{12}, \bar{S}_{13})$, i.e., when the available shells are enough to meet both the OEM and type-H REM's median demand but not enough to meet both the OEM and type-L REM's median demand, the impact of the uncertainty factor θ on the players' prices is unclear;

vi) When the available shells can meet all players' median demands, regardless of the REM's type, both the OEM and REM's optimal prices decrease as θ

increases, and the type-L REM and type-H REM share the same change rate. To put in detail, when $S \geq \left[\bar{S}_{13}, \infty\right)$, we have

$$\partial p_o^*/\partial\theta = -\frac{B_r C_o (u_{r_H} - u_{r_L})}{(-C_o C_r + 4B_o B_r)},$$
$$\partial p_{r_H}^*/\partial\theta = \partial p_{r_L}^*/\partial\theta = -\frac{1}{2}\frac{C_o C_r (u_{r_H} - u_{r_L})}{(-C_o C_r + 4B_o B_r)}.$$

BNE Quantities $\left[q_o^*, q_{r_H}^*, q_{r_L}^*\right]$

In Table 4.9 we present the partial derivatives of the two players' optimal quantities w.r.t. their remanufacturing costs and the uncertainty factor respectively and make a few interesting observations as follows:

$S \in$	$\left[0, \bar{S}_{11}\right)$	$\left[\bar{S}_{11}, \bar{S}_{12}\right)$	$\left[\bar{S}_{12}, \bar{S}_{13}\right)$	$\left[\bar{S}_{13}, \infty\right)$
$\partial q_o^*/\partial u_o$	0	$-$	$-$	$-$
$\partial q_{r_H}^*/\partial u_o$	0	$+$	$+$	$+$
$\partial q_{r_L}^*/\partial u_o$	0	$+$	$+$	$+$
$\partial q_o^*/\partial u_{r_H}$	0	0	$+$	$+$
$\partial q_{r_H}^*/\partial u_{r_H}$	0	0	$-$	$-$
$\partial q_{r_L}^*/\partial u_{r_H}$	0	0	$-$	$+$
$\partial q_o^*/\partial u_{r_L}$	0	0	0	$+$
$\partial q_{r_H}^*/\partial u_{r_L}$	0	0	0	$+$
$\partial q_{r_L}^*/\partial u_{r_L}$	0	0	0	$-$
$\partial q_o^*/\partial\theta$	0	0	$+/-$	$-$
$\partial q_{r_H}^*/\partial\theta$	0	0	$+/-$	$-$
$\partial q_o^*/\partial\theta$	0	0	$+/-$	$-$

Table 4.9: Sensitivity Analysis of the BNE Quantities $[q_o, q_{r_H}, q_{r_L}]$.

i) The OEM's unit cost u_o, the type-H REM's unit cost u_{r_H} and the type-L REM's unit cost u_{r_L} start to make effect on the players' optimal quantities when the available shells S is greater than $\bar{S}_{11}$, $\bar{S}_{12}$, and $\bar{S}_{13}$ respectively;

ii) When $S \geq \bar{S}_{11}$ and the OEM's unit cost increases, his own optimal quantity keeps decreasing and the REM's optimal quantities, for both type-L and type-H, keep increasing;

iii) When $S \geq \bar{S}_{12}$ and the type-H REM's unit cost increases, her optimal quantity keeps decreasing, the OEM's optimal quantity increases, and the type-L REM's optimal quantity decreases first and changes to increase when more shells becomes available;

iv) When $S \geq \bar{S}_{13}$ and the type-L REM's unit cost increases, her own optimal quantity decreases, the type-H REM and OEM's optimal quantities increase;

v) The impact of the uncertainty factor θ on the players' optimal quantities shares the same pattern with its impact on the players' optimal prices as we have just discussed above, i.e., no impact when $S \leq \bar{S}_{12}$, unclear impact for $S \in \left[\bar{S}_{12}, \bar{S}_{13}\right)$, and, when $S \geq \left[\bar{S}_{13}, \infty\right)$, all optimal quantities decrease as θ increases and the two types of REM share the same changing rate as

$$\partial q_{r_H}^*/\partial\theta = \partial q_{r_H}^*/\partial\theta = -\frac{1}{2}\frac{B_r C_o C_r (u_{r_L} - u_{r_H})}{(C_o C_r - 4B_o B_r)}.$$

BNE Profits $\left[\pi_o^*, \pi_{r_H}^*, \pi_{r_L}^*\right]$

Based on the above analysis on optimal prices and quantities, we present the partial derivatives of the two players' optimal profits w.r.t. their remanufacturing costs and the uncertainty factor individually in Table 4.10.

$S \in$	$\left[0, \bar{S}_{11}\right)$	$\left[\bar{S}_{11}, \bar{S}_{12}\right)$	$\left[\bar{S}_{12}, \bar{S}_{13}\right)$	$\left[\bar{S}_{13}, \infty\right)$
$\partial\pi_o^*/\partial u_o$	0	+/−	+/−	+/−
$\partial\pi_{r_H}^*/\partial u_o$	0	+/−	+	+
$\partial\pi_{r_L}^*/\partial u_o$	0	+/−	+	+
$\partial\pi_o^*/\partial u_{r_H}$	0	0	+	+/−
$\partial\pi_{r_H}^*/\partial u_{r_H}$	0	0	+/−	+/−
$\partial\pi_{r_L}^*/\partial u_{r_H}$	0	0	+/−	+
$\partial\pi_o^*/\partial u_{r_L}$	0	0	0	+
$\partial\pi_{r_H}^*/\partial u_{r_L}$	0	0	0	+
$\partial\pi_{r_L}^*/\partial u_{r_L}$	0	0	0	+/−
$\partial\pi_o^*/\partial\theta$	0	0	+/−	−
$\partial\pi_{r_H}^*/\partial\theta$	0	0	+/−	−
$\partial\pi_{r_L}^*/\partial\theta$	0	0	+/−	−

Table 4.10: Sensitivity Analysis of BNE Profits $[p_o, p_{r_H}, p_{r_L}]$.

Although not all the signs are clearly observable in Table 4.10, we can still see that:

i) When $S \geq \bar{S}_{12}$, the REM's profits increase as the OEM's unit cost increases;

ii) When $S \geq \bar{S}_{13}$, the OEM and type-H REM's profits increase as the type-L unit cost increases and all players' profits decrease as θ increases.

4.5.2 Numerical Study

In the above, we have given analytical results based on the closed-form BNE solutions under SAS1. In the following we conduct numerical study using Maple 12. Following Majumder and Groenevelt [58], we set both the OEM and REM's demand functions D_o, D_{r_H} and D_{r_L} in three market preference scenarios. For example, when we say the market prefers the OEM to the REM, we mean that: 1) when the OEM and the REM price identically, i.e., $p_o = p_{r_L}$ or $p_o = p_{r_H}$ or $p_o = p_{r_L} = p_{r_H}$, the market demand for the OEM's product is higher than the REM's product; and 2) when the OEM and the REM increase each own price by one unit, the reduction of OEM's market demand is less than the reduction of the REM's market demand. In detail, the three market preference scenarios are:

(1) The market prefers the OEM to the REM and the OEM and the REM's demand functions are:

$$\begin{cases} D_o = 10 - 2p_o + [\theta p_{r_L} + (1-\theta)\, p_{r_H}]\,, \\ D_{r_H} \;= 5 - 3p_{r_H} + p_o, \\ D_{r_L} \;= 5 - 3p_{r_L} + p_o; \end{cases}$$

(2) The market prefers the REM to the OEM and their demand functions are:

$$\begin{cases} D_o = 5 - 3p_o + [\theta p_{r_L} + (1-\theta)\, p_{r_H}]\,, \\ D_{r_H} \;= 10 - 2p_{r_H} + p_o, \\ D_{r_L} \;= 10 - 2p_{r_L} + p_o; \end{cases}$$

(3) The market prefers the OEM and the REM symmetrically and their demand functions are:

$$\begin{cases} D_o = 10 - 2p_o + [\theta p_{r_L} + (1-\theta)\, p_{r_H}]\,, \\ D_{r_H} \;= 10 - 2p_{r_H} + p_o, \\ D_{r_L} \;= 10 - 2p_{r_L} + p_o. \end{cases}$$

We also consider five possible values for each player's unit remanufacturing cost. They are: $u_o \in \{0.2, 0.4, 0.6, 0.8, 1.0\}$ for the OEM, $u_{r_H} \in \{0.3, 0.5, 0.7, 0.9, 1.1\}$ for the type-H REM and $u_{r_L} \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ for the type-L REM. In addition to that, we use eleven values for the uncertainty factor which are $\theta \in$ {0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0}. To facilitate our description, we use subscripts "SYM", "REM", and "OEM" to denote symmetric market, REM preferred market and OEM preferred market, respectively. We provide the numerical results in Table B.1-B.4 in Appendix B.2.7. Observations obtained from the numerical results are as following:

i) The OEM's optimal quantity reaches his median demand with a larger amount of available shells in a symmetric market than in an either REM-preferred or OEM-preferred market, i.e., $S_{1\_SYM} > \max(S_{1\_OEM}, S_{1\_REM})$;

ii) When the REM's expected unit cost — which we calculated as $\theta u_{r_L} + (1-\theta)u_{r_H}$ — is lower than the OEM's unit cost, the type-H REM's optimal quantity reaches her median demand in a REM-preferred market with a smaller amount of available shells than in an OEM-preferred market, i.e., $S_{2\_REM} < S_{2\_OEM}$, but it is the opposite, i.e., $S_{2\_REM} > S_{2\_OEM}$, when the REM's expected unit cost is very close to or even higher than the OEM's unit cost;

iii) The OEM's expected profit is higher if he believes the REM is more likely to have a high unit cost rather than a low unit cost, i.e., $\pi_o(\theta_1) \geqslant \pi_o(\theta_2)$ if $\theta_1 \leqslant \theta_2$;

iv) As the REM's expected unit cost increases, the impact of the uncertainty on the OEM's expected profit is weakened but the impact on the REM's profit, of either type-H or type-L, becomes more significant. In Figure 4.1 we present the fraction of changes in players' (expected) profits when the uncertainty factor θ changes from 0 to 1 when $u_o = 0.8$ and $(u_{r_H}, u_{r_L}) = (0.3, 0.1)$, $(0.5, 0.3)$, $(0.7, 0.5)$ and $(0.9, 0.1)$ respectively. We use this figure to illustrate the impact of the OEM's uncertainty on each player's (expected) profit given different market preference settings.

Each legend in Figure 4.1 is composed of two parts. The "SYM", "REM", and "OEM" before the underline are for symmetric market, REM preferred market and OEM preferred market respectively; the "o", "rH", and "rL" following the underline denote the OEM, type-H REM and type-L REM respectively. For example, "SYM_o" denotes the OEM's data points in a symmetric market. As shown in the right-side figure, the change of the REM's profit, for both type-L and type-H REM, increases most significantly when the OEM is preferred in the market, becomes weaker in a symmetric market and even further weaker in a market that prefers the REM. Moreover, the difference between the changes of type-H REM's profit and the changes of type-L REM's profit is most obvious when the market prefers the OEM, too (see data groups "OEM_rH" and "OEM_rL"), is not quite obvious in a symmetric market (see data groups "SYM_rH" and "SYM_rL"), and even shrink to zero when the market prefers REM herself (see data groups "REM_rH" and "REM_rL"). On the other hand, the change of OEM's profit drops most significantly when the market prefers the REM (see data group "REM_o"), less significantly when the market prefers the OEM himself (see data group "OEM_o"), and even further weak in a symmetric market (see data group "SYM_o") and even further weak . This observation implies that the impact of the OEM's uncertainty on either the OEM himself or the REM his competitor is most significant when the market prefers his/her competitor, while the OEM is the least impacted in a symmetric market but the REM is the least impacted in a market preferring herself.

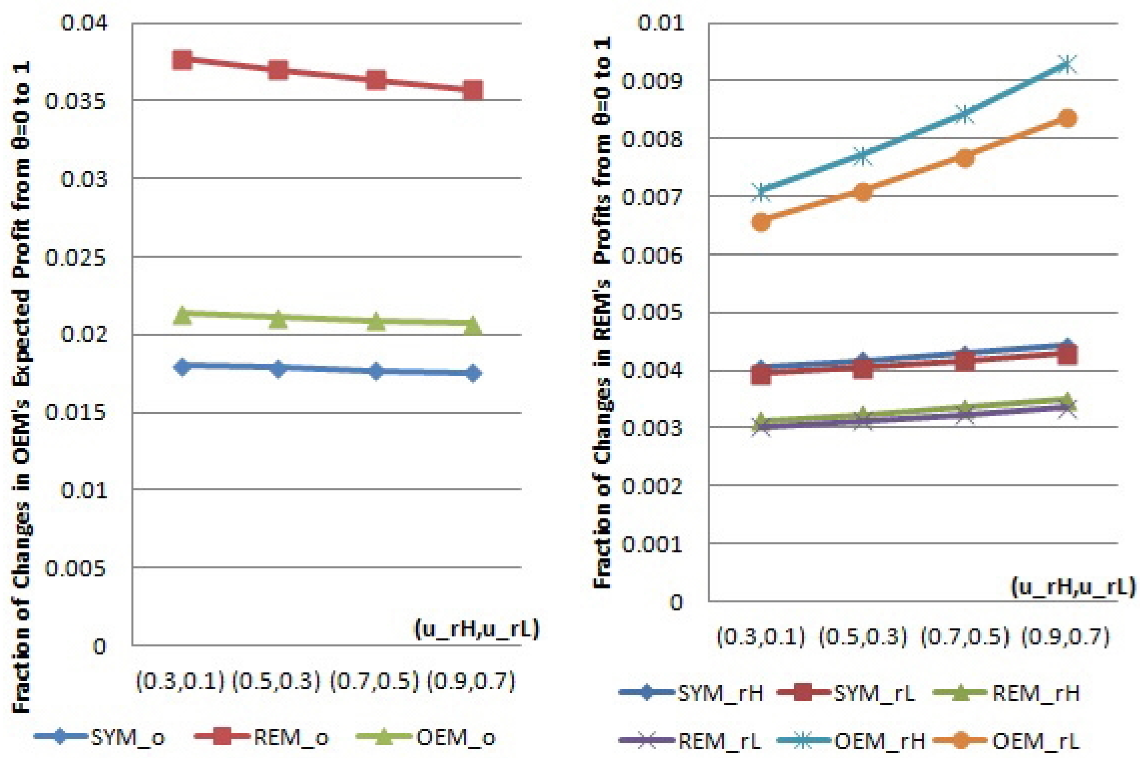

Figure 4.1: Comparison of the impact of the OEM's uncertainty (with $u_o = 0.8$) on both OEM and REM's (expected) profit given different market preference settings

4.5.3 Another Observation

Before closing up this section, we would like to bring forward a special observation we obtained in the process of sensitivity analysis.

To model the OEM's uncertainty, we know that ideally we should assume the REM's unit remanufacturing cost as a random variable following a continuous distribution. But modelling the uncertainty in this way requires integral operations to obtain a closed-form BNE solution and it often becomes intractable when the random variable's probability distribution is not as simple as a uniform distribution. In this chapter, we use a tri-vector $[\theta, u_{r_L}, u_{r_H}]$ to describe the OEM's uncertainty on the REM's unit cost. Actually, this type of tri-vector is also the feature of type-III game models (see Section 3.3, Chapter 3). In this way, we successfully mitigates the difficulty of integral operations which are necessary for modelling with a continuous probability distribution. However, the weakness of using this tri-vector method for uncertainty lies in its limitation on variance analysis, because, when we were doing the sensitivity analysis, we find it incapable to carry out variance analysis with this tri-vector method. Taking a further look at this tri-vector $[\theta, u_{r_L}, u_{r_H}]$, we realize that this tri-vector is actually a discrete distribution with two possible values for the REM's unit cost $u_r \in \{u_{r_L}, u_{r_H}\}$ and relevant probabilities as $P(u = u_{r_L}) = \theta$ and $P(u = u_{r_H}) = 1 - \theta$. This discrete distribution is even simpler than uniform distribution — the simplest continuous distribution. With this in mind, we think that this tri-vector method achieves tractable or closed-form optimal solutions by sacrificing the complicacy of the probability distribution of the REM's unit cost as the price.

4.6 Conclusion and Future Work

4.6.1 Conclusion

In this chapter, we study a remanufacturing competition problem between an OEM and a pure REM where the OEM has uncertainty about the REM's unit cost. We first formulate a static game with complete information for the OEM-REM remanufacturing competition problem with no uncertainty involved and prove the existence and uniqueness of the Nash Equilibrium solution. Then we extend to the static game with incomplete information to formulate the OEM-REM remanufacturing competition problem with the OEM's uncertainty on the REM's unit cost considered. We prove the existence of Bayesian Nash equilibrium to this model and obtain the closed-form BNE solution. By conducting sensitivity analysis with analytical results if possible and numerical study otherwise, we study the impact of such incomplete information, as well as both OEM and REM's remanufacturing unit costs, on their strategic decisions which include optimal prices, optimal remanufacturing quantities, and optimal profits. The results from sensitivity analysis reveal that the existence of such incomplete information won't affect both OEM and REM's strategic decisions when the number of available shells is not sufficient or, to put it in detail, is not large enough to cover both the OEM and type-H REM's median demand. Although the OEM's uncertainty about the REM's unit cost has no effect on the REM's entrance of remanufacturing production, the OEM could consider to keep his unit remanufacturing cost u_o at a lower level as a competitive strategy to hold back or to slow down the REM's entrance of remanufacturing production. When the number of available shells is large enough to meet both players' median demands (regardless of the REM's type), both players' optimal prices decrease as θ increases and the two types of REM share the same change rate. Another interesting insight we obtain is that such uncertainty's impact on either the OEM or REM is the most significant when his/her competitor is preferred in the market and the least when himself/herself is preferred in the market. We also realize that the tri-vector method we use for modelling uncertainty in this chapter, which is also the feature of type-III model in Chapter 3, actually achieves tractable or closed-form optimal solutions at the cost of sacrificing, to a certain degree, the appropriateness and complicacy of uncertainty modelling.

4.6.2 Future Work

The content of this chapter is near to the end, but our research work on the impact of incomplete information on remanufacturing competition needs to be carried on. In the following we list out the research work we plan to do in the future:

1) *BNE uniqueness*

   We have proved the existence of BNE solution under SAS1 and leave its uniqueness open. It would be better if we could prove the uniqueness of the BNE. We have a sense that the uniqueness can be proved by applying fixed-point theorem. We will work it out later.

2) *BNE under SAS2*

   We have done analysis on the BNE under SAS1, i.e., with the OEM has priority to access available shells. As we have discussed in Section 4.1, remanufacturing of a product often starts at smaller-sized local remanufacturing firms. Hence, it would be valuable to conduct BNE analysis under SAS2, i.e., with the REM has priority to access available shells. However, when the REM has the priority, the existence of incomplete information makes difference under SAS2. That is, we have $S_o = S - [\theta q_{r_L} + (1-\theta) q_{r_H}]$ in the setting of incomplete information while in the setting of complete information, we have $S_o = S - q_r$.

3) *Extension to a two-stage game*

   We have embedded the OEM's uncertainty on the REM's unit cost in a static game in which both the OEM and REM carry pure remanufacturing activity and the number of available shells S is set as a exogenously given parameter.

However, it is more practical to consider both original manufacturing and remanufacturing activities for the OEM. As Majumder and Groenevelt have done in [58], we can have the OEM do pure original manufacturing activity in the first stage, the OEM-REM competition in the second stage, and have the available shells for the second stage relevant with the OEM's original manufacturing activity in the first stage. With the closed-form BNE solution we have obtained in this chapter, we believe that embedding the OEM's uncertainty on the REM into a two-stage game, such as a Stackelberg game, would be a feasible and interesting research work to do.

4) *Uncertainty in market information*

We have assumed that the OEM's uncertainty lies on the REM's unit remanufacturing cost u_r and from the numerical results we have noticed the effect of market preference on the remanufacturing competition. This has inspired us with a new thought. That is, it may also be both interesting and feasible to study the impact of the OEM's uncertainty on the REM's market information by similar formulation. To put it in other words, we should consider the situation where the OEM may not know the REM's demand function parameters (A_r, B_r, C_r) with certainty. It will be interesting to know how such uncertainty may impact the competition behavior between the OEM and REM. Moreover, we notice that with the setting of uncertainty on (A_r, B_r, C_r), the market preference, which has been assumed in this chapter as exogenously given parameters, will become an endogenously given information. We believe the formulation and discussion will be very interesting.

5) *Other extensions*

We could also consider to apply the same modelling strategy to study the contracting behavior between OEMs and local remanufacturers. To do this we need to consider to apply cooperative game models (Shapley [82]).

Chapter 5

Concluding Remarks

5.1 Thesis Summary

Reverse logistics (RL) has been drawing increasing attention from both industrial practitioners and academic researchers. Industrial practitioners expect to obtain beneficial managerial insights to improve business performance. Academic researchers endeavor to reveal valuable insights by using proper modelling tools to open problems in RL. In this thesis, we have made contributions to RL with focus on customer returns, remanufacturing competition, and game theoretic applications.

