THE INERTIA GROUP OF SMOOTH 7-MANIFOLDS

By

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The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance of a thesis entitled "**The Inertia Group of Smooth 7-manifolds**" by **William Gollinger** in partial fulfillment of the requirements for the degree of **Master of Science**.

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Dedicated to my grandparents Robert and Mary Snyder,
who taught me to love science, inquiry, and Tetris

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Abstract

Let Θ_n be the group of h-cobordism classes of homotopy spheres, i.e. closed smooth manifolds which are homotopy equivalent to S^n, under connected sum. A homotopy sphere Σ^n which is not diffeomorphic to S^n is called "exotic." For an oriented smooth manifold M^n, the **inertia group** $I(M) \subset \Theta_n$ is defined as the subgroup of homotopy spheres such that $M\#\Sigma$ is orientation-preserving diffeomorphic to M. This thesis collects together a number of results on $I(M)$ and provides a summary of some fundamental results in Geometric Topology. The focus is on dimension 7, since it is the smallest known dimension with exotic spheres. The thesis also provides two new results: one specifically about 7-manifolds with certain S^1 actions, and the other about the effect of surgery on the homotopy inertia group $I_h(M)$.

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Introduction

In 1956, John Milnor [Mil56b] produced a class of 7-dimensional smooth manifolds that were pair-wise non-diffeomomorphic, yet all homeomorphic to the standard sphere S^7. This provided a counter-example to the Smooth Poincaré Conjecture.

The next major advance was the 1963 work of Milnor with Michel Kervaire [KM63], which studied these homotopy spheres in high dimensions. By treating the set of h-cobordism classes of homotopy spheres as a group Θ_n, under the operation of connected sum, they found that there exist exotic spheres in the vast majority of dimensions but also that in each dimension there are only finitely many.

A natural question arose: for a manifold M, if the connected sum is formed between M and a homotopy sphere Σ of the same dimension, does it produce a *new* smooth manifold? The collection of homotopy spheres which admit an orientation preserving diffeomorphism $M \to M\#\Sigma$ form the **inertia group** of M, denoted $I(M)$. There is a canonical topological identification $h_\Sigma \colon M \to M\#\Sigma$ which is identity outside of the attaching region; the subset of the inertia group consisting of spheres that admit a diffeomorphism homotopic to h_Σ is called the **homotopy inertia group** $I_h(M)$. When originally defined, it was not immediate that neither $I(M)$ or $I_h(M)$ are homotopy invariants.

The computation of $I(M)$ for an arbitrary M has proven to be exceedingly difficult, but there are cases where it is known. Some results provide sufficient conditions for $I(M) = 0$: for example, it is known that $I(S^p \times S^q) = 0$ when $p + q \geq 5$ [Sch71], and $I(\mathbb{C}P^n) = 0$ for $n \leq 8$ [Kaw68]. On the other hand for every n there is a manifold M^n such that $I(M) = \Theta_n$ [Win75]. There is no systematic method for approaching the problem in general, and as such many problems are still open. What is $I(\mathbb{C}P^n)$ when $n > 8$? When is $I_h(M) = I(M)$ or 0?

It is known that $I_h(M)$ is an h-cobordism invariant [Bru71], so a natural question is "If M and N are cobordant, how are $I_h(M)$ and $I_h(N)$ related?" Milnor [Mil61] studied the problem of finding a highly connected manifold in the cobordism class of any given manifold M^n, via elementary surgery theory. By choosing an embedding

$$f\colon S^k \times D^{n-k} \to M$$

one can form a new manifold

$$\chi(M, f) = M \setminus f(S^k \times 0) \cup (D^{k+1} \times S^{n-k-1})$$

using an appropriate attaching map. In fact, M and N are cobordant iff N can be obtained from M by a sequence of elementary surgeries. One new result from this thesis says something about this problem:

Theorem (3.4.1). *Let M^n be a smooth manifold and let $f\colon S^k \times \mathbb{D}^{n-k}$ be an embedding with $k < \min\{\lfloor \frac{n}{2} \rfloor, n-3\}$.*

Then $I_h(M) \subset I(\chi(M, f))$.

The other new result is about 7-manifolds with admitting semi-free S^1 actions:

Corollary (3.3.6). *Let M^7 be a closed smooth manifold satisfying $H^1(M; \mathbb{Z}/2) \cong 0$, and suppose it admits a semi-free S^1 action with fixed-point set F^5. Suppose further that the quotient space W is parallelizable, and that resulting S^1 bundle $\xi\colon M \setminus F \to W \setminus F$ has $e(\xi) \equiv 0 \pmod 2$ and $e^2(\xi) = 0$.*

Then $I(M) = 0$.

Chapter 1

Background

1.1 Basic Definitions and Concepts

It is assumed that the reader is familiar with general and algebraic topology. The main points will be highlighted in this section. Almost nothing will be proven.

We make the following notational conventions:

For a set A, $\mathcal{P}(A)$ denotes its power set.

$\mathbb{R}^n$ is n-dimensional Euclidean space, $\mathbb{D}^n$ is the unit disk, and S^n is the unit sphere, with their natural smooth structures and orientations. Define $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}$. $\mathbb{C}^n$ is n-dimensional complex space, and $\mathbb{H}^n$ is n-dimensional quaternionic space.

If $\mathbb{F}$ is one of $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, then $\mathbb{F}P^n$ is the associated projective space.

Topological spaces are typically denoted by X, Y, Z. For x a point in a topological space, the set of all open neighbourhoods is denoted by $\mathcal{U}_x$.

Smooth manifolds tend to be denoted by M^n, N^n, where n indicates the dimension; W is usually reserved for a coboundary. Homotopy spheres will be represented by Σ^n. *Unless otherwise stated, all smooth manifolds will be oriented and path-connected.*

A fibre bundle over a topological space B will be denoted $\xi\colon\ F \xrightarrow{i} E \xrightarrow{\rho} B$. The trivial $\mathbb{R}^n$ bundle over B will be denoted ϵ^n. For a smooth manifold M, the tangent bundle will be denoted τ_M. If k is a characteristic class of M then the expressions

$k(\tau_M)$ and $k(M)$ (in the appropriate cohomology group) will be interchangeable.

1.1.1 Topology and Manifolds

For a thorough treatment of General Topology, the reader is referred to [Mun75]. Recall some definitions:

Definition 1.1.1. *Let X be a set. A* **topology** *on X is a collection of subsets $\mathcal{T} \subset \mathcal{P}(X)$ with the following properties:*

1. $\forall A \subset \mathcal{T},\ \cup A \in \mathcal{T}$

2. $\forall A \subset \mathcal{T}$ *with* $|A| < \infty,\ \cap A \in \mathcal{T}$

A pair $(X, \mathcal{T})$, where $\mathcal{T}$ is a topology on X, is called a **topological space**.

A subset $U \subset X$ is **open** *if $U \in \mathcal{T}$. If $x \in U$ and U is open, U will be called an* **open neighbourhood of x**. *The* **neighbourhood system at x** *is*

$$\mathcal{U}_x := \{U \in \mathcal{T} \mid x \in U\}$$

Remark 1.1.2. *For any topological space $(X, \mathcal{T})$, taking $A = \emptyset \subset \mathcal{T}$ gives*

$$\cup\emptyset = \emptyset \in \mathcal{T} \text{ and } \cap\emptyset = X \in \mathcal{T}$$

Given a function $f\colon X \to Y$, it induces a function $f^{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$.

Definition 1.1.3. *If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, then a function $f\colon X \to Y$ is* **continuous** *if $\forall U \in \mathcal{T}_Y,\ f^{-1}(U) \in \mathcal{T}_X$.*

A continuous function f is an **open map** *if $\forall U \in \mathcal{T}_X,\ f(U) \in \mathcal{T}_Y$.*

A **topological (or continuous) embedding** *is an injective open map.*

A **homeomorphism** *is a surjective topological embedding.*

The **group of self-homeomorphisms** *$f\colon X \to X$ is denoted* $\mathrm{Top}(X)$

Definition 1.1.4. *Let $f, g\colon X \to Y$ be two continuous functions. f and g are* **homotopic** *(denoted $f \sim g$) if there is a* **homotopy** *from f to g, that is a continuous* *function* $F\colon X \times I \to Y$ *such that* $\forall x \in X,\ F(x,0) = f(x)$ *and* $F(x,1) = g(x)$.

A continuous function $f\colon X \to Y$ *is a* **homotopy equivalence** *if there is a continuous function* $g\colon Y \to X$ *such that* $fg \sim \mathrm{id}_Y$ *and* $gf \sim \mathrm{id}_X$.

For any spaces X *and* Y, *let* $[X, Y]$ *denote the set of homotopy-classes of maps from* X *to* Y.

Definition 1.1.5. *A collection of subsets* $\mathcal{B} \subset \mathcal{P}(X)$ *is a* **basis for a topology** *if*

1. $\cup\mathcal{B} = X$

2. $\forall B_1, B_2 \in \mathcal{B}, \forall x \in B_1 \cap B_2,\ \exists B_3 \in \mathcal{B}$ *such that* $x \in B_3 \subset B_1 \cap B_2$.

The **topology generated by** $\mathcal{B}$ *is* $\mathcal{T}_\mathcal{B} := \{\cup A \mid A \subset \mathcal{B}\}$.

A topological space $(X, \mathcal{T}_X)$ *is* **second countable** *if there is a basis* $\mathcal{B}$ *for a topology, where* $\mathcal{T}_\mathcal{B} = \mathcal{T}_X$ *and* $\mathcal{B}$ *is a countable set.*

Definition 1.1.6. *A topological space is* **Hausdorff** *if* $\forall x \neq y \in X$ *there are open nhds* U_x *and* U_y *(of* x *and* y *respectively) such that* $U_x \cap U_y = \emptyset$.

Definition 1.1.7. *A topological space* $(X, \mathcal{T})$ *is* **locally Euclidean of dimension n** *if* $\forall x \in X,\ \exists U \in \mathcal{U}_x$ *and a homeomorphism* $\phi\colon U \to V$ *where* V *is an open subset of* $\mathbb{R}^n_+$.

The pair (U, ϕ) *will be called a* **chart at x**. *A collection of charts which covers* X *will be called a* **topological atlas** $\mathcal{A}$. *The* **boundary** *of* X, *denoted* ∂X, *is the set of all* $x \in X$ *such that there is a chart* (U, ϕ) *at* x *where* $\phi(x) = (0, x_2, \ldots, x_n)$ *(for some* $x_2, \ldots, x_n \in \mathbb{R}$*).*

For any two charts $(U_i, \phi_i), (U_j, \phi_j) \in \mathcal{A}$ *there is a* **transition function**

$$\phi_{j,i} = \phi_j \circ (\phi_i|_{U_i \cap U_j})^{-1}\colon \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

Remark 1.1.8. *It is more than likely that* $U_i \cap U_j = \emptyset$. *In this case, the transition function* $\phi_{j,i}$ *is just* $\emptyset\colon \emptyset \to \emptyset$.

In any case, the transition function will always be a homeomorphism (this is vacuously true when $\phi_{j,i} = \emptyset$*).*

Definition 1.1.9. *A topological space is a* **topological manifold of dimension n** *if it is second countable, Hausdorff, and locally Euclidean of dimension n.*

A topological manifold is **closed** *if it is compact and without boundary, and is* **open** *if it is* non-*compact and without boundary.*

Remark 1.1.10. *If X is a topological manifold of dimension n with boundary, then ∂X is a topological manifold of dimension $n-1$.*

1.1.2 Fibre Bundles

A fibre bundle generalizes the tangent bundle of a smooth manifold. The standard reference on the theory of fibre bundles is [Ste99]. The basic definitions of **fibre bundle**, **local trivialization**, **structure group**, **bundle map**, **bundle equivalence**, etc., are to be found there. To define these notions properly there is a large amount of discussion which is not appropriate for a "review" section. Instead, important results and concepts are stated.

Theorem 1.1.11 (Homotopy Lifting Property)**.** *Let*

$$\xi\colon\ F \longrightarrow E \xrightarrow{\rho} B$$

be a fibre bundle, let $f\colon X \to B$ be a continuous function with lift $\tilde{f}\colon X \to E$, and let $F\colon X \times I \to B$ be continuous with $F(x,0) = f(x)$ for all $x \in X$.

Then there is a lift $\widetilde{F}\colon X \times I \to E$.

Corollary 1.1.12. *Any bundle over a contractible space is trivial.*

The following result could be called the "reconstruction theorem."

Theorem 1.1.13 ([Ste99, 3.2])**.** *Let X, Y be any topological spaces, G a topological group that acts on Y, and let $\{U_i\}_{i\in\Lambda}$ be an open covering of X. Suppose that for each pair $i, j \in \Lambda$ there is a continuous function $g_{ji}\colon U_j \cap U_i \to G$ so that for each $x \in U_i \cap U_j \cap U_k$ there is the relation $g_{kj}(x)g_{ji}(x) = g_{ki}(x)$.*

Then there is a bundle with base X, fibre Y and coordinate transformation group

G, with cocycles given by $\{g_{ji}\}_{i,j\in\Lambda}$. Furthermore, any two bundles over X with this information will be equivalent.

The spirit of this theorem is that to define a bundle it is enough to give local trivializations and a collection of coordinate transformations which "stitch together" the pieces.

Definition 1.1.14. *Let $\pi\colon E \to X$ be a bundle with fibre Y, and structure group G. Suppose G acts on another space Z. Then the* **associated bundle with fibre Z** *is the bundle given by Theorem 1.1.13 where the fibre Y is replaced by Z.*

The reconstruction theorem combined with the notion of "associated bundles" indicates that to classify bundles with structure group G it suffices to classify principal G-bundles.

The spirit of "stitching together trivial parts" almost immediately gives a classification of bundles over the sphere, since the two hemispheres are contractible.

Proposition 1.1.15. *Let G be a topological group. Then the equivalence classes of principal G-bundles over S^n are in natural bijective correspondence with $\pi_{n-1}(G)$.*

When the base space becomes more complicated, the situation is not as simple. However, there is a result with a simple statement:

Theorem 1.1.16. *Let X be a topological space, G a topological group. Then there is a space BG (called the* **classifying space** *of G) such that equivalence classes of principal G-bundles are in natural bijective correspondence with $[X, BG]$.*

In particular, there is a **universal bundle** *$G \to UG \to BG$ so that the homotopy class of the function $f : X \to BG$ corresponds to the bundle $f^*(UG)$*

Remark 1.1.17. *In Steenrod this is shown for G a compact Lie group [Ste99, §19.6]. The general case was handled by Milnor [Mil56a]*

1.1.3 Smooth Manifolds

For a reference on differentiable manifolds, the reader is referred to [Lee03] or [Kos93]

Definition 1.1.18. *Let $\mathcal{A}$ be a topological atlas for a topological manifold X of dimension n. $\mathcal{A}$ is a* **smooth atlas** *for X if every transition function is smooth (as a function between open subsets of $\mathbb{R}^n$).*

Two smooth atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are **compatible** *if $\mathcal{A}_1 \cup \mathcal{A}_2$ is also a smooth atlas.*

A **smooth structure** *is a smooth atlas which is maximal with respect to compatibility (that is, $\mathcal{A}$ is maximal if every $\mathcal{A}' \not\subset \mathcal{A}$ is not compatible).*

There is a non-standard but equivalent definition of a smooth structure, given in lectures by Milnor about Differential Topology [Mil07, p.146] (notes taken by Munkres). The spirit of this definition is that the smoothness of a manifold can be expressed in terms of the smooth real-valued functions defined on open subsets. (The two notions of "smooth structure" will be used interchangeably.)

Definition 1.1.19. *A* **smooth structure** *on X will be a collection $\mathcal{D}$ of real-valued functions $f\colon U \to \mathbb{R}$, defined on open subsets of X, with the following properties:*

1. *For each $p \in X$ there is a chart (U, ϕ) at p with the property that if f is a real-valued function defined on an open subset of U, then $f \in \mathcal{D}$ iff $f\phi^{-1}\colon \phi^{-1}(U) \subset \mathbb{R}^n \to \mathbb{R}$ is smooth. (Such a ϕ will be called a* **reference map at** *p.)*

2. *If f is a real-valued function defined on the open set $W \subset X$, U_i a collection of open subsets of W, and $U := \cup U_i$, then $f|_U \in \mathcal{D}$ iff $f|_{U_i} \in \mathcal{D}$ for each i. (In particular, if $U = X$ then a global function is smooth if it is smooth on a collection of open sub-domains which cover X.)*

The elements of a smooth structure will be called the **smooth functions on** *X.*

Now let X_1 and X_2 be manifolds with smooth structures $\mathcal{D}_1, \mathcal{D}_2$, and let $U \subset X_1$ open. It is said that $g : U \to X_2$ is a **smooth function on U** if for each $f_2 \in \mathcal{D}_2$ with domain W, the function $f_2 g$ is in $\mathcal{D}_1$, with possibly empty domain $g^{-1}(W)$. Any reference map is smooth from its domain to $\mathbb{R}^n$, which is given the trivial chart $(\mathbb{R}^n, \mathrm{id})$. It is readily shown that the composition of smooth functions is also smooth. A **smooth embedding** (usually just "embedding," sometimes "imbedding") is a smooth topological embedding whose inverse is smooth; a **diffeomorphism** is a surjective smooth embedding. The group of self-diffeomorphisms of a smooth manifold M is denoted $\mathrm{Diff}(M)$.

Definition 1.1.20. *Let M^n be a smooth manifold with smooth atlas $\mathcal{A}$ consisting of contractible open sets. Define the* **tangent bundle** *τ_M to be the $\mathbb{R}^n$ bundle given by the Reconstruction Theorem 1.1.13 as follows: let $\tau_M|_U = U \times \mathbb{R}^n$ for each $U \in \mathcal{A}$, and define the transition function for $U \cap V$ by $\mathrm{id}_{U\cap V} \times D(\phi_V \phi_U^{-1})$.*

By the theory of Morse [Mil63a] any smooth manifold admits a **handlebody decomposition**, and hence a **CW decomposition**.

Topological properties of the tangent bundle are often used to distinguish smooth manifolds.

Definition 1.1.21. *Let M^n be a smooth manifold.*

For $0 \leq k \leq n$ let $M_{(k)}$ be the k-skeleton of a CW decomposition, M^n is **k-parallelizable** *if $\tau_M|_{M_{(k)}} \cong \epsilon^n$.*

M^n is **parallelizable** *if it is n-parallelizable. It is* **parallelized** *if a particular trivialization is chosen $t\colon \tau_M \to M \times \mathbb{R}^n$.*

M^n is a **π-manifold** *(or* **s-parallelizable** *or* **stably parallelizable***) if*

$$\tau_M \oplus \epsilon^1 \cong \epsilon^{n+1}$$

Lemma 1.1.22 ([KM63, Lemma 3.3])**.** *Let M be an n-dimensional submanifold of S^{n+k} where $n < k$. Then M is a π-manifold iff its normal bundle is trivial.*

Lemma 1.1.23 ([KM63, Lemma 3.4])**.** *A connected manifold with non-vacuous boundary is a π-manifold iff it is parallelizable.*

Recall that a bundle with group G is classified by a map $f\colon M^n \to BG$. The structure group of an oriented manifold is reduced to SO_n by choosing a metric. SO_n has a double-cover, which is simply-connected if $n \geq 3$, called the **spin group**

$$\mathrm{Spin}_n \xrightarrow{2:1} SO_n$$

.

