RUSSELL’S PHILOSOPHICAL APPROACH TO LOGICAL ANALYSIS

BERTRAND RUSSELL'S PHILOSOPHICAL APPROACH TO LOGICAL ANALYSIS, 1897-1905.

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ABSTRACT

In what is supposed to have been a radical break with neo-Hegelian idealism, Bertrand Russell, alongside G.E Moore, advocated the analysis of propositions by their decomposition into constituent concepts and relations. Russell regarded this as a breakthrough for the analysis of the propositions of mathematics. However, it would seem that the decompositional-analytic approach is singularly unhelpful as a technique for the clarification of the concepts of mathematics. The aim of this thesis will be to clarify Russell's early conception of the analysis of mathematical propositions and concepts in the light of the philosophical doctrines to which his conception of analysis answered, and the demands imposed by existing mathematics on Russell's logicist program. Chapter 1 is concerned with the conception of analysis which emerged, rather gradually, out of Russell's break with idealism and with the philosophical commitments thereby entrenched. Chapter 2 is concerned with Russell's considered treatment of the significance of relations for analysis and the overturning of his "doctrine of internal relations" in his work on Leibniz. Chapter 3 is concerned with Russell's discovery of Peano and the manner in which it informed the conception of analysis underlying Russell's articulation of logicism for arithmetic and geometry in PoM. Chapter 4 is concerned with the philosophical and logical differences between Russell's and Frege's approaches to logical analysis in the logicist definition of number. Chapter 5 is concerned with connecting Russell's attempt to secure a theory of denoting, crucial to mathematical definition, to his decompositional conception of the analysis of propositions.

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LIST OF ABBREVIATIONS

LH	Bodemann, E. 1895. *Die Leibniz-Handschriften der Koniglichen offentlichen*. Hanover: Bibliothek zu Hannover.
PL	Bradley, F.H. 1967. *The Principles of Logic*. 2nd Ed. London: Oxford University Press.
A&R	1893. *Appearance and Reality: A Metaphysical Essay*. London: Swan Sonnenschein.
BLA	Frege, Gottlob. 1964. *The Basic Laws of Arithmetic*. Montgomery Furth, trans. Berkeley: University of California Press.
CO	"On Concept and Object," in *The Frege Reader,* Michael Beaney, Ed. Oxford: Blackwell, 181–93.
CP	1984. *Collected Papers on Mathematics, Logic and Philosophy*. B. McGuinness, Ed. Oxford/New York: Basil Blackwell.
FA	1980. *The Foundations of Arithmetic*. J.L. Austin, trans. Illinois: Northwestern University Press.
GG	1893–1903. *Grundgesetze der Arithmetik*. Jena: Verlag Hermann Pohle.
PMC	1980. *Philosophical and Mathematical Correspondence*. Gabriel, Gottfried, et al., Eds. Oxford: Basil Blackwell.
PW	1979. *Posthumous Writings*. Hans Hermes, Friedrich Kambartel, Friedrich Kaulbach, Eds. Chicago: University of Chicago Press.
G	Leibniz, G.W. 1965. *Die Philosophische Schriften von Leibniz,* 7 vols., C.I. Gerhardt (ed). Hildesheim: Olms.
I	Moore, G.E. 1900-1. "Identity," *Proceedings of the Aristotelian Society* vol. 1, 103-127.
N	1900. "Necessity," *Mind* vol. 9, no. 35, 289–304.
NJ	1899. "The Nature of Judgment," *Mind* vol. 8, no. 30, 176-93.
PE	1903. *Principia Ethica*. Cambridge: Cambridge University Press.

SW	Peano, Giuseppe. 1990. *Selected Works of Giuseppe Peano*. H.C. Kennedy, Ed. Toronto and London: University of Toronto Press.
AMR	Russell, Bertrand. 1898. “An Analysis of Mathematical Reasoning Being an Inquiry into the Subject-Matter, the Fundamental Conceptions, and the Necessary Postulates of Mathematics,” in Papers 2, 155–242.
AOG	1899. “The Axioms of Geometry,” in Papers 2, 390–415.
COR	1899. “The Classification of Relations,” in Papers 2, 136–146.
EAE	1898. “Are Euclid’s Axioms Empirical?” in Papers 2, 322–338.
EFG	1897. *An Essay on the Foundations of Geometry*. Cambridge: Cambridge University Press.
FIAM	“The Fundamental Ideas and Axioms of Mathematics,” in Papers 2, 261–305.
FN	1904. “Fundamental Notions,” in Papers 4, 111-259.
FUND	1905. “On Fundamentals,” in Papers 4, 359-413.
IMP	1919. *Introduction to Mathematical Philosophy*. London: G. Allen & Unwin.
LOR	1901. “The Logic of Relations with Some Applications to the Theory of Series,” in Papers 3, 310–49.
MTCA	1904. “Meinong’s Theory of Complexes and Assumptions,” in Papers 4, 431–474.
MPD	1959. *My Philosophical Development*. New York: Simon & Schuster.
OD	1905. “On Denoting,” in Papers 4, 414–427.
OF	1904. “On Functions,” in Papers 4, 96–110.
OMD	1903. “On Meaning and Denotation,” in Papers 4, 314-58.
OQAC	1898. “On Quantity and Allied Conceptions,” in Papers 2, 114–135.
PAD	1903. “Points about Denoting,” in Papers 4, 305-313.