In Chapter 2 we addressed a newsvendor problem with resalable customer returns and investigated the timing effect of both customer demand and customer returns on the newsvendor's inventory management. We first developed a basic model for a three-subperiod newsvendor problem with customer returns in which we have order quantity Q as the unique decision variable. With this basic model we investigated the concavity of the retailer's total expected profit in regards of order quantity Q. Then we further developed a general model in which, in addition to the order quantity Q, we have also two inventory thresholds Y_2 and Y_3 as decision variables. These two thresholds can be considered as an easy-to-follow reference for inventory control by a retailer who would like to have the option to return certain inventory to the supplier during the single period. With this general model, we conducted simulation and studied the timing effect of the portions of demand $[\alpha_1, \alpha_2, \alpha_3]$ and customer returns β on the retailer's inventory policy and obtained three major observations. Firstly, we observed that, in general, the earlier the major demand occurs, the lower the order quantity Q^* would be though the reduction is not quite significant. Also, for earlier and large demand, the expected total profit P^* is higher and the inventory level thresholds Y_2^* and Y_3^* are lower. This observation implies that, if there is opportunity for a retailer to manipulate the demand pattern, it's better to move major demand to an earlier subperiod so that the expected total profit can be improved accordingly. Secondly, we observed that, if the portions of demand $[\alpha_1, \alpha_2, \alpha_3]$ are "V" shaped, the order quantity Q^* won't be much different from the case with portions of demand evenly distributed but the total expected profit P^* decreases as the "V" valley goes deeper. From this observation we claimed that, if the portions of demand are estimated to be in a "V" shape, efforts to lessen the valley would benefit a retailer with a higher profit in case the cost of such an effort can be well balanced. Thirdly and also lastly in this chapter, regarding customer return pattern β, we found: 1) the expected total profit P^* drops significantly as the total return rate for sale in each subperiod increases, so we encouraged retailers to make effort to reduce customer returns; 2) since earlier customer returns result in a higher total expected profit, it would be better for the retailer to encourage customer to return, if they want to, as early as possible.

In Chapter 3 we explored the application of game theoretic models with incomplete information in inventory management, an essential operations management facet playing an important role in both supply chain management (SCM) and RL. We first presented a brief review of the static and dynamic games under complete information and illustrated the application of these two games in inventory management by using a single-period stochastic inventory problem with two competing newsvendors. Then we illustrated the Bayesian Nash equilibrium (BNE) solution concept for the static games under incomplete information with two competing newsvendors. In particular, we made the extension to a situation where both competitors have incomplete information on each other and obtained the BNE accordingly. At last, we illustrated the perfect Bayesian equilibrium (PBE) solution concept for the dynamic games under incomplete information with two competing newsvendors where the decision made by a newsvendor in the first stage is assumed as a discrete variable while the decision made by the other newsvendor in the second stage is continuous. We believe that the expository nature of this chapter may help researchers in inventory/SCM/RL gain easy access to the complicated notions related to the games played under incomplete information.

In Chapter 4, regarding remanufacturing — another important aspect in RL, we investigated the effect of uncertainty in remanufacturing competition between an original equipment manufacturer (OEM) and a pure remanufacturer (REM) in which the uncertainty lies in the OEM's information as to the REM's unit remanufacturing cost. We first formulated a static game with complete information for the OEM-REM remanufacturing competition problem with no uncertainty involved and prove the existence and uniqueness of the Nash equilibrium solution. Then we applied the type-III model in Chapter 3 to formulate the OEM-REM remanufacturing competition problem in which the OEM has uncertainty on the REM's unit cost. We proved the existence of BNE for this type-III game and obtained the closed-form BNE solution to this model. We conducted sensitivity analysis analytically, if possible, and numerically otherwise. The results reveal that the existence of such uncertainty won't affect both the OEM and REM's strategic decisions when the number of available shells is not large enough to cover both the OEM and type-H REM's median demand. Also we found that, although the OEM's uncertainty has no effect on the REM's entrance of remanufacturing production, the OEM could consider to keep his unit remanufacturing cost u_o at a lower level as a competitive strategy to hold back or to slow down the REM's entrance of remanufacturing production. Meanwhile, when the number of available shells is large enough to meet both players' median demands (regardless of the REM's type), both players' optimal prices decrease as θ increases and the REM's decreasing rate is independent with her type. Another interesting insight we obtained is that such uncertainty's impact on either the OEM or REM is the most significant when his/her competitor is preferred in the market and the least when himself/herself is preferred in the market.

5.2 Our Contributions

Here we would like to briefly summarize the major contributions we have made to the literature of RL in this thesis:

Regarding the field of RL, we bring forward a proper ordering of RL and a few other new terms related with SCM per their research scope.

Regarding a retailer's inventory management with resalable returns, we: (1) make the first effort investigating timing effects of customer demand and returns on a retailer's inventory management; (2) take both customer returns and return-to-supplier into consideration; (3) provide an easy-to-follow inter-period inventory management strategy for a retailer facing high volume of customer returns; and (4) obtain a few interesting managerial insights as to the profit performance of the retailer facing high volume of customer returns. Specifically, the retailer may obtain higher profits by having portions of demand (α) evenly distributed, decreasing customer return rates, or encouraging customers for early rather than late returns.

Regarding (non-cooperative) games with incomplete information, we: (1) provide a simplified exposition of applications of games of incomplete information to stochastic inventory management; and (2) present explicit methods for modelling games with incomplete information and computing corresponding equilibriums.

Regarding competition in remanufacturing with incomplete information, we: (1) study the competition in remanufacturing by considering the competitor's incomplete information on each other's feature information; (2) obtain the closed-form of Bayesian Nash equilibrium for the game model we construct; and (3) we provide a few crucial insights on the impact of incomplete information on remanufacturing competition. In detail, they are: (a) such incomplete information makes no impact when accessible shells are not sufficient; (b) the impact of such incomplete information on the REM is most significant in an OEM-preferred market and least in a REM-preferred market; and (c) the impact of such incomplete information on the OEM is most significant in an REM-preferred market but least in a symmetric market.

5.3 Thoughts for Future Work

Our thesis focuses on customer returns and remanufacturing — two important aspects in RL — with application of non-cooperative game theoretic models. As we have presented some future work regarding customer returns and remanufacturing at the end of Chapters 2 and 4 respectively, here we give a general direction for potential research regarding the application of non-cooperative game theoretic models with incomplete information in RL.

As discussed in Chapter 3, static and dynamic games with complete information are two game models widely applied in SCM and Nash equilibrium and Stackelberg equilibrium are the two corresponding solution concepts. Applications of static games with incomplete information in SCM are relatively limited. It becomes even scarce as to the application of dynamic games with incomplete information.

It has been well known that high degree of uncertainty is a particularity existing in multiple stages RL involves. As discussed in Chapter 2, retailers in the upstream directly face uncertainty in customer demand, quantity and quality of used products returned by customers, and ratio and pattern of customer returns. Furthermore, along with the increasing application of return policy between retailers and suppliers, the uncertainties faced by retailers are passed down to distributors or suppliers in the midstream. Also as discussed in Chapter 4, remanufacturing operations in the downstream experience high variability, too. Such uncertainty backward chain effect is similar to the well-known "bullwhip effect" in a forecast-driven distribution channel which refers to a trend of increasingly large swings in inventory in response to demand uncertainty as one looks backward at firms along the supply chain of a product. In summary, RL involves intensive interactions, either non-cooperative or cooperative, between actors scattered along the backward material flow in the presence of uncertainty.

We believe that static and dynamic games with incomplete information would find broader applications in RL analysis because of the intensive interactions as well as the high degree of uncertainty. The static game with incomplete information would be sufficient for cases when acting sequence between players is not an important issue. Otherwise, the dynamic game (with incomplete information) would be more appropriate to be applied. Since the complicacy of finding a PBE for a continuous game is equivalent to obtain solution to a decision variable which is a function of a variable rather than a numerical value, the space for applications of dynamic games would be limited though. However, there is still hope. As we have shown in Section 3.4, Chapter 3, finding a PBE for a dynamic game with incomplete information becomes possible if only the decision(s) to be made in the first stage can be described with a discrete variable, such as a limited set of marketing strategies or a limited number of possible costs or payoffs. Hence, as a general direction for potential research in the future, we believe that game theoretic models with incomplete information should find more extensive applications in the domain of RL.

Appendix A

Proofs for Chapter 2

In this Appendix, we provide proofs to the Lemmas in Chapter 2, the Newsvendor problem with resalable returns.

A.1 Proof of Lemma 2.1

Given the detail expressions of the first subperiod profit Π_1 in (2.2), we obtain the expected subperiod profit J_1 as

$$J_1 = E\left(\Pi_1\right) = \alpha_1 s \int_0^{z_1} x f\left(x\right) dx - \alpha_1 p \int_{z_1}^{\infty} x f\left(x\right) dx + \left(s+p\right) Q \int_{z_1}^{\infty} f\left(x\right) dx - cQ,$$

where $z_1 = Q/\alpha_1$. It follows that the first derivative of J_1 is

$$\frac{dJ_1}{dQ} = \left(s+p\right)\left[1 - F\left(z_1\right)\right] - c. \tag{A.1}$$

The values of the first derivative dJ_1/dQ at two ending points, one at $Q = 0$ and the other at $Q = \infty$, are respectively

$$\left.\frac{dJ_1}{dQ}\right|_{Q=0} = s + p - c > 0 \text{ and } \left.\frac{dJ_1}{dQ}\right|_{Q=\infty} = -c < 0.$$

Furthermore, the second derivative of J_1 is

$$\frac{d^2 J_1}{dQ^2} = -\frac{s+p}{\alpha_1} f\left(z_1\right) < 0.$$

Therefore, Lemma 2.1 is proved. ■

A.2 Proof of Lemma 2.2

Given the detail expressions of the second subperiod profit Π_2 in (2.3), we first obtain the relevant expected profit J_2 as

$$\begin{aligned} J_2 = E(\Pi_2) = & \int_0^{I_2/\alpha_2} [s\alpha_2 x - (s+b) R_2] f(x)\, dx \\ & + \int_{I_2/\alpha_2}^{\infty} [sI_2 - p(\alpha_2 x - I_2) - (s+b)R_2] f(x)\, dx, \end{aligned}$$

where I_2 and R_2 (as shown in (2.1)) are dependent on the relationship between Q and $\alpha_1 x$. From (2.1), for $x \in [0, Q/\alpha_1)$ we have intermediate variables as

$$\left\{ \begin{array}{l} R_2 = \beta_{12}\alpha_1 x, \\ I_2 = Q - (1-\beta_{12})\alpha_1 x, \\ S_2 = \min(\alpha_2 x, Q - (1-\beta_{12})\alpha_1 x), \end{array} \right.$$

and the subperiod profit as

$$\pi_2 = \left\{ \begin{array}{ll} s(\alpha_2 - \alpha_1\beta_{12})x, & x \in [0, Q/(\alpha_1 + \alpha_2 - \alpha_1\beta_{12})) \\ (s+p)(Q - \alpha_1 x) + p(\alpha_1\beta_{12} - \alpha_2)x, & x \in [Q/(\alpha_1 + \alpha_2 - \alpha_1\beta_{12}), \infty). \end{array} \right.$$

Similarly, for $x \in [Q/\alpha_1, \infty)$ we have intermediate variables as

$$\left\{ \begin{array}{l} R_2 = \beta_{12} Q, \\ I_2 = \beta_{12} Q, \\ S_2 = \min(\alpha_2 x, \beta_{12} Q), \end{array} \right.$$

and the subperiod profit as

$$\pi_2 = \left\{ \begin{array}{ll} s(\alpha_2 - \alpha_1\beta_{12})x, & x \in [0, Q/(\alpha_1 + \alpha_2 - \alpha_1\beta_{12})) \\ (s+p)(Q - \alpha_1 x) + p(\alpha_1\beta_{12} - \alpha_2)x, & x \in [Q/(\alpha_1 + \alpha_2 - \alpha_1\beta_{12}), \infty). \end{array} \right.$$

Also we know that when $\alpha_1\beta_{12} \le \alpha_2$, we have

$$\alpha_1 + \alpha_2 - \alpha_1\beta_{12} \ge \alpha_1 \text{ and } \frac{Q}{\alpha_1 + \alpha_2 - \alpha_1\beta_{12}} \le \frac{Q}{\alpha_1},$$

and when $\alpha_1\beta_{12} \ge \alpha_2$, we have

$$\alpha_1 \ge \alpha_2/\beta_{12} \text{ and } Q/\alpha_1 \le \beta_{12}Q/\alpha_2.$$

Therefore, we reorganize the above expressions and rewrite the second subperiod profit as:

If $\alpha_1\beta_{12} \leq \alpha_2$, then

$$\pi_2 = \begin{cases} s\left(\alpha_2 - \alpha_1\beta_{12}\right)x, & x \in \left[0, Q/\left(\alpha_1+\alpha_2-\alpha_1\beta_{12}\right)\right) \\ \left(s+p\right)\left(Q-\alpha_1 x\right) + p\left(\alpha_1\beta_{12}-\alpha_2\right)x, & x \in \left[Q/\left(\alpha_1+\alpha_2-\alpha_1\beta_{12}\right), Q/\alpha_1\right) \\ p\left(\beta_{12}Q - \alpha_2 x\right), & x \in \left[Q/\alpha_1, \infty\right); \end{cases}$$

and if $\alpha_1\beta_{12} \geq \alpha_2$, then

$$\pi_2 = \begin{cases} s\left(\alpha_2 - \beta_{12}\alpha_1\right)x, & x \in \left[0, Q/\alpha_1\right) \\ s\left(\alpha_2 x - \beta_{12}Q\right), & x \in \left[Q/\alpha_1, \beta_{12}Q/\alpha_2\right) \\ p\left(\beta_{12}Q - \alpha_2 x\right), & x \in \left[\beta_{12}Q/\alpha_2, \infty\right). \end{cases}$$

Correspondingly, the two possible expressions of the second subperiod expected profit (which we denote respectively as J_2^1 and J_2^2) are

$$\begin{aligned} J_2^1 = {} & \int_0^{z_2} s\left(\alpha_2-\alpha_1\beta_{12}\right)xf\left(x\right)dx + \int_{z_1}^{\infty} p\left(\beta_{12}Q-\alpha_2 x\right)f\left(x\right)dx \\ & + \int_{z_2}^{z_1}\left[\left(s+p\right)Q - s\alpha_1 x - p\left(\alpha_1+\alpha_2-\alpha_1\beta_{12}\right)x\right]f\left(x\right)dx, \qquad (\alpha_1\beta_{12} \leq \alpha_2) \\ J_2^2 = {} & \int_0^{z_1} s\left(\alpha_2-\alpha_1\beta_{12}\right)xf\left(x\right)dx + \int_{z_1}^{z_3} s\left(\alpha_2 x-\beta_{12}Q\right)f\left(x\right)dx \\ & + \int_{z_3}^{\infty} p\left(\beta_{12}Q-\alpha_2 x\right)f\left(x\right)dx, \qquad (\alpha_1\beta_{12} \geq \alpha_2) \end{aligned}$$

where $z_1 = Q/\alpha_1$, $z_2 = Q/\left(\alpha_1+\alpha_2-\alpha_1\beta_{12}\right)$, and $z_3 = \beta_{12}Q/\alpha_2$. The first derivatives of J_2^1 and J_2^2 are

$$\frac{dJ_2^1}{dQ} = \left(s+p\right)\left(F\left(z_1\right)-F\left(z_2\right)\right) + p\beta_{12}\left(1-F\left(z_1\right)\right), \ (\alpha_1\beta_{12} \leq \alpha_2); \text{ and} \tag{A.2}$$

$$\frac{dJ_2^2}{dQ} = -s\beta_{12}\left(F\left(z_3\right)-F\left(z_1\right)\right) + p\beta_{12}\left(1-F\left(z_3\right)\right), \ (\alpha_1\beta_{12} \geq \alpha_2). \tag{A.3}$$

Then the values of dJ_2^1/dQ and dJ_2^2/dQ at two ending points, one at $Q=0$ and the other at $Q=\infty$, can be found as:

$$\left.\frac{dJ_2}{dQ}\right|_{Q=0} = \left.\frac{dJ_2^1}{dQ}\right|_{Q=0} = \left.\frac{dJ_2^2}{dQ}\right|_{Q=0} = p\beta_{12} > 0,$$

$$\left.\frac{dJ_2}{dQ}\right|_{Q=\infty} = \left.\frac{dJ_2^1}{dQ}\right|_{Q=\infty} = \left.\frac{dJ_2^2}{dQ}\right|_{Q=\infty} = 0.$$

Lemma 2.2 is thus proved. ■

To prepare for the proof of Lemma 2.4, we also obtain the second derivative of J_2^1

and J_2^2 as

$$\frac{d^2 J_2^1}{dQ^2} = \frac{s + p\left(1 - \beta_{12}\right)}{\alpha_1} f\left(z_1\right) - \frac{s + p}{\alpha_1 + \alpha_2 - \alpha_1\beta_{12}} f\left(z_2\right), \ \left(\alpha_1\beta_{12} \le \alpha_2\right); \text{ and} \quad \text{(A.4)}$$

$$\frac{d^2 J_2^2}{dQ^2} = \frac{s\beta_{12}}{\alpha_1} f\left(z_1\right) - \frac{\left(s + p\right)\beta_{12}^2}{\alpha_2} f\left(z_3\right), \ \left(\alpha_1\beta_{12} \ge \alpha_2\right). \quad \text{(A.5)}$$

A.3 Proof to Lemma 2.3

By applying the same method as we have used for the second subperiod expected profit J_2, we can yield the third subperiod expected profit J_3. We have seen that J_2 has two possible expressions J_2^1 and J_2^2 (J_2^1 for $\alpha_1\beta_{12} \le \alpha_2$ and J_2^2 for $\alpha_1\beta_{12} \ge \alpha_2$). Having more complicated parameter settings for the third subperiod, we find six possible expressions for J_3 — the third subperiod expected profit. We denote them as J_3^k ($k = 1, \cdots, 6$) and present as follows.

First we need to define a group of intermediate parameters θ_k, $k = 1, \cdots, 5$, and three groups of intermediate variables, y_ℓ $\ell = 1, \cdots, 4$, z_m, $m = 1, 2, 3$, and a_n, $n = 1, \cdots, 10$. Each θ_k, $k = 1, \cdots, 5$, is a combination of portion of demand α_i and customer return rates β_{ij}; each y_ℓ, $\ell = 1, \cdots, 4$, and z_m, $m = 1, 2, 3$, is a function of the decision variable Q; and each a_n, $n = 1, \cdots, 10$, is a function of both the order quantity Q and the random demand X. In detail they are:

$$\begin{aligned}
\theta_1 &= \alpha_1\beta_{12} - \alpha_2, \\
\theta_2 &= \alpha_1\beta_{13} + \alpha_2\beta_{23} - \alpha_3, \\
\theta_3 &= \alpha_1\left(\beta_{13} + \beta_{12}\beta_{23}\right) - \alpha_3, \\
\theta_4 &= \beta_{13} + \beta_{12}\beta_{23} - \beta_{23}, \\
\theta_5 &= \alpha_2\left(\beta_{13} + \beta_{12}\beta_{23}\right) - \alpha_3\beta_{12};
\end{aligned} \quad \text{(A.6)}$$

$$\begin{aligned}
y_1 &= \frac{Q}{1 - \alpha_1\left(\beta_{12} + \beta_{13}\right) - \alpha_2\beta_{23}}, \\
y_2 &= \frac{\beta_{23}Q}{\alpha_3 - \alpha_1\left(\beta_{13} + \beta_{12}\beta_{23} - \beta_{23}\right)}, \\
y_3 &= \frac{\left(\beta_{12}\beta_{23} + \beta_{13}\right)Q}{\alpha_3}, \\
y_4 &= \frac{\left(\beta_{12} + \beta_{23}\right)Q}{\alpha_2\left(1 - \beta_{23}\right) + \alpha_3};
\end{aligned}$$

Case	Condition	J_3	Notation
1	$\theta_1 \leq 0,\ \theta_2 \leq 0$	$\int_0^{y_1} a_1 f(x)\,dx + \int_{y_1}^{z_2} a_2 f(x)\,dx + \int_{z_2}^{z_1} a_4 f(x)\,dx + \int_{z_1}^{\infty} a_6 f(x)\,dx$	J_3^1
2	$\theta_1 \leq 0,\ \theta_2 \geq 0,\ \theta_3 \geq 0$	$\int_0^{z_2} a_1 f(x)\,dx + \int_{z_2}^{z_1} a_3 f(x)\,dx + \int_{z_1}^{y_3} a_5 f(x)\,dx + \int_{y_3}^{\infty} a_6 f(x)\,dx$	J_3^2
3	$\theta_1 \leq 0,\ \theta_2 \geq 0,\ \theta_3 \leq 0$	$\int_0^{z_2} a_1 f(x)\,dx + \int_{z_2}^{y_2} a_3 f(x)\,dx + \int_{y_2}^{z_1} a_4 f(x)\,dx + \int_{z_1}^{\infty} a_6 f(x)\,dx$	J_3^3
4	$\theta_1 \geq 0,\ \theta_5 \geq 0$	$\int_0^{z_1} a_7 f(x)\,dx + \int_{z_1}^{z_3} a_9 f(x)\,dx + \int_{z_3}^{y_3} a_5 f(x)\,dx + \int_{y_3}^{\infty} a_6 f(x)\,dx$	J_3^4
5	$\theta_1 \geq 0,\ \theta_5 \leq 0,\ \theta_1 + \theta_2 \leq 0$	$\int_0^{y_1} a_7 f(x)\,dx + \int_{y_1}^{z_1} a_8 f(x)\,dx + \int_{z_1}^{z_3} a_{10} f(x)\,dx + \int_{z_3}^{\infty} a_6 f(x)\,dx$	J_3^5
6	$\theta_1 \geq 0,\ \theta_5 \leq 0,\ \theta_1 + \theta_2 \geq 0$	$\int_0^{z_1} a_7 f(x)\,dx + \int_{z_1}^{y_4} a_9 f(x)\,dx + \int_{y_4}^{z_3} a_{10} f(x)\,dx + \int_{z_3}^{\infty} a_6 f(x)\,dx$	J_3^6