Definition 1.1.24. *Let M^n be an oriented manifold, whose tangent bundle is classified by the map $f\colon M \to BSO_n$. Then M is a* **spin manifold** *if there is a lift*

$$\begin{tikzcd} & B\,\mathrm{Spin}_n \arrow[d] \\ M \arrow[r, "f"'] \arrow[ur, dotted, "\tilde{f}"] & BSO_n \end{tikzcd}$$

A **spin structure** *on M is a bundle corresponding to one such lift $\tilde{f}\colon M \to B\,\mathrm{Spin}_n$.*

Lemma 1.1.25 ([Mil65, lemma 1])**.** *If M is a spin manifold, then the spin structure is unique iff $H^1(M; \mathbb{Z}/2) = 0$.*

1.2 Algebraic Topology

1.2.1 Obstruction and Characteristic Classes

The problem of finding a section of a fibration is studied by obstruction theory. We will outline some main ideas, and refer the reader to [Ste99] and [Hu59] for a more thorough treatment. In the literature the development is done relative to some subcomplex of the base space, but for the sake of clarity we will omit that.

Let $F \longrightarrow E \xrightarrow{\rho} B$ be a fibre bundle. A CW structure on B might suggest that a section could be defined inductively, by extending it one skeleton at a time. Indeed for the base case, the 0-skeleton, constructing a section is trivial as long as F is not empty. Supposing that a section $s\colon B_{(n-1)} \to E$ has been constructed, the problem of extending s to $B_{(n)}$ becomes the problem of extending it over each n-cell.

Suppose C is an n-cell and $s\colon B_{(n-1)} \to E$ is a section of the subbundle over the $(n-1)$ skeleton. s is defined on $\partial C \cong S^{n-1}$, and so determines a map $\tilde{s}\colon S^{n-1} \to \rho^{-1}(C)$. Since C is contractible, by the homotopy lifting property of fibre bundles the contraction can be lifted, and so $\tilde{s}$ can be homotoped to a map into a single fibre F_x, i.e. $S^{n-1} \to F_x$. Then *s can be extended over C iff the induced element of $\pi_{n-1}(F_x)$ is trivial.*

Recall that the collection of n-cells generates the n-th chain group of cellular homology

$$C_n^{cell}(B) := H_n(B_{(n)}, B_{(n-1)}).$$

For each n-cell C, s determines an element of $\pi_{n-1}(F_x)$ for some $x \in C$, but when $\{\pi_{n-1}(F_x) \mid x \in B\}$ (or just $\{\pi_{n-1}(F)\}$ for short) is treated as a **bundle of coefficients** [Ste99, §31] then s induces a homomorphism

$$o(s)\colon C_n^{cell}(B) \to \{\pi_{n-1}(F)\}$$

In other words, s induces an element of $C^n_{cell}(B; \{\pi_{n-1}(F)\})$. In fact, $o(s)$ is a cocylcle.

Definition 1.2.1. *Let $F \longrightarrow E \xrightarrow{\rho} B$ be a fibre bundle with B a CW complex, and let $s\colon B_{(n-1)} \to E$ be a section over the $(n-1)$-skeleton. Define the* **n-th obstruction group** *as $H^n_{cell}(B; \{\pi_{n-1}(F)\})$. The* **obstruction cocycle of** *s is the element*

$$o(s) \in H^n_{cell}(B; \{\pi_{n-1}(F)\})$$

described above.

Given a section $s\colon B_{(n-1)} \to E$ which can be extended over $B_{(n)}$, let

$$\mathfrak{o}(s) := \{o(s')\} \subset C^{n+1}_{cell}(B; \{\pi_n(F)\})$$

where s' ranges over extensions of s to the n-skeleton.

Theorem 1.2.2 ([Ste99, 34.2])**.** *Suppose $s\colon B_{(n-1)} \to E$ is a section which can be* *extended over* $B_{(n)}$.

Then $\mathfrak{o}(s)$ *is a cohomology class, and* s *is extendable over* $B_{(n+1)}$ *iff* $\mathfrak{o}(s) = 0$.

Definition 1.2.3. *Suppose now that* F *is* $(k-1)$*-connected. Define the* **characteristic cohomology class** *of this bundle to be*

$$\mathfrak{o} \in H^{k+1}(B; \pi_k(F)),$$

the class of all $o(f)$ *such that* f *is a section of* $B_{(k)}$ *(i.e. an extension of* $\emptyset\colon \emptyset \to E$ *where by convention* $B_{(-1)} = \emptyset$*).*

A typical application of obstruction theory is the proof that every homotopy sphere is a π-manifold, as in [KM63, Theorem 3.1].

Closely related to obstruction theory is the theory of **characteristic classes** of vector bundles. The reader is referred to Milnor-Stasheff [MS74] for a concise development of the theory of **Stiefel-Whitney, Euler, Chern, and Pontryagin classes**.

We note that for a smooth manifold M with tangent bundle τ_M, the first Stiefel-Whitney class $\omega_1(\tau_M) \in H^1(M; \mathbb{Z}/2)$ represents the obstruction to defining an orientation on M. Furthermore, if M is oriented then it is a spin manifold (Definition 1.1.24) iff $\omega_2(M) = 0$.

A result of Kervaire relates characteristic classes to obstruction classes:

Lemma 1.2.4 ([Ker59, Lemma 1.1])**.** *Let* ξ *be a stable* G *bundle over a CW complex* K*, where* $G = SO_n, U_n$*. Let* f *be a section on* $K_{(q)}$*, where* $q = 4k$ *if* $G = SO_n$ *and* $q = 2r$ *if* $G = U_n$*. Then*

1. $p_k(\xi) = a_k(2k-1)!\mathfrak{o}(f)$ *if* $G = SO_n$
2. $c_r(\xi) = (r-1)!\mathfrak{o}(f)$ *if* $G = U_n$

where $a_k = 2$ *for* k *odd and* 1 *for* k *even.*

1.2.2 Signature and Theorems of Hirzebruch

The signature, an important invariant for studying $4k$-manifolds, comes from studying their intersection forms. The majority of material of this section comes from [MS74, Milnor-Stasheff].

Definition 1.2.5. *Let M^{2k} be a compact, oriented manifold with fundamental class $\mu \in H_{2k}(M, \partial M; \mathbb{Z})$, and let $\alpha, \beta \in H^k(M, \partial M; \mathbb{Z})$. Define the* **intersection** *of α and β as*

$$\lambda_M(\alpha, \beta) := \langle \alpha \cup \beta, \mu \rangle$$

The **intersection form** *is the bilinear form*

$$\lambda_M \colon H^k(M, \partial M; \mathbb{Z}) \times H^k(M, \partial M; \mathbb{Z}) \to H^{2k}(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$$

If $a_1, \ldots, a_n$ is a basis for $H^k(M, \partial M; \mathbb{Z})$, then λ_M is described by the matrix $(a_{i,j}) = \langle a_i \cup a_j, \mu \rangle$. If k is even then λ_M is symmetric, and so its matrix can be diagonalized over $\mathbb{Q}$.

Definition 1.2.6 ([MS74, p.224])**.** *Suppose k is even, and choose a basis for $H^k(M, \partial M; \mathbb{Q})$ so that the quadratic form $\tilde{\lambda}_M$ over these coefficients is diagonal. Then define the* **signature** *$\sigma(M)$ to be the number of positive diagonal entries minus the number of negative entries.*

Hirzebruch [Hir56] showed for a closed $4k$-manifold M, $\sigma(M)$ can be expressed as a polynomial in the Pontryagin numbers $\langle p_k(\tau_M), \mu \rangle$. To do this, he studied formal power series and "multiplicative sequences" of polynomials.

Definition 1.2.7. *For a commutative unital ring Λ, let $A^* = (A^0, A^1, A^2, \ldots)$ be a commutative, graded algebra; let A^Π be the ring of formal power series*

$$a = a_0 + a_1 + a_2 \ldots \quad a_i \in A^i$$

Consider a sequence of polynomials $K = \{K_1(x_1), K_2(x_1, x_2), K_3(x_1, x_2, x_3), \dots\}$. Given a series $a = 1 + a_1 + a_2 + \dots$, define

$$K(a) = 1 + K_1(a_1) + K_2(a_1, a_2) + \dots$$

Definition 1.2.8. *$K = \{K_1(x_1), K_2(x_1, x_2), \dots\} = \{K_n\}$ is a* **multiplicative sequence of polynomials** *if*

1. *Each $K_n(x_1, \dots, x_n)$ is homogenous of degree n if x_i is given degree i.*
2. *$K(ab) = K(a)K(b)$ for all such Λ-algebras A^* and $a, b \in A^{\Pi}$ with leading term 1.*

In particular, take $A^* = \Lambda[t]$. The following lemma due to Hirzebruch is crucial:

Lemma 1.2.9. *Given $f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + \dots \in \Lambda[t]$, there is one and only one multiplicative sequence $\{K_n\}$ with coefficients in Λ satisfying the condition*

$$K(1 + t) = f(t).$$

An equivalent condition is that the coefficient of x_1^n in each K_n is equal to λ_n

Definition 1.2.10. *Let $f(t) \in \Lambda[t]$ have leading term* 1. *Then the* **the multiplicative sequence belonging to f** *(or* **associated to f***) is the sequence given by Lemma 1.2.9.*

This result says that formal power series with coefficients in Λ and leading term 1 have an injective map into the set of multiplicative sequences. Moving from the formality back to the realm of manifolds, suppose that the variables are pontryagin classes:

Definition 1.2.11. *For a $4k$-manifold M with fundamental class μ, and any multiplicative sequence $K = \{K_n\}$ with rational coefficients, define the* **K-genus** *as*

$$K_k[M] := \langle K_k(p_1, \dots, p_k), \mu \rangle$$

Definition 1.2.12. *Let $\{L_k(p_1, \ldots, p_k)\}$ be the multiplicative sequence of polynomials belonging to the power series*

$$\frac{\sqrt{t}}{\tanh\sqrt{t}} = 1 + \frac{1}{3}t - \frac{1}{45}t^2 + \cdots + (-1)^{k-1}\frac{2^{2k}B_k}{(2k!)}t^k \ldots$$

(where B_k is the k-th Bernoulli number).

For example: $L_1 = \frac{1}{3}p_1,\ L_2 = \frac{1}{45}(7p_2 - p_1^2),\ L_3 = \frac{1}{945}(62p_3 - 13p_2p_1 + 2p_1^3), \ldots$

The following theorem is commonly referred to as the "Hirzebruch signature theorem":

Theorem 1.2.13 ([MS74, 19.4])**.** *For any smooth, closed, oriented $4k$-manifold M,*

$$\sigma(M) = L_k[M]$$

Corollary 1.2.14. *If M^8 is a smooth, closed, oriented 8 manifold, then*

$$\sigma(M) = \langle \frac{1}{45}(7p_2 - p_1^2), \mu \rangle$$

There is another series of interest to spin manifolds, $\hat{A}$:

Definition 1.2.15. *Let $\hat{A} = \hat{A}_n$ be the multiplicative sequence associated with $\frac{\sqrt{z}/2}{\sinh(\sqrt{z}/2)}$. Then the $\hat{A}$-**genus,** $\hat{A}[X]$, is defined as $\hat{A}_k(p_1, \ldots, p_k)[X]$.*

Later on, in Section 2.4 we will see an invariant which is based on the $\hat{A}$-genus, and the fact that it is integral for spin manifolds:

Theorem 1.2.16 (Hirzebruch)**.** *Let X be a closed, smooth, oriented spin manifold of dimension $4k$. Then the $\hat{A}$-genus is an integer; if k is odd, then $\hat{A}[X]$ is even.*

1.2.3 Pontryagin-Thom and Stable Homotopy

The theory of exotic spheres is inexorably linked to the so-called "J-homomorphism" (or Hopf-Whitehead homomorphism), which relates the homotopy of spheres to the homotopy of orthogonal groups. It is remarkable that such a fundamentally smooth phenomenon can be described by something which is homotopic in nature. What bridges this apparent gap is the so-called "Pontryagin-Thom" construction, which is summarized in this section. Most of the following is from [Kos93, Kosinski].

Definition 1.2.17. *Let M^{n+k} be a manifold (without boundary, for simplicity) and let V^k be a submanifold* with trivial normal bundle *and tubular neighbourhood N. A* **trivialization** *(or* **framing***) of the normal bundle is a choice of diffeomorphism $t\colon N \to V \times \mathbb{R}^n$.*

Construction 1.2.18. *(Pontryagin-Thom construction)*

Let (V^k, t) be a trivialized submanifold of M^{n+k}, with trivialized normal bundle N. Compose t with projection onto the $\mathbb{R}^n$ factor, giving $\pi t\colon N \to \mathbb{R}^n$ (a diffeomorphism on each fibre). Compactify $\mathbb{R}^n$ by adding $*$, so that πt becomes a map to S^n. Extend this map continuously by sending the complement of N to $*$, "forgetting" about everything outside of N. The resulting continuous surjection is called the **Pontryagin-Thom construction***:*

$$p(V^k, t) : M^{n+k} \to S^n$$

Theorem 1.2.19 ([Kos93, IX,5.5])**.** *Let M be compact and closed. Then "p" gives a bijection between $\Omega^k(M)$ (the set of compact framed k-dimensional submanifolds of M modulo framed cobordism) and $[M^{n+k}, S^n]$.*

That is, every continuous map $f : M \to S^n$ is homotopic to $p(V^k, t)$ for some framed submanifold (V^k, t), and this framed submanifold is unique up to framed cobordism.

Corollary 1.2.20. *Let $M = S^{k+n}$. Then the above theorem says that every class in $\pi_{n+k}(S^n)$ is represented by some $p(V^k, t)$ for $V^k \subset S^{n+k}$ a framed submanifold.*

This construction gives an alternative way to define the suspension homomorphism. Given $f : S^{n+k} \to S^n$, f is homotopic to $p(V^k, t)$ for $V^k \subset S^{n+k}$. The standard embedding $S^{n+k} \subset \mathbb{R}^{n+k+1}$ adds a trivial factor to the normal bundle of

V and extends the trivialization t to some t'. Then define the **suspension** of f as $Sf := p(V^k, t') : S^{n+k+1} \to S^{n+1}$ (well-defined since p is bijective). This induces a group homomorphism

$$S : \pi_{k+n}(S^n) \to \pi_{k+n+1}(S^{n+1})$$

Proposition 1.2.21. *(Freudenthal Suspension Theorem)*
S is surjective if $n > k$ and bijective *if $n > k + 1$*

Definition 1.2.22. *Let $n > k + 1$. Then the* **k-th stable homotopy group** *is defined as $\Pi_k := \pi_{k+n}(S^n)$.*

As above, Π_k is isomorphic to the framed cobordism group $\Omega^k(S^{n+k})$. This group has a special subgroup of interest, namely framed k-spheres $\Omega^k_S(S^{n+k})$. In the future these will be denoted Ω^{fr}_k and S^{fr}_k respectively, to eliminate any reference to n.

For another application of the Pontryagin-Thom construction, consider the standard embeddings $S^k \subset \mathbb{R}^{k+1} \subset \mathbb{R}^{n+k}$, with tubular nhd N. The normal bundle of S^k in $\mathbb{R}^{n+k}$ is then isomorphic to the normal bundle ν from $S^k \subset \mathbb{R}^{k+1}$, plus $n-1$ trivial bundles. Choosing a trivialization $t : N \to S^k \times \mathbb{R}^n$ of ν gives a map

$$p(S^k, t) : \mathbb{R}^{n+k} \to S^n$$

Compactifying $\mathbb{R}^{n+k}$ extends the domain of $p(S^k, t)$, yielding a map which is better suited for homotopy theory:

$$\tilde{p}(S^k, t) : S^{n+k} \to S^n$$

Thus a trivialization t of N induces a homotopy class, say $J(t) \in \pi_{n+k}(S^n)$. A framing of the normal bundle of S^k in $\mathbb{R}^{n+k}$ can be given by a section of the corresponding Stiefel bundle with fiber $V_{n,n} = SO_n$, i.e. a map $S^k \to SO_n$, and so they are parametrized (up to equivalence) by $\pi_k(SO_n)$. Hence the preceding construction sets up a function

$$J : \pi_k(SO_n) \to \pi_{n+k}(S^n)$$

This function is a group homomorphism, and has the property that $\operatorname{Im} J = S^k_{fr} \subset \Pi_k$ (when $n > k + 1$). Since the sphere is the only compact, closed, connected manifold in dimension 1, it follows that $J : \pi_1(SO_n) \to \pi_{n+1}(S^n)$ is surjective for $n > 2$. Furthermore, for $n = 2$ then J is an isomorphism $\pi_1(SO_2) \cong \pi_3(S^2) \cong \mathbb{Z}$.

For $n > k + 1$ the codomain of J becomes the stable homotopy group. There is stability in the domain of J as well, since $\pi_k(SO_n) \cong \pi_k(SO_m)$ for $n, m > k + 1$. In fact, this gives a homomorphism for each k

$$J_k : \pi_k(SO) \to \Pi_k$$

The image of J_k is S^k_{fr}. Let $S^k \subset \mathbb{R}^{n+k}$ have a normal bundle N trivialized by t, corresponding to an element of $\pi_k(SO_n)$. Then t is in the kernel of J_k iff $S^k \subset \mathbb{R}^{n+k+1}_+$ bounds a manifold whose normal bundle has a trivialization extending t.

Definition 1.2.23. *Let M^n be a closed s-parallelizable manifold, let $k > n + 1$, and choose an imbedding $i\colon M \to S^{n+k}$ (which exists and is unique up to isotopy). Since M is an s-parallelizable submanifold of S^{n+k} its rank k normal bundle is trivial. For each trivialization t, there is a map $p(M, t)\colon S^{n+k} \to S^k$, or in other words an element of Π_n.*

Define $p(M) := \{p(M, t)\} \subset \Pi_n$

1.3 Operating on Smooth Manifolds

There are some standard manifold operations, used to take one or more smooth manifolds and produce another one. The first operation studied in this section is a way of taking two manifolds and attaching them along a common submanifold. The second is a way of modifying a given manifold so as to reduce a particular homotopy group.

1.3.1 Attaching Manifolds

One of the elementary constructions in Differential Topology is that of "Attaching Manifolds along a submanifold." Kosinski [Kos93] gives a standard treatment of this procedure.

Construction 1.3.1 ($M_1 \#_N M_2$)**.** *Let N^n be an n-dimensional smooth manifold and let M_1, M_2 be $(n+k)$-dimensional manifolds, all oriented and possibly with non-empty boundary.*

Let D^{n+k} be a closed Riemannian k-disk bundle over N and let E be the associated open disk bundle; let D_∂ and E_∂ denote their restrictions to ∂N. Identifying N with the zero section of E, let $E_0 = E \setminus N$.