Papers 1	1990. *The Collected Papers of Bertrand Russell, Volume 1: Cambridge Essays, 1888-99*. Blackwell, Kenneth, Brink, Andrew, Griffin, Nicholas, Rempel, Richard A, and Slater, John G., Eds. London: G. Allen & Unwin.
Papers 2	1990. *The Collected Papers of Bertrand Russell, Volume 2: Philosophical Papers, 1896-99*. Griffin, Nicholas and Lewis, Albert C., Eds. London: Routledge.
Papers 3	1993. *The Collected Papers of Bertrand Russell, Volume 3: Toward the "Principles of Mathematics", 1900-02*. Moore, G.H., Ed. London: Routledge.
Papers 4	1994. *The Collected Papers of Bertrand Russell, Volume 4: Foundations of Logic, 1903-05*. Urquhart, Alasdair and Lewis, Albert. C., Eds. London: Routledge.
PoL	2005. *A critical exposition of the philosophy of Leibniz: with an appendix of leading passages*. London: Routledge.
PoM	*1996. The Principles of Mathematics*. New York: W.W. Norton and Co.
RIW	1901. "Recent Italian Work on the Foundations of Mathematics," in Papers 3, 350–62.
RW	1901. "Recent Work on the Principles of Mathematics," in Papers 3, 363–379.
CPLP	Russell, Bertrand and Couturat, Louis. 2001. *Correspondance sur la Philosophie, la Logique et la Politique avec Louis Couturat*. A.F. Schmid, Ed. Paris: Kimé.
PM	1970. Russell, Bertrand and Whitehead, A.N. *Principia Mathematica to *56*. Cambridge: Cambridge University Press.

INTRODUCTION

"An idea which can be defined, or a proposition which can be proved, is of only subordinate philosophical interest."[1] –Bertrand Russell

In what is supposed to have been a radical break with neo-Hegelian idealism, Russell, alongside G.E Moore, advocated the analysis of propositions by their decomposition into constituent concepts and relations. Russell regarded this as a breakthrough for the analysis of the propositions of mathematics. Historically, the decompositional approach to analysis has not entailed any particular conception of logic or even any special logical techniques for carrying out the analyses of propositions. Russell construes his logicist project as an elaborate refutation of Kant, but Kant's analyses are decompositional, though he rarely strayed from the subject-predicate logic. Moreover, it would seem, in fact, that the decompositional-analytic approach is singularly unhelpful as a technique for the clarification of the concepts of mathematics. Numbers, for instance, are individual terms on Russell's early decompositional-analytic approach, but they are indefinable. Though Russell supplemented it by a changing amalgam of the various techniques that he appropriated from Boole, Whitehead, Peano, and Frege, as they became available to him, and by his own logic of relations in the fall of 1900, the decompositional conception of analysis was preserved from the period preceding the

[1] PoL, 201.

articulation of logicism through the theory of descriptions which would enable him to dispense with his problematic theory of classes. In retaining the decompositional-analytic approach, Russell rejected Frege's function-argument form of analysis, which had been developed, complete with quantification theory, to facilitate Frege's logicization of arithmetic, and which was available to Russell when he wrote PoM. The aim of this thesis will be to clarify Russell's conception of analysis in the light of the philosophical doctrines to which it answered, and the demands imposed by existing mathematics on Russell's early logicist program.

Since the scope of the following thesis is broad enough as it is, and its arguments are complicated in places, I shall not introduce any additional content here. I shall instead give a brief synopsis of the developments that I shall treat at length in the chapters which follow.