Table A.1: The six possible expressions of J_3 – the third subperiod expected profit.

$$
\begin{aligned}
z_1 &= \frac{Q}{\alpha_1}, \\
z_2 &= \frac{Q}{\alpha_1 + \alpha_2 - \alpha_1\beta_{12}}, \\
z_3 &= \frac{\beta_{12}Q}{\alpha_1};
\end{aligned}
$$

and

$$
\begin{aligned}
a_1 &= vQ + v\left[\alpha_1\left(\beta_{12} + \beta_{13}\right) + \alpha_2\beta_{23} - 1\right] + s\left(\alpha_3 - \alpha_1\beta_{13} - \alpha_2\beta_{23}\right)x, \\
a_2 &= \left(s + p\right)Q + s\left[\alpha_1\left(\beta_{12} - 1\right) - \alpha_2\right]x + p\left[\alpha_1\left(\beta_{12} + \beta_{13}\right) + \alpha_2\beta_{23} - 1\right]x, \\
a_3 &= \left(v - s\right)\beta_{23}Q + \left(v - s\right)\left[\alpha_1\left(\beta_{13} + \beta_{12}\beta_{23} - \beta_{23}\right) - \alpha_3\right]x, \\
a_4 &= p\beta_{23}Q + p\left[\alpha_1\left(\beta_{13} + \beta_{12}\beta_{23} - \beta_{23}\right) - \alpha_3\right]x, \\
a_5 &= \left(v - s\right)\left(\beta_{12}\beta_{23} + \beta_{13}\right)Q - \left(v - s\right)\alpha_3 x, \\
a_6 &= p\left(\beta_{12}\beta_{23} + \beta_{13}\right)Q - p\alpha_3 x, \\
a_7 &= \left(v - s\beta_{13}\right)Q + s\left(\alpha_3 - \alpha_2\beta_{23}\right)x + v\left[\alpha_1\left(\beta_{12} + \beta_{13}\right) + \alpha_2\beta_{23} - 1\right]x, \\
a_8 &= \left[s\left(1 - \beta_{13}\right) + p\right]Q + s\left[\alpha_1\left(\beta_{12} + \beta_{13} - 1\right) - \alpha_2\right]x + p\left[\alpha_1\left(\beta_{12} + \beta_{13}\right) + \alpha_2\beta_{23} - 1\right]x, \\
a_9 &= \left[v\left(\beta_{12} + \beta_{13}\right) - s\beta_{13}\right]Q + s\left(\alpha_3 - \alpha_2\beta_{23}\right)x + v\left[\alpha_2\left(\beta_{23} - 1\right) - \alpha_3\right]x, \\
a_{10} &= \left[s\beta_{12} + p\left(\beta_{12} + \beta_{13}\right)\right]Q - s\alpha_2 x + p\left[\alpha_2\left(\beta_{23} - 1\right) - \alpha_3\right]x.
\end{aligned}
$$

Given the above intermediate parameters and variables, we present the six possible expressions of J_3 — the expected profit of the third subperiod — in Table A.1.

Correspondingly, the six possible expressions for the first derivative of J_3 are

$$\begin{aligned}\frac{dJ_3^1}{dQ} &= vF(y_1) + (s+p)(F(z_2) - F(y_1)) \\ &\quad + p\beta_{23}(F(z_1) - F(z_2)) + p(\beta_{12}\beta_{23} + \beta_{13})(1 - F(z_1)), \qquad \text{(A.7)} \\ \frac{dJ_3^2}{dQ} &= vF(z_2) + (v-s)\beta_{23}(F(z_1) - F(z_2)) \\ &\quad + \frac{v-s}{\beta_{12}\beta_{23} + \beta_{13}}(F(y_3) - F(z_1)) + p(\beta_{12}\beta_{23} + \beta_{13})(1 - F(y_3)), \qquad \text{(A.8)} \\ \frac{dJ_3^3}{dQ} &= vF(z_2) + (v-s)\beta_{23}(F(y_2) - F(z_2)) \\ &\quad + p\beta_{23}(F(z_1) - F(y_2)) + p(\beta_{12}\beta_{23} + \beta_{13})(1 - F(z_1)), \qquad \text{(A.9)} \\ \frac{dJ_3^4}{dQ} &= (v - s\beta_{13})F(z_1) + [v(\beta_{12} + \beta_{23}) - s\beta_{13}](F(z_3) - F(z_1)) \\ &\quad + (v-s)(\beta_{12}\beta_{23} + \beta_{13})(F(y_3) - F(z_3)) + p(\beta_{12}\beta_{23} + \beta_{13})(1 - F(y_3)), \qquad \text{(A.10)} \\ \frac{dJ_3^5}{dQ} &= (v - s\beta_{13})F(y_1) + [s(1 - \beta_{13}) + p](F(z_1) - F(y_1)) \\ &\quad + [(s+p)\beta_{12} + p\beta_{13}](F(z_3) - F(z_1)) + p(\beta_{12}\beta_{23} + \beta_{13})(1 - F(z_3)), \qquad \text{(A.11)} \\ \frac{dJ_3^6}{dQ} &= (v - s\beta_{13})F(z_1) + [v(\beta_{12} + \beta_{23}) - s\beta_{13}](F(y_4) - F(z_1)) \\ &\quad + [s\beta_{12} + p(\beta_{12} + \beta_{13})](F(z_3) - F(y_4)) + p(\beta_{12}\beta_{23} + \beta_{13})(1 - F(z_3)). \qquad \text{(A.12)}\end{aligned}$$

Then it follows that the possible values of dJ_3/dQ at two ending points, one at $Q = 0$ and the other at $Q = \infty$, are:

When $Q = 0$,

$$\left.\frac{dJ_3}{dQ}\right|_{Q=0} = \left.\frac{dJ_3^1}{dQ}\right|_{Q=0} = \left.\frac{dJ_3^2}{dQ}\right|_{Q=0} = \cdots = \left.\frac{dJ_3^6}{dQ}\right|_{Q=0} = p(\beta_{12}\beta_{23} + \beta_{13}) > 0$$

and when $Q = \infty$,

$$\begin{aligned}\left.\frac{dJ_3}{dQ}\right|_{Q=\infty} &= \left.\frac{dJ_3^1}{dQ}\right|_{Q=\infty} = \left.\frac{dJ_3^2}{dQ}\right|_{Q=\infty} = \left.\frac{dJ_3^3}{dQ}\right|_{Q=\infty} = v > 0,\ (\theta_1 \leq 0), \text{ and} \\ \left.\frac{dJ_3}{dQ}\right|_{Q=\infty} &= \left.\frac{dJ_3^4}{dQ}\right|_{Q=\infty} = \left.\frac{dJ_3^5}{dQ}\right|_{Q=\infty} = \left.\frac{dJ_3^6}{dQ}\right|_{Q=\infty} = v - s\beta_{13},\ (\theta_1 \geq 0).\end{aligned}$$

We have $v - s\beta_{13} \leq 0$ if $v/s \leq \beta_{13}$ and $\alpha_1\beta_{12} \geq \alpha_2$. Therefore, the proof to Lemma 2.3 is completed. ■

Also to prepare for the proof of Lemma 2.4, we obtain the six possible expressions for the second derivative of J_3 as

$$\frac{d^2J_3^1}{dQ^2} = \frac{p(\beta_{23}-\beta_{12}\beta_{23}-\beta_{13})}{\alpha_1} f(z_1) + \frac{s+p(1-\beta_{23})}{\alpha_1+\alpha_2-\alpha_1\beta_{12}} f(z_2) + \frac{(v-s-p)}{1-\alpha_1(\beta_{12}+\beta_{13})-\alpha_2\beta_{23}} f(y_1), \tag{A.13}$$

$$\frac{d^2J_3^2}{dQ^2} = \frac{(v-s)(\beta_{23}-\beta_{12}\beta_{23}+\beta_{13})}{\alpha_1} f(z_1) + \frac{v(1-\beta_{23})+s\beta_{23}}{\alpha_1+\alpha_2-\alpha_1\beta_{12}} f(z_2) + \frac{(\beta_{12}\beta_{23}+\beta_{13})(v-s-p)}{\alpha_3} f(y_3), \tag{A.14}$$

$$\frac{d^2J_3^3}{dQ^2} = \frac{p(\beta_{23}-\beta_{12}\beta_{23}-\beta_{13})}{\alpha_1} f(z_1) + \frac{v(1-\beta_{23})+s\beta_{23}}{\alpha_1+\alpha_2-\alpha_1\beta_{12}} f(z_2) + \frac{(v-s-p)\beta_{23}^2}{\alpha_3-\alpha(\beta_{13}+\beta_{12}\beta_{23}-\beta_{23})} f(y_2), \tag{A.15}$$

$$\frac{d^2J_3^4}{dQ^2} = \frac{v(1-\beta_{12}-\beta_{23})}{\alpha_1} f(z_1) + \frac{(s\beta_{12}\beta_{23}-v\beta_{13})\beta_{12}}{\alpha_1} f(z_3) + \frac{(v-s-p)(\beta_{12}\beta_{23}+\beta_{13})^2}{\alpha_1} f(y_3), \tag{A.16}$$

$$\frac{d^2J_3^5}{dQ^2} = \frac{(s+p)(1-\beta_{12}-\beta_{13})}{\alpha_1} f(z_1) + \frac{(s+p-p\beta_{23})\beta_{12}^2}{\alpha_1} f(z_3) + \frac{v-s-p}{1-\alpha_1(\beta_{12}+\beta_{13})-\alpha_2\beta_{23}} f(y_1), \tag{A.17}$$

$$\frac{d^2J_3^6}{dQ^2} = \frac{v(1-\beta_{12}-\beta_{23})}{\alpha_1} f(z_1) + \frac{(s+p-p\beta_{23})\beta_{12}^2}{\alpha_1} f(z_3) + \frac{(v-s-p)(\beta_{13}+\beta_{23})}{[\alpha_2(1-\beta_{23})+\alpha_3]/(\beta_{12}+\beta_{23})} f(y_4). \tag{A.18}$$

A.4 Proof to Lemma 2.4

The total expected profit J is the sum of the three subperiods' expected profits J_i, $i = 1, 2, 3$. The first derivative of the total expected profit dJ/dQ is simply the sum of the first derivative of each subperiod's expected profit dJ_i/dQ, $i = 1, 2, 3$, i.e.,

$$dJ/dQ = \sum_{i=1}^{3} dJ_i/dQ.$$

We then have the value of dJ/dQ at $Q = 0$ as

$$\left.\frac{dJ}{dQ}\right|_{Q=0} = \sum_{i=1}^{3} \left.\frac{dJ_i}{dQ}\right|_{Q=0} = s + p - c + p\beta_{12} + p(\beta_{12}\beta_{23} + \beta_{13}) > 0$$

and the value at $Q = \infty$ as

$$\left.\frac{dJ}{dQ}\right|_{Q=\infty} = \sum_{i=1}^{3} \left.\frac{dJ_i}{dQ}\right|_{Q=\infty} = \begin{cases} -c + v < 0, & \text{if } \theta_1 \leq 0 \\ -c + v - s\beta_{13} < 0, & \text{if } \theta_1 \geq 0. \end{cases}$$

It is clear that dJ/dQ is positive at $Q = 0$ and negative at $Q = \infty$. This indicates that there should exist at least one point of Q at which the value of dJ/dQ would be zero. We denote this point as $\tilde{Q}$, then we have $dJ/dQ|_{Q=\tilde{Q}} = 0$.

Also as shown from (A.13) to (A.18), we have six possible expressions (cases) for the second derivative of the total expected profit d^2J/dQ^2 and the condition of each case are identical with that of the counterpart in Table A.1. We need to examine one by one the sufficient condition for the negativity of each possible expression.

We first look at Case 1. The condition of Case 1 is $\theta_1 \leq 0$ and $\theta_2 \leq 0$ (See Case 1 in Table A.1). Under that condition, we have d^2J/dQ^2 as

$$\begin{aligned} \text{Case 1: } \frac{d^2J}{dQ^2} &= \frac{d^2\left(J_1 + J_2^1 + J_3^1\right)}{dQ^2} \\ &= \left[-\frac{s+p}{\alpha_1} f\left(z_1\right)\right] + \left[\frac{s + p\left(1-\beta_{12}\right)}{\alpha_1} f\left(z_1\right) - \frac{s+p}{\alpha_1 + \alpha_2 - \alpha_1\beta_{12}} f\left(z_2\right)\right] \\ &+ \left[\frac{p\left(\beta_{23} - \beta_{12}\beta_{23} - \beta_{13}\right)}{\alpha_1} f\left(z_1\right) + \frac{s + p\left(1-\beta_{23}\right)}{\alpha_1 + \alpha_2 - \alpha_1\beta_{12}} f\left(z_2\right)\right. \\ &\left.+ \frac{\left(v-s-p\right)}{1 - \alpha_1\left(\beta_{12} + \beta_{13}\right) - \alpha_2\beta_{23}} f\left(y_1\right)\right] \\ &= \frac{\left(\beta_{23} - \beta_{12}\beta_{23} - \beta_{13} - \beta_{12}\right)p}{\alpha_1} f\left(z_1\right) + \frac{-\beta_{23}p}{\alpha_1 + \alpha_2 - \alpha_1\beta_{12}} f\left(z_2\right) \\ &+ \frac{v-s-p}{1 - \alpha_1\left(\beta_{12} + \beta_{13}\right) - \alpha_2\beta_{23}} f\left(y_1\right). \end{aligned}$$

Since $-\beta_{23}p < 0$ and $v - s - p < 0$, it is obvious that the last two terms in the above expression are negative, one including $f\left(z_2\right)$ and the other including $f\left(y_1\right)$. It follows that if only the first term (which includes $f\left(z_1\right)$) is negative, we will have d^2J/dQ^2 be negative. Therefore, the sufficient condition for d^2J/dQ^2 being negative in condition of Case 1 is:

$$\beta_{23} - \beta_{12}\beta_{23} - \beta_{13} - \beta_{12} < 0. \tag{A.19}$$

Here we need to point out that (A.19) is a sufficient but not necessary condition for the strict concavity of the total expected profit in condition of Case 1. The reason is that d^2J/dQ^2, as the sum of one positive term and two negative terms, is very likely to be negative even when $\beta_{23} - \beta_{12}\beta_{23} - \beta_{13} - \beta_{12}$ is positive.

Similarly, we can find the expressions of d^2J/dQ^2 in condition of Case 2 and Case

3 as, in respective,

$$\text{Case 2: } \frac{d^2 J}{dQ^2} = \frac{d^2 \left(J_1 + J_2^1 + J_3^2\right)}{dQ^2}, \ (\theta_1 \leq 0, \ \theta_2 \geq 0 \ \& \ \theta_3 \geq 0)$$
$$\text{Case 3: } \frac{d^2 J}{dQ^2} = \frac{d^2 \left(J_1 + J_2^1 + J_3^3\right)}{dQ^2}, \ (\theta_1 \leq 0, \ \theta_2 \geq 0 \ \& \ \theta_3 \leq 0)$$

and the sufficient conditions for the strict concavity in condition of each case as, respectively,

$$\text{Case 2: } (v - s)(\beta_{23} - \beta_{12}\beta_{23} - \beta_{13}) - p < 0,$$
$$\text{Case 3: } \beta_{23} - \beta_{12}\beta_{23} - \beta_{13} - \beta_{12} < 0.$$

We now turn to Case 4. We know that, in condition of Case 4 ($\theta_1 \geq 0$ & $\theta_5 \geq 0$), $d^2 J/dQ^2$ in detail is:

$$\begin{aligned}
\text{Case 4: } \frac{d^2 J}{dQ^2} &= \frac{d^2 \left(J_1 + J_2^2 + J_3^4\right)}{dQ^2} \\
&= \left[-\frac{s+p}{\alpha_1} f(z_1)\right] + \left[-\frac{s\beta_{12}}{\alpha_1} f(z_1) - \frac{(s+p)\beta_{12}^2}{\alpha_1} f(z_3)\right] \\
&+ \left[\frac{v(1 - \beta_{12} - \beta_{23})}{\alpha_1} f(z_1) + \frac{(s\beta_{12}\beta_{23} - v\beta_{13})\beta_{12}}{\alpha_1} f(z_3)\right. \\
&+ \left.\frac{(v - s - p)(\beta_{12}\beta_{23} + \beta_{13})^2}{\alpha_1} f(y_3)\right] \\
&= \frac{(v-s)(1 - \beta_{12}) - p - v\beta_{23}}{\alpha_1} f(z_1) + \frac{s(\beta_{23} - 1)\beta_{12} - v\beta_{13} - p\beta_{12}}{\alpha_1/\beta_{12}} f(z_3) \\
&+ \frac{(v - s - p)(\beta_{12}\beta_{23} + \beta_{13})^2}{\alpha_1} f(y_3)
\end{aligned}$$

The coefficients in front of $f(z_3)$ and $f(y_3)$ are both negative. For example, as to the one in front of $f(z_1)$, we see that $(v - s)(1 - \beta_{12}) - p - v\beta_{23}$ is less than $(v - s) - p$ which is apparently negative. Therefore we conclude that, in condition of Case 4, the expected total profit is strictly concave with no other condition than the case condition (i.e., $\theta_1 \geq 0$ & $\theta_5 \geq 0$).

Similarly, we obtain d^2J/dQ^2 in condition of Case 5 and Case 6, respectively, as

$$\begin{aligned}
\text{Case 5: } \frac{d^2J}{dQ^2} &= \frac{d^2\left(J_1+J_2^2+J_3^5\right)}{dQ^2} \\
&= \frac{-s\beta_{13}-p\left(\beta_{12}+\beta_{13}\right)}{\alpha_1}f\left(z_1\right)+\frac{-p\beta_{12}^2\beta_{23}}{\alpha_1}f\left(z_3\right) \\
&+\frac{v-s-p}{1-\alpha_1\left(\beta_{12}+\beta_{13}\right)-\alpha_2\beta_{23}}f\left(y_1\right), \\
\text{Case 6: } \frac{d^2J}{dQ^2} &= \frac{d^2\left(J_1+J_2^2+J_3^6\right)}{dQ^2} \\
&= \frac{(v+s)\left(\beta_{12}-1\right)-p-v\beta_{23}}{\alpha_1}f\left(z_1\right)+\frac{-p\beta_{12}^2\beta_{23}}{\alpha_1}f\left(z_3\right) \\
&+\frac{(v-s-p)\left(\beta_{12}+\beta_{13}\right)^2}{\alpha_2\left(1-\beta_{23}\right)+\alpha_3}f\left(y_4\right).
\end{aligned}$$

By applying the same approach as we have used for Case 4, we find that the expected total profit is strictly concave in condition of Case 5 and Case 6 as well. The proof to Lemma 2.4 is thus completed. ■

Appendix B

Proofs and Solutions in Chapter 4

In this Appendix, we provide proofs and solution derivations in Chapter 4.

B.1 Regarding the Remanufacturing Competition Problem with Complete Information

B.1.1 Illustrating the OEM's optimal solution $(p_o^*, q_o^*)^c$ in (4.6)

We use Figure B.1 to illustrate the optimal solution in $(p_o^*, q_o^*)^c$ in (4.6).

There are two planes in Figure B.1. The lower plane shows the relation between the OEM's remanufacturing quantity q_o and sale price p_o, so we call it the q-p plane; the upper sub-figure shows the relation between his profit π_o and sale price p_o so we call it the π-p plane. As the q-p plane shows, the demand function $D_o(u_o, p_r)$ is valid in the domain $[u_o, P_o(p_r)]$, i.e., given the REM's price p_r, the OEM's max and min (or zero) demand occurs at u_o and $P_o(p_r)$ respectively and $P_o(p_r) = \arg_{p_o}[D_o(p_o, p_r) = 0] = (A_o + C_o p_r)/B_o$. So, without limitation of shell accessibility, i.e., when $S_o = \infty$, the OEM's profit would be maximized at $\mathrm{T_M}$ in the π-p plane and the optimal solution is $(p_o^*, q_o^*)^c|_{S_o=\infty} = ((u_o + P_o)/2, D_o^m)$.