Let $f_i' : D \to M_i$ be embeddings such that $f_i'(D_\partial) \subset \partial M_i$, the images are transverse to the respective boundaries, f_1' is orientation preserving, and f_2' is orientation reversing. Let f_i denote the restriction to E_0. Denote by $\tilde{M}_i$ the manifold $M_i \setminus f_i'(N)$.

Define $\alpha_E : E_0 \to E_0$ by $\alpha_E(v_x) = (1 - ||v_x||)\frac{v_x}{||v_x||}$; α_E should be thought of as turning E_0 inside out.

Finally, construct $M_1 \#_N M_2$ by forming a quotient of $\tilde{M}_1 \coprod \tilde{M}_2$ using the orientation preserving diffeomorphism $f_2 \alpha_E f_1^{-1} : f_1(E_0) \to f_2(E_0)$

Equivalently, $M_1 \#_N M_2$ can be defined as the result of the following topological pushout diagram (see Appendix A)

$$\begin{tikzcd}
 & \tilde{M}_1 \arrow[dr, "k"] & \\
E_0 \arrow[ur, "f_1"] \arrow[dr, "f_2\alpha_E"'] & & M(f_1, f_2) \\
 & \tilde{M}_2 \arrow[ur, "l"'] &
\end{tikzcd}$$

Proposition 1.3.2. *The topological manifold produced by Construction 1.3.1 is a smooth manifold.*

Proof. By Lemma A.2.1, the topological pushout is actually a smooth pushout. □

For the sake of notation, it is often assumed that f_2 is actually orientation preserving and that it factors as $f_2 = g\alpha_E$. If the embeddings involved are being emphasized, the attached manifold will be denoted by $M(f_1, f_2)$.

The smooth structure of $M(f_1, f_2)$ is invariant up to isotopy of the embeddings. Furthermore, $M_1 \#_N (M_2 \#_{N'} M_3) = (M_1 \#_N M_2) \#_{N'} M_3$ when the embeddings of D and D' are disjoint. (See, for example, Appendix A.)

Example 1.3.3. *As a special case let N be a 0-dimensional manifold, i.e. a one-point set, and let $g_i \colon N \to M_i$ $i = 1, 2$ be embeddings into the interiors of M_1 and M_2 respectively. The resulting manifold $M_1 \#_N M_2$ is called the* **connected sum** *$M_1 \# M_2$. Since there is a blanket assumption of path-connectedness, the connected sum does not depend on the g_i.*

There is an analogous construction for when N is a submanifold of the boundary:

Construction 1.3.4 ([Kos93, VI.5])**.** *Let M_1, M_2 be $(n + k + 1)$-dimensional manifolds. Let N be a closed compact n-manifold, and let ξ be a rank k vector bundle over N with total space E; identify N with the 0-section of E.*

Let h_1, h_2 be embeddings of E into $\partial M_1, \partial M_2$. Using a collared nhd of $\partial(M_i)$ in M_i, extend h_i to $\bar{h}_i \colon E \times \mathbb{R}_+ \to M_i$. Let $E_0 = E \times \mathbb{R}_+ \setminus (N \times 0)$, and let $\widetilde{M_i} = M_i \setminus \bar{h}_i(E_0)$

Define $\alpha_E \colon E_0 \to E_0$ analogously to the above.

Construct $M_1 \natural_N M_2$ as the quotient of $M_1 \coprod M_2$ by the diffeomorphism

$$\bar{h}_2 \alpha_E \bar{h}_1^{-1} \colon \bar{h}_1(E_0) \to \bar{h}_2(E_0)$$

Just as above, this construction can be interpreted as a topological pushout so that Lemma A.2.1 applies to say that $M_1 \natural_N M_2$ has a natural smooth structure.

Example 1.3.5. *Suppose again that N is a 0-manifold, but this time embedded into the boundaries of M_1 and M_2. The resulting manifold is called the* **boundary sum** *$M_1 \natural M_2$.*

Example 1.3.6. *Suppose M_1 and M_2 have boundaries which are orientation-reversing diffeomorphic*

$$\phi\colon \partial M_1 \to -\partial M_2$$

Then by taking $N = \partial X_1$ and the diffeomorphisms $h_i\colon N \to (-1)^{i+1}\partial M_i$, the boundary attaching operation produces a closed smooth manifold $M_1 \natural_N M_2$. But there is an obvious way to topologically attach these manifolds, by forming that quotient $M_1 \cup_\phi M_2$. This operation results in a topological manifold which is homeomorphic to $M_1 \natural_N M_2$, so we will often use $M_1 \cup_\phi M_2$ to actually mean the smooth manifold construction.

1.3.2 Elementary Surgery

A. Wallace and R. Thom initiated the study of "surgery" on smooth manifolds. Milnor's 1961 paper "A Procedure for Killing Homotopy Groups of Differentiable Manifolds"[Mil61] carefully analyzes the effects of Thom's "spherical modifications" on the topology of a manifold.

Construction 1.3.7. *Let $n = p + q + 1$ and let M^n be a smooth manifold. Consider a smooth orientation preserving embedding*

$$f : S^p \times D^{q+1} \to M^{q+p+1}$$

The smooth manifold $\chi(M, f)$ is defined as follows:

First take the manifolds $M_0 = M \setminus f(S^p \times 0)$ and $E = \mathring{D}^{p+1} \times S^q$. Then define two embeddings:

1. $\mathring{f}\colon S^p \times (\mathring{D}^{q+1} \setminus 0) \to M_0$*, the restriction of f, and*
2. $s\colon S^p \times (\mathring{D}^{q+1} \setminus 0) \to E$ *by $s(u, rv) = (ru, v)$ for $u \in S^q, v \in S^p, r \in (0, 1)$.*

The **modified (or surgered) manifold** *$\chi(M, f)$ is defined as the pushout of $\mathring{f}$ and s (as in Appendix A).*

Definition 1.3.8. *Two manifolds of the same dimension are* **χ-equivalent** *if they are connected by a finite sequence of surgeries.*

Theorem 1.3.9 ([Mil61, Theorem 1]). *Two compact oriented manifolds without boundary are χ-equivalent iff they are cobordant.*

The main application of this procedure is a reduction of the groups π_k and H_k.

Definition 1.3.10. *Let $\lambda \in \pi_p(M)$. An embedding $f\colon S^p \to M$* **represents** *λ if $\lambda = f_*(\iota)$, where ι is a generator of $\pi_p(S^p \times D^{n-p})$.*

Lemma 1.3.11 ([Mil61, Lemma 3]). *Let f_0 be any continuous map in λ, and that $n \geq 2p+1$. Then there exists and embedding $S^p \times D^{n-p} \to M$ representing λ iff $f_*(\tau_M)$ is trivial.*

Theorem 1.3.12 ([Mil61, Theorem 2]). *Let M be a π-manifold of dimension $n \geq 2p+1$. Then any homotopy class is $\pi_p(M)$ is represented by an embedding f such that $\chi(M, f)$ is a π-manifold.*

Corollary 1.3.13. *Any compact π-manifold of dimension n is cobordant to a π-manifold which is $\lfloor \frac{n}{2} - 1 \rfloor$-connected.*

Lemma 1.3.14 ([Mil61, Lemma 7]). *Let M be m-parallelizable of dimension $2m$ with m even. For $\beta \in H_m(M)$, let $f_0 : S^m \to M$ be an imbedding representing β. Then the normal bundle of $f_0(S^m)$ is trivial iff the intersection number $\langle \beta, \beta \rangle$ is zero.*

Recall that the signature is a cobordism invariant, so that it does not change with surgeries.

Theorem 1.3.15 ([Mil61, Theorem 4]). *Let M be m-parallelizable and $(m-1)$-connected of dimension $2m$ where $m \neq 2$ is even. Suppose the intersection form of M has signature 0, and that ∂M has no homology in dimensions m and $m-1$. Then M is χ-equivalent to an m-connected manifold.*

1.3.3 Plumbing

Construction 1.3.16. *Let M^m, N^n be manifolds, let ξ_M be an n-disk bundle over M and let ξ_N be an m-disk bundle over N. M and N have contractible neighbourhoods U_M, U_N so that*

$$\xi_M|_{U_M} \cong U_M \times \mathbb{D}^n \quad \text{and} \quad \xi_N|_{U_N} \cong U_N \times \mathbb{D}^m$$

Let $\phi_M \colon U_M \to \mathbb{D}^m$ and $\phi_N \colon U_N \to \mathbb{D}^n$, and let $s \colon \mathbb{D}^m \times \mathbb{D}^n \to \mathbb{D}^n \times \mathbb{D}^m$ send (x, y) to (y, x). Then, identify the total spaces of ξ_M and ξ_N by the diffeomorphism

$$U_M \times \mathbb{D}^n \xrightarrow{\phi_M \times \mathrm{id}} \mathbb{D}^m \times \mathbb{D}^n \xrightarrow{s} \mathbb{D}^n \times \mathbb{D}^m \xrightarrow{\phi_N \times \mathrm{id}} U_N \times \mathbb{D}^m$$

The result will be a topological manifold "with corners," which can be smoothed. We will say that the resulting smooth manifold is **plumbed** *from ξ_M and ξ_N*

Example 1.3.17. *Take k copies of the closed manifold M^m, and let G be a connected graph with k vertices $v_1, \ldots, v_k$ labelled by integers. For each $1 \leq i \leq n$, let ξ_i be the m-disk bundle over M^m with Euler number v_i. Then, starting with adjacent vertices v_i and v_j, plumb together ξ_i and ξ_j and inductively plumb adjacent bundles with the resulting manifold.*

1.4 E_{4k} and G_{4k}

There is a general construction for producing a manifold of dimension $4k$ with signature 8.

Definition 1.4.1 (E_{4k})**.** *Take 8 copies of the tangent disk bundle ξ over S^{2k}. Plumb them according to the following graph:*

$$\begin{tikzcd}[arrows=dash]
2 \arrow[r] & 2 \arrow[r] & 2 \arrow[r] & 2 \arrow[r] & 2 \arrow[r] \arrow[d] & 2 \arrow[r] & 2 \\
 & & & & 2 & &
\end{tikzcd}$$

After smoothing corners, the resulting manifold will be called the E_{4k} manifold.

E_{4k} is parallelizable, has signature 8, and the boundary of E_{4k} is a homotopy sphere if $k > 1$. For an alternate description and proofs of these assertions, see [Mil07, Differentiable Manifolds which are Homotopy Spheres].

Kervaire and Milnor investigated a minimum signature for almost parallelizable $4n$-manifolds.

Let j_n be the order of the finite cyclic group $J\pi_{4n-1}(SO_m)$ in the stable range $m > 4n$. Let $a_n = 2$ for n odd and 1 for n even.

Theorem 1.4.2 ([MK60, Theorems 1,2])**.** *If M^{4n} is almost parallelizable, the $p_n[M]$ is divisible by $j_n a_n (2n-1)!$.*

Furthermore, there exists an almost parallelizable manifold G_{4n} with this Pontryagin number.

Theorem 1.4.3 ([MK60, 457])**.**

$$\sigma(G_{4n}) = 2^{2n-1}(2^{2n-1} - 1)B_n j_n a_n / n$$

The index of any almost parallelizable $4n$-manifold is a multiple of this number.

In particular, $\sigma(G_8) = 224$.

Chapter 2

Homotopy Spheres

Differential Topology seeks to distinguish between smooth manifolds who are homotopy-equivalent. A reasonable place to start would be with the simplest closed manifolds: spheres.

Definition 2.0.1. *A (smooth)* **homotopy sphere** *of dimension n is a closed smooth manifold M which homotopy equivalent to the standard sphere S^n.*

An **exotic sphere** *is a homotopy sphere which is not diffeomorphic to S^n.*

The Topological Poincaré Conjecture states that every homotopy sphere is homeomorphic to S^n. This "conjecture" has recently been verified in all dimensions. It was known classically for $n = 1, 2$, and proven in high dimensions ($n > 4$) as a consequence of Smale's h-cobordism theorem [Sma61]. The case $n = 4$ follows from Freedman's classification of topological 4-manifolds, and $n = 3$ follows from Thurston's Geometrization Conjecture, proven by Perelman. (It is interesting to note that the proofs are not direct, but rather the result follows from very powerful results about all manifolds of a given dimension.)

On the other hand, the Smooth Poincaré Conjecture, which asserts that every homotopy n-sphere is *diffeomorphic* to S^n, is false, as was first demonstrated by Milnor [Mil56b] in dimension 7. There are certain high dimensions for which the conjecture is true, but whether it is true in arbitrarily high dimensions remains unknown.

A useful fact about homotopy spheres is that they are always stably parallelizable.

Theorem 2.0.2 ([KM63, Theorem 3.1]). *Every homotopy sphere Σ^n is a π-manifold.*

Proof. This can be seen by studying the obstruction to trivializing the bundle

$$\tau(\Sigma) \oplus \epsilon^1.$$

For this bundle to be trivial, there need to be $n+1$ independent nowhere-vanishing sections, or equivalently a section of the associated Stiefel bundle with fibre $V_{n+1,n+1} = O_{n+1}$. Since homotopy spheres can be oriented, this reduces to looking for a section of the associated SO_{n+1} bundle. Σ^n is $(n-1)$-connected, so the only obstruction is

$$\mathfrak{o}_n(\Sigma) \in H^n\big(\Sigma; \pi_{n-1}(SO_{n+1})\big)$$

Pass to the stable limit $\pi_{n-1}(SO)$ and apply Bott Periodicity: when $n \geq 2$

$$\begin{aligned}\pi_{n-1}(SO) = &\mathbb{Z} \ if \ n \equiv 0, 4 \ mod \ 8 \\ &\mathbb{Z}_2 \ if \ n \equiv 1, 2 \ mod \ 8 \\ &0 \ otherwise\end{aligned}$$

In the first case, write $n = 4k$. Work of Kervaire and Milnor [MK60] has shown that $\mathfrak{o}_n$ can be identified with a multiple of the k-th Pontryagin class $p_k(\tau \oplus \epsilon^1) = p_k(\tau)$. Hirzebruch's signature theorem expresses the signature $\sigma(\Sigma)$ as a polynomial in the pontryagin numbers, but since Σ^n is $(4k-1)$-connected it must be the case that $\sigma = a_k p_k(\Sigma)$. The vanishing of $H^{2k}(\Sigma)$ now says that there is no intersection form, so $0 = \sigma = p_k(\Sigma)$. Therefore $\mathfrak{o}_n = 0$.

The second case uses the homomorphism $J_{n-1} : \pi_{n-1}(SO) \to \pi_{n-k-1}(S^k)$ defined above. Rohlin showed that $J_{n-1}(\mathfrak{o}_n) = 0$, while it was shown by Adams [AW65] that J_{n-1} is injective when n is congruent to 1 or 2 modulo 8, so $\mathfrak{o}_n = 0$.

The final case trivially has $\mathfrak{o}_\mathfrak{n} = 0$. Hence the obstruction always vanishes.

□

2.1 Dimension 7

This section summarizes the work of Milnor in "On Manifolds Homeomorphic to the 7-Sphere." For a more complete development, the reader is referred to the original source [Mil56b].

Lemma 2.1.1. *Let $\xi = \{S^3 \to E \xrightarrow{\rho} S^4\}$ be a 3-sphere bundle over the 4-sphere with euler number e. Then E is a homotopy sphere iff $e = \pm 1$.*

Proof. E is clearly simply connected, so if it has the same cohomology as S^7 they will be homotopy equivalent. Recall the Gysin sequence for vector a bundle with total space E' and base B:

$$\cdots \to H^i(B) \xrightarrow{\cup e} H^{i+n}(B) \xrightarrow{\rho^*} H^{i+n}(E'_0) \to H^{i+1}(B) \xrightarrow{\cup e} \ldots$$

where $E'_0 = E' \setminus B$. E is a sphere bundle, but if E' is the associated $\mathbb{R}^4$ bundle then E'_0 retracts onto its sphere bundle, which is naturally isomorphic to E. Thus there is the sequence

$$\cdots \to H^i(S^4) \xrightarrow{\cup e} H^{i+4}(S^4) \xrightarrow{\rho_0^*} H^{i+4}(E) \to H^{i+1}(S^4) \xrightarrow{\cup e} \ldots$$

For $i \neq -1, 0, 3, 4$ (i.e. $i + 4 \neq 3, 4, 7, 8$) $H^i(S^4) \cong H^{i+1}(S^4) \cong 0$ and hence $H^{i+4}(E) \cong H^{i+4}(S^4)$. Using $i = -4$ yields $H^0(E) \cong \mathbb{Z}$, and $H^{i+4}(E) \cong 0$ for any other value of i aside from $-1, 0, 3, 4$, so these are the only cases remaining.

Using $i = 4$ trivially gives $H^8(M^7_k) \cong 0$ and $i = 3$ gives $H^7(E) \cong H^4(S^4) \cong \mathbb{Z}$.

For the last two cases, consider this part of the sequence:

$$0 \to H^3(E) \to H^0(S^4) \xrightarrow{\cup e} H^4(S^4) \to H^4(E) \to 0$$

The proof up to this point has made no assumptions about e. If $e = \pm 1$ then $\cup e : H^0(S^4) \to H^4(S^4)$ is an isomorphism, so by exactness $H^3(E) \cong H^4(E) \cong 0$. Combined with the above results, $H^i(E) \cong \mathbb{Z}$ when $i = 0, 7$ and $\cong 0$ for all other values, hence has the same cohomology as S^7. Conversely, if $H^3(E) \cong H^4(E) \cong 0$ then $\cup e$ is an isomorphism, and so e must be ± 1.

□

The structure group of an S^3 bundle over an orientable smooth manifold can be reduced to SO_4 (after choosing a metric). Since the base is a sphere S^4, by Steenrod [Ste99, §18] they are determined up to bundle isomorphism by a homotopy class of continuous maps from the equator into the structure group. Hence they are classified by $\pi_3(SO_4) \cong \mathbb{Z} \oplus \mathbb{Z}$ [Ste99, §22]. For the purposes of this discussion we will use the parametrization given by Milnor [Mil56b, p.402]:

Definition 2.1.2. *Given a pair $(h, j) \in \mathbb{Z} + \mathbb{Z}$, define the function $f_{h,j} : S^3 \to SO(4)$ by $f_{h,j}(q) \cdot q' = q^h q' q^j$, where $\mathbb{R}^4$ is identified with $\mathbb{H}$.*

Definition 2.1.3. *Let $\xi_{h,j}$ be the S^3 bundle over S^4 corresponding to the function $f_{i,j}$*

Lemma 2.1.4. $e(\xi_{h,j}) = h + j$

Corollary 2.1.5. *If $h + j = 1$, then the entire space $\xi_{h,j}$ is a homotopy sphere. Thus there is a parametrization of some homotopy 7-spheres by pairs (h, j) such that $h + j = 1$.*

Definition 2.1.6. *For integers h, j such that $h + j = 1$, let $k = h - j$ and $M_k = E(\xi_{h,j})$.*

Milnor [Mil56b] distinguished members of this class using a fundamental geometric result:

Theorem 2.1.7. *Every oriented 7-manifold bounds an oriented 8-manifold.*

Proof. The oriented cobordism group Ω_7^{SO} is trivial [Tho54, IV.13], hence every oriented 7-manifold bounds. □

Let B^8 be a manifold bounded by M_k (compatibly oriented) and consider the long exact cohomology sequence of the pair (B, M_k). Part of it is as follows:

$$\cdots \leftarrow H^4(M_k) \leftarrow H^4(B) \xleftarrow{i} H^4(B, M_k) \leftarrow H^3(M_k) \leftarrow \dots$$

Since M_k is a homotopy sphere, the groups on the ends are 0, and so i is an isomorphism. In fact, i will be an isomorphism whenever $H^3(M) = H^4(M) = 0$, so let M be any 7-manifold with this property, and let B^8 be a cobounding 8-manifold.