Chapter 1 is concerned with the conception of analysis which emerged out of Russell's break with idealism in 1898 and with the philosophical commitments thereby entrenched. Having pointed out that the anti-psychologistic positions which mark the advent of early analytic philosophy have in common the view that the proposition is the basic unit of analysis, I articulate Russell's theory of terms and briefly outline the developments which are crucial to understanding Russell's break with idealism and the role which decompositional analysis was to play. I frame the problem of the current thesis in terms of the question of how Russell's decompositional conception of analysis and the attendant theory of terms were supposed to facilitate analyses in *mathematics*. To lay crucial groundwork, I discuss Russell's earlier attempt to give the conditions for space as the form of externality by means of "purely logical" transcendental arguments in EFG and his broader project of answering the Kantian question "how is pure mathematics possible?" in AMR, which would be subsequently transformed into the question "what axioms allow mathematics to be true?" I characterize the manner in which the antinomy of the spatial point, which first arises in EFG where indiscriminable points are required for a relational account of space, is generalized, in AMR, to the contradiction of relativity which holds in all of the sciences. The contradiction of relativity rests on Russell's doctrine of internal relations—namely, that relations have their grounds in adjectives (properties) of the relata— which must be dispensed with if analyses in mathematics are to be possible. I turn, then, to Russell's doctrine of internal relations and the manner in which he overturned it.

Chapter 2 is concerned with Russell's considered treatment of the significance of relations for analysis and the role which his work on Leibniz played in overturning his doctrine of internal relations. I interpret Russell's work on Leibniz and surrounding texts in some detail to show that Russell did not regard relations as reducible to adjectives prior to his work on Leibniz. The crucial argument of my second chapter is one intended to show that Russell did not merely adopt his external view of relations from Moore as he claims. Rather, in PoL, Russell pressed Moore's anti-Bradleian thesis that "the logical idea is not an adjective" to its conclusion, arguing in favour of the primitive diversity of logical subjects. The diversity of logical subjects served as the model for the externality of relations in Russell's arguments from COR and figures centrally in Russell's conception of analysis leading up to his 1905 theory of descriptions. Next, I consider the manner in which Russell arrived at his intensional view of relations, which is important for understanding Russell's early conception of analysis and his logicist definitions. Russell arrived at his definition of number, for instance, by supplementing Peano's symbolic logic with his intensional logic of relations in LOR, which Russell drafted in 1900 and revised in 1901.

Chapter 3 is concerned with Russell's discovery of Peano and the manner in which "the new symbolic logic" informed the conception of analysis underlying Russell's articulation of logicism for arithmetic and geometry in PoM. The first section of the chapter is intended to show that Russell's logicism is not a formal device, but was answerable to the demands of existing mathematics. It is pointed out that Russell takes implicit definitions in the various branches of mathematics to be legitimate definitions and explicit definitions are afforded only a marginal role. In the second section, I consider the "if-thenist" position—i.e., the conception of logicism on which the statements of mathematics are conditionals whereby the axioms in the antecedents imply the theorems in the consequents. On Coffa's account, "if-thenism" is applicable to Russell's conception of geometry, but not to his conception of arithmetic in PoM, since Russell's explicit definitions in arithmetic are not captured by conditionals. I attempt to address Griffin's claim that Coffa has a misguided conception of the nature of Russell's conditionals. On Griffin's account, Russell's conditionals are not implications of theorems by axioms, but rather are of precisely the sort which Coffa attributes to Peano, viz. propositions in which the antecedents in universally quantified implications determine the range of variables in the corresponding consequents, i.e., "for all x, if x is a then φx". I argue that while Russell may have adopted "if-thenism" for geometry in 1900, by the time he articulates his logicist thesis in May 1901, Russell has not only adopted the conception of implications which Coffa attributes to Peano ("for all x, if x is a then φx"), but has supplemented his implicit definitions with explicit definitions in arithmetic whose role it is to give existence theorems for the classes defined. In the light of the Contradiction, it would seem that only the implicit definitions are valid, but if the real advantage of logicism is, as Russell claims, that it makes existing mathematics true, then it would seem that Russell's logicism, formulated according to the requirements of the various branches of mathematics, does not dissolve into a mere formal apparatus. Nevertheless, applications of arithmetic seem to require the explicit definitions by which numbers are identified with classes, and these definitions are undermined by the Contradiction. Since it is to logicism in the face of the Contradiction that I turn next, I conclude the chapter with an account of the manner in which Russell initially construed the Contradiction.