The OEM's optimization problem with limitation of shell accessibility is a bit more complex. When accessible shells S_o is less than D_o^m, as $S_o = S_1$ in the q-p plane, the OEM's remanufacturing quantity is a piecewise function composed of two pieces: the horizontal line $q_o = S_1$ and the portion of the demand function that is below the level $q_o = S_1$. Correspondingly, his profit line is a piecewise curve composed of two pieces as well: the curve between points $\mathrm{T_0}$ and $\mathrm{T_1}$ and the straight line between points $\mathrm{T_1}$ and $\mathrm{T_F}$ in the π-p plane. So the maximum profit is realized at point $\mathrm{T_1}$ and the optimal solution is correspondingly located at $(p_o^*, q_o^*)^c|_{S_o=S_1}$. When accessible shells S_o is higher than D_o^m, as $S_o = S_2$ in the q-p plane, the OEM's remanufacturing quantity is again a piecewise function composed of the horizontal line $q_o = S_2$ and

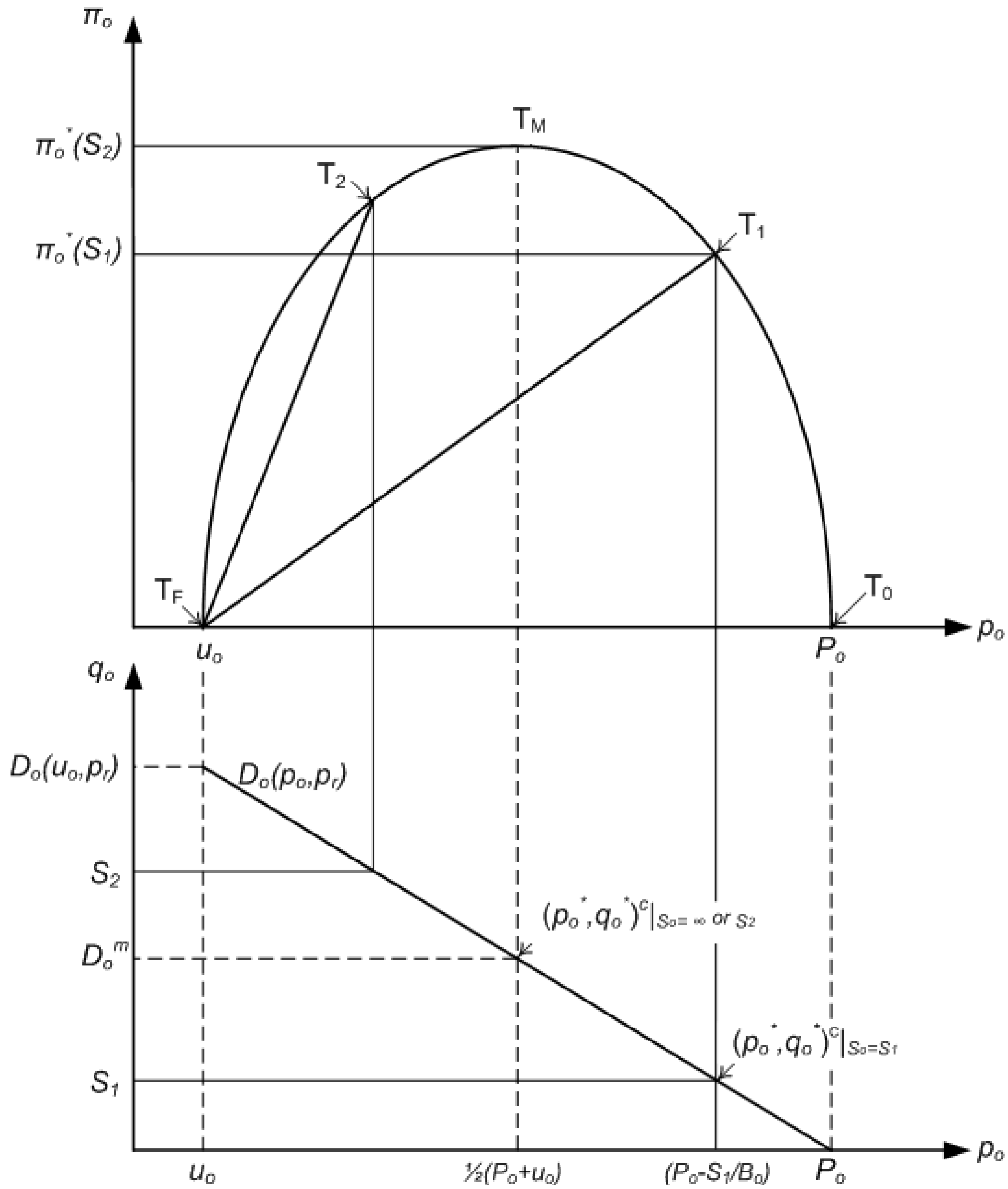

Figure B.1: The optimal solutions to optimization problem Θ_o^c when $S_o = S_1$ and $S_o = S_2$.

the portion of the demand function that is below the level $q_o = S_2$. Correspondingly, his profit curve is a piecewise function composed of the curve between points T_0 and T_2 and the straight line between points T_2 and T_F in the π-p plane. In this case, the maximum profit is realized at the point T_M and the optimal solution is located at the point of median demand which is marked as $(p_o^*, q_o^*)^c |_{S_o=S_2}$ in the q-p plane. This is also the optimal solution with no limitation of shell accessibility. Hence, the OEM's optimal solution will be fixed at $(p_o^*, q_o^*)^c |_{S_o=S_2}$ for any S_o that is greater or equal to D_o^m.

B.1.2 Proof of Theorem 4.1

Proof of existence

Assume that the OEM chooses a price p_o within domain $[0, P_o]$ where P_o is upper bound of p_o, given p_r. Let $p_r = f_1(p_o)$ and $q_r = f_2(p_o)$ be the REM's best response to p_o. Both $f_1(p_o)$ and $f_2(p_o)$ are functions of p_o whose detailed expression are available in (4.12). Also let $g_1(p_r, q_r)$ be the OEM's price as his best response to p_r and q_r. The detailed expression of $g_1(p_r, q_r)$ is identical with the expression of p_o available in (4.6). As shown in (4.12) and (4.6), $f_1(p_o)$ and $f_2(p_o)$ are continuous functions mapping p_o to p_r and q_r respectively and $g_1(p_r, q_r)$ is a continuous function mapping (p_r, q_r) to p_o. Therefore, $g_1[f_1(p_o), f_2(p_o)]$ is a continuous function mapping from the domain of p_o to itself. By applying Brouwer's fixed-point theorem [35, page 45], there must exist a fixed point $\hat{p}_o$ which satisfies $g_1[f_1(\hat{p}_o), f_2(\hat{p}_o)] = \hat{p}_o$.

Then we define $\hat{p}_r = f_1(\hat{p}_o)$ and $\hat{q}_r = f_2(\hat{p}_o)$. If it is under SAS1, let $(\hat{p}_o, \hat{q}_o, \hat{p}_r, \hat{q}_r)$ solve $\Theta_o^c(\hat{p}_o, \hat{q}_o \mid \hat{p}_r, S)$ and $\Theta_r^c(\hat{p}_r, \hat{q}_r \mid \hat{p}_o, S - \hat{q}_o)$ together; if it is under SAS2, let $(\hat{p}_o, \hat{q}_o, \hat{p}_r, \hat{q}_r)$ solve $\Theta_r^c(\hat{p}_r, \hat{q}_r \mid \hat{p}_o, S)$ and $\Theta_o^c(\hat{p}_o, \hat{q}_o \mid \hat{p}_r, S - \hat{q}_r)$ together. Thus, ($\hat{p}_o$, $\hat{q}_o$, $\hat{p}_r$, $\hat{q}_r$) forms a NE.

The proof of the existence of the NE in Theorem 4.1 is thus completed. This proof is similar to the proof of existence for the Theorem 1 in [58]. ■

Proof of uniqueness

The uniqueness of the Nash Equilibrium is equivalent to the uniqueness of the fixed point $\hat{p}_o$ as mentioned above. We proof the uniqueness of the fixed point $\hat{p}_o$ by contradiction.

We assume that $\hat{p}_o \in [0, P_o]$ is a fixed point. So we have $g_1[f_1(\hat{p}_o), f_2(\hat{p}_o)] = \hat{p}_o$. Now we assume there is another point $\mathring{p}_o \in [0, P_o]$ and $\mathring{p}_o = \hat{p}_o + \varepsilon$, $\varepsilon > 0$. If $\mathring{p}_o$ is also a fixed point, we should be able to verify $g_1[f_1(\mathring{p}_o), f_2(\mathring{p}_o)] = \mathring{p}_o$ as well. In the following we will verify this with SAS2.

Firstly, we look at the difference of the REM's best responses to the two points $\mathring{p}_o$ and $\hat{p}_o$. According to (4.12), it is not difficult to get Δf_1 and Δf_2 — the difference of the REM's best responses $f_1(\cdot)$ and $f_2(\cdot)$ — as:

$$\begin{aligned}\Delta f_1 &= f_1(\mathring{p}_o) - f_1(\hat{p}_o)\\ &= \begin{cases} \frac{A_r+C_r\mathring{p}_o-S}{B_r} - \frac{A_r+C_r\hat{p}_o-S}{B_r} = \frac{C_r}{B_r}\varepsilon, \text{ if } 0 \le S < D_r^m(\hat{p}_o), \\ \frac{A_r+C_r\mathring{p}_o-S}{B_r} - \frac{1}{2}\left(u_r + \frac{A_r+C_r\hat{p}_o}{B_r}\right), \text{ if } D_r^m(\hat{p}_o) \le S < D_r^m(\mathring{p}_o), \\ \frac{1}{2}\left(u_r + \frac{A_r+C_r\mathring{p}_o}{B_r}\right) - \frac{1}{2}\left(u_r + \frac{A_r+C_r\hat{p}_o}{B_r}\right) = \frac{C_r}{2B_r}\varepsilon, \text{ if } S > D_r^m(\mathring{p}_o), \end{cases}\end{aligned}$$

and

$$\Delta f_2 = f_2(\mathring{p}_o) - f_2(\hat{p}_o) = \begin{cases} S - S = 0, & \text{if } 0 \le S < D_r^m(\hat{p}_o), \\ S - D_r^m(\hat{p}_o), & \text{if } D_r^m(\hat{p}_o) \le S < D_r^m(\mathring{p}_o), \\ D_r^m(\mathring{p}_o) - D_r^m(\hat{p}_o) = \frac{1}{2}C_r\varepsilon & \text{if } S > D_r^m(\mathring{p}_o). \end{cases}$$

Therefore, given $\hat{p}_o$ and $\mathring{p}_o$ as we have just defined, the range of the difference between the REM's best responses can be yielded as:

$$\frac{C_r}{2B_r}\varepsilon \le \Delta f_1 \le \frac{C_r}{B_r}\varepsilon \tag{B.1}$$

and

$$0 \le \Delta f_2 \le \frac{1}{2}C_r\varepsilon. \tag{B.2}$$

Secondly, we look at the difference of the OEM's best response $g_1(\cdot)$ to the REM's best responses $(f_1(\hat{p}_o), f_2(\hat{p}_o))$ and $(f_1(\mathring{p}_o), f_2(\mathring{p}_o))$. For the convenience of illustration, we let $\hat{p}_r = f_1(\hat{p}_o)$, $\hat{q}_r = f_2(\hat{p}_o)$, $\mathring{p}_r = f_1(\mathring{p}_o)$ and $\mathring{q}_r = f_2(\mathring{p}_o)$. According to the results in the first step, given $\hat{p}_o < \mathring{p}_o$, we have $\hat{p}_r < \mathring{p}_r$, $\hat{q}_r \le \mathring{q}_r$. Correspondingly, $S - \hat{q}_r \ge S - \mathring{q}_r$. Again, to facilitate our discussion, we denote $S - \hat{q}_r$ and $S - \mathring{q}_r$ as $\hat{S}_o$ and $\mathring{S}_o$ respectively. Hence, we have $\hat{S}_o \ge \mathring{S}_o$.

According to (4.6), we yield Δg_1 — the difference of the OEM's best responses — as

$$\begin{aligned}\Delta g_1 &= g_1(f_1(\mathring{p}_o), f_2(\mathring{p}_o)) - g_1(f_1(\hat{p}_o), f_2(\hat{p}_o))\\ &= \begin{cases} \frac{A_o+C_o\mathring{p}_r-(S-\mathring{q}_r)}{B_o} - \frac{A_o+C_o\hat{p}_r-(S-\hat{q}_r)}{B_o}, \text{ if } \mathring{S}_o \in [0, D_o^m(\mathring{p}_r)) \ \&\ \hat{S}_o \in [0, D_o^m(\hat{p}_r)) \\ \frac{A_o+C_o\mathring{p}_r-(S-\mathring{q}_r)}{B_o} - \frac{1}{2}\left(u_o + \frac{A_o+C_o\hat{p}_r}{B_o}\right), \text{ if } \mathring{S}_o \in [0, D_o^m(\mathring{p}_r)) \ \&\ \hat{S}_o \in [D_o^m(\hat{p}_r), \infty) \\ \frac{1}{2}(u_o + \frac{A_o+C_o\mathring{p}_r}{B_o}) - \frac{1}{2}(u_o + \frac{A_o+C_o\hat{p}_r}{B_o}), \text{ if } \mathring{S}_o \in [D_o^m(\mathring{p}_r), \infty) \ \&\ \hat{S}_o \in [D_o^m(\hat{p}_r), \infty) \end{cases}\\ &= \begin{cases} \frac{C_o}{B_o}\Delta f_1 - \frac{1}{B_o}\Delta f_2, \text{ if } \mathring{S}_o \in [0, D_o^m(\mathring{p}_r)) \ \&\ \hat{S}_o \in [0, D_o^m(\hat{p}_r)), \\ \frac{A_o+C_o\mathring{p}_r-(S-\mathring{q}_r)}{B_o} - \frac{1}{2}\left(u_o + \frac{A_o+C_o\hat{p}_r}{B_o}\right), \text{ if } \mathring{S}_o \in [0, D_o^m(\mathring{p}_r)) \ \&\ \hat{S}_o \in [D_o^m(\hat{p}_r), \infty), \\ \frac{C_o}{2B_o}\Delta f_1, \text{ if } \mathring{S}_o \in [D_o^m(\mathring{p}_r), \infty) \ \&\ \hat{S}_o \in [D_o^m(\hat{p}_r), \infty). \end{cases}\end{aligned}$$

It follows that Δg_1 would be ranged within

$$\Delta g_1 \in \left[\underline{\Delta g_1}, \overline{\Delta g_1}\right] \triangleq \left[\frac{C_o}{2B_o}\Delta f_1, \frac{C_o}{B_o}\Delta f_1 - \frac{1}{B_o}\Delta f_2\right].$$

Given (B.1) and (B.2), we yield

$$\frac{1}{2}\left(\frac{C_r}{B_r} - \frac{C_r}{B_o}\right)\varepsilon \leq \overline{\Delta g_1} \leq \frac{C_o C_r}{B_o B_r}\varepsilon$$

and

$$\frac{1}{4}\frac{C_o C_r}{B_o B_r}\varepsilon \leq \underline{\Delta g_1} \leq \frac{1}{2}\frac{C_o C_r}{B_o B_r}\varepsilon.$$

To put the above together, we get

$$\frac{1}{4}\frac{C_o C_r}{B_o B_r}\varepsilon \leq \underline{\Delta g_1} \leq \Delta g_1 \leq \overline{\Delta g_1} \leq \frac{C_o C_r}{B_o B_r}\varepsilon,$$

which can be further simplified as

$$\frac{1}{4}\frac{C_o C_r}{B_o B_r}\varepsilon \leq \Delta g_1 \leq \frac{C_o C_r}{B_o B_r}\varepsilon. \tag{B.3}$$

In Section 4.3.2, we have assumed $B_o > C_r > 0$ and $B_r > C_o > 0$. Therefore it is obvious that $C_o C_r / B_o B_r < 1$ and $\Delta g_1 < \varepsilon$. Based on that and by the definition of Δg_1, we can evaluate $g_1\left(f_1\left(\mathring{p}_o\right), f_2\left(\mathring{p}_o\right)\right)$ as

$$g_1\left(f_1\left(\mathring{p}_o\right), f_2\left(\mathring{p}_o\right)\right) = g_1\left(f_1\left(\hat{p}_o\right), f_2\left(\hat{p}_o\right)\right) + \Delta g_1 = \hat{p}_o + \Delta g_1 < \hat{p}_o + \varepsilon = \mathring{p}_o,$$

i.e.,

$$g_1\left(f_1\left(\mathring{p}_o\right), f_2\left(\mathring{p}_o\right)\right) < \mathring{p}_o \text{ or } g_1\left(f_1\left(\mathring{p}_o\right), f_2\left(\mathring{p}_o\right)\right) \neq \mathring{p}_o$$

This is contradict with our assumption that $g_1\left(f_1\left(\mathring{p}_o\right), f_2\left(\mathring{p}_o\right)\right) = \mathring{p}_o$ if $\mathring{p}_o$ is another fixed point.

The proof of the uniqueness of the NE under SAS2 is thus completed. The uniqueness of the NE under SAS1 can be proofed in the same way by starting with assuming two fixed points for the REM's price p_r. ■

B.1.3 Derivation of the NE

We first solve (Θ_o^c, Θ_r^c) with SAS1 in which the OEM has the priority to access available shells. With SAS1, we have $S_o = S$ and $S_r = S - q_o$. Hence, under each possible value of S, we solve (4.5) with $S_o = S$ and (4.11) with $S_r = S - q_o$ simultaneously. Having optimal solutions to (4.5) and (4.11) in (4.6) and (4.12) respectively, we can obtain the Nash equilibrium for each valid value of S.

(1) When $0 \leq S < D_o^m$,

When the value of S drops within this range, we have $S_o < D_o^m = \frac{1}{2}D_o(u_o, p_r)$ as well since $S_o = S$ with SAS1. Referring to the OEM's optimal solution in (4.6), we have

$$q_o = S \text{ and } p_o = P_o - S_o/B_o = (A_o + C_o p_r - S)/B_o. \tag{B.4}$$

With $q_o = S$, we yield $S_r = S - q_o = 0$. Then, referring to the REM's optimal solution in (4.12), we obtain

$$q_r = 0 \text{ and } p_r = P_r = (A_r + C_r p_o)/B_r. \tag{B.5}$$

The expressions of p_o, q_o, p_r, and q_r in (B.4) and (B.5) are all functions of each other. Hence, by solving them as a set of linear equations, we obtain the NE as

$$[p_o^*, q_o^*, p_r^*, q_r^*]_{11}^c = \begin{bmatrix} \frac{A_o B_r + A_r C_o - B_r S}{B_o B_r - C_o C_r} \\ S \\ \frac{A_r B_o + A_o C_r - C_r S}{B_o B_r - C_o C_r} \\ 0 \end{bmatrix}^{\mathrm{T}}. \tag{B.6}$$

Note that this NE expression applies to any value of S until it exceeds $\bar{S}_{11}^c = D_o^m = (1/2)D_o(u_o, p_r)$ which is a function of the REM's price p_r. By applying the expression of p_r in (B.6) to the expression of $\bar{S}_{11}^c$, we yield

$$\bar{S}_{11}^c = \frac{B_o(A_o B_r - B_o u_o B_r + C_o u_o C_r + A_r C_o)}{2B_o B_r - C_o C_r}. \tag{B.7}$$

That is, when $S \in [0, \bar{S}_{11}^c)$, the NE for the remanufacturing competition problem with complete information is in the form of (B.6).

(2) When $\bar{S}_{11}^c \leq S < D_o^m + D_r^m$

When the available shells S is greater than $\bar{S}_{11}^c$, by referring to (4.6) we see that the OEM will maximize his profit at

$$q_o = \frac{1}{2}D_o(u_o, p_r) \text{ and } p_o = \frac{1}{2}(u_o + P_o) \tag{B.8}$$

and, correspondingly, the REM will take over the leftover shells and optimizes her profit at

$$q_r = S - q_o \text{ and } p_r = \frac{A_r + C_r p_o}{B_r} - \frac{S - q_o}{B_r}. \tag{B.9}$$

Solving p_o, q_o, p_r, and q_r in (B.8) and (B.9) altogether, we obtain the NE as

$$[p_o^*, q_o^*, p_r^*, q_r^*]_{12}^c = \begin{bmatrix} \frac{A_oB_r+A_rC_o+B_ou_oB_r-B_ou_oC_o-C_oS}{2B_oB_r-C_oC_r-B_oC_o} \\ \frac{B_o(A_oB_r+A_rC_o-B_ou_oB_r+C_ou_oC_r-C_oS)}{2B_oB_r-C_oC_r-B_oC_o} \\ \frac{A_oB_o+A_oC_r+2A_rB_o-B_o^2u_o+B_ou_oC_r-2B_oS}{2B_oB_r-C_oC_r-B_oC_o} \\ \frac{-A_oB_oB_r-A_rB_oC_o+B_o^2u_oB_r-B_ou_oC_oC_r+2B_oB_rS-C_oC_rS}{2B_oB_r-C_oC_r-B_oC_o} \end{bmatrix}^{\mathrm{T}}. \tag{B.10}$$

Again, this NE expression applies to any value of S that is higher than $\bar{S}_{11}^c$ until it exceeds $\bar{S}_{12}^c = D_o^m + D_r^m = (1/2)D_o(u_o, p_r) + (1/2)D_r(p_o, u_r)$, which is a function of both players' price p_o and p_r. By applying the expressions of p_o and p_r in (B.10) to the expression of $\bar{S}_{12}^c$, we obtain

$$\begin{aligned}\bar{S}_{12}^c = & \frac{2A_oB_oB_r+A_oB_rC_r+2A_rB_oB_r+A_rB_oC_o-2B_oB_r^2u_r-2B_o^2B_ru_o}{4B_oB_r-C_oC_r} \\ & + \frac{+B_oB_rC_ou_r+B_oB_rC_ru_o+B_oC_oC_ru_o+B_rC_oC_ru_r}{4B_oB_r-C_oC_r}.\end{aligned} \tag{B.11}$$

(3) When $S > \bar{S}_{12}^c$

When S is greater than $\bar{S}_{12}^c$, we see that the OEM will still maximize his profit at

$$q_o = \frac{1}{2}D_o(u_o, p_r) \text{ and } p_o = \frac{1}{2}(u_o + P_o), \tag{B.12}$$

and the REM will maximize her profit at

$$q_r = \frac{1}{2}D_r(p_o, u_r) \text{ and } p_r = \frac{1}{2}(u_r + P_r). \tag{B.13}$$

Solving p_o, q_o, p_r, and q_r in (B.12) and (B.13), we can obtain the NE as

$$[p_o^*, q_o^*, p_r^*, q_r^*]_{13}^c = \begin{bmatrix} \frac{2A_oB_r+A_rC_o+2B_oB_ru_o+B_rC_ou_r}{4B_oB_r-C_oC_r} \\ \frac{B_o(2A_oB_r+A_rC_o-2B_oB_ru_o+B_rC_ou_r+C_oC_ru_o)}{4B_oB_r-C_oC_r} \\ \frac{A_oC_r+2A_rB_o+2B_oB_ru_r+B_oC_ru_o}{4B_oB_r-C_oC_r} \\ \frac{B_r(A_oC_r+2A_rB_o-2B_oB_ru_r+B_oC_ru_o+C_oC_ru_r)}{4B_oB_r-C_oC_r} \end{bmatrix}^{\mathrm{T}}. \tag{B.14}$$

This NE expression applies to any value of S that is greater than $\bar{S}_{12}^c$.