Definition 2.1.8. *Let $\mu = [B, M]$ denote the fundamental class of $H^8(B, M)$. Let $q(B) := \langle (i^{-1}p_1)^2, \mu\rangle$.*

Theorem 2.1.9 ([Mil56b],Theorem 1)**.** *Let M bound B^8. Then $2q(B) - \sigma(B)$ is well-defined (up to choice of B) modulo 7.*

Proof. Chose B_1 and B_2 bounded by M. Attach these manifolds along their common boundary to form C^8, and let $q(C) = \langle p_1^2, [C]\rangle$. C is a closed manifold, so by the Hirzebruch signature theorem

$$\sigma(C) = \frac{1}{45}(7\langle p_2, [C]\rangle - \langle p_1^2, [C]\rangle)$$

Rearranging and simplifying, this implies that $2\langle p_1^2, [C]\rangle - \sigma(C) \equiv 0 \pmod 7$. But $\sigma(C) = \sigma(B_1) - \sigma(B_2)$ and $q(C) = q(B_1) - q(B_2)$ [Mil56b, Lemma 1], so

$$2q(B_1) - \sigma(B_1) \equiv 2q(B_2) - \sigma(B_2) \pmod 7$$

□

Definition 2.1.10. *Let $\lambda(M) = [2q(B) - \sigma(B)] \pmod 7$, where B^8 is any manifold bounded by M.*

Corollary 2.1.11 ([Mil56b], Corollary 1)**.** *If $\lambda(M) \not\equiv 0 \pmod 7$, then $H^4(B) \neq 0$ for any cobounding 8-manifold B. In particular, M is not the standard sphere S^7. If $\lambda(M_k) \not\equiv 0 \pmod 7$, then M_k is an exotic sphere.*

Lemma 2.1.12 ([Mil56b], Lemma 3)**.** *Let ι denote the standard generator for $H^4(S^4)$. Then $p_1(\xi_{h,j}) = \pm 2(h - j)\iota$.*

Proof. $p_1(\xi_{h,j})$ is linear in h and j, and if the orientation of the fibre S^3 is reversed this class will not change. It follows that $p_1(\xi_{h,j}) = p_1(\xi_{-h,-j})$. Thus $p_1 = c(h-j)\iota$ for some constant c. When $k = h - j = 1$ the associated disk bundle is diffeomorphic to $\mathbb{H}P^2$ with an 8-disk removed [Mil56b, p403], and it is known [Hir53] that $p_1(\mathbb{H}P^2)$ is twice a generator of $H^4(\mathbb{H}P^2)$. Hence $c = \pm 2$. □

Lemma 2.1.13 ([Mil56b]. Lemma 4)**.** $\lambda(M_k) \equiv k^2 - 1 \pmod 7$

Proof. $\lambda(M_k)$ can be computed using *any* cobounding manifold B_k^8, so take the associated disk bundle. Then B_k retracts onto the base space S^4, so the generator of $H^4(B_k)$ is given by $\alpha = \rho^*(\iota)$. Orientations can be chosen so that $\sigma(B_k) = 1$.

The tangent bundle of B_k is given by $\tau(B_k) = \rho^*(\tau(S^4)) \bigoplus \rho^*(\xi_{h,j})$; that is, the tangent bundle of B_k is the sum of the bundle of vectors tangent to the base space with the bundle of vectors tangent to the fibre. $p_1(S^4) = 0$, so by the Whitney product formula

$$p_1(B_k) = \rho^*(p_1(\xi_{h,j})) = \rho^*(2(h-j)\iota) = 2k\alpha$$

Thus $q(B_k) = 4k^2$ and $\lambda(M_k) = 2(4k^2) - 1 \equiv k^2 - 1 \pmod 7$

□

Corollary 2.1.14. *If* $k^2 \neq 1 \pmod 7$ *then* M_k *is an exotic sphere.*

2.2 Θ_n

The spheres Milnor constructed in dimension 7 are given as the total space of an S^3 bundle over S^4, but not every homotopy sphere appears in this way: the connected sum of 2 homotopy spheres is again a homotopy sphere, but not necessarily the total space of a spherical fibration over a sphere. Regardless of the form they take, the collection of *all* homotopy spheres of a given dimension (up to h-cobordism) has been successfully studied as an algebraic object in the famous Kervaire-Milnor paper "Groups of Homotopy Spheres. I" [KM63]. This section summarizes the work done there.

Definition 2.2.1. *Let $n \in \mathbb{N}$. Define Θ_n to be the set of all homotopy spheres of dimension n, modulo h-cobordism.*

Lemma 2.2.2 ([KM63, Theorem 1.1])**.** *Θ_n is an abelian group under connected sum, $\#$.*

Proof. Appendix A, for example, shows that # is well-defined, associative, and commutative.

Given homotopy spheres Σ, Σ' of dimension n, $\Sigma\#\Sigma'$ is simply connected by the Van Kampen theorem, and from Meyer-Vietoris it is seem that the homology vanishes is all dimensions but 0 and n. Thus $\Sigma\#\Sigma'$ is again a homotopy sphere of dimension n.

For any n-manifold $M\#S^n \cong_+ M$, so S^n is the identity element for each n.

For any homotopy sphere Σ, let $\bar{\Sigma}$ denote the same manifold with reversed orientation. $\Sigma\#\bar{\Sigma}$ bounds a contractible manifold [KM63, Lemma 2.4], which happens iff $\Sigma\#\bar{\Sigma}$ is h-cobordant to the standard sphere [KM63, Lemma 2.3]. Hence inverses are given by reversing the orientation. □

This group is related to the stable homotopy group. Recall $p(M^n) \subset \Pi_n$ from Definition 1.2.23.

Lemma 2.2.3 ([KM63, Lemma 4.5])**.** *$p(S^n)$ is a subgroup of Π_n. For any homotopy sphere Σ^n, $p(\Sigma)$ is a coset of this subgroup. Thus the correspondence $\Sigma \to p(\Sigma)$ defines a homomorphism*

$$p' \colon \Theta_n \to \Pi_n/p(S^n)$$

[KM63, Lemma 4.2] indicates that the kernel of this homomorphism consists exactly of the homotopy n-spheres which bound parallelizable $(n+1)$-manifolds.

Definition 2.2.4. *Let bP_{n+1} denote the kernel of the homomorphism p'.*

Corollary 2.2.5. *Θ_n/bP_{n+1} is isomorphic to a subgroup of $\Pi_n/p(S^n)$, and therefore is finite.*

Theorem 2.2.6. $bP_{2k+1} = 0$ *[KM63, Theorem 5.1]*

bP_{4k} is finite cyclic. [KM63, Corollary 7.6]

bP_{4k+2} is either 0 or cyclic of order 2. [KM63, Theorem 8.5]

Hence bP_{n+1} is finite for all n.

Corollary 2.2.7. *Θ_n is finite for all n.*

Combined with the h-cobordism theorem, this says that for $n \geq 5$ there are finitely many diffeomorphism classes of homotopy n-spheres. In the case $n = 4$, $\Theta_4 = 0$ but since $n = 4$ is too small to apply the h-cobordism theorem this only says that any 2 homotopy 4-spheres are h-cobordant. In fact, at the time of writing it is not known if there are any exotic 4-spheres.

2.3 Γ^n

There is another interpretation of homotopy spheres, which is equivalent in high dimensions.

Definition 2.3.1. *Define* $\Gamma^n := \pi_0 \operatorname{Diff}(S^{n-1})$

Proposition 2.3.2. *Define a map $\Gamma^n \to \Theta_n$ by $[f] \to S^{n-1} \cup_f S^{n-1} = \Sigma(f)$*
Then if $n \geq 5$ this map is an isomorphism.

There are more equivalent interpretations of this group:

Proposition 2.3.3. *The above definition is equivalent to the following:*

1. *The group of concordance classes of diffeomorphisms of S^{n-1}, under composition.*

2. *The group of concordance classes* $\operatorname{rel} \partial D^{n-1}$ *of diffeomorphisms of D^{n-1} which are* id *on ∂D^{n-1}*

In any case, for $\sigma \in \Gamma^n$ the corresponding diffeomorphism will be denoted h_σ.

2.4 Some Invariants for Smooth Manifolds

Since "homeomorphic" does not imply "diffeomorphic" it was necessary for mathematicians, as we saw with λ in Section 2.1, to invent new invariants that were diffeomorphism invariants *but not homotopy invariants.* In the paper "Differentiable Manifolds Which are Homotopy Spheres" [Mil07, p65] Milnor made another invariant λ', based on a minimum signature for certain $4k$-manifolds. Eells and Kuiper [EK62] gave a succinct description of Milnor's "λ" invariants in terms of Hirzebruch's signature theorem, and also give their own invariant μ based on Hirzebruch's "$\hat{A}$ integrality" theorem. This section is primarily results from these two papers.

In Section 2.1 we saw that Milnor [Mil59] had use a (mod 7) invariant $\lambda(M)$ for 7-manifolds in order to distinguish between homotopy spheres. This invariant generalizes to higher dimensions:

Definition 2.4.1 ([EK62, §2])**.** *A smooth $(4k-1)$-manifold M satisfies the* **λ-conditions** *if it is closed, oriented, and its rational homology groups satisfy*

$$H^{2k}(M;\mathbb{Q}) = 0, \;\; H^{4i}(M;\mathbb{Q}) = 0 \;\; (0 < i < k)$$

Just as in [Mil59, Milnor], this cohomology assumption ensures that the "Pontryagin class" $j^{-1}p_i(W)$ in $H^{4i}(W, M;\mathbb{Q})$ is well-defined.

Let L_k denote the k-th polynomial associated with $\frac{z^{1/2}}{\tanh z^{1/2}}$ (see [Hir56] or [MS74, §19]) so that for a closed $4k$-manifold W

$$\sigma(W) = \langle L_k\big(p_1(W), \ldots, p_{k-1}(W), p_k(W)\big), [W]\rangle$$

Let $s_k = L_k(0, ..., 0, 1)$ be the coefficient of p_k, and for the sake of brevity let

$$L_k(p_1, \ldots, p_{k-1}, 0)[W] = \langle L_k\big(j^{-1}p_1(W), \ldots, j^{-1}p_{k-1}(W), 0\big), [W, M]\rangle$$

Definition 2.4.2 ([EK62, §2 (2)]). *For a $(4k-1)$-manifold M satisfying the λ-conditions and W any appropriate boundary, define $\lambda(M)$ as*

$$\lambda(M) \equiv \frac{\sigma(W) - L_k(p_1, p_2, \ldots, p_{k-1}, 0)[W]}{s_k} \pmod 1$$

The well-definedness of this invariant is proven using Hirzebruch's signature theorem, much like in Section 2.1.

In "Differentiable Manifolds Which are Homotopy Spheres" Milnor defined a different invariant λ', but this invariant is defined specifically for homology spheres.

Definition 2.4.3 ([EK62, §2]). *A smooth homology $(4k-1)$-sphere M satisfies the* **λ'-conditions** *if it is closed, oriented, and bounds a compact parallelizable $4k$-manifold W.*

Definition 2.4.4. *Let $\sigma_k = \gcd\{\sigma(W)\}$, where W varies over all almost-parallelizable $4k$-manifolds without boundary (as studied in [Mil07, p65])*

In the same paper, Milnor shows that if W is almost-parallelizable and $\partial W \neq \emptyset$, then $\sigma(W) \pmod{\sigma_k}$ is a diffeomorphism invariant of ∂W. Furthermore, $\sigma(W)$ is divisible by 8 whenever ∂W is a homology sphere.

Definition 2.4.5. *Let M^{4k-1} be a smooth homology sphere satisfying the λ'-conditions. Define*

$$\lambda'(M) \equiv \frac{\sigma(W)}{8} \pmod{\frac{\sigma_k}{8}}$$

The λ invariants are based on the Hirzebruch signature theorem for closed manifolds, which can be stated as

$$L_k(p_1, \ldots, p_k)[X^{4k}] = \sigma(X)$$

where the L_k are the polynomials associated with $\frac{\sqrt{z}}{\tanh \sqrt{z}}$. Recall $\hat{A}_k$, the k-th polynomial associated with $\frac{\sqrt{z}/2}{\sinh(\sqrt{z}/2)}$. Recall also that the $\hat{A}$-genus is an integer for *spin* $4k$-manifolds.

Definition 2.4.6 ([EK62, §3]). *A smooth $(4k-1)$-manifold M satisfies the* μ**-conditions** *if it is closed, oriented, and bounds a compact spin $4k$-manifold W such that*

1. *The homomorphisms*

$$j^*\colon H^{2k}(W,M;\mathbb{Q}) \to H^{2k}(W;\mathbb{Q}),\ j^*\colon H^{2i}(W,M;\mathbb{Q}) \to H^{2k}(W;\mathbb{Q})\ (0<i<k),$$

   from the long exact sequence of the pair (W,M), are isomorphisms.

2. *The inclusion homomorphism $i^*\colon H^1(W;\mathbb{Z}/2) \to H^1(M;\mathbb{Z}/2)$ is a surjection.*

The first condition ensures that Pontryagin classes from W can be pulled back to (W,M). The second condition is assumed because it is something which is true for every coboundary W when $H^1(M;\mathbb{Z}/2)=0$.

Before defining μ, set $a_k := \frac{4}{3+(-1)^k}$, $t_k := \frac{\hat{A}_k(0,\ldots,0,1)}{L_k(0,\ldots,0,1)}$, and

$$N_k(p_1,\ldots,p_{k-1})[X] := \hat{A}_k(p_1,\ldots,p_{k-1},0)[X] - t_k L_k(p_1,\ldots,p_{k-1},0)[X]$$

Definition 2.4.7. *Let M^{4k-1} satisfy the μ conditions, with given co-boundary W^{4k}. Define*

$$\mu(W,M) = \frac{N_k(p_1,\ldots,p_{k-1})[W] + t_k\sigma(W)}{a_k}$$

Let $\mu(M) \equiv \mu(W,M) \pmod 1$.

Theorem 2.4.8 ([EK62, p.97]). $\frac{N_k(p_1,\ldots,p_{k-1})[W]+t_k\sigma(W)}{a_k}$ *is independent of W. Thus $\mu(M)$ only depends on the smooth structure of M.*

Proposition 2.4.9 ([EK62, p.99]). *Let ι be one of the invariants λ, λ' or μ, and suppose M^{4k-1} and N^{4k-1} satisfy the ι-conditions. Then*

1. $\iota(M) = -\iota(M)$

2. $\iota(M\#N) = \iota(M) + \iota(N)$

3. *If M and N are h-cobordant, then $\iota(M) = \iota(N)$*

In [EK62, §4], μ is compared to λ'.

Proposition 2.4.10. *Let M be a closed, smooth, oriented homology $(4k-1)$-sphere. If M has a parallelizable co-boundary W, then both $\lambda'(M)$ and $\mu(M)$ are defined. The invariant λ' gives a differentiable classification of such spaces M which is at least as fine as that given by μ. For dimensions $4k-1$ where $1 \leq k \leq 5$, both invariants give the same information, and this is expressed by the formula*

$$\frac{8\lambda'(M)}{\sigma_k} \equiv -\mu(M) \pmod 1$$

for $1 \leq k \leq 5$.

The main advantages of μ over λ' are that μ is easier to compute and has a larger domain of definition.

This thesis is interested in dimension 7, which is discussed in [EK62, §6]. For any 7-manifold M satisfying the μ-conditions,

$$\mu(M) \equiv \frac{p_1^2[W] - 4\sigma[W]}{2^7 \cdot 7} \pmod 1$$

On the other hand, when M also satisfies the λ conditions

$$\lambda(M) \equiv \frac{p_1^2[W] - 4\sigma[W]}{7} \pmod 1$$

Recall the family of S^3 bundles M_k^7 from Section 2.1.

Proposition 2.4.11 ([EK62, §6 (13)])**.**

$$\mu(M_{2h-1}^7) \equiv \frac{h(h-1)}{56} \pmod 1$$

In particular, $\mu(M_3^7) \equiv \frac{1}{28} \pmod 1$ when $h = 2$. [KM63, Kervarie-Milnor] had shown that $\Theta_7 \cong \mathbb{Z}/28$, so combining this fact with the additivity from Proposition 2.4.9 it follows that M_3^7 is a generator Θ_7. Thus

Theorem 2.4.12 ([EK62, §6 p103])**.** *Two smooth* 7*-manifolds which are homeomorphic to the* 7*-sphere are diffeomorphic iff they have the same* μ *values. Of the* 28 *diffeomorphism classes, precisely 16 occur as total spaces of* S^3 *bundles over* S^4*.*

Corollary 2.4.13 ([EK62, §6 p103])**.** *If* M^7 *is any manifold satisfying the* μ *conditions, then the underlying topological manifold admits at least 28 different differentiable structures.*

Chapter 3

The Inertia Group, $I(M)$

3.1 Definition and Variations

Definition 3.1.1. *Let M^n be a smooth, oriented n-dimensional manifold. The* **inertia group** $I(M) \subset \Theta_n$ *is defined as the set of $\Sigma \in \Theta_n$ for which there exists an orientation preserving diffeomorphism $\phi\colon M \to M\#\Sigma$*

The exact origin of this definition is unknown to the author. Tamura [Tam62] gave explicit inertial spheres for certain 3-sphere bundles over S^4, but without using this terminology.

Lemma 3.1.2. *The inertia group is a group.*

Proof. First, this set is non-empty since $M \cong_+ M\#S^n$ for any n-manifold M.

It must be shown that a connected sum of inertial spheres is inertial, i.e. if $\phi_1\colon M \cong_+ M\#\Sigma_1$ and $\phi_2\colon M \cong_+ M\#\Sigma_2$, then there needs to be an orientation preserving diffeomorphism $M \cong_+ M\#(\Sigma_1\#\Sigma_2)$. Arranging that Σ_1 and Σ_2 are attached using disjoint disks, ϕ_1 can be "surgered" to produce a diffeomorphism

$$(\phi_1)_\#\colon M\#\Sigma_2 \cong_+ (M\#\Sigma_1)\#\Sigma_2$$

There is a natural diffeomorphism $\alpha : (M\#\Sigma_1)\#\Sigma_2 \to M\#(\Sigma_1\#\Sigma_2)$, so $\alpha \circ (\phi_1)_\# \circ \phi_2$ is the required diffeomorphism.