Chapter 4 is concerned with the philosophical and logical differences between Russell's and Frege's approaches to advancing a logicist definition of number in the face of the Contradiction. It has been assumed, more or less correctly, that in defining the numbers as classes of similar classes in LOR, Russell had independently discovered Frege's definition of number as the value-ranges correlated with extensionally equivalent functions. The differences between Russell's and Frege's definitions might be thought to be exhausted by metaphysical or epistemological concerns about the objects defined by abstraction principles or the manner in which these are apprehended. I contend, however, that crucial differences in the logics to which number statements were supposed to be reduced leave it doubtful whether they had the same definition. Not only does Russell first put forth his view from within an intensional logic of relations, but as his logic of relations collapses into the intensional logic of propositional functions, Russell rejected a solution to the Contradiction on which Fregean functions were fundamental. These differences, I claim, are central to understanding Russell's conception of the logicist definition of mathematical objects and are not exhausted by divergence in the manner of conceiving the ontological status of abstracta. These contentions are elaborated in the chapter which follows, in connection with denoting complexes, which presented intractable problems for Russell's conception of analysis until they were eliminated in 1905.

Chapter 5 is concerned with connecting Russell's attempt to secure a theory of denoting, crucial to mathematical definition, to his decompositional conception of the analysis of propositions. In the first section, I point out the manner in which his problematic 1903 theory of denoting was at odds with his decompositional conception of analysis, on which the proposition is to be regarded as the basic element of analysis, the nature of the consitutent terms being determined by their manner of occurrence within it. In the second section, I try to establish the connection between Russell's pronouncement, in 1904, that propositional functions are more fundamental than mathematical functions and the approaches he took to dispensing with his earlier theory of denoting. I argue that Russell's reasons for explicitly denying functions the role which they had in Frege's project, not only led him to adhere to the PoM view on which propositional functions were granted preeminence, but also led him to treat mathematical functions as denoting complexes containing variables. I conclude that the theory of descriptions, which permitted the logicist definitions to be carried out without the introduction of classes as entities, preserves Russell's conception of analysis.

CHAPTER 1: ANALYSIS AND THE DECOMPOSITION OF IDEALISM

1.1 CONTEXTUALIZING RUSSELL'S BREAK WITH IDEALISM

Anti-psychologism in logic, in its various incarnations, was a commonly held position prior to Russell's break with idealism, though there was considerable disagreement as to what the position entailed.[2] In the second half of the 19th century, a number of logical works had exhibited antagonism toward views which made the laws of thought, the propositions of logic, or 'logical ideas' dependent upon psychological processes.[3] F.H Bradley, to whose views Moore's and Russell's new realist philosophy was opposed, and Gottlob Frege, had, at nearly the same time, written important logical works which aimed to divest logic of psychologism.[4] Both Bradley and Frege targeted J. S. Mill's associationist psychology, maintaining that ideas could not be treated naturalistically, as mental occurrences, if there was to be any logical account of how they are used in judgments and inferences.[5] This parallel was recognized by Richard Wollheim, who regarded Bradley's rejection of psychologism as "…one of the …very

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[2] Griffin and Godden characterize (metaphysical) psychologism in logic as "…the claim that the subject matter of logic is, at least in some essential respect, psychological in nature" [Griffin and Godden 2009]. What complicates matters is that many of the objections leveled against this form of psychologism were leveled at those who intended their doctrines to be anti-psychologistic and yet were supposed to subscribe to this thesis tacitly.

[3] George Boole, *The Laws of Thought*; Bolzano, *Wissenschaftslehre*, 1837; Gottlob Frege, *Grundlagen der Arithmetik*, 1884; Edmund Husserl, in *Philosophie der Arithmetik. Logische und psychologische Untersuchungen*, 1891.

[4] F.H Bradley in *The Principles of Logic* in 1883 and Gottlob Frege in *Grundlagen der Arithmetic* in 1884.

[5] In his 1884 *Grundlagen der Arithmetik*, Frege attempted to distinguish the origins of a belief from the ultimate grounds for its justification and logical laws from laws of thought. In particular, he rejected Mill's psychologistic philosophy of mathematics, on which numbers were properties of aggregates and counting required aggregative thought.

few links that bind him to the more eminent or advanced amongst his philosophical contemporaries. A striking parallel can be drawn between his strictures on the state of British Logic in his day and, for instance, what was being said…by Gottlob Frege" [Wollheim 1956, 25]. Frege, who had criticized Husserl's treatment of logic as being, in the first instance, a theory of judgment [Frege 1894],[6] characterized psychologism in his 1897 paper, "Logic", as the view that "…a thought (a judgment as it is usually called) is something psychological like an idea" [Frege 1897, 143]. It is perhaps an anti-psychologistic conception of the nature of judgments—the structure and existence of propositions or thoughts—which distinguishes the brand of anti-psychologism with which early analytic philosophy is often associated.[7] The view which Russell takes towards the nature and analysis of propositions, that is, both towards their structure and existence and towards the nature and manner of occurrence of their constituents, I hope to show, is the theme linking crucial developments in Russell's early work.