B.2 Regarding the Remanufacturing Competition Problem with Incomplete Information

B.2.1 Proof of Lemma 4.1

By the definition of demand functions in Section 4.4.3, it is easy to yield $D_{r_H}(p_o, u_{r_H}) < D_{r_L}(p_o, u_{r_L})$ for any valid p_o because $u_{r_L} < u_{r_H}$. In the same way, we have $D_{r_H}(p_o^*, u_{r_H}) <$

$D_{r_L}(p_o^*, u_{r_L})$ as well since p_o^* is a valid value of p_o.

The proof of Lemma 4.1 is completed. ■

B.2.2 Proof of Lemma 4.2

According to (4.24), we find that, given p_o:

(i) If $S_r < \frac{1}{2}D_{r_H}(p_o, u_{r_H})$, then

$$\begin{cases} p_{r_L}^* = p_{r_H}^* = P_r - \frac{S_r}{B_r}, \\ q_{r_L}^* = q_{r_H}^* = S_r, \end{cases}$$

(ii) If $\frac{1}{2}D_{r_H}(p_o, u_{r_H}) < S_r < \frac{1}{2}D_{r_L}(p_o, u_{r_L})$, then

$$\begin{cases} p_{r_L}^* = \frac{A_r - S_r + C_r p_o}{B_r}, \ p_{r_H}^* = \frac{1}{2}(u_{r_H} + P_r), \\ q_{r_L}^* = S_r, \ q_{r_H}^* = \frac{1}{2}D_{r_H}(p_o, u_{r_H}), \end{cases} \Rightarrow \begin{cases} p_{r_L}^* < p_{r_H}^* \\ q_{r_L}^* > q_{r_H}^* \end{cases}$$

(iii) If $S_r > \frac{1}{2}D_{r_L}(p_o, u_{r_L})$, then

$$\begin{cases} p_{r_L}^* = \frac{1}{2}(u_{r_L} + P_r), \ p_{r_H}^* = \frac{1}{2}(u_{r_H} + P_r), \\ q_{r_L}^* = \frac{1}{2}D_{r_L}(p_o, u_{r_L}), q_{r_H}^* = \frac{1}{2}D_{r_H}(p_o, u_{r_H}), \end{cases} \Rightarrow \begin{cases} p_{r_L}^* < p_{r_H}^* \\ q_{r_L}^* > q_{r_H}^* \end{cases}$$

We also know that $S_r = S - q_o^*$ in SAS1 and $S_r = S$ in SAS2. That is, the type-L REM and type-H REM always share the same S_r. Hence, given p_o, both (a) and (b) in Lemma 4.2 hold true under both SAS1 and SAS2.

The proof of Lemma 4.2 is completed. ■

B.2.3 Proof of Lemma 4.3

By the definition of demand functions in Section 4.3.2, $D_o(p_o, p_r) = A_o - B_o p_o + C_o p_r$ is a linear function of p_r. Since by assumption $C_o > 0$ and by Lemma 4.2, $p_{r_L}^* \leq p_{r_H}^*$, then it is straightforward to yield $D_o\left(p_o, p_{r_L}^*\right) \leq D_o\left(p_o, p_{r_H}^*\right)$ and $D_o\left(u_o, p_{r_L}^*\right) \leq D_o\left(u_o, p_{r_H}^*\right)$. The proof of Lemma 4.3 is completed. ■

B.2.4 Illustrating the OEM's Optimal Solution $(p_o^*, q_o^*)_1$ under SAS1

We have $S_o = S$ under SAS1, so here $\Theta_o(p_o, q_o \mid p_{r_L}, p_{r_H}, \theta, S_o)$ can be re-written as

$$\max \pi_o = \{\theta \ [\min(S, D_o^c(p_o, p_{r_L}))] + (1-\theta)[\min(S, D_o^c(p_o, p_{r_H}))]\}(p_o - u_o). \quad \text{(B.15)}$$

As Lemma 4.2 states, we have $p_{r_L}^* \leq p_{r_H}^*$ given p_o and S_r. Hence, we need to move forward our investigation of $\Theta_o(p_o, q_o \mid p_{r_L}, p_{r_H}, \theta, S_o)$ by assuming $p_{r_L} = p_{r_H}$ and $p_{r_L} < p_{r_H}$ individually.

In case that $p_{r_L} = p_{r_H} \equiv p_r$, then by definition we have $D_o^c(p_o, p_{r_L}) = D_o^c(p_o, p_{r_H}) \equiv D_o(p_o, p_r)$, so $\Theta_o(p_o, q_o \mid p_{r_L}, p_{r_H}, \theta, S_o)$ in (B.15) can be further simplified into

$$\max \pi_o = [\min(S, D_o^c(p_o, p_r))] (p_o - u_o)$$

which is exactly identical with (4.5) — the OEM's optimization problem in the formulation with complete information. Therefore, in case that $p_{r_L} = p_{r_H} \equiv p_r$, the optimal solution to $\Theta_o(p_o, q_o \mid p_{r_L}, p_{r_H}, \theta, S_o)$ can be found in (4.6).

In case that $p_{r_L} < p_{r_H}$, by Lemma 4.2 we have $D_o^c(p_o, p_{r_L}) < D_o^c(p_o, p_{r_H})$ so we use Figure B.2 to facilitate our illustration. Similar to Figure B.1, there are two planes Figure B.2, the π-p plane in the above and the q-p plane in the below.

When there is no limitation of shell accessibility, i.e., $S = \infty$, (B.15) becomes

$$\max\ \pi_o = \{\theta\ D_o^c(p_o, p_{r_L}) + (1-\theta)\, D_o^c(p_o, p_{r_H})\}(p_o - u_o),$$

which is equivalent to

$$\max\ \pi_o = [D_o(p_o, \theta p_{r_L} + (1-\theta)\, p_{r_H})](p_o - u_o). \tag{B.16}$$

Referring to Figure B.2, the demand function $D_o(p_o, \theta p_{r_L} + (1-\theta)\, p_{r_H})$ in (B.16) is shown as the bold straight line in the q-p plane and the profit curve π_o in (B.16) is shown as the bold curve in the π-p plane. It is obvious that without limitation of shell accessibility, the OEM's profit π_o is maximized at T_M in the π-p plane and the optimal solution is marked as $(p_o^*, q_o^*)\,|_{S=\infty}$ in the q-p plane. This optimal solution actually applies to both shell accessibility scenarios. It is reasonable because, when $S = \infty$, both players can access as many shells as they need no matter who has the higher priority for shell accessibility, i.e., in this case, there is no difference between the two shell accessibility schemes.

With limitation of shell accessibility, i.e., when $S < \infty$, the optimal solution to (4.25) under SAS1 is a bit more complex.

As the q-p plane in Figure B.2 shows, given each valid value of accessible shells S, the OEM's remanufacturing quantity is a piecewise function composed of three pieces. For example, when $S = S_1$, the remanufacturing quantity line is composed of three pieces — T_0-T_{11}, T_{11}-T_{12} and T_{12}-T_{1F} in the q-p plane. T_0-T_{11} maps to the price range within which the OEM's demand is less than the accessible shells S regardless of the type of REM he is competing with; T_{11}-T_{12} maps to the price range within which the OEM's demand has gone beyond the accessible shells if the REM is of type H while his demand is still lower than S if the REM is of type L; T_{12}-T_{1F} maps to the price range in which the OEM's demand has gone beyond the accessible shells regardless of the REM's type. Correspondingly, the OEM's profit given each value of S is a piecewise curve composed of three pieces, too. For example, as shown in the π-p plane, when $S = S_1$, the profit curve goes through T_0-T_{11}-T_{12}-T_F, the maximal profit occurs between T_{11} and T_{12}, and the optimal solution locates at $(p_o^*, q_o^*)_1\,|_{S=S_1}$

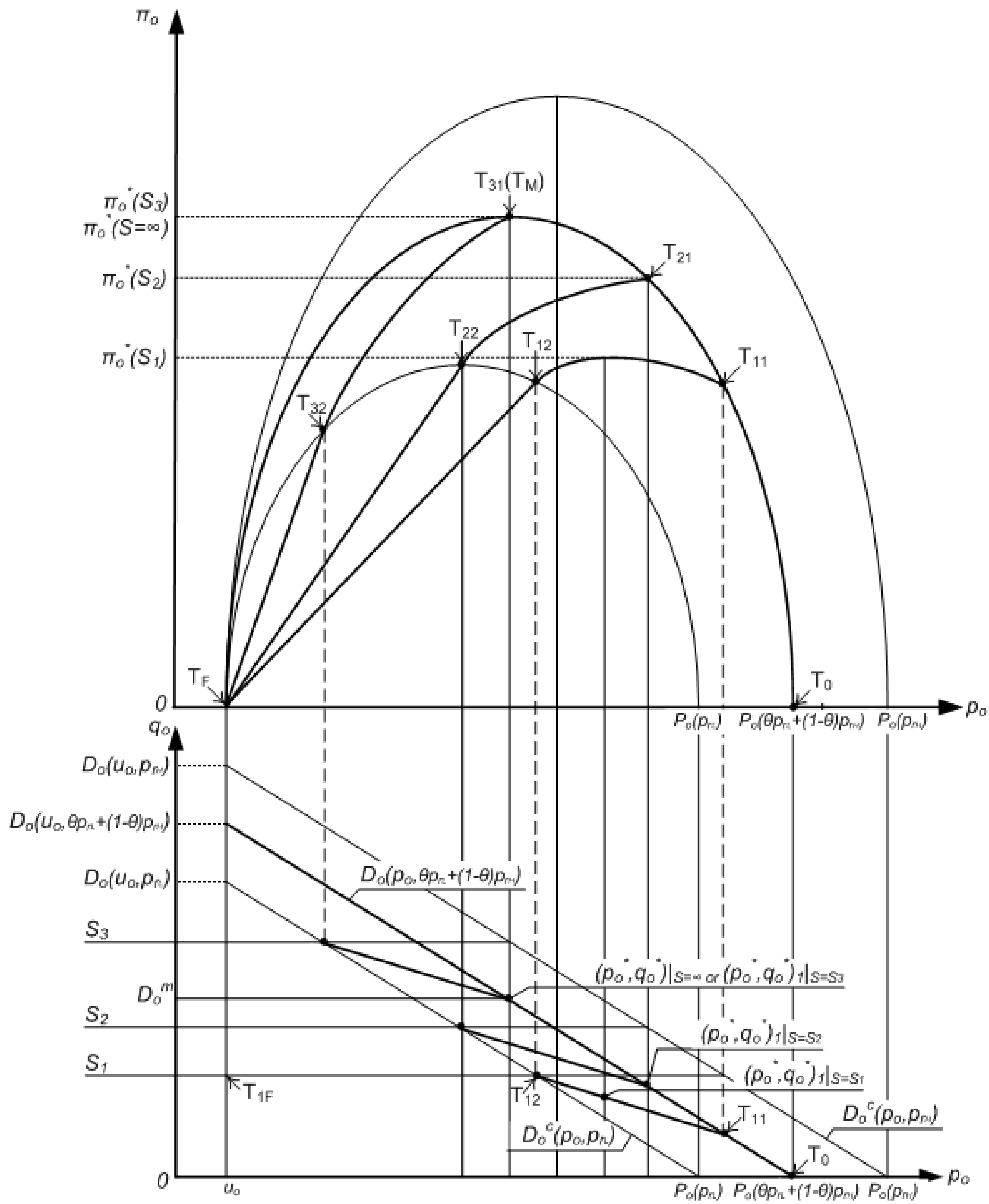

Figure B.2: The optimal solutions to the OEM's optimization problem with SAS1.

in the q-p plane; when $S = S_2$, the profit curve goes through $\mathrm{T_0}$-$\mathrm{T_{21}}$-$\mathrm{T_{22}}$-$\mathrm{T_F}$, the maximal profit occurs at $\mathrm{T_{21}}$, and the optimal solution locates at $(p_o^*, q_o^*)_1\,|_{S=S_2}$; when $S = S_3$, the profit curve goes through $\mathrm{T_0}$-$\mathrm{T_{31}}(\mathrm{T_M})$-$\mathrm{T_{32}}$-$\mathrm{T_F}$, the maximal profit occurs at $\mathrm{T_{31}}(\mathrm{T_M})$, and the optimal solution locates at $(p_o^*, q_o^*)_1\,|_{S=S_3 \text{ or } \infty}$. Actually, we notice that the maximal profit occurs at $\mathrm{T_{31}}(\mathrm{T_M})$ for any $S \geq S_3$ in the q-p plane.

Fortunately, the OEM's real situation under SAS1 is a bit simpler than the illustration in Figure B.2 if we take the REM's response into account. As stated in Lemma 4.2 part (b), the two types of REM would take the same price and same quantity when her accessible shells S_r is not high enough. Meanwhile, the OEM under SAS1 won't leave any accessible shells to the REM if only using up all available shells can maximize his profit. Accordingly, the cases when $S = S_1$ or S_2, as shown in the q-p plane of Figure B.2, won't occur at all since the OEM would use up all available shells and, as a result of that, the two types of REM would take $p_{r_L} = p_{r_H} = P_r\left(p_o\right)$ and $q_{r_L} = q_{r_H} = 0$, their demand function will overlap with each other, and the OEM's optimization problem would be in the form of (4.5). This case will hold true if only $S \in [0, D_o^m]$.

Furthermore, still by Lemma 4.2 part (b), we have $p_{r_L}^* = p_{r_H}^* = P_r - \frac{S_r}{B_r}$ and $q_{r_L}^* = q_{r_H}^* = S_r$ if only $S_r \leq D_{r_H}^m\left(p_o\right)$. This implies that the two types of REM will still share the same demand function and result the OEM with the optimal solution in the form of (4.5) again when $S \in [D_o^m, D_o^m + D_{r_H}^m\left(p_o\right)]$. Therefore, what Figure B.2 describes is actually the case when $S \in [D_o^m + D_{r_H}^m\left(p_o\right), \infty)$ for it is in this range of S that the two types of REM would take different prices and their demand functions and profit curves won't no longer duplicate. It would be as $S = S_3$ in the q-p plane, the OEM's profit would maximize at $\mathrm{T_M}$ in the π-p plane and his optimal solution would be $(p_o^*, q_o^*)_1\,|_{S=S_3}$ as shown in the q-p plane.

B.2.5 Proof of Theorem 4.2

The proof of BNE existence with SAS1 is similar to the proof of NE existence in B.1.2.

Assume that the OEM chooses a price p_o within domain $[0, P_o]$ where $P_o = [A_o + C_o\left(\theta p_{r_L} + (1-\theta)\, p_{r_H}\right)]/B_o$ is the upper bound of p_o given p_{r_L} and p_{r_H}. Let $p_{r_L} = f_{1L}\left(p_o\right)$ and $q_{r_L} = f_{2L}\left(p_o\right)$ be the type-L REM's best response to p_o and $p_{r_H} = f_{1H}\left(p_o\right)$ and $q_{r_H} = f_{2H}\left(p_o\right)$ be the type-H REM's best response to p_o. $f_{1L}\left(p_o\right)$, $f_{2L}\left(p_o\right)$, $f_{1H}\left(p_o\right)$, and $f_{2H}\left(p_o\right)$ are all functions of p_o whose detailed expressions are available in (4.24). Let $g_1\left(p_{r_L}, q_{r_L}, p_{r_H}, q_{r_H}\right)$ be the OEM's price as his best response to $p_{r_L}, q_{r_L}, p_{r_H}$ and q_{r_H}. The detailed expression of $g_1\left(p_{r_L}, q_{r_L}, p_{r_H}, q_{r_H}\right)$ is available as the expression of p_o in Table 4.4. As shown in (4.24) and Table 4.4, $f_{1L}\left(p_o\right)$, $f_{2L}\left(p_o\right)$, $f_{1H}\left(p_o\right)$ and $f_{2H}\left(p_o\right)$ are continuous functions mapping p_o to p_{r_L}, q_{r_L}, p_{r_H}, and q_{r_H} respectively and $g_1\left(p_{r_L}, q_{r_L}, p_{r_H}, q_{r_H}\right)$ is a continuous function mapping $\left(p_{r_L}, q_{r_L}, p_{r_H}, q_{r_H}\right)$ to p_o. Therefore, $g_1\left[f_{1L}\left(p_o\right), f_{2L}\left(p_o\right), f_{1H}\left(p_o\right), f_{2H}\left(p_o\right)\right]$ is a continuous function mapping from the domain of p_o to itself. Still by applying Brouwer's fixed-point theorem [35, page 45], there must exist a fixed point $\hat{p}_o$ which satisfies

$$g_1[f_{1L}(\hat{p}_o), f_{2L}(\hat{p}_o), f_{1H}(\hat{p}_o), f_{2H}(\hat{p}_o)] = \hat{p}_o.$$

Then we define $\hat{p}_{r_L} = f_{1L}(\hat{p}_o)$, $\hat{q}_{r_L} = f_{2L}(\hat{p}_o)$, $\hat{p}_{r_H} = f_{1H}(\hat{p}_o)$, and $\hat{q}_{r_H} = f_{2H}(\hat{p}_o)$. Under SAS1 we let $(\hat{p}_o, \hat{q}_o, \hat{p}_{r_L}, \hat{q}_{r_L}, \hat{p}_{r_H}, \hat{q}_{r_H})$ solve $\Theta_o(\hat{p}_o,\ \hat{q}_o \mid \hat{p}_{r_L},\ \hat{p}_{r_H},\ S)$, $\Theta_{r_L}(\ \hat{p}_{r_L},\ \hat{q}_{r_L} \mid \hat{p}_o,\ S - \hat{q}_o\ )$, and $\Theta_{r_H}(\hat{p}_{r_H}, \hat{q}_{r_H} \mid \hat{p}_o, S - \hat{q}_o)$ together. The solution $(\hat{p}_o,\ \hat{q}_o,\ \hat{p}_{r_L},\ \hat{q}_{r_L},\ \hat{p}_{r_H},\ \hat{q}_{r_H})$ forms a BNE.

The proof of the existence of the BNE in Theorem 4.2 is thus completed.

B.2.6 Derivation of the BNE under SAS1

To solve $(\Theta_o, (\Theta_{r_L}, \Theta_{r_H}))$ with SAS1, we will solve (4.25) with $S_o = S$ and (4.22) and (4.23) with $S_r = S - q_o$ simultaneously for each possible and valid value of S. The closed-form optimal solution to (4.25) is available in Table 4.4 and the closed-form optimal solution to (4.22) and (4.23) are available in (4.24). In the following, we will illustrate in detail how to solve (4.25) with (4.22) and (4.23) simultaneously to obtain the BNE for each possible amount of S under SAS1.