To verify that inverses remain in $I(M)$, if $\Sigma \in I(M)$ then

$$M\#\bar{\Sigma} \cong_+ (M\#\Sigma)\#\bar{\Sigma} \cong_+ M\#(\Sigma\#\bar{\Sigma}) \cong_+ M\#S^n \cong_+ M$$

so $\bar{\Sigma} \in I(M)$. □

Stolz [Sto85, pVII] gives an interpretation which justifies the term "inertia." If $\mathcal{M}_n^k$ denotes the set of diffeomorphism classes of $(k-1)$-connected n-manifolds, then there is an action

$$\Theta_n \times \mathcal{M}_n^k \to \mathcal{M}_n^k$$

given by connected sum. Then for a fixed $M \in \mathcal{M}_n^k$, the group $I(M)$ is the isotropy (or inertia) subgroup of this action. Furthermore, two manifolds M and N are in the same orbit iff they are almost diffeomorphic (that is, $M - \{x_0\} \cong_+ N - \{x_1\}$ for some $x_0 \in M$ and $x_1 \in N$.)

Proposition 3.1.3. *$I(M)$ is not a homotopy invariant.*

Proof. Let Σ^{10} denote the generator of the 3-component of Θ_{10}. Then $I(S^3 \times \Sigma^{10}) = \Theta_{13}$ [Kaw69a, Corollary 2].

On the other hand, if $p + q \geq 5$ then $I(S^p \times S^q) = 0$ [Kaw69a, Corollary 3] and hence $I(S^3 \times S^{10}) = 0 \neq I(S^3 \times \Sigma^{10})$. □

There is a refined version of $I(M)$ where the diffeomorphisms are required to "look like" a standard topological identification $M \to M\#\Sigma$. $M\#\Sigma$ can be described by removing a disk D from M and attaching $\mathbb{D}^n$ to M with an orientation preserving diffeomorphism σ of the boundary S^{n-1}. Any orientation preserving diffeomorphism of S^{n-1} has degree 1, so σ is *homotopic* to $\mathrm{id}_{S^{n-1}}$. Using this homotopy it is possible to construct a homeomorphism $M\#S^n \cong M\#\Sigma$ which is identity on $M \setminus D$ (using the construction of Proposition A.3.3, for example), where $M\#S^n$ is naturally diffeomorphic to M. This gives a homeomorphism $h_\Sigma\colon M \cong M\#\Sigma$ which is "identity" outside the attaching region.

Definition 3.1.4. *If M^n is an n-manifold and Σ^n a homotopy sphere, then let*

$$h_\Sigma \colon M \to M\#\Sigma$$

be the standard topological identification [Bru71, Appendix II].

Define the **homotopy inertia group** *$I_h(M)$ to be the set of all $\Sigma \in I(M)$ who admit a diffeomorphism $M \to M\#\Sigma$ which is homotopic to h_Σ.*

Proposition 3.1.5. *$I_h(M)$ is also not a homotopy invariant.*

Proof. Brumfiel [Bru71, Remark II.10] shows that $I_h(S^1 \times \mathbb{C}P^3) = I(S^1 \times \mathbb{C}P^3) = \mathbb{Z}/7 \subset \Theta_7$. He also notes that there is a class of manifolds P_j^6 which are all homotopy equivalent to $\mathbb{C}P^3$, but $I_h(S^1 \times P_j^6) = 0$ when $j \equiv 1 \pmod 7$. □

Proposition 3.1.6 ([Fra84, Corollary 6.2])**.** *$I_h(M)$ is an h-coboridsm invariant when the dimension of M is ≥ 8*

The following two results show there is a significant discrepancy between these two groups:

Theorem 3.1.7 ([Win71, Theorem 2.10])**.** *For every n, there exists a smooth, closed, simply-connected n-manifold M^n such that $I(M) = \Theta_n$.*

Theorem 3.1.8 (`arXiv:0912.4874v1 [math.DG]` (Taylor's Theorem, 3.2))**.** *If M is a smooth, closed, oriented manifold of dimension $4k-1 \geq 7$, then*
$I_h(M) \cap bP_{4k}$ has index ≥ 2 in bP_{4k}.

In particular, $I_h(M) \neq \Theta_{4k-1}$.

Corollary 3.1.9. *There exists a smooth, closed, simply-connected manifold in each dimension $4k-1 \geq 7$ such that $I_h(M) \neq I(M)$.*

These are not all of the variants that have been defined. Recall Γ^n from Section 2.3, the group of diffeomorphisms of D^{n-1} which are id on ∂D^{n-1}, modulo concordance rel ∂D^{n-1}. For $\sigma \in \Gamma^n$, h_σ denotes the corresponding diffeomorphism of D^{n-1}. Levine [Lev70] makes use of the following definition:

Definition 3.1.10. *For M an n-manifold, let $I_1(M)$ be the set of all $\sigma \in \Gamma^{n+1}$ such that the diffeomorphism which is h_σ on an n-disk $D \subset M$ and* id *everywhere else, is concordant to* id.

This definition is related to the usual definition as follows:

Proposition 3.1.11 ([Lev70, Proposition 1])**.**

$$I_1(M) = I(M \times S^1)$$

Frame [Fra82] gave a variation of the inertia group, which he used for studying the inertia group of fibre bundles:

Definition 3.1.12. *Fix $i \in \mathbb{N}$ and let $* \in S^i$. Define $\tilde{I}(M^n \times S^i)$ to be the set of homotopy spheres Σ such that there is a diffeomorphism $G\colon (M \times S^i)\#\Sigma \to M \times S^i$ with $G|_{M\times *} = \mathrm{id}$.*

3.2 Collected Results

3.2.1 Miscellaneous

One simple result about $I_h(M)$ is the following:

Proposition 3.2.1 ([Sch71, Proposition 1.2])**.** *Let M be a closed smooth manifold of dimension $n \geq 7$. If M imbeds in $\mathbb{R}^{n+1}$ then $I_h(M) = 0$.*

Naoum [Nao73] gave a similar result for $I(M)$:

Theorem 3.2.2 ([Nao73])**.** *Let M^n be a smooth closed manifold of dimension ≥ 7 such that $H_i(M^n;\mathbb{Z})$ is cyclic for each $i \geq 0$.*

If there exists a smooth embedding $f\colon M^n \to S^{n+1}$ and a smooth, simply-connected submanifold $W \subset S^{n+1}$ such that $f(M) = \partial W$, then $I(M) = 0$.

Corollary 3.2.3 ([Nao73])**.** *Let M^n be a simply connected, parallelizable manifold with cyclic homology groups. Let T^{2n-1} be the total space of the tangent sphere bundle of M.*

Then $I(T^{2n-1}) = 0$.

Some results are concerning the types of homotopy spheres in $I(M)$, based on properties of M:

Lemma 3.2.4 ([Kaw69b, Lemma 9.1])**.** *Let M^n be simply-connected and spin.*

Then $I(M)$ only contains homotopy spheres bounding spin manifolds.

Theorem 3.2.5 ([Fra82, Theorem 1])**.** *Let $M^n \to C \to N^i$ be a smooth fibre bundle with fibre M^n and base manifold N^i.*

Then $\tilde{I}(M \times S^i) \subset I(C)$.

The following result concerns the inertia group of a fibre bundle:

Theorem 3.2.6 ([Fra82, Theorem 3])**.** *Let $n + 2 < i$, for i fixed.*

Then $I_h(M \times S^i) \subset \tilde{I}(M \times S^i)$, and so $I_h(M \times S^i) \subset I(C)$ for any fibre bundle $M^n \to C \to S^i$.

Kawakubo proved some results about complex projective spaces.

Theorem 3.2.7 ([Kaw68, Theorem 1])**.** *If $n \leq 8$, then $I(\mathbb{C}P^n) = 0$*

Proposition 3.2.8 ([Kaw69b, Proposition 9.2])**.** *For all $k \geq 1$,*

$$I(\mathbb{C}P^{4k+1}) \neq \Theta_{8k+2}$$

More generally, Conrad [Con73] studied manifolds in the homotopy type of $\mathbb{C}P^n$.

3.2.2 Connected Sums

It is not always the case that $I(M\#N) = I(M) + I(N)$, but there are some known results about the inertia group of connected sums:

Proposition 3.2.9. *For all n-manifolds M and N,*

$$I(M\#N) \supset I(M) + I(N)$$

In particular, $I(M\#N)$ also contains the subgroup $I(M) \cap I(N)$.

Proof. If $\Sigma \in I(M)$ then there is a diffeomorphism $\phi\colon M \cong_+ M\#\Sigma$ which wlog may be chosen to fix the disk where $M\#N$ is formed. Then ϕ may be modified to a diffeomorphism $\phi'\colon M\#N \cong_+ (M\#\Sigma)\#N$ in the obvious way. Then $\Sigma \in I(M\#N)$ by associativity and commutativity of $\#$.

Similarly for $\Sigma \in I(N)$. □

Proposition 3.2.10 ([Fra84, Proposition 3.2])**.** *For all n-manifolds N and M with $n \geq 5$, $I_h(M) \subset I_h(N\#M)$*

Proposition 3.2.11 ([Fra84, Corollary 4.2])**.** *Let M^n be 2-connected with $n \geq 7$. Then for all m, $I(M) = I(M\#m(S^2 \times S^{n-2}))$*

Lemma 3.2.12 ([Fra84, Lemma 4.1])**.** *If M^n and N^n are h-cobordant with $n \geq 5$ then for sufficiently large m there is a diffeomorphism $M\#m(S^2\times S^{n-2}) \cong N\#m(S^2\times S^{n-2})$*

Lemma 3.2.13 ([Fra84, Corollary 6.1])**.** *If $n \geq 8$, then*

$$I_h(M^n) = I_h(M^n\#m(S^2 \times S^{n-2}))$$

Corollary 3.2.14. *If M^n and N^n are h-cobordant with $n \geq 8$, then $I_h(M) = I_h(N)$*

3.2.3 Constructions

The first example known to the author of a manifold with a non-trivial inertia group was given in Tamura's 1962 paper "Sur les sommes connexes de certaines variétés différentiables" [Tam62]. Here, much like in Milnor [Mil56b], he considered a class of 8-manifolds $\bar{B}_{m,n}$ which are D^4 bundles over S^4, and their associated 3-sphere bundles $B_{m,n}$. These bundles correspond to characteristic maps $m\rho + n\sigma \in \pi_3(SO_4)$, where $\rho(q)\cdot q' = qq'q^{-1}$ and $\sigma(q)\cdot q' = qq'$ (compare Definition 2.1.2). Tamura's parametrization is such that the Euler number is given by n. He shows

Theorem 3.2.15 ([Tam62])**.**

$$B_{mn+\frac{n^3+n}{2}\ ,\ 1-n^2} \# B_{m,1} \cong_+ B_{m+\frac{n^3+n}{2}\ ,\ 1-n^2}$$

One corollary is that $B_{m,1}$ is homeomorphic to S^7 for every m. Milnor [Mil56b] showed more directly that the bundles with Euler number ± 1 are exactly those homeomorphic to S^7. More relevant to this discussion is the following:

Corollary 3.2.16 ([Tam62])**.**

$$B_{m,0} \# B_{m-1,1} \cong_+ B_{m,0}$$

In other words, $B_{m-1,1} \in I(B_{m,0})$ for every m. Moreover, Tamura points out that $B_{m-1,1}$ is a generator of the group of homotopy 7-spheres, and so

Corollary 3.2.17. *$I(B_{m,0}) = \Theta_7$ for every m.*

Although Tamura was not writing in terms of "inertia group" or $I(M)$ (or a French equivalent), this corollary opened the door to this field of research.

Recall (Proposition 2.3.3) the group $\Gamma^n = \pi_0(\mathrm{Diff}(S^{n-1}))$, which is naturally isomorphic to Θ_n when $n \geq 5$. Levine [Lev70] gives another interpretation: the members are equivalence classes of diffeomophisms of D^{n-1} which fix the boundary, modulo concordance rel ∂D^{n-1}.

Construction 3.2.18 ([Lev70, §7])**.** *Let $n, k \in \mathbb{N}$, $\sigma \in \Gamma^{n+1}$, $\tau \in \Gamma^{k+1}$, $\alpha \in \pi_n(SO_k)$, $\beta \in \pi_k(SO_n)$.*

Let h_σ represent σ as a diffeomorphism of S^n, which in particular fixes a hemisphere $D \subset S^n$, and let h_τ represent τ as a diffeomorphism of D^k which is id *on an nhd N of the boundary S^{k-1}.*

Let $f\colon (S^n, D) \to (SO_k, e)$ and $g\colon (D^k, S^{k-1}) \to (SO_n, e)$ represent α and β respectively.

Define two diffeomorphisms of $S^n \times D^k$

$$d_1(x, y) = (h_\sigma(x), f(x)(y))$$
$$d_2(x, y) = (g(y)(x), h_\tau(y))$$

Let $d = d_1^{-1}d_2^{-1}d_1d_2$*. By construction, there is an interior disk* $D_0 \subset S^n \times D^k$ *such that* $d = \mathrm{id}$ *outside of* D_0*.*

Finally define $\delta(\sigma, \alpha; \tau, \beta) \in \Gamma^{n+k+1}$ *to be the sphere represented by* $d|_{D_0}$*.*

These spheres also admit a representation through a "plumbing" construction. Let X_1 be the D^{k+1} bundle over Σ_σ corresponding to $S(\alpha)$ and X_2 be the D^{n+1} bundle over Σ_τ corresponding to $S(\beta)$. Each of these bundles has a subbundle diffeomorphic to $D^{n+1}\times D^{k+1}$ so form X_δ as a quotient space of $X_1 \dot{\bigcup} X_2$ by identifying these subbundles. Then $\delta(\sigma, \alpha; \tau, \beta)$ is represented by ∂X_δ.

When $\sigma = \tau = 0$, this is the same as Milnor's construction from [Mil59].

Theorem 3.2.19 ([Lev70, Theorem 8.1])**.** *Let* M *be a closed, smooth* $(n + k + 1)$*-manifold and suppose that* Σ_σ *is embedded in* M *with normal bundle associated to* $\alpha \in \pi_n(SO_k)$*. Then, for any* $\tau \in \Gamma^{k+1}$*,* $\beta \in \pi_k(SO_n)$*, we have:*

$$\delta(\sigma, \alpha; \tau, \beta) \in I(M)$$

Moreover, Levine shows that if T is a tubular nhd of Σ_σ in M and if $M\#\Sigma_\delta$ is formed at a disk in the interior of T, then $M\#\Sigma_\delta$ is diffeomorphic to M via a diffeomorphism that is id outside of $\overline{M - T}$.

In order to show that the Σ_δ spheres are not standard, Levine [Lev70, §10] computed their μ values.

Definition 3.2.20. *For* $r, s \geq 1$ *let*

$$\mu_{r,s} = \frac{a_r a_s B_r B_s (2^{2r} - 1)(2^{2s} - 1)}{16 a_{r+s} rs (2^{2r+2s-1} - 1)} (mod\ 1)$$

where B_r *is the* r*-th Bernoulli number,* $a_r = 1$ *if* r *is even,* 2 *if* r *is odd.*

In particular, $\mu_{1,1} = \frac{1}{112}$.

For $\alpha \in \pi_n(SO_k)$, its stable suspension into $\pi_n(SO) \cong \mathbb{Z}$ determines a unique non-negative integer denoted $|\alpha|$. If $n \geq 2k + 1$ then $|\alpha| = 0$.

Proposition 3.2.21. *[Lev70, Proposition 5] If $\delta = \delta(\sigma, \alpha; \tau, \beta)$, then*

$$\mu(\delta) = \mu_{r,s}|\alpha||\beta|$$

Example 3.2.22 ([Lev70, Example 2])**.** *Assume $n = 4r - 1$, $k = 4s - 1$, and let $\lambda \in H_{n+1}(M^{n+k+1}; \mathbb{Z})$ be represented by an embedded sphere.*

Then the order of $I(M)$ is a multiple of the denominator of

$$\frac{\epsilon_{r,s}\mu_{r,s}(p_r(M) \cdot \lambda)}{a_r(2r-1)!}$$

where $\epsilon_{r,s} = 2$ if $r = s = 1, 2$ or $r = 3, s = 4$, and $\epsilon_{r,s} = 1$ otherwise.

Corollary 3.2.23. *If $s < 2r < 4s$, there exists a k-sphere bundle M over S^{n+1} such that the order of $I(M)$ is a multiple of the denominator of $\epsilon_{r,s}\epsilon_{s,r}\mu_{r,s}$*

3.2.4 Products of Spheres

The problem of computing the inertia group, as well as the related problem of classifying smooth structures, is notoriously difficult. One case where there exist complete results is the case of products of spheres.

Theorem 3.2.24 ([Sch71, Theorem A])**.** *Let $p + q \geq 5$. Then $I(S^p \times S^q) = 0$.*

Moreover, Kawakubo solved the problem of computing $I(S^p \times \Sigma^q)$ where $\Sigma^q \in \Theta_q$. [Kaw69a, Theorem C] gives the inertia group of $S^p \times \Sigma^{q+1}$ in terms of a pairing, which he calls $K_1 \colon \pi_p(SO_{q+1}) \times \Gamma^{q+1} \to \Theta^{p+q+1}$

Definition 3.2.25. *Let $0 < p < q$, $h \in \pi_p(SO_{q+1})$, $r \in \Gamma^{q+1}$, and let $F : S^p \times S^q \to S^p \times S^q$ be defined by $F(x, y) = (x, rh(x)r^{-1}(y))$.*

Let $K_1(h, r) := D^{p+1} \times S^q \cup_F S^p \times D^{q+1}$.

Theorem 3.2.26 ([Kaw69a, Theorem C])**.** *Let $p \neq q$, $p + q \geq 5$.*

Then $I(S^p \times \Sigma^{q+1}) = K_1\big(\pi_p(SO_{q+1}), \Sigma^{q+1}\big)$.

3.2.5 π-manifolds

The study of $I(M)$ for π-manifolds is closely related to the image $p(M) \subset \Pi_n$ (or $p'(M) \subset \Pi_n / \operatorname{Im} J$), as studied by Kosinski [Kos67].

Definition 3.2.27. *Let* $I'(M) = I(M)/(I(M) \cap bP_{n+1})$.
Let $\rho(M) = |p'(M)|$.

$I'(M)$ acts on $p'(M)$: for $\alpha \in p'(M)$ and $\Sigma \in I(M)$, define $\Sigma \cdot \alpha = \alpha + p'(\Sigma)$.

Lemma 3.2.28 ([Kos67, Lemma 1.3])**.** *$I'(M)$ acts without fixed points on $p'(M)$. Thus the order of $I'(M)$ is not larger than $\rho(M)$. In particular, if $\rho(M) = 1$ then $I(M) \subset bP_{n+1}$.*

Theorem 3.2.29 ([Kos67, Theorem 3.1])**.** *Let M be a $2n$-dimensional π-manifolds, $n > 3$, $n \neq 7$. Then $\rho(M) = 1$ in each of the following cases:*

1. *M is $(n-1)$-connected*
2. *M is $(n-2)$-connected, and $n \equiv 3, 5, 6, 7 \pmod 8$*
3. *M is $(n-3)$-connected, and $n \equiv 6, 7 \pmod 8$*
4. *M is $(n-4)$-connected, and $n \equiv 7 \pmod 8$*

In particular, $I(M) = 0$ in each of these cases because $bP_{2n+1} = 0$ according to Section 2.2.