It is to Bradley's conception of the nature and composition of the judgment that Moore's and Russell's new logic is opposed. In his *Principles of Logic*, Bradley attempted to arrive at a logical notion of meaning, maintaining, against Mill's

[6] There, Frege also criticizes Husserl for espousing an equivocal notion of 'idea', treating concepts and objects as sometimes subjective and sometimes objective.

[7] In his 1884 *Grundlagen der Arithmetik*, Frege famously articulated his "context principle", according to which words do not have meaning in isolation, but have meaning according to the place they occupy in significant propositions. What he meant is controversial and in the folio of Russell's notes on the *Grundlagen*, Russsell's reaction to Frege's articulation of the context principle is a single interrogation mark: "?" [Linsky 2004, 31].

psychologism, that the meaning of a sign, that is, the ideal content[8] or the logical idea in a judgment, taken apart from the sign, has nothing to do with any images with which it may be associated.[9] However, in characterizing logical ideas as distinct from mental occurrences, Bradley maintained that an ideal content must be regarded as that part of the content of 'signs of existence other than themselves' which is "…cut off, fixed by the mind, and considered apart from the existence of the sign" [PL, 8].[10] In his 1899 paper "On the Nature of Judgment" (henceforth, NJ), Moore vehemently rejects this characterization of the logical idea, protesting that if meaning were thus abstracted from the content of our ideas, as mental occurrences, truth and falsity would depend on the relation of our ideas to reality [NJ, 177].[11] The logical idea or, as Moore puts it, a concept constituting a judgment: "…is not a mental fact, nor any part of a mental fact" [NJ, 179].[12] Russell echoes this view in PoM, where he admonishes Bradley on the grounds that "meaning" "…is a notion confusedly compounded of logical and psychological elements…," where "[t]he confusion is largely due…to the notion that *words* occur in propositions, which in turn is due to the notion that propositions are essentially mental

[8] Bradley inherited the notion of ideal content from Hermann Lotze's chapter on 'The World of Ideas', contained in Book III of his *Logic* [Lotze 1884, 434-449].
[9] Marion Mathieu argues that, while Bradley's *Principles of Logic* aimed at divesting the 'logical idea' of associationist psychology, it did not directly target Mill's claim that the laws of thought are exclusively psychological, but attacked Hamilton's 'great axiom' that all human knowledge is relative or phenomenal and Mill's stronger version of that axiom [Mathieu 2008].
[10] This distinction between logical ideas and mental occurrences, and a distinction between a judgment and a proposition as a thought is found in Bolzano 1837 S22; S52; S291.
[11] Moore's argument is not a *reductio*, but the thesis that truth depends on a relation between our ideas and reality is the target of Moore's anti-skeptical arguments against the mental status of concepts.
[12] Moore writes: "It is indifferent to [the nature of concepts] whether anybody thinks them…They are incapable of change; and the relation into which they enter with the knowing subject implies no action or reaction" [NJ, 179].

and are to be identified with cognitions" [PoM, 47]. In attributing to Bradley the view that meanings are fixed by abstraction from the total content of the sign, Moore held that the abstraction itself requires a prior and psychological judgment, and so on, *ad infinitum*.[13] Arguably, Moore and, by extension, Russell misunderstood Bradley's position. Consider Bradley's 1899 response to Moore:

> **[Moore's criticism] seems to be that the separation of meaning from existence required for judgment presupposes a previous judgment. Well certainly it *may* do so—-a psychological judgment, that is, but then again it may not…I suppose my phrase 'cut off' etc. has been taken to imply…a previous idea. I never meant this [Baldwin 1993, 14].**

Thomas Baldwin points out that, for Bradley, the total content of a sign cannot be identified with its meaning, since distinct signs may have the same meaning, but the meaning of a sign can be identified by its role in a judgment "…and especially [by]our treatment of some of them as true or false" [Baldwin 1993, 13]. According to Moore, there are graver problems, however, with Bradley's notion that judgment requires a separation of meaning from existence and it is worth briefly considering Moore's broader criticism.