(1) When the accessible shells S is less than half of the OEM's maximum demand given the REM's price p_{r_L} if she is type L or p_{r_H} if she is type H, i.e.,

$$0 \le S < \frac{1}{2} D_o(u_o, (p_{r_L}, p_{r_H})),$$

the OEM will use up all the accessible shells and leave no shell to either type of REM. That means, in this case,

$$q_{r_L} = q_{r_H} = 0 \text{ and } p_{r_L} = p_{r_H} = P_r. \tag{B.17}$$

Having the two types of REM sharing the same quantity and price in (B.17), the OEM's optimization problem in (4.25) becomes

$$\max\left[\min\left(S, D_o(p_o, P_r)\right)\right](p_o - u_o)$$

which is exactly the optimization problem in (4.5) with $p_r = P_r$ and its optimal solution is available in (4.6). Since $S < \frac{1}{2}D_o(u_o, (p_{r_L}, p_{r_H})) = \frac{1}{2}D_o(u_o, P_r)$, the OEM will take

$$q_o^* = S \text{ and } p_o^* = P_o - \frac{S}{B_o} \tag{B.18}$$

Therefore, solve (p_o, q_o), (p_{r_L}, q_{r_L}), and (p_{r_H}, q_{r_H}) in (B.17) and (B.18), the closed-form BNE of $(\Theta_o, (\Theta_{r_L}, \Theta_{r_H}))$ in this case can be found as

$$\left[p_o^*, q_o^*, p_{r_L}^*, q_{r_L}^*, p_{r_H}^*, q_{r_H}^*\right]_{11} = \begin{bmatrix} \frac{A_oB_r+A_rC_o-B_rS}{B_oB_r-C_oC_r} \\ S \\ \frac{A_oC_r+A_rB_o-C_rS}{B_oB_r-C_oC_r} \\ 0 \\ \frac{A_oC_r+A_rB_o-C_rS}{B_oB_r-C_oC_r} \\ 0 \end{bmatrix}^{\mathrm{T}} \tag{B.19}$$

This BNE applies to any value of S until it exceeds $\bar{S}_{11} = (1/2)D_o\left(u_o, (p_{r_L}, p_{r_H})\right)$ where $p_{r_L} = p_{r_H} = P_r$. By applying the expression of p_{r_L} and p_{r_H} in (B.19) to the expression of $\bar{S}_{11}$, we yield

$$\bar{S}_{11} = \frac{B_o\left(A_oB_r + A_rC_o - B_oB_ru_o + C_oC_ru_o\right)}{2B_oB_r - C_oC_r}. \tag{B.20}$$

That is, when $0 \leq S < \bar{S}_{11}$, the BNE for the remanufacturing competition problem with in complete information under SAS1 is in the form of (B.19).

(2) When the accessible shells S is greater than $\bar{S}_{11}$ but less than the sum of half of the OEM's max demand and half of the type-H REM's max demand, i.e.,

$$\frac{1}{2}D_o\left(u_o, (p_{r_L}, p_{r_H})\right) \leq S < \frac{1}{2}D_o\left(u_o, (p_{r_L}, p_{r_H})\right) + \frac{1}{2}D_{r_H}\left(p_o, u_{r_H}\right)$$

by Lemma 4.1 we get $S < \frac{1}{2}D_o\left(u_o, (p_{r_L}, p_{r_H})\right) + \frac{1}{2}D_{r_L}\left(p_o, u_{r_L}\right)$ as well. So the two types of REM would maximize her profit at

$$q_{r_L} = q_{r_H} = S - q_o \text{ and } p_{r_L} = p_{r_H} = P_r - \frac{S - q_o}{B_r}. \tag{B.21}$$

Similar to the above, the OEM's optimization problem in this case is transformed to the one in (4.5) again. Referring to its optimal solution in (4.6), we have the OEM maximize his profit at

$$q_o = \frac{1}{2}D_o\left(u_o, (p_{r_L}, p_{r_H})\right) \text{ and } p_o = \frac{1}{2}\left(u_o + P_o\right) \tag{B.22}$$

Solving (p_o, q_o), (p_{r_L}, q_{r_L}), and (p_{r_H}, q_{r_H}) in (B.22) and (B.21) altogether, we can obtain the BNE in this case as

$$\left[p_o^*, q_o^*, p_{r_L}^*, q_{r_L}^*, p_{r_H}^*, q_{r_H}^*\right]_{12} = \tag{B.23}$$
$$\begin{bmatrix} \frac{A_oB_r+A_rC_o+B_oB_ru_o-B_oC_ou_o-C_oS}{2B_oB_r-B_oC_o-C_oC_r} \\ \frac{B_o(A_oB_r+A_rC_o-B_oB_ru_o+C_oC_ru_o-C_oS)}{2B_oB_r-B_oC_o-C_oC_r} \\ \frac{A_oB_o+A_oC_r+2A_rB_o-B_o^2u_o+B_oC_Ru_o-2B_oS}{2B_oB_r-B_oC_o-C_oC_r} \\ \frac{-A_oB_oB_r-A_rB_oC_o+B_o^2B_ru_o+2B_oB_rS-B_oC_oC_ru_o-C_oC_rS}{2B_oB_r-B_oC_o-C_oC_r} \\ \frac{A_oB_o+A_oC_r+2A_rB_o-B_o^2u_o+B_oC_Ru_o-2B_oS}{2B_oB_r-B_oC_o-C_oC_r} \\ \frac{-A_oB_oB_r-A_rB_oC_o+B_o^2B_ru_o+2B_oB_rS-B_oC_oC_ru_o-C_oC_rS}{2B_oB_r-B_oC_o-C_oC_r} \end{bmatrix}^{\mathrm{T}}.$$

Again, this BNE expression applies to any value of S that is higher than $\bar{S}_{11}$ until it exceeds $\bar{S}_{12} = \frac{1}{2}D_o\left(u_o, \left(p_{r_L}, p_{r_H}\right)\right) + \frac{1}{2}D_{r_H}\left(p_o, u_{r_H}\right)$. By inserting the expressions of p_o^*, $p_{r_L}^*$, and $p_{r_H}^*$ in (B.23) to the expression of $\bar{S}_2$, we yield

$$\begin{aligned} \bar{S}_{12} = & \frac{2A_oB_oB_r+A_oB_rC_r+2A_rB_oB_r+A_rB_oC_o-2B_oB_r^2u_r-2B_o^2B_ru_o}{4B_oB_r-C_oC_r} \\ & + \frac{+B_oB_rC_ou_r+B_oB_rC_ru_o+B_oC_oC_ru_o+B_rC_oC_ru_r}{4B_oB_r-C_oC_r}. \end{aligned} \tag{B.24}$$

It means that when $\bar{S}_{11} \le S < \bar{S}_{12}$, the BNE in (B.23) applies.

(3) When S is greater than $\bar{S}_{12}$ but less than the sum of half of the OEM's max demand and half of the type-L REM's max demand, i.e.,

$$\bar{S}_{12} \le S < \frac{1}{2}D_o\left(u_o, \left(p_{r_L}, p_{r_H}\right)\right) + \frac{1}{2}D_{r_L}\left(p_o, u_{r_L}\right),$$

the two types of REM would response differently. That is, the type-L REM and the type-H REM will take, respectively,

$$q_{r_L} = S - q_o \text{ and } p_{r_L} = P_r - \frac{S - q_o}{B_r}, \tag{B.25}$$

and

$$q_{r_H} = \frac{1}{2}D_{r_H}\left(p_o, u_{r_H}\right) \text{ and } p_{r_H} = \frac{1}{2}\left(u_{r_H} + P_r\right). \tag{B.26}$$

Moreover, we can see that, in this case of S, the OEM's profit is maximized at

$$q_o = \frac{1}{2}D_o\left(u_o, \left(p_{r_L}, p_{r_H}\right)\right) \text{ and } p_o = \frac{1}{2}\left[u_o + \theta P_o\left(p_{r_L}\right) + (1-\theta)\, P_o\left(p_{r_H}\right)\right]. \tag{B.27}$$

Solving (B.25), (B.26), and (B.27) together, we can obtain the BNE in this case as well. Due to the lengthy expression of these BNE solutions, we write it as:

$$\left[p_o^*, q_o^*, p_{r_L}^*, q_{r_L}^*, p_{r_H}^*, q_{r_H}^*\right]_{13} = \tag{B.28}$$

$$\begin{bmatrix} \left(\frac{2A_oB_r+2B_rB_ou_o-2B_ou_oC_o\theta-B_ru_{r_H}C_o\theta-2SC_o\theta+A_rC_o+A_rC_o\theta+B_rr_{r_H}C_o}{4B_oB_r-2B_oC_o\theta-C_oC_r(1+\theta)}\right), \\ \left(\frac{B_o(2A_oB_r+A_rC_o(1+\theta)-2B_oB_ru_o+B_rC_o(1-\theta)u_{r_H}+C_oC_r(1+\theta)u_o-2C_o\theta S)}{4B_oB_r-2B_oC_o\theta-C_oC_r(1+\theta)}\right), \\ \left(\begin{array}{l} \frac{2A_oB_oB_r+2A_oB_rC_r+4A_rB_oB_r+A_rB_oC_o(1-\theta)-2B_o^2B_ru_o-4B_oB_rS+B_oB_rC_o(1-\theta)u_{r_H}}{B_r[4B_oB_r-2B_oC_o\theta-C_oC_r(1+\theta)]} \\ +\frac{2B_oB_rC_ru_O+B_oC_oC_r(1-\theta)u_o+B_rC_oC_r(1-\theta)u_{r_H}+C_oC_r(1-\theta)S}{B_r[4B_oB_r-2B_oC_o\theta-C_oC_r(1+\theta)]} \end{array}\right), \\ \left(\begin{array}{l} \frac{-2A_oB_oB_r-A_rB_oC_o(1+\theta)+2B_o^2B_ru_O-B_oB_rC_o(1-\theta)u_{r_H}}{4B_oB_r-2B_oC_o\theta-C_oC_r(1+\theta)} \\ +\frac{4B_oB_rS-B_oC_oC_r(1+\theta)u_o-C_oC_r(1+\theta)S}{4B_oB_r-2B_oC_o\theta-C_oC_r(1+\theta)} \end{array}\right), \\ \left(\begin{array}{l} \frac{A_oB_rC_r+2A_rB_oB_r-A_rB_oC_o\theta+2B_oB_r^2u_{r_H}}{B_r[4B_oB_r-2B_oC_o\theta-C_oC_r(1+\theta)]} \\ +\frac{B_oB_rC_ru_o-B_oB_rC_o\theta u_{r_H}-B_oC_oC_r\theta u_o-B_rC_oC_r\theta u_{r_H}-C_oC_r\theta S}{B_r[4B_oB_r-2B_oC_o\theta-C_oC_r(1+\theta)]} \end{array}\right), \\ \left(\begin{array}{l} \frac{A_oB_rC_r+2A_rB_oB_r-A_rB_oC_o\theta-2B_oB_r^2u_{r_H}}{4B_oB_r-2B_oC_o\theta-C_oC_r(1+\theta)} \\ +\frac{B_oB_rC_ru_o+B_oB_rC_o\theta u_{r_H}-B_oC_oC_r\theta u_o+B_rC_oC_ru_{r_H}-C_oC_r\theta S}{4B_oB_r-2B_oC_o\theta-C_oC_r(1+\theta)} \end{array}\right). \end{bmatrix}^{\mathrm{T}}.$$

Similarly, this BNE expression applies to any value of S that is higher than $\bar{S}_{12}$ until it exceeds $\bar{S}_{13} = \frac{1}{2}D_o\left(u_o, \left(p_{r_L}, p_{r_H}\right)\right) + \frac{1}{2}D_{r_L}\left(p_o, u_{r_L}\right)$. By inserting the expressions of p_o^*, $p_{r_L}^*$, and $p_{r_H}^*$ in (B.28) to the expression of $\bar{S}_{13}$, we yield

$$\begin{aligned} \bar{S}_{13} = & \frac{4A_OB_OB_R+2A_OB_RC_R+4A_RB_OB_R+2A_RB_OC_O-4B_O^2B_Rr_O-4B_OB_R^2r_{RL}}{4B_OB_R-C_OC_R} \\ + & \frac{2B_OB_RC_O[\theta r_{RL}+(1-\theta)r_{RH}]+2B_OB_RC_Rr_O+2B_OC_OC_Rr_O+B_RC_OC_R[(1+\theta)r_{RL}+(1-\theta)r_{RH}]}{4B_OB_R-C_OC_R} \end{aligned} . \tag{B.29}$$

That is, when $\bar{S}_{12} < S < \bar{S}_{13}$, the BNE in (B.28) applies.

(4) When S is greater than $\bar{S}_{13}$, the two types of REM would take

$$q_{r_L} = \frac{1}{2}D_{r_L}\left(p_o, u_{r_L}\right) \text{ and } p_{r_L} = \frac{1}{2}\left(u_{r_H} + P_r\right) \tag{B.30}$$

and

$$q_{r_H} = \frac{1}{2}D_{r_H}\left(p_o, u_{r_H}\right) \text{ and } p_{r_H} = \frac{1}{2}\left(u_{r_H} + P_r\right) \tag{B.31}$$

respectively and the OEM would take

$$q_o = \frac{1}{2}D_o\left(u_o, p_{r_L}, p_{r_H}\right) \text{ and } p_o = \frac{1}{2}\left[u_o + \theta P_o\left(p_{r_L}\right) + (1-\theta)\,P_o\left(p_{r_H}\right)\right]. \tag{B.32}$$

Solving (B.31), (B.30) and (B.32), we can obtain the BNE as

$$\left[p_o^*, q_o^*, p_{r_L}^*, q_{r_L}^*, p_{r_H}^*, q_{r_H}^*\right]_{14} = \tag{B.33}$$
$$\begin{bmatrix} \frac{2A_oB_r+A_rC_o+2B_oB_ru_o+B_rC_o\left[\theta u_{r_L}+(1-\theta)u_{r_H}\right]}{4B_oB_r-C_oC_r} \\ \frac{B_o\left\{2A_oB_r+A_rC_o-2B_oB_ru_o+B_rC_o\left[\theta u_{r_L}+(1-\theta)u_{r_H}\right]+C_oC_ru_o\right\}}{4B_oB_r-C_oC_r} \\ \frac{2A_oC_r+4A_rB_o+4B_oB_ru_{r_L}+2B_oC_ru_o+C_oC_r\left[(1-\theta)u_{r_H}-(1-\theta)u_{r_L}\right]}{2(4B_oB_r-C_oC_r)} \\ \frac{B_r(2A_oC_r+4A_rB_o-4B_oB_ru_{r_L}+2B_oC_ru_o+C_oC_r\left[(1+\theta)u_{r_L}+(1-\theta)u_{r_H}\right]}{2(4B_oB_r-C_oC_r)} \\ \frac{2A_oC_r+4A_rB_o+4B_oB_ru_{r_H}+2B_oC_ru_o+C_oC_r\theta\left(u_{r_L}-u_{r_H}\right)}{2(4B_oB_r-C_oC_r)} \\ \frac{B_r\left\{2A_oC_r+4A_rB_o-4B_oB_ru_{r_H}+2B_oC_ru_o+C_oC_r\left[2u_{r_H}+\theta\left(u_{r_L}-u_{r_H}\right)\right]\right\}}{2(4B_oB_r-C_oC_r)} \end{bmatrix}^{\mathrm{T}}.$$

This BNE expression applies to any value of S that is greater than $\bar{S}_{13}$.

B.2.7 Numerical Results for Sensitivity Analysis

				Symmetric						REM Preferred						OEM Preferred					
u_o	u_{r_H}	u_{r_L}	θ	$\bar{S}_{11}$	$\bar{S}_{12}$	$\bar{S}_{13}$	π_o	π_{r_H}	π_{r_L}	$\bar{S}_{11}$	$\bar{S}_{12}$	$\bar{S}_{13}$	π_o	π_{r_H}	π_{r_L}	$\bar{S}_{11}$	$\bar{S}_{12}$	$\bar{S}_{13}$	π_o	π_{r_H}	π_{r_L}
0.2		0.1		8.4000		13.1333	21.1684		21.9564	5.1818		9.2609	4.4461		15.7287	6.1818		9.3087	15.0533		4.8686
0.2	0.3	0.1	1	8.4000	13.0000	13.1333	21.1684	20.6510	21.9564	5.1818	9.1217	9.2609	4.4461	14.6270	15.7287	6.1818	9.0739	9.3087	15.0533	4.1342	4.8686
0.2	0.3	0.1	0.9	8.4000	13.0000	13.1400	21.2031	20.6596	21.9652	5.1818	9.1217	9.2670	4.4588	14.6317	15.7336	6.1818	9.0739	9.3152	15.0820	4.1373	4.8719
0.2	0.3	0.1	0.8	8.4000	13.0000	13.1467	21.2378	20.6682	21.9740	5.1818	9.1217	9.2730	4.4716	14.6364	15.7385	6.1818	9.0739	9.3217	15.1107	4.1403	4.8752
0.2	0.3	0.1	0.7	8.4000	13.0000	13.1533	21.2726	20.6767	21.9829	5.1818	9.1217	9.2791	4.4843	14.6411	15.7434	6.1818	9.0739	9.3283	15.1394	4.1434	4.8785
0.2	0.3	0.1	0.6	8.4000	13.0000	13.1600	21.3074	20.6853	21.9917	5.1818	9.1217	9.2852	4.4971	14.6458	15.7482	6.1818	9.0739	9.3348	15.1681	4.1465	4.8819
0.2	0.3	0.1	0.5	8.4000	13.0000	13.1667	21.3422	20.6939	22.0006	5.1818	9.1217	9.2913	4.5099	14.6505	15.7531	6.1818	9.0739	9.3413	15.1968	4.1495	4.8852
0.2	0.3	0.1	0.4	8.4000	13.0000	13.1733	21.3771	20.7025	22.0094	5.1818	9.1217	9.2974	4.5227	14.6552	15.7580	6.1818	9.0739	9.3478	15.2256	4.1526	4.8885
0.2	0.3	0.1	0.3	8.4000	13.0000	13.1800	21.4120	20.7110	22.0182	5.1818	9.1217	9.3035	4.5355	14.6599	15.7629	6.1818	9.0739	9.3543	15.2544	4.1557	4.8919
0.2	0.3	0.1	0.2	8.4000	13.0000	13.1867	21.4469	20.7196	22.0271	5.1818	9.1217	9.3096	4.5483	14.6646	15.7678	6.1818	9.0739	9.3609	15.2832	4.1588	4.8952
0.2	0.3	0.1	0.1	8.4000	13.0000	13.1933	21.4818	20.7282	22.0359	5.1818	9.1217	9.3157	4.5612	14.6694	15.7727	6.1818	9.0739	9.3674	15.3121	4.1618	4.8985
0.2	0.3	0.1	0	8.4000	13.0000	13.2000	21.5168	20.7368	22.0448	5.1818	9.1217	9.3217	4.5741	14.6741	15.7775	6.1818	9.0739	9.3739	15.3410	4.1649	4.9019
0.2	3.0		0	8.4000	13.0000		21.5168	20.7368		5.1818	9.1217		4.5741	14.6741		6.1818	9.0739		15.3410	4.1649	
0.4		0.1		8.2286		13.0000	19.9712		22.3112	4.9091		9.0261	3.7749		16.0227	6.0000		9.1696	14.0220		5.0024
0.4	0.3	0.1	1	8.2286	12.8667	13.0000	19.9712	20.9952	22.3112	4.9091	8.8870	9.0261	3.7749	14.9105	16.0227	6.0000	8.9348	9.1696	14.0220	4.2576	5.0024
0.4	0.3	0.1	0.9	8.2286	12.8667	13.0067	20.0049	21.0038	22.3201	4.9091	8.8870	9.0322	3.7866	14.9153	16.0276	6.0000	8.9348	9.1761	14.0496	4.2607	5.0058
0.4	0.3	0.1	0.8	8.2286	12.8667	13.0133	20.0387	21.0125	22.3290	4.9091	8.8870	9.0383	3.7983	14.9200	16.0326	6.0000	8.9348	9.1826	14.0773	4.2638	5.0091
0.4	0.3	0.1	0.7	8.2286	12.8667	13.0200	20.0724	21.0211	22.3379	4.9091	8.8870	9.0443	3.8101	14.9248	16.0375	6.0000	8.9348	9.1891	14.1050	4.2669	5.0125
0.4	0.3	0.1	0.6	8.2286	12.8667	13.0267	20.1063	21.0298	22.3468	4.9091	8.8870	9.0504	3.8219	14.9295	16.0424	6.0000	8.9348	9.1957	14.1327	4.2701	5.0159
0.4	0.3	0.1	0.5	8.2286	12.8667	13.0333	20.1401	21.0384	22.3558	4.9091	8.8870	9.0565	3.8336	14.9343	16.0473	6.0000	8.9348	9.2022	14.1605	4.2732	5.0193
0.4	0.3	0.1	0.4	8.2286	12.8667	13.0400	20.1740	21.0471	22.3647	4.9091	8.8870	9.0626	3.8455	14.9391	16.0523	6.0000	8.9348	9.2087	14.1882	4.2763	5.0226
0.4	0.3	0.1	0.3	8.2286	12.8667	13.0467	20.2078	21.0557	22.3736	4.9091	8.8870	9.0687	3.8573	14.9438	16.0572	6.0000	8.9348	9.2152	14.2160	4.2794	5.0260
0.4	0.3	0.1	0.2	8.2286	12.8667	13.0533	20.2418	21.0644	22.3825	4.9091	8.8870	9.0748	3.8691	14.9486	16.0621	6.0000	8.9348	9.2217	14.2439	4.2825	5.0294
0.4	0.3	0.1	0.1	8.2286	12.8667	13.0600	20.2757	21.0730	22.3914	4.9091	8.8870	9.0809	3.8810	14.9533	16.0671	6.0000	8.9348	9.2283	14.2717	4.2856	5.0328
0.4	0.3	0.1	0	8.2286	12.8667	13.0667	20.3097	21.0817	22.4004	4.9091	8.8870	9.0870	3.8929	14.9581	16.0720	6.0000	8.9348	9.2348	14.2996	4.2888	5.0361
0.4	0.3			8.2286	12.8667		20.3097	21.0817		4.9091	8.8870		3.8929	14.9581		6.0000	8.9348		14.2996	4.2888	
0.4		0.5		8.2286	12.7333		20.3097		21.0817	4.9091		8.8870	3.8929		14.9581	6.0000		8.9348	14.2996		4.2888
0.4	0.5	0.0	1	8.2286	12.7333	12.8667	20.3097	19.8030	21.0817	4.9091	8.7478	8.8870	3.8929	13.8842	14.9581	6.0000	8.7000	8.9348	14.2996	3.6014	4.2888
0.4	0.5	0.3	0.9	8.2286	12.7333	12.8733	20.3437	19.8114	21.0903	4.9091	8.7478	8.8930	3.9048	13.8887	14.9628	6.0000	8.7000	8.9413	14.3275	3.6042	4.2919
0.4	0.5	0.3	0.8	8.2286	12.7333	12.8800	20.3777	19.8198	21.0990	4.9091	8.7478	8.8991	3.9167	13.8933	14.9676	6.0000	8.7000	8.9478	14.3555	3.6071	4.2950
0.4	0.5	0.3	0.7	8.2286	12.7333	12.8867	20.4118	19.8282	21.1077	4.9091	8.7478	8.9052	3.9286	13.8979	14.9723	6.0000	8.7000	8.9543	14.3834	3.6099	4.2981
0.4	0.5	0.3	0.6	8.2286	12.7333	12.8933	20.4459	19.8366	21.1163	4.9091	8.7478	8.9113	3.9405	13.9025	14.9771	6.0000	8.7000	8.9609	14.4114	3.6128	4.3012
0.4	0.5	0.3	0.5	8.2286	12.7333	12.9000	20.4800	19.8450	21.1250	4.9091	8.7478	8.9174	3.9525	13.9071	14.9819	6.0000	8.7000	8.9674	14.4395	3.6157	4.3044
0.4	0.5	0.3	0.4	8.2286	12.7333	12.9067	20.5141	19.8534	21.1337	4.9091	8.7478	8.9235	3.9645	13.9117	14.9866	6.0000	8.7000	8.9739	14.4675	3.6185	4.3075
0.4	0.5	0.3	0.3	8.2286	12.7333	12.9133	20.5483	19.8618	21.1423	4.9091	8.7478	8.9296	3.9765	13.9163	14.9914	6.0000	8.7000	8.9804	14.4956	3.6214	4.3106
0.4	0.5	0.3	0.2	8.2286	12.7333	12.9200	20.5825	19.8702	21.1510	4.9091	8.7478	8.9357	3.9885	13.9208	14.9961	6.0000	8.7000	8.9870	14.5237	3.6243	4.3137
0.4	0.5	0.3	0.1	8.2286	12.7333	12.9267	20.6168	19.8786	21.1597	4.9091	8.7478	8.9417	4.0006	13.9254	15.0009	6.0000	8.7000	8.9935	14.5518	3.6271	4.3169
0.4	0.5	0.3	0	8.2286	12.7333	12.9333	20.6510	19.8870	21.1684	4.9091	8.7478	8.9478	4.0126	13.9300	15.0057	6.0000	8.7000	9.0000	14.5800	3.6300	4.3200
0.4	0.5			8.2286	12.7333		20.6510	19.8870		4.9091	8.7478		4.0126	13.9300		6.0000	8.7000		14.5800	3.6300	
0.6		0.1		8.0571		12.8667	18.8089		22.6689	4.6364		8.7913	3.1586		16.3194	5.8182		9.0304	13.0272		5.1381
0.6	0.3	0.1	1	8.0571	12.7333	12.8667	18.8089	21.3422	22.6689	4.6364	8.6522	8.7913	3.1586	15.1968	16.3194	5.8182	8.7957	9.0304	13.0272	4.3828	5.1381
0.6	0.3	0.1	0.9	8.0571	12.7333	12.8733	18.8416	21.3509	22.6779	4.6364	8.6522	8.7974	3.1693	15.2016	16.3244	5.8182	8.7957	9.0370	13.0538	4.3860	5.1415
0.6	0.3	0.1	0.8	8.0571	12.7333	12.8800	18.8744	21.3596	22.6868	4.6364	8.6522	8.8035	3.1800	15.2064	16.3294	5.8182	8.7957	9.0435	13.0805	4.3891	5.1449
0.6	0.3	0.1	0.7	8.0571	12.7333	12.8867	18.9072	21.3684	22.6958	4.6364	8.6522	8.8096	3.1908	15.2112	16.3343	5.8182	8.7957	9.0500	13.1072	4.3923	5.1483
0.6	0.3	0.1	0.6	8.0571	12.7333	12.8933	18.9400	21.3771	22.7048	4.6364	8.6522	8.8157	3.2015	15.2160	16.3393	5.8182	8.7957	9.0565	13.1339	4.3955	5.1517
0.6	0.3	0.1	0.5	8.0571	12.7333	12.9000	18.9728	21.3858	22.7138	4.6364	8.6522	8.8217	3.2123	15.2208	16.3443	5.8182	8.7957	9.0630	13.1607	4.3986	5.1551
0.6	0.3	0.1	0.4	8.0571	12.7333	12.9067	19.0057	21.3945	22.7228	4.6364	8.6522	8.8278	3.2231	15.2256	16.3493	5.8182	8.7957	9.0696	13.1875	4.4018	5.1586
0.6	0.3	0.1	0.3	8.0571	12.7333	12.9133	19.0386	21.4032	22.7318	4.6364	8.6522	8.8339	3.2340	15.2304	16.3542	5.8182	8.7957	9.0761	13.2143	4.4049	5.1620
0.6	0.3	0.1	0.2	8.0571	12.7333	12.9200	19.0715	21.4120	22.7408	4.6364	8.6522	8.8400	3.2448	15.2352	16.3592	5.8182	8.7957	9.0826	13.2411	4.4081	5.1654
0.6	0.3	0.1	0.1	8.0571	12.7333	12.9267	19.1044	21.4207	22.7498	4.6364	8.6522	8.8461	3.2557	15.2400	16.3642	5.8182	8.7957	9.0891	13.2680	4.4113	5.1688
0.6	0.3	0.1	0	8.0571	12.7333	12.9333	19.1374	21.4294	22.7588	4.6364	8.6522	8.8522	3.2665	15.2448	16.3691	5.8182	8.7957	9.0957	13.2949	4.4144	5.1722
0.6	0.3			8.0571	12.7333		19.1374	21.4294		4.6364	8.6522		3.2665	15.2448		5.8182	8.7957		13.2949	4.4144	