Kosinski [Kos67, §4] considers a particular case of the pairing

$$\pi_m(SO_{n+1}) \times \pi_n(SO_{m+1}) \to \theta_{n+m+1}$$

given in [Mil59]. Let $\gamma, \gamma' \in \pi_{n-1}(SO_n)$, and let $M(\gamma, \gamma')$ be a handlebody of dimension $2n$ with $\{e, e'\}$ a basis of $H_n(M)$, where the normal bundles of these classes have characteristic maps $\alpha(e) = \gamma$ and $\alpha(e') = \gamma'$, and such that $M(\gamma, \gamma')$ has intersection matrix

$$\mathcal{M} = \begin{bmatrix} e \cdot e & 1 \\ (-1)^n & e' \cdot e' \end{bmatrix}$$

Let $\Sigma(\gamma, \gamma') = \partial M(\gamma, \gamma')$. Then $\Sigma(\gamma, \gamma')$ is a homotopy $(n-1)$-sphere iff $\det(\mathcal{M}) = \pm 1$.

Definition 3.2.30 ([Kos67, Theorem 6.1])**.** *Let $\Sigma_{n-1} \subset \Theta_{2n-1}$ be the set of homotopy spheres that can be constructed using the above method.*

Theorem 3.2.31 ([Kos67, Theorem 6.1])**.** *Let M be a $(2n-1)$-dimensional $(n-2)$-connected π-manifold, $n \geq 3$. Then*

$I(M) \subset \Sigma_{n-1}$ *if* $n \not\equiv 0 \pmod 4$ *and* $I(M) \subset bP_{2n}$ *if* $n \equiv 0 \pmod 4$

3.2.6 Highly Connected 7-Manifolds

This section is based on [Wil75, Wilkens]. Here he considers $(m-1)$-connected $(2m+1)$-manifolds where $m = 3, 7$, like in [Wil72]. Of particular interest to this thesis is the case $m = 3$ (i.e. $n = 2m + 1 = 7$).

Lemma 3.2.32 ([Wil75, §2])**.** *If M is an $(m-1)$-connected $(2m+1)$-manifold where $m = 3, 7$, then M is m-parallelizable and the only obstruction to stable parallelizability is a class*

$$\hat{\beta} \in H^{m+1}(M)$$

Proof. $\pi_{m-1}(SO) \cong 0$ for $m = 3, 7$, so there is no obstruction in $H^m(M; \pi_{m-1}(SO))$ to trivializing the tangent bundle over the m-skeleton.

In the next dimension, $\pi_m(SO) \cong \mathbb{Z}$ and so the obstruction $\hat{\beta}$ to trivializing the tangent bundle over the $(m+1)$-skeleton really lives in $H^{m+1}(M)$.

Finally, the only other possible obstruction would be in $H^{2m+1}(M; \pi_{2m}(SO))$, which is 0 since $\pi_{2m}(SO) = 0$. □

Lemma 3.2.33 ([Ker59, Lemma 1.1])**.**

$$p_1(M) = 2\hat{\beta} \ (if \ m = 3) \quad p_2(M) = 6\hat{\beta} \ (if \ m = 7)$$

Recall the **torsion product** of two groups $G * H$: roughly speaking, $*$ detects the common torsion between G and H [Mun84, p331].

Theorem 3.2.34 ([Wil75, Theorem 1]). *Let M be a closed $(m-1)$-connected $(2m+1)$-manifold ($m = 3$ or 7).*

If $\hat{\beta}$ is of finite order, then $I(M) \cong 0$.

If $H^{m+1}(M) * \Theta_{2m+1} \cong 0$ and r is the largest integer dividing $\hat{\beta}$ then $I(M)$ consists exactly of those elements of bP_{2m+2} divisible by $r/4$.

In particular, when $m = 3$ the second condition says that $H^4(M)$ has no elements of order 2 or 7. In Wilkens' classification paper [Wil72] he shows that for any finitely generated abelian group G and even element $\hat{\beta} \in G$, these exists a manifold M^7 with $H^4(M) \cong G$ and with tangential invariant $\hat{\beta}$. Furthermore, [Wil72, Theorem 3] shows that the manifolds M^7 with $H^4(M)$ torsion free are classified up to almost-diffeomorphism by $(H^4, \hat{\beta})$. It follows that M admits exactly $|\Theta_7/I(M)|$ smooth structures

Corollary 3.2.35 ([Wil75, p.538]). *Every subgroup of Θ_7 appears as the inertia group of some manifold M^7. This remains true if M is required to have $H^4(M) \cong \mathbb{Z}$.*

Furthermore, if r is any divisor of 28 and $r \geq 2$, then there exists a manifold M with $H^4(M) \cong \mathbb{Z}$ and admits exactly r smooth structures.

In [Wil75, §6], Wilkens shows that "$I(M_1 \# M_2) = I(M_1) + I(M_2)$" is not true in general.

Proposition 3.2.36 ([Wil75, §6]). *There exists 2-connected 7-manifolds P_1, P_2, P_3 such that.*

1. $I(P_1) = I(P_2) = I(P_3) = I(P_2 \# P_3) = 0$
2. $I(P_1 \# P_2) \cong \mathbb{Z}/7$, $I(P_1 \# P_3) \cong \mathbb{Z}/4$
3. $I(P_1 \# P_2 \# P_3) = \Theta_7 \cong \mathbb{Z}/28$

(There is an analogous result in dimension 15, but with bP_{16} instead of Θ_{15}.)

3.3 7-Manifolds with certain S^1 actions

(The proofs of this section are due to Ian Hambleton and the author.)

Question 3.3.1. *Let M^7 be a smooth, oriented manifold which admits a smooth semi-free S^1 action. Suppose further that the fixed point set $F \subset M$ has codimension 2. What can be said about $I(M)$?*

Using the fact that F has codimension 2, it is possible to translate the problem into another context, namely that of M bounding a D^2 bundle over a 6-manifold W with boundary F. ([Hsi64, Hsiang] demonstrates this for the case of a circle action on S^n, but the argument is similar.)

Let $\nu(F)$ be an open tubular nhd of F in M, so that the action of S^1 is free on $M \setminus \nu(F)$. $\nu(F)$ can be chosen so that in that neighbourhood the action is by rotation in the fibres, and so the quotient $\tilde{\nu}(F)$ is diffeomorphic to $F \times [0,1)$. It follows that the quotient of M by the S^1 action is a 6-manifold W, with $\partial W \cong F$. Furthermore, $M \setminus \nu(F) \to W \setminus \tilde{\nu}(F)$ is a circle bundle, and every S^1 bundle is associated to a D^2 bundle, so there is a D^2 bundle ξ over W with total space U^8 such that

$$\partial U = \partial E(\xi|_{W \setminus F}) \cup E(\xi|_F) \cong (M \setminus \nu(F)) \cup \nu(F) = M$$

Conversely, let U^8 be the total space of a D^2 bundle ξ over the 6-manifold W with boundary F^5. There is a natural S^1 action on U^8 given by rotating the fibres, whose fixed point set is the 0-section. This action preserves the boundary $\partial U = \partial E(\xi|_{W \setminus F}) \cup E(\xi|_F)$, so we can consider the restricted action. The fixed points of the restricted action will be the 0-section of $E(\xi|_F)$, or in other words F, and therefore has codimension 2.

Notation. *For the remainder of the section, fix the following notation:*

W^6 is a smooth, oriented manifold with non-empty boundary F^5.

ξ is a disk bundle over W with projection π, ξ_0 its restriction to F.

U^8 is the total space of ξ, D the total space of ξ_0.

Finally, $M^7 = \partial U$. (Note that $M = S(\xi) \cup D$)

Lemma 3.3.2. *Let W be parallelizable, $e(\xi)$ even and $e^2(\xi) = 0$.*

Then U is a spin manifold with $p_1(\tau_U) = 0$.

Proof. The characteristic classes of τ_U are computed using the equation

$$\tau_U = \pi^*(\tau_W) \oplus \pi^*(\xi)$$

Since $e(\xi)$ is even, $\omega_2(\xi) \equiv e(\xi) \equiv 0 \pmod 2$ [MS74, §9.2], so by the Whitney product formula and the triviality of τ_W it follows that $\omega_2(\tau_U) = 0$.

For any oriented $2k$-plane bundle η, $p_k(\eta) = e^2(\eta)$ [MS74, §15.8], and so $p_1(\xi) = e^2(\xi) = 0$. Again it follows that $p_1(\tau_U) = 0$. $\square$

Lemma 3.3.3. *If $H^1(F;\mathbb{Z}/2) = 0 = H^1(W;\mathbb{Z}/2)$, then $H^1(M;\mathbb{Z}/2) = 0$.*

Proof. (In this proof, all coefficients will be $\mathbb{Z}/2$.)

Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^1(W) \arrow[r] \arrow[d] & H^1(S(\xi)) \arrow[r] \arrow[d, "j^*"] & H^0(W) \arrow[r, "\cup e(\xi)"] \arrow[d, "i^*"] & H^2(W) \arrow[r] \arrow[d, "i^*"] & \dots \\
0 \arrow[r] & H^1(F) \arrow[r] & H^1(\partial S(\xi)) \arrow[r] & H^0(F) \arrow[r, "\cup e(\xi_0)"] & H^2(F) \arrow[r] & \dots
\end{tikzcd}$$

where the horizontal rows are the Gysin sequences of $S(\xi)$ and $S(\xi_0) = \partial S(\xi)$ respectively, i^* is the map induced from $i\colon F \to W$, and j^* is induced from $j\colon \partial S(\xi) \to S(\xi)$.

The claim is that j^* is injective. This follows from the observations that i^* is injective on H^0 and that $H^1(W) = 0$, plus a little diagram chasing.

Now look at the Mayer-Vietoris sequence

$$0 \longrightarrow H^1(M) \longrightarrow H^1(S(\xi)) \oplus H^1(D) \xrightarrow{h} H^1(S(\xi_0)) \longrightarrow \dots$$

Since $H^1(D) = H^1(F) = 0$, the homomorphism h is really j^*, which is injective. Hence $H^1(M) = 0$. $\square$

Lemma 3.3.4. *Let X_1, X_2 be spin 8-manifolds with $\partial X_1 = -\partial X_2 = N$, and let $Z = X_1 \cup_N X_2$. Suppose further that $p_1(\tau_{X_1}) = 0 = p_1(\tau_{X_2})$ and $H^1(N;\mathbb{Z}/2) = 0$.*

Then Z is spin and $p_1^2(\tau_Z) = 0$.

Proof. The spin structures on X_1 and X_2 induce spin structures on N. In order for Z to inherit a spin structure, these spin structure on N must be equal. According to Milnor [Mil65, Lemma 1] the spin structure on N is unique iff $H^1(N;\mathbb{Z}/2) = 0$, which was assumed. Thus Z is spin.

To compute $p_1^2(\tau_Z)$, consider this portion of the Mayer-Vieotris sequence:

$$H^3(N) \xrightarrow{\delta} H^4(Z) \longrightarrow H^4(X_1) \oplus H^4(X_2)$$

By assumption the restriction of p_1 to each X_i is 0, so by exactness $\exists \alpha \in H^3(N)$ such that $\delta(\alpha) = p_1(\tau_Z)$. Consider a careful description of the map δ:

$$\begin{tikzcd}
H^3(X_1, N) \arrow[d, "k_1^*"] & H^3(N) \arrow[l, "\delta_1"'] \arrow[r, "\delta_2"] \arrow[d, "\delta"] & H^3(X_2, N) \arrow[d, "k_2^*"] \\
H^4(Z, X_2) \arrow[r, "j_2^*"] & H^4(Z) & H^4(Z, X_1) \arrow[l, "j_1^*"']
\end{tikzcd}$$

where the δ_i's and j_i's are from long exact sequences of pairs, and k_i^* are the inverses of excision isomorphisms. Thus we can write

$$p_1(\tau_Z) \cup p_1(\tau_Z) = \delta(\alpha) \cup \delta(\alpha) = (j_2^* k_1^* \delta_1(\alpha)) \cup (j_1^* k_2^* \delta_2(\alpha))$$

which must be 0 by commutativity of the following diagram:

$$\begin{tikzcd}
H^4(Z, X_1) \times H^4(Z, X_2) \arrow[r, "\cup"] \arrow[d, "j_1^* \times j_2^*"] & H^8(Z, X_1 \cup X_2) = 0 \arrow[d] \\
H^4(Z) \times H^4(Z) \arrow[r, "\cup"] & H^8(Z)
\end{tikzcd}$$

Thus $p_1^2(\tau_Z) = 0$. □

Now the above results will be used to help prove the main result:

Theorem 3.3.5. *Using the notation of this section, assume W^6 is parallelizable, $H^1(W;\mathbb{Z}/2) = 0 = H^1(F;\mathbb{Z}/2)$, $e(\xi)$ is even and $e^2(\xi) = 0$.*

Then $I(M) = 0$.

Proof. Let $\Sigma^7 \in I(M)$, with orientation-preserving diffeomorphism $\phi\colon M\#\Sigma \to M$ (note that ϕ is also an orientation-*reversing* diffeomorphism from $M\#\Sigma$ to $-M$).

There is a preferred generator of Θ_7 given by $\Sigma_0 = \partial E_8$, where E_8 is the parallelizable 8-manifold constructed by Milnor [Mil07, p65] to have $\sigma(E_8) = 8$. If $[\Sigma] = k[\Sigma_0]$ then Σ bounds $V^8 := \natural^k E_8$, the k-fold boundary sum.

Construct the manifold Z as follows: First, form the boundary sum $U \natural V$. Its boundary is $M\#\Sigma$ so it can be attached to $-U$ using ϕ to form an oriented closed manifold. Let $Z = (U\natural V) \cup_\phi (-U)$. Then by Novikov additivity its signature is

$$\sigma(Z) = \sigma(U\natural V) - \sigma(U) = \sigma(U) + \sigma(V) - \sigma(U) = \sigma(V) = k\sigma(E_8) = 8k$$

On the other hand, Lemma 3.3.2 shows that U is spin and $p_1(\tau_U) = 0$. Since E_8 is parallelizable so is V, and a Mayer-Vietoris argument shows

$$H^i(U\natural V) \cong H^i(U) \oplus H^i(V)$$

It follows that $\omega_2(U\natural V) = 0$ and $p_1(U\natural V) = 0$, since the restriction of these classes to each of the summands are 0. Lemma 3.3.3 shows $H^1(M;\mathbb{Z}/2) = 0$, and so Lemma 3.3.4 applies to give $\omega_2(\tau_Z) = 0$ and $p_1^2(\tau_Z) = 0$.

Z is closed and $\omega_2(\tau_Z) = 0$, so $Z \in \Omega_8^{spin} \cong \mathbb{Z} \oplus \mathbb{Z}$. Generators are known [Mil63b]: the quaternionic projective plane $\mathbb{H}P^2$ and the manifold from [MK60, Theorem 2] which we denote G_8. Since $p_1^2(\tau_{\mathbb{H}P^2}) \neq 0$ [Szc64, Corollary 2.3] and $p_1^2(\tau_Z) = 0$, it must be the case that Z is cobordant to the connected sum of a number of copies of G_8, and hence $\sigma(G_8)$ divides $\sigma(Z) = 8k$. [MK60, p.457] shows $\sigma(G_8) = 224$, so $224 = 32 \cdot 7$ divides $8k$. Thus 28 divides k, so in fact $\Sigma = k\Sigma_0$ is diffeomorphic to S^7.

Therefore $I(M) = 0$. □

Translating back to the context of circle-actions, we have:

Corollary 3.3.6. *Let M^7 have a semi-free S^1 action with fixed-point set F^5 satisfying $H^1(F;\mathbb{Z}/2) \cong 0$. Suppose further that the quotient space W is parallelizable, $H^1(W;\mathbb{Z}/2) \cong 0$ and that resulting bundle $\xi\colon M \setminus \nu(F) \to W \setminus \tilde{\nu}(F)$ has $e(\xi) \equiv 0 \pmod 2$ and $e^2(\xi) = 0$. Then $I(M) = 0$.*

Remark 3.3.7. *Most results from the literature about the inertia group of* 7*-manifolds are about* 2*-connected ones. No assumptions have been made here about the connectedness of the manifolds.*

Example 3.3.8. *1. S^1 has trivial tangent bundle, and so $W = S^1 \times D^5$ is a parallelizable* 6*-manifold with a boundary $F = S^1 \times S^4$ (in particular $\pi_1(W) \neq 0$). The product bundle $W \times D^2$ has Euler class 0, and so $\partial(W \times D^2) \cong \partial(S^1 \times D^7) = S^1 \times S^6$ has trivial inertia group. (This was already proven in [Sch71, Schultz] as a special case of the main theorem "$I(S^n \times S^m) = 0$ for $n + m \geq 5$".)*

2. Let N^k be any parallelizable manifold of dimension $k \leq 5$, without boundary. Then $N \times D^{6-k}$ is a parallelizable 6-manifold with boundary, and $M^7 = N \times S^{7-k}$ has trivial inertia group.

To construct more interesting examples, we use Wall's classification of closed simply-connected 6-manifolds (Appendix B) to make a suitable W^6 with boundary, and then choose an even element of $H^2(W)$ whose cup-square is 0 for the Euler class of a disk bundle.

According to the classification theorem (Theorem B.0.3), to produce a closed, simply-connected spin manifold with torsion-free homology it is enough to specify: finitely-generated free-abelian groups $H = H^2(W)$, $G = H^3(W)$, a trilinear form $\mu \colon H \times H \times H \to \mathbb{Z}$, an even element $\bar{\omega}_2 \in H$, and a homomorphism $p_1 \colon H \to \mathbb{Z}$, all of whom satisfying certain relations.

For Theorem 3.3.5 we need a parallelizable manifold W' with non-empty boundary. If W is a closed, smooth, simply-connected manifold with $p_1(W) = 0$ and $\omega_2(W) = 0$, and D is the interior of an embedded 6-disk, then $W' = W \setminus D$ will be parallelizable. Thus we will specify $p_1 = 0$, and $\bar{\omega}_2$ any even element of H. We also need $H^1(W'; \mathbb{Z}/2) = 0 = H^1(\partial W'; \mathbb{Z}/2)$, but this is guaranteed by $\pi_1(W) = 0$ and $\partial W' \cong S^5$. We may choose H and G to be any finitely-generated free abelian groups.