In his first Fellowship Dissertation (1896-7), Moore had expressed a debt to Bradley to whom he felt he "owe[d] his conception of the fundamental problems of Metaphysics," but by the second Fellowship Dissertation (1897-8), had rejected neo-Hegelian idealism completely. In his 1897 Fellowship Dissertation, Moore had begun to

[13] Moore writes: "[M]y question is, whether we can thus cut off a part of the character of our ideas, and attribute that part to something else, unless we already know, in part at least, what is the character of the idea from which we are to cut off the part in question. If not, then we have already made a judgment with regard to the character of *our* idea..." [Baldwin 1993, 13].

develop a criticism, which he clarified in the second Fellowship Dissertation and which Russell subsequently adopted in PoL, of the conflation of psychological considerations as to the constitution of the mind, the origins of knowledge, or the conditions for belief into considerations about what is true or objective. In the 1897 Dissertation, "The Metaphysical Basis of Ethics," which consists in a Bradleian treatment of Kant's ethics, Moore writes:

> **It is perhaps impossible to dispense with the term 'rational' for what is true or objective, especially after its full adoption by Hegel; but it is extremely important to avoid confusing the 'rational' in this sense which is the fundamental one for Kant's system, with the 'rational' in the sense of that which implies the psychological faculty of making judgments and inferences. The distinction between what is true and what is only believed (although only a 'rational' being can believe) is one which cannot be either done away or bridged over [Baldwin and Preti, 63].**

Presumably, Moore believed the separation between what is true and objective from the psychological requirements of judgment to be compatible with his Bradleian metaphysics. However, in revising his conception of judgments in 1897 to 1898, Moore arrived at his new realist position on the nature and proper constituents of judgment expressed in his second Fellowship Dissertation.[14] In his 1898 Fellowship Dissertation and in "On the Nature of Judgment," Moore intends to "…show… that the 'idea used in judgment' is not part of the content of our ideas, nor produced by any action of our minds, and that hence truth and falsehood are not dependent on the relation of *our* ideas to reality" [NJ, 177].[15] Truth and falsity are to be regarded as immediate properties of propositions and "[w]hat

[14] Moore kept the title, "The Metaphysical Basis of Ethics," for the second dissertation.

[15] Moore's italicization of 'our' is meant to attack the notion of a *conceptus communis* or *gemeinsamer Begrif*, which is also a Kantian notion, where an idea or *vorstellung* becomes a logical idea or common concept by means of the analytical unity of consciousness.

kind of relation makes a proposition true, what false, cannot be further defined, but must be immediately recognized" [NJ, 180]. On September 11, 1898, Moore confusedly relates the "chief discovery" of his second dissertation to Russell:

> **My chief discovery which shocked me a good deal when I made it, is expressed in the form that an existent is a proposition. I see now that I might have put this more mildly. Of course, by an existent must be understood an existent existent—-not what exists, but that + its existence [RA].[16]**

On September 13, 1898, Russell responded:

> **I am curious to know how a really thorough account of Kant might be written. I fear Caird's[17] hair will stand on end when he hears that an existent is a proposition. I think your expression needlessly paradoxical, but I imagine I agree with what you mean [RA].**

What Moore meant is not clearly conveyed by the letter, but the gist of the view is that what is known in an existential judgment is not an existent to which the judgment refers, but rather an existential proposition, constituted by the concept whose existence is concerned (the existent) and the concept of existence predicated of it (its existence).[18] In other words, the world is not made up of existents, but of the propositions which assert existence of them. In the same letter, Moore articulates, with greater clarity, the

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[16] At the end of 1898, Moore introduces the distinction between universals and particulars: the former never exist, but have being, and the latter do sometimes exist. Existents are no longer identified with true existential propositions. He subsequently holds that, whereas universals are identical if indiscernible, particulars can merely differ numerically [Moore 1901–5, 402].

[17] In his book The Philosophy of Kant, Edward Caird had taken issue with Kant for not following out the consequences of his own principles, retaining the 'antithesis' of the world of experience and the world of ideas. T.H Green's review of Edward Caird's *The Philosophy of Kant*, Works, iii, 137; cited in Hylton 1990.

[18] In NJ, Moore writes: "A proposition is constituted by any number of concepts together with a specific relation between them...And this description will also apply to those uses where there appears to be a reference to existence. Existence is itself a concept; it is something which we mean; and the great body of propositions, in which existence is joined to other concepts or syntheses of concepts, are simply true or false according to the relation in which it stands to them..." [NJ, 180].

conception of the nature of the propositions which he espouses in the second fellowship dissertation and which is the cornerstone of his and Russell's new logic:

> **I carefully state that a proposition is not to be understood as any thought or words, but the concepts + their relation of which we think. It is only propositions in this sense, which can be true and from which inference can be made. Truth, therefore does not depend upon any relation between ideas and reality, nor even between concepts and reality, but is an inherent property of the whole formed by certain concepts and their relations [RA].**

Moore has followed out the consequences of his earlier view that the distinction between what is true and what is believed cannot be bridged over. Bosanquet, who finds it difficult to take the dissertation seriously, remarks that "[i]t is necessary no doubt to distinguish, in the process and products of cognition, between their nature as knowledge and their psychological genesis, [b]ut the theory here propounded seems to reduce the world of truth to an immutable framework of hypostatised 'propositions' or 'Concepts' in relations, which are indeed possible objects of thought, but are entities not dependent upon thought nor partaking of any character which distinctively belongs to thought" [Baldwin and Preti 2011, 245]. With this anti-idealist conception of the nature and proper constituents of propositions, Bradley's theory of judgment comes under attack for reasons which outstrip the question of the extent to which his "logical ideas" are mere ideas and to what extent they are veritably concepts.