Table B.1: Numerical Results for Sensitivity Analysis - 1/4

				Symmetric						REM Preferred						OEM Preferred					
u_o	u_{r_H}	u_{r_L}	θ	$\bar{S}_{11}$	$\bar{S}_{12}$	$\bar{S}_{13}$	π_o	π_{r_H}	π_{r_L}	$\bar{S}_{11}$	$\bar{S}_{12}$	$\bar{S}_{13}$	π_o	π_{r_H}	π_{r_L}	$\bar{S}_{11}$	$\bar{S}_{12}$	$\bar{S}_{13}$	π_o	π_{r_H}	π_{r_L}
0.6		0.3		8.0571		12.7333	19.1374		21.4294	4.6364		8.6522	3.2665		15.2448	5.8182		8.7957	13.2949		4.4144
0.6	0.5	0.3	1	8.0571	12.6000	12.7333	19.1374	20.1401	21.4294	4.6364	8.5130	8.6522	3.2665	14.1605	15.2448	5.8182	8.5609	8.7957	13.2949	3.7166	4.4144
0.6	0.5	0.3	0.9	8.0571	12.6000	12.7400	19.1704	20.1486	21.4382	4.6364	8.5130	8.6583	3.2774	14.1651	15.2496	5.8182	8.5609	8.8022	13.3218	3.7195	4.4176
0.6	0.5	0.3	0.8	8.0571	12.6000	12.7467	19.2035	20.1570	21.4469	4.6364	8.5130	8.6643	3.2884	14.1697	15.2544	5.8182	8.5609	8.8087	13.3487	3.7224	4.4208
0.6	0.5	0.3	0.7	8.0571	12.6000	12.7533	19.2365	20.1655	21.4556	4.6364	8.5130	8.6704	3.2993	14.1743	15.2592	5.8182	8.5609	8.8152	13.3757	3.7253	4.4239
0.6	0.5	0.3	0.6	8.0571	12.6000	12.7600	19.2696	20.1740	21.4644	4.6364	8.5130	8.6765	3.3102	14.1790	15.2640	5.8182	8.5609	8.8217	13.4027	3.7282	4.4271
0.6	0.5	0.3	0.5	8.0571	12.6000	12.7667	19.3028	20.1824	21.4731	4.6364	8.5130	8.6826	3.3212	14.1836	15.2688	5.8182	8.5609	8.8283	13.4297	3.7311	4.4303
0.6	0.5	0.3	0.4	8.0571	12.6000	12.7733	19.3359	20.1909	21.4818	4.6364	8.5130	8.6887	3.3322	14.1882	15.2736	5.8182	8.5609	8.8348	13.4568	3.7340	4.4334
0.6	0.5	0.3	0.3	8.0571	12.6000	12.7800	19.3691	20.1994	21.4906	4.6364	8.5130	8.6948	3.3432	14.1929	15.2784	5.8182	8.5609	8.8413	13.4839	3.7370	4.4366
0.6	0.5	0.3	0.2	8.0571	12.6000	12.7867	19.4023	20.2078	21.4993	4.6364	8.5130	8.7009	3.3542	14.1975	15.2832	5.8182	8.5609	8.8478	13.5110	3.7399	4.4398
0.6	0.5	0.3	0.1	8.0571	12.6000	12.7933	19.4355	20.2163	21.5081	4.6364	8.5130	8.7070	3.3653	14.2021	15.2880	5.8182	8.5609	8.8543	13.5381	3.7428	4.4429
0.6	0.5	0.3	0	8.0571	12.6000	12.8000	19.4688	20.2248	21.5168	4.6364	8.5130	8.7130	3.3763	14.2068	15.2929	5.8182	8.5609	8.8609	13.5653	3.7457	4.4461
0.6	0.5			8.0571	12.6000		19.4688	20.2248		4.6364	8.5130		3.3763	14.2068		5.8182	8.5609		13.5653	3.7457	
0.6		0.5		8.0571		12.6000	19.4688		20.2248	4.6364		8.5130	3.3763		14.2068	5.8182		8.5609	13.5653		3.7457
0.6	0.7	0.5	1	8.0571	12.4667	12.6000	19.4688	18.9728	20.2248	4.6364	8.3739	8.5130	3.3763	13.1607	14.2068	5.8182	8.3261	8.5609	13.5653	3.1053	3.7457
0.6	0.7	0.5	0.9	8.0571	12.4667	12.6067	19.5021	18.9810	20.2333	4.6364	8.3739	8.5191	3.3874	13.1651	14.2114	5.8182	8.3261	8.5674	13.5924	3.1079	3.7486
0.6	0.7	0.5	0.8	8.0571	12.4667	12.6133	19.5354	18.9892	20.2418	4.6364	8.3739	8.5252	3.3985	13.1696	14.2160	5.8182	8.3261	8.5739	13.6197	3.1106	3.7515
0.6	0.7	0.5	0.7	8.0571	12.4667	12.6200	19.5688	18.9974	20.2502	4.6364	8.3739	8.5313	3.4096	13.1741	14.2207	5.8182	8.3261	8.5804	13.6469	3.1132	3.7544
0.6	0.7	0.5	0.6	8.0571	12.4667	12.6267	19.6021	19.0057	20.2587	4.6364	8.3739	8.5374	3.4208	13.1785	14.2253	5.8182	8.3261	8.5870	13.6742	3.1159	3.7574
0.6	0.7	0.5	0.5	8.0571	12.4667	12.6333	19.6356	19.0139	20.2672	4.6364	8.3739	8.5435	3.4319	13.1830	14.2300	5.8182	8.3261	8.5935	13.7015	3.1185	3.7603
0.6	0.7	0.5	0.4	8.0571	12.4667	12.6400	19.6690	19.0221	20.2757	4.6364	8.3739	8.5496	3.4431	13.1875	14.2346	5.8182	8.3261	8.6000	13.7288	3.1212	3.7632
0.6	0.7	0.5	0.3	8.0571	12.4667	12.6467	19.7025	19.0303	20.2842	4.6364	8.3739	8.5557	3.4543	13.1919	14.2392	5.8182	8.3261	8.6065	13.7562	3.1239	3.7661
0.6	0.7	0.5	0.2	8.0571	12.4667	12.6533	19.7360	19.0386	20.2927	4.6364	8.3739	8.5617	3.4655	13.1964	14.2439	5.8182	8.3261	8.6130	13.7835	3.1265	3.7690
0.6	0.7	0.5	0.1	8.0571	12.4667	12.6600	19.7695	19.0468	20.3012	4.6364	8.3739	8.5678	3.4767	13.2009	14.2485	5.8182	8.3261	8.6196	13.8109	3.1292	3.7720
0.6	0.7	0.5	0	8.0571	12.4667	12.6667	19.8030	19.0550	20.3097	4.6364	8.3739	8.5739	3.4879	13.2053	14.2532	5.8182	8.3261	8.6261	13.8384	3.1319	3.7749
0.6	0.7			8.0571	12.4667		19.8030	19.0550		4.6364	8.3739		3.4879	13.2053		5.8182	8.3261		13.8384	3.1319	
0.8		0.1		7.8857		12.7333	17.6814		23.0294	4.3636		8.5565	2.5971		16.6189	5.6364		8.8913	12.0690		5.2755
0.8	0.3	0.1	1	7.8857	12.6000	12.7333	17.6814	21.6921	23.0294	4.3636	8.4174	8.5565	2.5971	15.4858	16.6189	5.6364	8.6565	8.8913	12.0690	4.5099	5.2755
0.8	0.3	0.1	0.9	7.8857	12.6000	12.7400	17.7132	21.7009	23.0385	4.3636	8.4174	8.5626	2.6068	15.4907	16.6239	5.6364	8.6565	8.8978	12.0946	4.5131	5.2790
0.8	0.3	0.1	0.8	7.8857	12.6000	12.7467	17.7449	21.7097	23.0475	4.3636	8.4174	8.5687	2.6166	15.4955	16.6289	5.6364	8.6565	8.9043	12.1203	4.5163	5.2824
0.8	0.3	0.1	0.7	7.8857	12.6000	12.7533	17.7767	21.7184	23.0566	4.3636	8.4174	8.5748	2.6263	15.5003	16.6339	5.6364	8.6565	8.9109	12.1460	4.5195	5.2859
0.8	0.3	0.1	0.6	7.8857	12.6000	12.7600	17.8085	21.7272	23.0656	4.3636	8.4174	8.5809	2.6361	15.5052	16.6389	5.6364	8.6565	8.9174	12.1717	4.5227	5.2894
0.8	0.3	0.1	0.5	7.8857	12.6000	12.7667	17.8404	21.7360	23.0747	4.3636	8.4174	8.5870	2.6459	15.5100	16.6439	5.6364	8.6565	8.9239	12.1975	4.5259	5.2928
0.8	0.3	0.1	0.4	7.8857	12.6000	12.7733	17.8722	21.7448	23.0837	4.3636	8.4174	8.5930	2.6557	15.5149	16.6490	5.6364	8.6565	8.9304	12.2233	4.5291	5.2963
0.8	0.3	0.1	0.3	7.8857	12.6000	12.7800	17.9041	21.7536	23.0928	4.3636	8.4174	8.5991	2.6655	15.5197	16.6540	5.6364	8.6565	8.9370	12.2491	4.5323	5.2998
0.8	0.3	0.1	0.2	7.8857	12.6000	12.7867	17.9361	21.7624	23.1019	4.3636	8.4174	8.6052	2.6754	15.5246	16.6590	5.6364	8.6565	8.9435	12.2749	4.5355	5.3032
0.8	0.3	0.1	0.1	7.8857	12.6000	12.7933	17.9680	21.7712	23.1109	4.3636	8.4174	8.6113	2.6852	15.5294	16.6640	5.6364	8.6565	8.9500	12.3008	4.5387	5.3067
0.8	0.3	0.1	0	7.8857	12.6000	12.8000	18.0000	21.7800	23.1200	4.3636	8.4174	8.6174	2.6951	15.5343	16.6690	5.6364	8.6565	8.9565	12.3267	4.5419	5.3102
0.8	0.3			7.8857	12.6000		18.0000	21.7800		4.3636	8.4174		2.6951	15.5343		5.6364	8.6565		12.3267	4.5419	
0.8		0.3		7.8857		12.6000	18.0000		21.7800	4.3636		8.4174	2.6951		15.5343	5.6364		8.6565	12.3267		4.5419
0.8	0.5	0.3	1	7.8857	12.4667	12.6000	18.0000	20.4800	21.7800	4.3636	8.2783	8.4174	2.6951	14.4395	15.5343	5.6364	8.4217	8.6565	12.3267	3.8336	4.5419
0.8	0.5	0.3	0.9	7.8857	12.4667	12.6067	18.0320	20.4885	21.7888	4.3636	8.2783	8.4235	2.7050	14.4441	15.5391	5.6364	8.4217	8.6630	12.3526	3.8366	4.5451
0.8	0.5	0.3	0.8	7.8857	12.4667	12.6133	18.0641	20.4971	21.7976	4.3636	8.2783	8.4296	2.7149	14.4488	15.5439	5.6364	8.4217	8.6696	12.3786	3.8395	4.5483
0.8	0.5	0.3	0.7	7.8857	12.4667	12.6200	18.0961	20.5056	21.8064	4.3636	8.2783	8.4357	2.7249	14.4535	15.5488	5.6364	8.4217	8.6761	12.4045	3.8425	4.5515
0.8	0.5	0.3	0.6	7.8857	12.4667	12.6267	18.1282	20.5141	21.8152	4.3636	8.2783	8.4417	2.7348	14.4582	15.5536	5.6364	8.4217	8.6826	12.4305	3.8455	4.5548
0.8	0.5	0.3	0.5	7.8857	12.4667	12.6333	18.1604	20.5227	21.8240	4.3636	8.2783	8.4478	2.7448	14.4628	15.5585	5.6364	8.4217	8.6891	12.4566	3.8484	4.5580
0.8	0.5	0.3	0.4	7.8857	12.4667	12.6400	18.1925	20.5312	21.8328	4.3636	8.2783	8.4539	2.7548	14.4675	15.5633	5.6364	8.4217	8.6957	12.4826	3.8514	4.5612
0.8	0.5	0.3	0.3	7.8857	12.4667	12.6467	18.2247	20.5398	21.8416	4.3636	8.2783	8.4600	2.7648	14.4722	15.5682	5.6364	8.4217	8.7022	12.5087	3.8543	4.5644
0.8	0.5	0.3	0.2	7.8857	12.4667	12.6533	18.2569	20.5483	21.8505	4.3636	8.2783	8.4661	2.7748	14.4769	15.5731	5.6364	8.4217	8.7087	12.5348	3.8573	4.5676
0.8	0.5	0.3	0.1	7.8857	12.4667	12.6600	18.2892	20.5569	21.8593	4.3636	8.2783	8.4722	2.7849	14.4816	15.5779	5.6364	8.4217	8.7152	12.5609	3.8602	4.5708
0.8	0.5	0.3	0	7.8857	12.4667	12.6667	18.3214	20.5654	21.8681	4.3636	8.2783	8.4783	2.7949	14.4862	15.5828	5.6364	8.4217	8.7217	12.5871	3.8632	4.5741
0.8	0.5			7.8857	12.4667		18.3214	20.5654		4.3636	8.2783		2.7949	14.4862		5.6364	8.4217		12.5871	3.8632	