For Wall's classification it remains to specify the trilinear pairing μ, and for our theorem we need an even element $e \in H^2(W')$ whose cup-square is 0. Specifying

μ is equivalent to specifying the cup-product on $H^2(W)$: we set $\alpha \cup \beta = 0$ for all $\alpha, \beta \in H^2(W)$. Then the invariants will satisfy the relations of Theorem B.0.3, and any element $e \in H^2(W')$ will satisfy $e^2 = 0$. Then if we choose any even element of $H^2(W'; \mathbb{Z})$ for the Euler class of a disk bundle ξ with total space U^8, the manifold $M^7 = \partial U$ will have $I(M) = 0$. In summary:

Example 3.3.9. *Specify the closed, smooth, simply-connected manifold W^6 using the following data (according to Wall's classification):*

1. $H^2(W) = \mathbb{Z}^k$, $H^3(W) = \mathbb{Z}^r$, *for any* $k, r \geq 0$
2. $\cup \colon H^2(W) \times H^2(W) \to H^4(W)$ *is trivial*
3. $p_1(W) = 0$, *and* $\bar{\omega}_2(W)$ *any even element of* $H^2(W)$

Let $D \subset W$ be the interior of an embedded 6-disk and let $W' = W \setminus D$. Let $e \in H^2(W') \cong H^2(W)$ be even, let U^8 be the total space of the disk bundle over W' with Euler class e, and let $M^7 = \partial U$. Then $I(M) = 0$.

The author expects that one can construct examples with non-trivial cup-product if more care is taken when defining μ.

Question 3.3.10. *Suppose more generally that M^n bounds a parallelizable manifold U^{n+1}. Is it possible to conclude that $I(M) = 0$? If not, what about $I(M) \cap bP_{n+1}$?*

3.4 Homotopy Inertia Group and Surgery

Recall the *homotopy inertia group* $I_h(M) \subset I(M)$, which is the set of all homotopy spheres Σ with a diffeomorphism $\phi \colon M \cong_+ M \# \Sigma$ which is homotopic to the standard homeomorphism h_Σ.

Theorem 3.4.1. *Let M^n be a smooth manifold and $f \colon S^k \times \mathbb{D}^{n-k} \to M$ be an embedding with $k < \min\{\lfloor \frac{n}{2} \rfloor, n-3\}$. Then $I_h(M) \subset I(\chi(M, f))$.*

Proof. Let $\Sigma \in I_h(M)$, $\phi\colon M \to M\#\Sigma$ a diffeomorphism homotopic to h_Σ. Assume wlog that the attaching disk for Σ is disjoint from the image of f.

A homotopy-inertial diffeomorphism $\chi(M,f) \cong_+ \chi(M,f)\#\Sigma$ will given based on ϕ. By lemma A.1.1 (Appendix A), if $\phi : M \cong_+ N$ then there is an induced diffeomorphism

$$\chi(\phi)\colon \chi(M,f) \to \chi(N,\phi f)$$

Here $N = M\#\Sigma$, so $\chi(M,f) \cong_+ \chi(M\#\Sigma,\phi f)$.
Corollary A.3.2 (also Appendix A) gives us a natural diffeomorphism $\alpha\colon \chi(M\#\Sigma, pf) \to \chi(M,f)\#\Sigma$ (where p is the pushout projection $\tilde{M} \to M\#\Sigma$). To finish the proof, the gap must be filled with a diffeomorphism

$$h_\phi\colon \chi(M\#\Sigma,\phi f) \to \chi(M\#\Sigma, pf)$$

It suffices to show that ϕf is isotopic to pf.

$\phi \sim h_\Sigma$ by assumption, so $\phi f \sim h_\Sigma f = pf$ (p and h_Σ are both essentially identity away from the attaching region of Σ, which was assumed to be disjoint from $\operatorname{Im} f$). Restricting both sides to $S^k \times 0$ gives two homotopic embeddings in less than the middle dimension, so the homotopy induces an embedding

$$F : (S^k \times 0) \times I \to (M\#\Sigma) \times I$$

which is a concordance between ϕf and pf. Since the embeddings have codimension ≥ 3, they are isotopic by [Hud70, Theorem 2.1]. Then the isotopy lifts to the rest of the disk bundle $S^k \times D^{n-k}$, so ϕf is isotopic to pf and so there is the desired diffeomorphism h_ϕ.

Thus the composite

$$\chi(\phi)_\#\colon\ \chi(M,f) \xrightarrow{\chi(\phi)} \chi(M\#\Sigma,\phi f) \xrightarrow{h_\phi} \chi(M\#\Sigma,pf) \xrightarrow{\alpha} \chi(M,f)\#\Sigma$$

is the desired diffeomorphism. Hence $I_h(M) \subset I(\chi(M,f))$ $\square$

Remark 3.4.2. *The author suspects that the conclusion of the theorem can be improved to "$I_h(M) \subset I_h(\chi(M,f))$", but has yet to produce a proof.*

This result can be combined with Milnor's elementary surgery theory [Mil61, Theorem 2].

Corollary 3.4.3. *Let M be a compact π-manifold of dimension n. Then M is cobordant to a $\lfloor \frac{n}{2} - 1 \rfloor$-connected manifold M' such that $I_h(M) \subset I(M')$.*

Corollary 3.4.4. *Let M be a compact π-manifold of dimension 7. Then M is cobordant to a 2-connected 7-manifold M' such that $I_h(M) \subset I(M')$.*

There is a special case when $k = 0$ and M has two connected components M_1 and M_2. $S^0 = \{1, -1\}$ is the only disconnected sphere, and if f embeds the components of $S^0 \times \mathbb{D}^n$ into different components of M, then $\chi(M, f) = M_1 \# M_2$. Then we saw a result of Frame (Proposition 3.2.10) says that if $n \geq 5$ then $I_h(M) \subset I_h(\chi(M, f))$.

Appendix A

Pushouts in the Smooth Category

A.1 General Pushouts

Let $\mathcal{C}$ be a category, A, B, C objects in $\mathcal{C}$, and $f : A \to B,\ g : A \to C$ morphisms. Define the **pushout** of f and g to be a commutative diagram A, B, C, D

$$\begin{tikzcd}
 & B \arrow[dr, "k"] \arrow[drr, bend left, "k'"] & & \\
A \arrow[ur, "f"] \arrow[dr, "g"'] & & D \arrow[r, dashed, "h"] & D' \\
 & C \arrow[ur, "l"'] \arrow[urr, bend right, "l'"'] & &
\end{tikzcd}$$

with the special property that if D' is any other object in $\mathcal{C}$ and $k' : B \to D'$, $l' : C \to D'$ are morphisms such that $k'f = l'g$, then there is a *unique* morphism $h : D \to D'$ that "factors" the other two maps (that is $k' = hk$ and $l' = hl$). If $k'f = l'g$, k' and l' will often be called "compatible."

If $h : D \to E$ is any morphism, then the compatibility of hk and hl ensures a unique morphism $h' : D \to E$ such that $h'k = hk$ and $h'l = hl$. But h has this property, so by uniqueness $h = h'$. In other words, to check that two morphisms

$h, h' : D \to E$ are equal, it suffices to verify $h'k = hk$ and $h'l = hl$. In particular, in order to deduce that $h : D \to D$ is equal to id_D it is enough to check $hk = k$ and $hl = l$.

If the pushout exists it is unique up to natural isomorphism, since if D' were another pushout of $B \overset{f}{\leftarrow} A \overset{g}{\to} C$ then there would be a map $h' : D' \to D$ such that $k = h'k'$ and $l = h'l'$. Then $h'h$ has the property that $k = h'k' = h'hk$ and also $l = h'hl$, hence $h'h = \mathrm{id}_D$; similarly $hh' = \mathrm{id}_{D'}$, so h is an isomorphism with inverse h'.

If the defining property is phrased slightly different, it says that to construct a morphism $h : D \to D'$, then it suffices to choose any compatible maps $k' : B \to D'$, $l' : C \to D'$ to get a unique $\mathcal{C}$-morphism *for free*. This is especially useful if $\mathcal{C}$ is the category of smooth manifolds and smooth functions. This principle will be used many times, as it almost magically produces natural isomorphisms between objects that are naturally isomorphic by intuition. For example, the following very useful general fact shows how equivalences piece together:

Lemma A.1.1. *Suppose we have two pushouts, and isomophisms h_1, h_2, h_3 as in the diagram*

$$\begin{tikzcd}
& B \arrow[rrr, dashed, "h_1"] \arrow[dl, "k"'] & & & B' \arrow[dr, "k'"] & \\
D & & A \arrow[ul, "f"'] \arrow[dl, "g"] \arrow[r, dashed, "h_3"] & A' \arrow[ur, "f'"] \arrow[dr, "g'"'] & & D' \\
& C \arrow[ul, "l"] \arrow[rrr, dashed, "h_2"'] & & & C' \arrow[ur, "l'"'] &
\end{tikzcd}$$

Suppose further that $f'h_3 = h_1 f$ and $g'h_3 = h_2 g$. Then there is a natural isomorphism $D \to D'$.

Proof. $k'h_1$ and $l'h_2$ are compatible since $k'h_1 f = k'f'h_3 = l'g'h_3 = l'h_2 g$, so there is a unique morphism $h : D \to D'$ such that $k'h_1 = hk$ and $l'h_2 = hl$. Similarly, we get an $h' : D' \to D$ such that $kh_1^{-1} = h'k'$ and $lh_2^{-1} = h'l'$. Then $h'hk = h'k'h_1 = kh_1^{-1}h_1 = k$

and $h'hl = h'l'h_2 = lh_2^{-1}h_2 = l$, and so $h'h = \mathrm{id}_D$; similarly $hh' = \mathrm{id}_{D'}$. Thus h is an isomorphism and $h' = h^{-1}$. $\square$

Now it is shown that if the object C can be used in two different pushout diagrams, and if after taking the first pushout the other pushout is possible, then the order in which the pushouts are performed does not change the result.

Lemma A.1.2. *Consider the following diagram, composed of pushout squares:*

$$\begin{tikzcd}
 & B \arrow[dr, "k_1"] \arrow[drr, bend left, "l_4"] & & \\
A \arrow[ur, "f"] \arrow[dr, "g"'] & & D_1 \arrow[r, dashed, "\beta"] \arrow[ddr, "k_3"] & D_4 \arrow[dd, dashed, bend left, "\alpha'"] \\
 & C \arrow[ur, "l_1"] \arrow[dr, "l_2"] & & \\
A' \arrow[ur, "g'"] \arrow[dr, "f'"'] & & D_2 \arrow[r, dashed, "\beta'"] \arrow[uur, "k_4"] & D_3 \arrow[uu, dashed, "\alpha"] \\
 & B' \arrow[ur, "k_2"] \arrow[urr, bend right, "l_3"'] & &
\end{tikzcd}$$

Then D_3 and D_4 are naturally isomorphic.

Proof. First of all, the pushout squares give the following relations:

$$k_1 f = l_1 g \quad ; \quad l_2 g' = k_2 f' \quad ; \quad l_4 f = k_4 l_2 g \quad ; \quad l_3 f' = k_3 l_1 g'$$

The third relation says there is a unique $\beta : D_1 \to D_4$ such that $\beta k_1 = l_4$ and $\beta l_1 = k_4 l_2$.

The fourth relation says there is a unique $\beta' : D_2 \to D_3$ such that $\beta' l_2 = k_3 l_1$ and $\beta' k_2 = l_3$.

Since $k_4 k_2 f' = k_4 l_2 g' = \beta l_1 g'$, there is a unique $\alpha : D_3 \to D_4$ such that $\alpha l_3 = k_4 k_2$ and $\alpha k_3 = \beta$.

Since $k_3 k_1 f = \beta' l_2 g$, there is a unique $\alpha' : D_4 \to D_3$ such that $\alpha' l_4 = k_3 k_1$ and

$\alpha' k_4 = \beta'$.

Now, to show that $\alpha'\alpha = \mathrm{id}_{D_3}$ it suffices to show $\alpha'\alpha k_3 = k_3$ and $\alpha'\alpha l_3 = l_3$. $\alpha'\alpha l_3 = \alpha' k_4 k_2 = \beta' k_2 = l_3$, but $\alpha'\alpha k_3 = \alpha'\beta$, for which there are no relations. However, $\alpha'\beta l_1 = \alpha' k_4 l_2 = \beta' l_2 = k_3 l_1$ and $\alpha'\beta k_1 = \alpha' l_4 = k_3 k_1$, so $\alpha'\beta = k_3$. Hence $\alpha'\alpha = \mathrm{id}_{D_3}$. Similarly $\alpha\alpha' = \mathrm{id}_{D_4}$, so α is a $\mathcal{C}$-isomorphism with inverse α'.

□

In the category of Sets the pushout always exists: the pushout of f and g is the quotient of the disjoint union $B \coprod C$ under the relation $f(a) \sim g(a)$ (for all $a \in A$). Indeed, this description is captured by a decomposition of the pushout diagram in terms of constructions called "co-product" and "co-equalizer," manifested in Set as the disjoint union and a quotient by the image of two functions (see Mac Lane [ML98]):

$$
\begin{tikzcd}
& B \arrow[d, "\mathrm{i}"] \arrow[dr, dashed, "k = pi"] & \\
A \arrow[ur, "f"] \arrow[r, dashed, "if", "jg"'] \arrow[dr, "g"'] & B \coprod C \arrow[r, "\mathrm{p}"] & D \\
& C \arrow[u, "\mathrm{j}"] \arrow[ur, dashed, "l = pj"'] &
\end{tikzcd}
$$

More precisely, if i and j are the natural inclusions into the disjoint union, then the pushout D is the quotient of $B \coprod C$ by the equivalence relation

$$\Delta \cup \{(if(a), jg(a)) \mid a \in A\}$$

$k = pi$ and $l = pj$ are sometimes called the "projection maps," and are always injective since the images of i and j are disjoint.

Now suppose A, B, C have topologies and f, g are continuous. Then there is a natural topology on $B \coprod C$ given by "U is open iff $i^{-1}(U)$ and $j^{-1}(U)$ are open in B and C respectively," so that i and j are continuous and moreover open. If any other set were added to this topology then one of i or j would not be continuous anymore, so it is the *largest* topology making them both continuous, i.e. the **weak topology** induced by the two Set-functions. The quotient set can then inherits the weak topology induced by p; a set $U \subset D$ is open iff $p^{-1}(U)$ is open, making p an open continuous function. Then clearly k and l are continuous and open, as compositions of continuous open maps; since they are injective it follows that they are homeomorphisms onto their images in D. This topology on D is actually also the weak topology induced by k and l: $U \subset D$ is open iff $k^{-1}(U)$ and $l^{-1}(U)$ are open. Thus pushouts always exists in the category of Topological spaces and continuous functions.

The goal is to extend this idea to smooth manifolds, but topological manifolds present problems of their own: if A, B, C are topological manifolds (second-countable, locally Euclidean, Hausdorff) in general the pushout will not be a manifold. Second-countability is preserved under pushout, but it is easy to break Hausdorff and Locally Euclidean even when $B = C = \mathbb{R}$. If $A = 0$ and f, g, are inclusions, then D will be a space that has a central point with 4 rays emanating from it where there is no Euclidean reference frame (but is still Hausdorff). On the other hand if $A = \mathbb{R} \setminus 0$ and f, g are still inclusions, then D is the so-called "line with two origins" where the two origins cannot be separated by disjoint open sets (although it is locally Euclidean). But in certain cases it is possible: if, for instance, there is a manifold N and homeomorphisms $f : N \to \partial B$, $g : N \to \partial C$, then the pushout of f and g will be a topological manifold. If A, B, C are endowed with orientations, and f and g are both orientation preserving, then D receives a canonical orientation making k and l orientation preserving.

A.2 Smooth Pushouts

If the objects are smooth, then even more care must be taken. In many cases the topological manifold can be "smoothed" but it isn't natural (e.g. exotic spheres), and in fact Kervarie [Ker60] showed there exist topological manifolds with no smooth structure, and do not even have the same homotopy type as a smooth manifold. The following lemma can be used to conclude that a Topological pushout is in fact a Smooth pushout:

Lemma A.2.1. *Let X^n be a topological manifold. Let M_1, M_2 be smooth open n-manifolds and let $g_1 : M_1 \to X$ and $g_2 : M_2 \to X$ be boundary preserving continuous functions such that*

1. *Each g_i is a Topological embedding.*
2. $\cup g_i(M_i) = X$
3. $\forall i, j \in \{1, 2\}$ $g_i^{-1} g_j : g_j^{-1} g_i(M_i) \to g_i^{-1} g_j(M_j)$ *is smooth.*

Then there is a unique smooth structure on X making g_1 and g_2 Smooth embeddings.

Moreover, if X' is any smooth manifold and there are two smooth maps $h_1 : M_1 \to X'$ and $h_2 : M_2 \to X'$ such that

$$h_1(x_1) = h_2(x_2) \iff g_1(x_1) = g_2(x_2)$$

then there is a unique (smooth) $h : X \to X'$ such that $h_1 = hg_1$ and $h_2 = hg_2$.

Proof. The proof will use the definition of "smooth structure" given in Definition 1.1.19.

Let $\mathcal{D}_1, \mathcal{D}_2$ be the smooth structures on M_1 and M_2. A smooth structure $\mathcal{D}$ on X is naturally defined by taking all partial functions $f \colon X \to \mathbb{R}$ such that $\exists f_1 \in M_1, f_2 \in M_2$ with $f|_{g_i(M_i)} = f_i g_i^{-1}$ (these functions are defined on $g_i(M_i)$, an open subset by assumption). It must be verified that this is a smooth structure.

Let $p \in X$. By assumption 2, $p \in g_i(M_i)$ for $i = 1$ or 2. Let's say 1. Since M_1 is a smooth manifold, there is a chart (U, ϕ) at $g_1^{-1}(p)$ with the property "$f_1 \in \mathcal{D}_1$ iff $f_1 \phi^{-1}$ is smooth," so take the chart $(g_1(U), \phi g_1^{-1})$ at p. Then if f is a real valued function defined on an open subset of $g_1(U)$, it must be checked that $f \in \mathcal{D}$ iff $f(\phi g_1^{-1})^{-1}$ is smooth. If $f \in \mathcal{D}$, then $f|_{g_i(M_i)} = f_i g_i^{-1}$ for each i (and for some $f_i \in \mathcal{D}_i$). If $i = 1$ then $f(\phi g_1^{-1})^{-1} = f_1 g_1^{-1}(\phi g_1^{-1})^{-1} = f_1 g_1^{-1} g_1 \phi^{-1} = f_1 \phi^{-1}$, which is smooth. If $i = 2$ then $f_2 g_2^{-1} g_1 \phi^{-1}$, which is smooth by assumption 3. Conversely, if it is know that $f g_1 \phi^{-1}$ is smooth, then $f g_1 \in \mathcal{D}_1$, so since $f = (f g_1) g_1^{-1}$ it is shown that $f \in \mathcal{D}$. Thus $(g_1(U), \phi g_1^{-1})$ is an appropriate chart at p. If instead p was in $g_2(M_2)$, then the argument is the same since $g_1^{-1} g_2 = (g_2^{-1} g_1)^{-1}$ is also smooth.