On Bradley's view, judgment is the act of assertion by which an ideal content (or meaning in the strict sense of a logical idea) is referred to a reality beyond itself. In the *Principles of Logic*, Bradley writes: "In the act of assertion we transfer this adjective to, and unite it with, a real substantive. And we perceive at the same time, that the relation thus set up is neither made by the act, nor merely holds within it or by right of it, but is real both independent of and beyond it" [PL, 14]. It is clear that Bradley intends that uniting a property with a substantive is in no way constituted by the mental act of judgment or the association of ideas. However, the relation of predication is not a proper constituent of the judgment at all, but, by referring the abstract meaning or logical idea to a real substantive, it points to a reality beyond the judgment. While judgment for Bradley is not, as it was traditionally conceived, the act of conjoining mutually independent ideas by means of the copula, what deserves emphasis in this theory is that judgments are not composed out of mutually independent ideas at all. Rather, in its true form, a judgment ascribes a property to its true subject, the Absolute.[19] In this vein, Bradley claims that all judgments are categorical in that they affirm something of reality, but that they are all at once hypothetical in that they cannot do so unconditionally [PL, 104]. On Stewart Candlish's account, this twofold nature of judgment is made intelligible by recognizing that, for Bradley, all judgments are of the form "'Reality is such that if anything is S then it is P" [PL, 623; Candlish 2007].[20] For Bradley, neither the logical ideas in a judgment nor judgments themselves are independent entities and, insofar as they require

[19] Chalmers and Griffin write: "Bradley makes his position clearer in one of the Terminal Essays appended to the second edition of the Logic. There he distinguishes between ordinary judgments, where the subject is what he calls 'a selected reality', i.e., 'a limited aspect and portion of the universe', and the higher level, where the subject is Reality (or the Absolute)" [Chalmers and Griffin 1997, 47 and 47n].
[20] Bradley writes: "Our 'S is P' affirms really that Reality is such that S is P" [PL, 630]. Russell uses Bradley's example "if anything is arsenic, then it poisons", to illustrate the nature of hypothetical judgments in EFG, S6.

abstraction, judgments themselves cannot be considered to be unconditionally true or inferences fully valid [PL, 10],[21] the latter being merely "…the ideal self-development of an object taken as real" [PL, 428, 456]. James Allard argues that Bradley's contention that all judgments have the logical form "Reality is such that if anything is S, then it is P" is intended to resolve the difficulties involved in the substitutivity of identicals within an intensional conception of judgment [Allard 2005, 77].[22] On Bradley's intensional view of judgment, the extension of a term is its denotation [PL, 193 n2] and the intension is its ideal content or meaning [PL, 168], which is universal and which does not denote uniquely. Since judgment has an ineliminable intensional aspect which precludes the inter-substitution of co-extensive parts, Reality as a whole must be invoked as the only object (logical subject) that can be uniquely denoted in (intensional) judgments [Allard 2005, 80][23]. Whether or not Allard's interpretation is correct, we shall see toward the end of the present chapter that, in reading Bradley's PL, Russell was especially concerned with the difficulty of supplying unique reference by means of adjectives which are universal and with the distinction between identity of content and numerical identity. In

[21] Bradley is not concerned with the psychological process of abstraction or with ideas regarded as psychical events, images, or series of symbols, but with ideas as universal meanings fixed by the mind and taken as adjectives to be referred to some subject, but indifferent to any particular existent and, in this sense, an abstraction of the understanding.

[22] The instance with which Russell famously contends in OD is the substitution of "Scott" for "the author of Waverly", in such a case as "George IV wished to know whether Scott is the author of Waverly", where George IV clearly does not wish to know whether Scott is Scott.

[23] Allard writes: "If [Bradley] can find a way for judgments to denote a single individual in the actual world and denote no other individuals in possible worlds, then he can preserve substitutivity within judgmental contexts. Such judgments would have to identify uniquely a single individual...Bradley argues that only one subject can be denoted uniquely in this way—reality as a whole. This provides a rationale for his claim that all judgments must refer to reality and must have the logical form 'Reality is such that S is P'" [Allard 2005, 80].

subsequent chapters, we shall see that these concerns, and the related difficulty of the substitution of identicals *salva veritate* where meaning and denotation are distinct, motivated crucial developments in Russell's early work.