Table B.2: Numerical Results for Sensitivity Analysis - 2/4

				Symmetric						REM Preferred						OEM Preferred					
u_o	u_{r_H}	u_{r_L}	θ	$\bar{s}_{11}$	$\bar{s}_{12}$	$\bar{s}_{13}$	π_o	π_{r_H}	π_{r_L}	$\bar{s}_{11}$	$\bar{s}_{12}$	$\bar{s}_{13}$	π_o	π_{r_H}	π_{r_L}	$\bar{s}_{11}$	$\bar{s}_{12}$	$\bar{s}_{13}$	π_o	π_{r_H}	π_{r_L}
0.8		0.5		7.8857		12.4667	18.3214		20.5654	4.3636		8.2783	2.7949		14.4862	5.6364		8.4217	12.5871		3.8632
0.8	0.7	0.5	1	7.8857	12.3333	12.4667	18.3214	19.3028	20.5654	4.3636	8.1391	8.2783	2.7949	13.4297	14.4862	5.6364	8.1870	8.4217	12.5871	3.2123	3.8632
0.8	0.7	0.5	0.9	7.8857	12.3333	12.4733	18.3537	19.3110	20.5740	4.3636	8.1391	8.2843	2.8050	13.4342	14.4909	5.6364	8.1870	8.4283	12.6133	3.2150	3.8662
0.8	0.7	0.5	0.8	7.8857	12.3333	12.4800	18.3860	19.3193	20.5825	4.3636	8.1391	8.2904	2.8151	13.4387	14.4956	5.6364	8.1870	8.4348	12.6395	3.2177	3.8691
0.8	0.7	0.5	0.7	7.8857	12.3333	12.4867	18.4184	19.3276	20.5911	4.3636	8.1391	8.2965	2.8252	13.4432	14.5003	5.6364	8.1870	8.4413	12.6658	3.2204	3.8721
0.8	0.7	0.5	0.6	7.8857	12.3333	12.4933	18.4508	19.3359	20.5996	4.3636	8.1391	8.3026	2.8354	13.4477	14.5050	5.6364	8.1870	8.4478	12.6920	3.2231	3.8750
0.8	0.7	0.5	0.5	7.8857	12.3333	12.5000	18.4832	19.3442	20.6082	4.3636	8.1391	8.3087	2.8455	13.4523	14.5097	5.6364	8.1870	8.4543	12.7183	3.2258	3.8780
0.8	0.7	0.5	0.4	7.8857	12.3333	12.5067	18.5156	19.3525	20.6168	4.3636	8.1391	8.3148	2.8557	13.4568	14.5143	5.6364	8.1870	8.4609	12.7447	3.2285	3.8810
0.8	0.7	0.5	0.3	7.8857	12.3333	12.5133	18.5481	19.3608	20.6253	4.3636	8.1391	8.3209	2.8659	13.4613	14.5190	5.6364	8.1870	8.4674	12.7710	3.2312	3.8839
0.8	0.7	0.5	0.2	7.8857	12.3333	12.5200	18.5806	19.3691	20.6339	4.3636	8.1391	8.3270	2.8761	13.4658	14.5237	5.6364	8.1870	8.4739	12.7974	3.2340	3.8869
0.8	0.7	0.5	0.1	7.8857	12.3333	12.5267	18.6131	19.3774	20.6425	4.3636	8.1391	8.3330	2.8863	13.4703	14.5284	5.6364	8.1870	8.4804	12.8238	3.2367	3.8899
0.8	0.7	0.5	0	7.8857	12.3333	12.5333	18.6457	19.3857	20.6510	4.3636	8.1391	8.3391	2.8966	13.4748	14.5331	5.6364	8.1870	8.4870	12.8502	3.2394	3.8929
0.8	0.7			7.8857	12.3333		18.6457	19.3857		4.3636	8.1391		2.8966	13.4748		5.6364	8.1870		12.8502	3.2394	
0.8		0.7		7.8857		12.3333	18.6457		19.3857	4.3636		8.1391	2.8966		13.4748	5.6364		8.1870	12.8502		3.2394
0.8	0.9	0.7	1	7.8857	12.2000	12.3333	18.6457	18.1604	19.3857	4.3636	8.0000	8.1391	2.8966	12.4566	13.4748	5.6364	7.9522	8.1870	12.8502	2.6459	3.2394
0.8	0.9	0.7	0.9	7.8857	12.2000	12.3400	18.6783	18.1684	19.3940	4.3636	8.0000	8.1452	2.9068	12.4609	13.4793	5.6364	7.9522	8.1935	12.8767	2.6483	3.2421
0.8	0.9	0.7	0.8	7.8857	12.2000	12.3467	18.7109	18.1764	19.4023	4.3636	8.0000	8.1513	2.9171	12.4652	13.4839	5.6364	7.9522	8.2000	12.9032	2.6508	3.2448
0.8	0.9	0.7	0.7	7.8857	12.2000	12.3533	18.7435	18.1845	19.4106	4.3636	8.0000	8.1574	2.9274	12.4696	13.4884	5.6364	7.9522	8.2065	12.9297	2.6533	3.2475
0.8	0.9	0.7	0.6	7.8857	12.2000	12.3600	18.7762	18.1925	19.4189	4.3636	8.0000	8.1635	2.9377	12.4739	13.4929	5.6364	7.9522	8.2130	12.9563	2.6557	3.2502
0.8	0.9	0.7	0.5	7.8857	12.2000	12.3667	18.8089	18.2006	19.4272	4.3636	8.0000	8.1696	2.9481	12.4783	13.4974	5.6364	7.9522	8.2196	12.9828	2.6582	3.2529
0.8	0.9	0.7	0.4	7.8857	12.2000	12.3733	18.8416	18.2086	19.4355	4.3636	8.0000	8.1757	2.9584	12.4826	13.5019	5.6364	7.9522	8.2261	13.0094	2.6606	3.2557
0.8	0.9	0.7	0.3	7.8857	12.2000	12.3800	18.8744	18.2166	19.4438	4.3636	8.0000	8.1817	2.9688	12.4870	13.5064	5.6364	7.9522	8.2326	13.0361	2.6631	3.2584
0.8	0.9	0.7	0.2	7.8857	12.2000	12.3867	18.9072	18.2247	19.4522	4.3636	8.0000	8.1878	2.9792	12.4913	13.5110	5.6364	7.9522	8.2391	13.0627	2.6655	3.2611
0.8	0.9	0.7	0.1	7.8857	12.2000	12.3933	18.9400	18.2327	19.4605	4.3636	8.0000	8.1939	2.9896	12.4957	13.5155	5.6364	7.9522	8.2457	13.0894	2.6680	3.2638
0.8	0.9	0.7	0	7.8857	12.2000	12.4000	18.9728	18.2408	19.4688	4.3636	8.0000	8.2000	3.0000	12.5000	13.5200	5.6364	7.9522	8.2522	13.1161	2.6705	3.2665
0.8	0.9			7.8857	12.2000		18.9728	18.2408		4.3636	8.0000		3.0000	12.5000		5.6364	7.9522		13.1161	2.6705	
1.0		0.1		7.7143		12.6000	16.5888		23.3928	4.0909		8.3217	2.0906		16.9210	5.4545		8.7522	11.1474		5.4148
1.0	0.3	0.1	1	7.7143	12.4667	12.6000	16.5888	22.0448	23.3928	4.0909	8.1826	8.3217	2.0906	15.7775	16.9210	5.4545	8.5174	8.7522	11.1474	4.6387	5.4148
1.0	0.3	0.1	0.9	7.7143	12.4667	12.6067	16.6195	22.0537	23.4019	4.0909	8.1826	8.3278	2.0993	15.7824	16.9261	5.4545	8.5174	8.7587	11.1721	4.6420	5.4183
1.0	0.3	0.1	0.8	7.7143	12.4667	12.6133	16.6503	22.0625	23.4110	4.0909	8.1826	8.3339	2.1080	15.7873	16.9311	5.4545	8.5174	8.7652	11.1967	4.6452	5.4218
1.0	0.3	0.1	0.7	7.7143	12.4667	12.6200	16.6811	22.0714	23.4202	4.0909	8.1826	8.3400	2.1168	15.7922	16.9362	5.4545	8.5174	8.7717	11.2214	4.6485	5.4253
1.0	0.3	0.1	0.6	7.7143	12.4667	12.6267	16.7119	22.0802	23.4293	4.0909	8.1826	8.3461	2.1256	15.7971	16.9413	5.4545	8.5174	8.7783	11.2462	4.6517	5.4288
1.0	0.3	0.1	0.5	7.7143	12.4667	12.6333	16.7428	22.0891	23.4384	4.0909	8.1826	8.3522	2.1344	15.8020	16.9463	5.4545	8.5174	8.7848	11.2709	4.6549	5.4323
1.0	0.3	0.1	0.4	7.7143	12.4667	12.6400	16.7736	22.0980	23.4476	4.0909	8.1826	8.3583	2.1432	15.8069	16.9514	5.4545	8.5174	8.7913	11.2957	4.6582	5.4359
1.0	0.3	0.1	0.3	7.7143	12.4667	12.6467	16.8045	22.1068	23.4567	4.0909	8.1826	8.3643	2.1520	15.8118	16.9564	5.4545	8.5174	8.7978	11.3205	4.6614	5.4394
1.0	0.3	0.1	0.2	7.7143	12.4667	12.6533	16.8355	22.1157	23.4658	4.0909	8.1826	8.3704	2.1609	15.8166	16.9615	5.4545	8.5174	8.8043	11.3454	4.6647	5.4429
1.0	0.3	0.1	0.1	7.7143	12.4667	12.6600	16.8664	22.1246	23.4750	4.0909	8.1826	8.3765	2.1697	15.8215	16.9666	5.4545	8.5174	8.8109	11.3702	4.6680	5.4464
1.0	0.3	0.1	0	7.7143	12.4667	12.6667	16.8974	22.1334	23.4841	4.0909	8.1826	8.3826	2.1786	15.8264	16.9716	5.4545	8.5174	8.8174	11.3951	4.6712	5.4499
1.0	0.3			7.7143	12.4667		16.8974	22.1334		4.0909	8.1826		2.1786	15.8264		5.4545	8.5174		11.3951	4.6712	
1.0		0.3		7.7143		12.4667	16.8974		22.1334	4.0909		8.1826	2.1786		15.8264	5.4545		8.5174	11.3951		4.6712
1.0	0.5	0.3	1	7.7143	12.3333	12.4667	16.8974	20.8228	22.1334	4.0909	8.0435	8.1826	2.1786	14.7212	15.8264	5.4545	8.2826	8.5174	11.3951	3.9525	4.6712
1.0	0.5	0.3	0.9	7.7143	12.3333	12.4733	16.9284	20.8314	22.1423	4.0909	8.0435	8.1887	2.1875	14.7259	15.8313	5.4545	8.2826	8.5239	11.4200	3.9555	4.6745
1.0	0.5	0.3	0.8	7.7143	12.3333	12.4800	16.9595	20.8400	22.1512	4.0909	8.0435	8.1948	2.1964	14.7306	15.8362	5.4545	8.2826	8.5304	11.4450	3.9585	4.6777
1.0	0.5	0.3	0.7	7.7143	12.3333	12.4867	16.9906	20.8486	22.1600	4.0909	8.0435	8.2009	2.2054	14.7354	15.8411	5.4545	8.2826	8.5370	11.4700	3.9615	4.6810
1.0	0.5	0.3	0.6	7.7143	12.3333	12.4933	17.0217	20.8572	22.1689	4.0909	8.0435	8.2070	2.2143	14.7401	15.8460	5.4545	8.2826	8.5435	11.4950	3.9645	4.6842
1.0	0.5	0.3	0.5	7.7143	12.3333	12.5000	17.0528	20.8658	22.1778	4.0909	8.0435	8.2130	2.2233	14.7448	15.8509	5.4545	8.2826	8.5500	11.5200	3.9675	4.6875
1.0	0.5	0.3	0.4	7.7143	12.3333	12.5067	17.0840	20.8744	22.1867	4.0909	8.0435	8.2191	2.2323	14.7495	15.8558	5.4545	8.2826	8.5565	11.5451	3.9705	4.6908
1.0	0.5	0.3	0.3	7.7143	12.3333	12.5133	17.1152	20.8830	22.1956	4.0909	8.0435	8.2252	2.2413	14.7543	15.8607	5.4545	8.2826	8.5630	11.5701	3.9735	4.6940
1.0	0.5	0.3	0.2	7.7143	12.3333	12.5200	17.1464	20.8916	22.2044	4.0909	8.0435	8.2313	2.2503	14.7590	15.8656	5.4545	8.2826	8.5696	11.5953	3.9765	4.6973
1.0	0.5	0.3	0.1	7.7143	12.3333	12.5267	17.1776	20.9003	22.2133	4.0909	8.0435	8.2374	2.2594	14.7637	15.8705	5.4545	8.2826	8.5761	11.6204	3.9795	4.7006
1.0	0.5	0.3	0	7.7143	12.3333	12.5333	17.2089	20.9089	22.2222	4.0909	8.0435	8.2435	2.2684	14.7684	15.8754	5.4545	8.2826	8.5826	11.6456	3.9825	4.7038
1.0	0.5			7.7143	12.3333		17.2089	20.9089		4.0909	8.0435		2.2684	14.7684		5.4545	8.2826		11.6456	3.9825	

Table B.3: Numerical Results for Sensitivity Analysis - 3/4

				Symmetric						REM Preferred						OEM Preferred					
u_o	u_{r_H}	u_{r_L}	θ	$\bar{S}_{11}$	$\bar{S}_{12}$	$\bar{S}_{13}$	π_o	π_{r_H}	π_{r_L}	$\bar{S}_{11}$	$\bar{S}_{12}$	$\bar{S}_{13}$	π_o	π_{r_H}	π_{r_L}	$\bar{S}_{11}$	$\bar{S}_{12}$	$\bar{S}_{13}$	π_o	π_{r_H}	π_{r_L}
1.0		0.5		7.7143		12.3333	17.2089		20.9089	4.0909		8.0435	2.2684		14.7684	5.4545		8.2826	11.6456		3.9825
1.0	0.7	0.5	1	7.7143	12.2000	12.3333	17.2089	19.6356	20.9089	4.0909	7.9043	8.0435	2.2684	13.7015	14.7684	5.4545	8.0478	8.2826	11.6456	3.3212	3.9825
1.0	0.7	0.5	0.9	7.7143	12.2000	12.3400	17.2402	19.6439	20.9175	4.0909	7.9043	8.0496	2.2775	13.7060	14.7732	5.4545	8.0478	8.2891	11.6708	3.3240	3.9855
1.0	0.7	0.5	0.8	7.7143	12.2000	12.3467	17.2715	19.6523	20.9261	4.0909	7.9043	8.0557	2.2866	13.7106	14.7779	5.4545	8.0478	8.2957	11.6960	3.3267	3.9885
1.0	0.7	0.5	0.7	7.7143	12.2000	12.3533	17.3029	19.6606	20.9348	4.0909	7.9043	8.0617	2.2957	13.7151	14.7826	5.4545	8.0478	8.3022	11.7212	3.3294	3.9915
1.0	0.7	0.5	0.6	7.7143	12.2000	12.3600	17.3343	19.6690	20.9434	4.0909	7.9043	8.0678	2.3049	13.7197	14.7873	5.4545	8.0478	8.3087	11.7465	3.3322	3.9945
1.0	0.7	0.5	0.5	7.7143	12.2000	12.3667	17.3657	19.6774	20.9520	4.0909	7.9043	8.0739	2.3140	13.7242	14.7921	5.4545	8.0478	8.3152	11.7718	3.3349	3.9976
1.0	0.7	0.5	0.4	7.7143	12.2000	12.3733	17.3971	19.6857	20.9607	4.0909	7.9043	8.0800	2.3232	13.7288	14.7968	5.4545	8.0478	8.3217	11.7971	3.3377	4.0006
1.0	0.7	0.5	0.3	7.7143	12.2000	12.3800	17.4286	19.6941	20.9693	4.0909	7.9043	8.0861	2.3324	13.7334	14.8015	5.4545	8.0478	8.3283	11.8225	3.3405	4.0036
1.0	0.7	0.5	0.2	7.7143	12.2000	12.3867	17.4601	19.7025	20.9779	4.0909	7.9043	8.0922	2.3416	13.7379	14.8063	5.4545	8.0478	8.3348	11.8479	3.3432	4.0066
1.0	0.7	0.5	0.1	7.7143	12.2000	12.3933	17.4916	19.7108	20.9866	4.0909	7.9043	8.0983	2.3508	13.7425	14.8110	5.4545	8.0478	8.3413	11.8733	3.3460	4.0096
1.0	0.7	0.5	0	7.7143	12.2000	12.4000	17.5232	19.7192	20.9952	4.0909	7.9043	8.1043	2.3601	13.7470	14.8157	5.4545	8.0478	8.3478	11.8987	3.3487	4.0126
1.0	0.7			7.7143	12.2000		17.5232	19.7192		4.0909	7.9043		2.3601	13.7470		5.4545	8.0478		11.8987	3.3487	
1.0		0.7		7.7143		12.2000	17.5232		19.7192	4.0909		7.9043	2.3601		13.7470	5.4545		8.0478	11.8987		3.3487
1.0	0.9	0.7	1	7.7143	12.0667	12.2000	17.5232	18.4832	19.7192	4.0909	7.7652	7.9043	2.3601	12.7183	13.7470	5.4545	7.8130	8.0478	11.8987	2.7448	3.3487
1.0	0.9	0.7	0.9	7.7143	12.0667	12.2067	17.5548	18.4913	19.7276	4.0909	7.7652	7.9104	2.3693	12.7227	13.7516	5.4545	7.8130	8.0543	11.9242	2.7473	3.3515
1.0	0.9	0.7	0.8	7.7143	12.0667	12.2133	17.5864	18.4994	19.7360	4.0909	7.7652	7.9165	2.3786	12.7271	13.7562	5.4545	7.8130	8.0609	11.9497	2.7498	3.3542
1.0	0.9	0.7	0.7	7.7143	12.0667	12.2200	17.6180	18.5075	19.7443	4.0909	7.7652	7.9226	2.3879	12.7315	13.7607	5.4545	7.8130	8.0674	11.9752	2.7523	3.3570
1.0	0.9	0.7	0.6	7.7143	12.0667	12.2267	17.6497	18.5156	19.7527	4.0909	7.7652	7.9287	2.3972	12.7359	13.7653	5.4545	7.8130	8.0739	12.0007	2.7548	3.3597
1.0	0.9	0.7	0.5	7.7143	12.0667	12.2333	17.6814	18.5238	19.7611	4.0909	7.7652	7.9348	2.4066	12.7403	13.7698	5.4545	7.8130	8.0804	12.0263	2.7573	3.3625
1.0	0.9	0.7	0.4	7.7143	12.0667	12.2400	17.7132	18.5319	19.7695	4.0909	7.7652	7.9409	2.4159	12.7447	13.7744	5.4545	7.8130	8.0870	12.0519	2.7598	3.3653
1.0	0.9	0.7	0.3	7.7143	12.0667	12.2467	17.7449	18.5400	19.7779	4.0909	7.7652	7.9470	2.4253	12.7491	13.7790	5.4545	7.8130	8.0935	12.0775	2.7623	3.3680
1.0	0.9	0.7	0.2	7.7143	12.0667	12.2533	17.7767	18.5481	19.7862	4.0909	7.7652	7.9530	2.4347	12.7534	13.7835	5.4545	7.8130	8.1000	12.1032	2.7648	3.3708
1.0	0.9	0.7	0.1	7.7143	12.0667	12.2600	17.8085	18.5562	19.7946	4.0909	7.7652	7.9591	2.4441	12.7578	13.7881	5.4545	7.8130	8.1065	12.1289	2.7673	3.3736
1.0	0.9	0.7	0	7.7143	12.0667	12.2667	17.8404	18.5644	19.8030	4.0909	7.7652	7.9652	2.4535	12.7622	13.7927	5.4545	7.8130	8.1130	12.1546	2.7698	3.3763
1.0	0.9			7.7143	12.0667		17.8404	18.5644		4.0909	7.7652		2.4535	12.7622		5.4545	7.8130		12.1546	2.7698	
1.0		0.9		7.7143		12.0667	17.8404		18.5644	4.0909		7.7652	2.4535		12.7622	5.4545		7.8130	12.1546		2.7698
1.0	1.1	0.9	1	7.7143	11.9333	12.0667	17.8404	17.3657	18.5644	4.0909	7.6261	7.7652	2.4535	11.7718	12.7622	5.4545	7.5783	7.8130	12.1546	2.2233	2.7698
1.0	1.1	0.9	0.9	7.7143	11.9333	12.0733	17.8722	17.3735	18.5725	4.0909	7.6261	7.7713	2.4630	11.7760	12.7666	5.4545	7.5783	7.8196	12.1803	2.2255	2.7723
1.0	1.1	0.9	0.8	7.7143	11.9333	12.0800	12.0800	17.3814	18.5806	4.0909	7.6261	7.7774	2.4724	11.7802	12.7710	5.4545	7.5783	7.8261	12.2061	2.2278	2.7748
1.0	1.1	0.9	0.7	7.7143	11.9333	12.0867	17.9361	17.3893	18.5887	4.0909	7.6261	7.7835	2.4819	11.7845	12.7754	5.4545	7.5783	7.8326	12.2319	2.2300	2.7773
1.0	1.1	0.9	0.6	7.7143	11.9333	12.0933	17.9680	17.3971	18.5969	4.0909	7.6261	7.7896	2.4914	11.7887	12.7798	5.4545	7.5783	7.8391	12.2577	2.2323	2.7798
1.0	1.1	0.9	0.5	7.7143	11.9333	12.1000	18.0000	17.4050	18.6050	4.0909	7.6261	7.7957	2.5009	11.7929	12.7842	5.4545	7.5783	7.8457	12.2836	2.2345	2.7824
1.0	1.1	0.9	0.4	7.7143	11.9333	12.1067	18.0320	17.4129	18.6131	4.0909	7.6261	7.8017	2.5105	11.7971	12.7886	5.4545	7.5783	7.8522	12.3094	2.2368	2.7849
1.0	1.1	0.9	0.3	7.7143	11.9333	12.1133	18.0641	17.4207	18.6213	4.0909	7.6261	7.8078	2.5200	11.8013	12.7930	5.4545	7.5783	7.8587	12.3353	2.2390	2.7874
1.0	1.1	0.9	0.2	7.7143	11.9333	12.1200	18.0961	17.4286	18.6294	4.0909	7.6261	7.8139	2.5296	11.8056	12.7974	5.4545	7.5783	7.8652	12.3613	2.2413	2.7899
1.0	1.1	0.9	0.1	7.7143	11.9333	12.1267	18.1282	17.4365	18.6375	4.0909	7.6261	7.8200	2.5392	11.8098	12.8018	5.4545	7.5783	7.8717	12.3872	2.2435	2.7924
1.0	1.1	0.9	0	7.7143	11.9333	12.1333	18.1604	17.4444	18.6457	4.0909	7.6261	7.8261	2.5488	11.8140	12.8062	5.4545	7.5783	7.8783	12.4132	2.2458	2.7949
1.0	1.1			7.7143	11.9333		18.1604	17.4444		4.0909	7.6261		2.5488	11.8140		5.4545	7.5783		12.4132	2.2458	

Table B.4: Numerical Results for Sensitivity Analysis - 4/4

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