Now to check condition 2 of a smooth structure. Let f be a real-valued function defined on $W \subset X$, $\{U_\lambda\}$ a collection of open subsets of W, $U = \cup U_\lambda$. It must be verified that $f|_U \in \mathcal{D}$ iff $\forall \lambda \; f|_{U_\lambda} \in \mathcal{D}$. Rewrite the definition of $\mathcal{D}$ for $f|_U$ and $f|_{U_\lambda}$:

$$f|_U \in \mathcal{D} \iff \forall i \; \exists f_{i,U} \colon g_i^{-1}(U) \to \mathbb{R} \; (\in \mathcal{D}_i) \; st \; (f|_U)|_{g_i(M_i)} = f|_{U \cap g_i(M_i)} = f_{i,U} g_i^{-1}$$

$$f|_{U_\lambda} \in \mathcal{D} \iff \forall i \; \exists f_{i,\lambda} \colon g_i^{-1}(U_\lambda) \to \mathbb{R} \; (\in \mathcal{D}_i) \; st \; = f|_{U_\lambda \cap g_i(M_i)} = f_{i,\lambda} g_i^{-1}$$

Then it follows from the fact that $f_{i,\lambda} = f_{i,U}|_{U_\lambda}$ and that condition 2 holds in $\mathcal{D}_i$.

Given this smooth structure $\mathcal{D}$, the g_i's are smooth almost by definition. Let $f \in \mathcal{D}$: then for each i there is an $f_i \in \mathcal{D}_i$ such that $f|_{g_i(M_i)} = f_i g_i^{-1}$. Then $f g_1 = f_1 \in \mathcal{D}_1$ so g_1 is smooth, and $f g_2 = f_2 \in \mathcal{D}_2$ so g_2 is smooth.

Now the "moreover" part is proved before uniqueness. Let $h_1 : M_1 \to X'$ and $h_2 : M_2 \to X'$ such that $h_1(x_1) = h_2(x_2) \iff g_1(x_1) = g_2(x_2)$. Then there is only one possible Set-map $h : X \to X'$ such that $h_1 = h g_1$ and $h_2 = h g_2$, defined piecewise: for $x \in X$, if $x = g_i(x_i)$ let $h(x) = h_i(x_i)$. If $x = g_1(x_1) = g_2(x_2)$ then by assumption $h_1(x_1) = h_2(x_2)$ so $h(x)$ is well defined. To verify that it is smooth, take a function $f' \in \mathcal{D}'$ (the smooth structure on X') and check that $f'h \in \mathcal{D}$. Notice that $f'h|_{h_i(M_i)} = f' h_i g_i^{-1}$. Since each h_i is smooth $f' h_i \in \mathcal{D}_i$, and so by definition $f'h|_{h_i(M_i)} \in \mathcal{D}$ for each i, thus $f'h|_{h_1(M_1) \cup h_2(M_2)} = f'h \in \mathcal{D}$.

Now uniqueness: let $\mathcal{D}'$ be another smooth structure on X such that the g_i's are all smooth. Then by the above argument a smooth map from X to X' can be defined in pieces: on M_1 take g_1 and on M_2 take g_2. Then $g_1(x_1) = g_2(x_2)$ iff $g_1(x_1) = g_2(x_2)$, so they define a unique smooth map $h : X \to X'$ such that $g_1 = h g_1$ and $g_2 = h g_2$. As functions on sets, this means $h = \mathrm{id}_X$, so $\mathrm{id}_X : (X, \mathcal{D}) \to (X, \mathcal{D}')$ is smooth: that is, $\mathcal{D}' \subset \mathcal{D}$. Let $f \in \mathcal{D} \setminus \mathcal{D}'$. It follows that there is some point $p \in X$ so that according to the chart (U', ϕ') at p given by $\mathcal{D}'$, $f\phi'^{-1}$ is not smooth. But in the chart (U, ϕ) at p given by $\mathcal{D}$, $f\phi^{-1}$ is smooth (since $f \in \mathcal{D}$). Since each reference map is an embedding wrt that smooth structure, the composition $f(\phi'^{-1}\phi')\,\mathrm{id}_X\,\phi^{-1} = (f\phi'^{-1})(\phi'\,\mathrm{id}_X\,\phi^{-1})$ is smooth from $\mathbb{R}^n$ to $\mathbb{R}^n$. This is a contradiction since $(f\phi'^{-1})$ is not smooth. Hence $\mathcal{D} = \mathcal{D}'$ □

Remark A.2.2. 1. *The smooth structure $\mathcal{D}$ on X induces a smooth structure on the boundary via $\partial\mathcal{D} := \{f|_{\partial X} \mid f \in \mathcal{D}\}$. The remarkable point is that this smooth structure is equal to the smooth structure induced by $g_1|_{\partial M_1}$ and $g_2|_{\partial M_2}$ pushing forward $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_2$.*

2. *Suppose M_1, M_2, and X are given orientations. Then if the "transition functions" $g_i^{-1}g_j$ are orientation preserving, they induce a unique orientation of X making g_1 and g_2 orientation preserving.*

A Topological pushout hardly ever produces a Topological manifold, and even more infrequently does it naturally produce a Smooth manifold. As such, constructions in the field of Differential Topology must be very delicate, as there is much more structure than intuition would suggest. Many standard constructions are careful enough that they in fact produce Smooth pushout diagrams. Some of these are investigated in the next section.

A.3 Operating on Manifolds, Revisited

The standard manifold operations manifest themselves as pushout diagrams. This fact is used in the main portion of the thesis, and is proven in this section.

A.3.1 Attaching Manifolds

In this section, standard manifold operations are interpreted as pushout diagrams: attaching manifolds along a submanifold, and surgery.

Theorem A.3.1. *The operation of attaching oriented manifolds along a submanifold produces a smooth pushout diagram, with a canonical orientation.*

Furthermore, if N has non-empty boundary (necessarily embedded transversely into ∂M_1 and ∂M_2), then $\partial M(f_1, f_2)$ is the pushout of $f_1|_{D_\partial}$ and $f_2|_{D_\partial}$.

Proof. The attaching process in definition A.3.1 as a topological pushout, with projections k and l. To prove the Theorem, it must be verified that k and l meet the conditions of Lemma A.2.1. As a set pushout k and l cover X, and as discussed in the paragraph on Topological pushout, k and l are homeomorphisms onto their images. It remains to check condition 3, that $k^{-1}l$ (defined on $l(\tilde{M}_2) \cap k(\tilde{M}_1)$) is a diffeomorphism. This is immediate since $k^{-1}l = f_1 f_2^{-1}$.

□

A similar proof applies to the operation of attaching along submanifolds of the boundary.

Expressing this operation as a pushout formalizes some standard proofs. For example:

Corollary A.3.2. *Let M_1, M_2, M_3 be smooth m-manifolds, N an n-dimensional smooth manifold with $(m-n)$-disk bundle D, N' an n'-dimensional smooth manifold with $(m'-n')$-disk bundle D'. Let $f_1 : D \to M_1, g_1 : D \to M_2$ and $f_2 : D' \to M_2, g_2 : D' \to M_3$ be embeddings such that $g_1(D) \cap f_2(D') = \emptyset$. Then there is a natural diffeomorphism*

$$\alpha : (M_1 \#_N M_2) \#_{N'} M_3 \cong_+ M_1 \#_N (M_2 \#_{N'} M_3)$$

Proof. First, some abbreviations. For the disk bundle D, let E be the open disk bundle and let E_0 be E without the zero-section N (similarly for D'). Let $\tilde{M}_1 = M_1 \setminus f_1(N), \tilde{M}_3 = M_3 \setminus g_2(N')$, and let $\tilde{M}_2 = M_2 \setminus (g_1(N) \dot{\cup} f_2(N'))$. Then there is the following diagram of smooth pushouts:

$$\begin{tikzcd}
 & E_0 \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & E'_0 \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\
\tilde{M}_1 \arrow[dr, "k_1"] \arrow[ddrrr, bend right, "k_4"'] & & \tilde{M}_2 \arrow[dl, "l_1"'] \arrow[dr, "k_2"] & & \tilde{M}_3 \arrow[dl, "l_2"'] \arrow[ddlll, bend left, "l_3"] \\
 & M_1\#_N M_2 \setminus l_1 g_1(N') \arrow[d, "k_3"] \arrow[drr, dashed, "\beta"] & & M_2\#_{N'} M_3 \setminus k_2 f_2(N) \arrow[d, "l_4"'] & \\
 & (M_1\#_N M_2)\#_{N'} M_3 \arrow[rr, dashed, "\alpha"'] & & M_1\#_N(M_2\#_{N'} M_3) &
\end{tikzcd}$$

The result follows immediately from Lemma A.1.2. $\square$

As it stands, this result is incomplete since there is the extra assumption that the imbeddings in M_2 have disjoint images, so that successive pushouts could be taken. The next result will show that this assumption can be relaxed a bit (and fully in some cases) without losing generality. More specifically, what if the embeddings can be isotoped away from each? Does isotoping the embeddings change the smooth structure of the pushout in any way? Let N, D, M_1, M_2, f_1, f_2, etc be as above. Let Φ_1 and Φ_2 be ambient isotopies of M_1 and M_2. In particular, for every i, t the function $\Phi_i^t f_i : D \to M_i$ is an embedding and $\Phi_i^0 = \mathrm{id}_{M_i}$. Using these isotopies, one can construct a smoothly varying family of smooth manifolds starting with $M(f_1, f_2)$ and ending with $M(\Phi_1^1 f_1, \Phi_2^1 f_2)$.

The basic building blocks will be $M_1 \times I$ and $M_2 \times I$. Since $\Phi_i : M_i \times I \to M_i$ is smooth, so is the function $\bar{\Phi}_i : M_i \times I \to M_i \times I$ sending (x, t) to $(\Phi_i(x, t), t)$. Since Φ_i is an isotopy, $\bar{\Phi}_i$ will be an embedding. Furthermore, $D \times I$ has a natural disk bundle structure over $N \times I$, and $F_i : D \times I \to M_i \times I$ sending (x, t) to $(\Phi_i^t f_i(x), t)$

is an embedding. Let $\mathcal{M}(\Phi_1, \Phi_2)$ be the resulting smooth pushout:

$$\begin{tikzcd}
& \tilde{M}_1 \times I \arrow[dr, "K"] & \\
E_0 \times I \arrow[ur, "F_1"] \arrow[dr, "F_2"'] & & \mathcal{M}(\Phi_1, \Phi_2) \\
& \tilde{M}_2 \times I \arrow[ur, "L"'] &
\end{tikzcd}$$

Proposition A.3.3. *In the situation described in the preceding paragraph,*

1. *$\mathcal{M}(\Phi_1, \Phi_2)$ has boundary diffeomorphic to $M(f_1, f_2) \cup M(\Phi_1^1 f_1, \Phi_2^1 f_2)$*

2. *For each $t \in [0, 1]$ there is a natural homotopy equivalence $\iota_t : M(\Phi_1^t f_1, \Phi_2^t f_2) \to \mathcal{M}(\Phi_1, \Phi_2)$.*

3. *There is a natural diffeomorphism $h : \mathcal{M}(\Phi_1, \Phi_2) \to M(f_1, f_2) \times I$ given in terms of the two isotopies.*

Proof. 1. This is given by Theorem A.3.1.

2. According to Lemma A.1.1 it suffices to give homotopy equivalences $h_1 : \tilde{M}_1 \to \tilde{M}_1 \times I$, $h_2 : \tilde{M}_2 \to \tilde{M}_2 \times I$ and $h_3 : E_0 \to E_0 \times I$ such that $h_1 f_1 = F_1 h_3$ and $h_2 f_2 = F_2 h_3$. For each of the domain spaces, simply send x to (x, t); these are homotopy equivalences. Then for any $x \in E_0$, we see $F_1 h_3(x) = F_1(x, t) = (f_1(x), t) = h_1 f_1(x)$, and similarly $F_2 h_3(x) = h_2 f_2(x)$. Then Lemma A.1.1 gives us ι_t.

3. We will use the same method as part 2, except using diffeomorphisms instead of homotopy equivalences. For this we will need to know that $M(f_1, f_2) \times I$ is naturally diffeomorphic to the pushout of $\tilde{M}_1 \times I$ and $\tilde{M}_2 \times I$ with the embeddings $f_1 \times \mathrm{id}_I$ and $f_2 \times \mathrm{id}_I$.
   Supposing we know that, we can write the diffeomorphims:
   Define $h_1 : \tilde{M}_1 \times I \to (M_1 \setminus \Phi_1^1 f_1(N)) \times I$ by sending (y, t) to $(\Phi_1^t(y), t)$ (clearly a diffeomorphism), and similarly for h_2; let $h_3 = \mathrm{id}_{E_0 \times I}$. Now we simply verify

that $F_1h_3(x,t) = F_1(x,t) = (\Phi_1^t f_1(x), t) = h_1(f_1 \times \mathrm{id}_I)(x,t)$, and similarly that $F_2h_3 = h_2(f_2 \times \mathrm{id}_I)$. Hence there is an isomorphism in the smooth category by Lemma A.1.1, i.e. a diffeomorphism.

□

Corollary A.3.4. *If (Φ_1, Φ_2) is a pair of isotopies on M_1 and M_2, then for all t there is a natural diffeomorphism $\Phi_\#^t : M(f_1, f_2) \to M(\Phi_1^t f_1, \Phi_2^t f_2)$. Let $\Phi_\# = \Phi_\#^1$.*
If these Φ's are homotopies, then replace "diffeomorphism" with "homotopy equivalence."

Proof. Take $\Phi_\#^t$ to be the composition of the following embeddings/diffeomorphisms:

$$M(f_1, f_2) \to M(f_1, f_2) \times t \overset{h^{-1}}{\to} \mathcal{M}(\Phi_1, \Phi_2) \overset{\iota_t^{-1}}{\to} M(\Phi_1^t f_1, \Phi_2^t f_2)$$

□

(Hence the associativity lemma can be generalized to embeddings that can be isotoped away from each other.)

This construction contains a bit more information. Observe that if the Φ_i's were only assumed to be homotopies, then the topological pushout is still valid even if it is not guaranteed to be smooth anymore (or even a manifold). Nevertheless part 2 still holds, part 3 will always hold with a homotopy equivalence, and if the Φ_i^1's are families of homeomorphisms then part 3 will give a homeomorphism. Regardless, if Φ_i^1 are both embeddings then we get a natural homotopy equivalence between $M(f_1, f_2)$ and $M(\Phi_1^1 f_1, \Phi_2^1 f_2)$. If Φ_1^t is identity for all t, then careful examination of the maps involved shows that the homotopy equivalence is pieced together using id on $M_1 \setminus f_1(N)$. The phrase that is usually used is "The spaces are equivalent through an isomorphism which is identity outside of the attaching region in M_1."

A.3.2 Surgery

Thom's "spherical modifications" can also be interpreted as pushouts. Given an embedding $f : S^k \times \mathbb{D}^{n-k} \to M^n$, the restriction to $S^k \times E_0^{n-k}$ gives an embedding into

$\tilde{M} := M \setminus f(S^k \times 0)$. There is also an embedding

$$s : S^k \times E_0^{n-k} \to \mathbb{D}^{k+1} \times S^{n-k-1}$$

sending $(u, \theta v)$ to $(\theta u, v)$ for $u \in S^k$, $v \in S^{n-k-1}$, and $0 < \theta < 1$. Consider the Topological pushout, often denoted $\chi(M, f) := M \#_{S^k \times \mathbb{D}^{n-k}} (\mathbb{D}^{k+1} \times S^{n-k-1})$

$$\begin{tikzcd}
 & \tilde{M} \arrow[dr, "p"] & \\
S^k \times E_0^{n-k} \arrow[ur, "f"] \arrow[dr, "s"'] & & \chi(M, f) \\
 & \mathbb{D}^{k+1} \times S^{n-k-1} \arrow[ur, "l"'] &
\end{tikzcd}$$

Theorem A.3.5. *The above diagram is a Smooth pushout diagram.*

Proof. Follows from Lemma A.2.1. $\square$

Appendix B

Wall's Classification of Simply-Connected 6-manifolds

The classification of simply-connected 6-manifolds was done mostly by Wall [Wal66].

Theorem B.0.1 ([Wal66, Theorem 1]). *Let M be a closed, smooth, simply-connected 6-manifold. Then*

$$M \cong M_1 \# M_2$$

where $H_3(M_1)$ is finite and $M_2 \cong \#^k S^3 \times S^3$.

Define the sentence (H) ="The homology of M is torsion-free and $\omega_2(M) = 0$."

Theorem B.0.2 ([Wal66, Theorem 2]). *Let M satisfy (H) and $H_3(M) = 0$. Then M can be obtained from S^6 by doing surgery on disjoint embeddings $g_i \colon S^3 \times D^3 \to S^6$.*

In other words, any 6-manifold M satisfying (H) and $H_3(M) = 0$ is the boundary of a 7-dimensional handlebody.

For the purposes of classification, if a manifold satisfies (H) then by [Wal66, Theorem 1] wlog $H_3(M) = 0$. Thus if M is simply connected the remaining homology groups will be in dimensions $0, 2, 4$ and 6.

By Poincaré duality and the Universal Coefficient Theorem, $H_2(M)$ determines

$H^4(M), H_4(M)$ and $H^2(M)$. In particular, $H^4(M) = \mathrm{Hom}(H_4(M), \mathbb{Z})$. Thus if $H = H^2(M) \cong H_4(M)$ and $\widehat{H} = H^4(M) \cong \mathrm{Hom}(H, \mathbb{Z})$, then the cap product induces a bilinear form

$$\cup \colon H \times H \to \widehat{H}$$

or equivalently a trilinear form

$$\mu \colon H \times H \times H \to \mathbb{Z}$$

It follows that μ is symmetric, and μ and H determine the entire homology and cohomology structure [Wal66, §3].

The Wu class $1+v_2$ determines the Stiefel-Whitney classes: $\omega_2 = v_2$, $\omega_4 = v_2^2 = \omega_2^2$, and all others vanish. The only integral characteristic classes are $p_1 \in H^4(M) \cong \widehat{H}$, and the Euler class which is determined by the homology [Wal66, §3].

Theorem B.0.3 ([Wal66, Theorem 3])**.** *The invariants of a closed, smooth, simply-connected* 6*-manifold with torsion-free homology can be described as:*

1. *Two free-abelian groups* $H = H^2(W)$, $G = H^3(W)$
2. *A symmetric trilinear map* $\mu \colon H \times H \times H \to \mathbb{Z}$ *(induced by cup product)*
3. *A homomorphism* $p_1 \colon H \to \mathbb{Z}$
4. *An element* $\omega_2 \in H \otimes \mathbb{Z}/2$*, the reduction of some* $\bar{\omega}_2 \in H$*.*

These invariants satisfy the relations:

1. *For* $x, y, \in H$, $\mu(x, y, x + y + \bar{\omega}_2) \equiv 0 \pmod 2$
2. *For* $x \in H$, $p_1(x) \equiv \mu(x, \bar{\omega}_2, \bar{\omega}_2) \pmod 4$ *and* $p_1(x) \equiv \mu(x, x, x) \pmod 3$

Moreover, these invariants give a diffeomorphism classification of closed, smooth, simply-connected 6*-manifolds satisfying* (H)*.*

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