Moore's new logic departs radically from the notion that a judgment ascribes an adjective, abstracted from the reality in which it is grounded, to a substantive which has an existence apart from the judgment. On Moore's view, there is nothing to distinguish a substantive from a collection of adjectives (properties) and there is nothing apart from its role in a judgment that makes a concept an adjective. Whereas Bradley treats this as grounds for dispensing with "things" or independently subsisting entities,[24] Moore regards this as grounds for dispensing with the view that judgment involves a connection between logical ideas and the reality underneath them. There is, on Moore's account, nothing more ultimate to which a judgment refers than the concepts which are its ultimate constituents. Existents are to be identified with true existential propositions which assert a necessary connection between concepts and do not depend, for their truth, on psychological conditions for certainty or on conformity between our concepts and reality. Thus, the separation of meaning from existence that is supposed, on Bradley's theory, to

[24] In *Appearance and Reality*, Bradley argues that a thing without properties is unintelligible, but if a thing is an aggregate of properties, relations would need to relate them into a unity, but are incapable of doing so, and the reality of things and relations are denied.

be required for judgment, is utterly dissolved in Moore's new realist philosophy.[25] In NJ, Moore writes:

> **[T]he existential judgment, which is presupposed in Kant's reference to experience or in Mr. Bradley's reference to reality, has turned out to be…merely a necessary combination of concepts, for the necessity of which we can seek no ground… A concept is not in any intelligible sense an 'adjective'…for we must, if we are to be consistent, describe what appears to be most substantive as no more than a collection of such supposed adjectives: and thus, in the end, the concept turns out to be the only substantive or subject, and no one concept either more or less an adjective than any other... The nature of the judgment is more ultimate than either [our mind or the world], and less ultimate only than the nature of its constituents—the nature of the concept or logical idea** [NJ, 193].

On Moore's new logic, judgments are non-linguistic, mind-independent complex entities constituted by self-subsistent concepts and the necessary relations between them. On Moore's new realism, this anti-psychologistic conception of the proposition and its constituents is accompanied by the peculiar tenets that truth and falsity are immediate properties of propositions and concepts, as non-linguistic, extra-mental entities, enter directly into propositions as their constituents.[26] If a proposition is regarded as roughly akin to a state of affairs, it is not inconceivable that its truth and falsity depend on whether

[25] "The opposition of concepts to existents disappears, since an existent is seen to be nothing but a concept or complex of concepts standing in a unique relation to the concept of existence. Even an existential proposition…seems to lose its strangeness, when it is remembered that a proposition is here to be understood…as the combination of concepts which is affirmed" [NJ, 180].

[26] It is worth noting that if depsychologizing the proposition were all that was required for a solution to problems given rise to by traditional accounts of the nature and truth conditions of the proposition, Frege's view, wherein the sense of a complete proposition is a thought (*gendanke*) whose reference is a truth-value, would have sufficed, for thoughts and their constituents are thoroughly non-psychologistic, non-mental entities. However, as we shall see, Fregean senses give rise to problems of analysis similar to those given rise to by Russell's own denoting concepts and both prove an obstacle to resolving the contradiction.

that state of affairs obtains and that the constituents of that state of affairs should enter directly into the state of affairs they constitute.[27]

Moore's article NJ was lifted directly from his Fellowship Dissertation [Baldwin 1993, 6n7],[28] from which we may conclude that the views articulated there were available to Russell in November of 1898, since it was then that he read Moore's dissertation.[29] However, in trying to establish the timing or precise nature of Moore's influence on Russell, the historical record is not especially illuminating. What is clear is that Russell had arrived at a position, similar to Moore's, on the nature and proper constituents of judgment in his 1898 manuscript, AMR, where he articulated his theory of terms. On this theory, judgments are complex entities composed of terms and anything may be counted as a term which can be taken as the logical subject in a proposition [AMR, 167]. On Russell's theory, terms have a peculiar sort of being, not constituted by their being objects of thought. Rather, Russell says, "[i]t is true, in fact, that there are such terms; and when we say this, we do not intend merely to assert a psychological fact" [AMR, 169]. "Terms" in Russell's terminology are the non-psychological constituents of propositions akin to what Moore calls "concepts". The differences between Moore's theory of

[27] The world is made up of concepts in relations constituting true propositions, which are actual states of affairs, but since any possible combination of concepts has being, the realm of being also includes non-actual states of affairs.
[28] This is established by a careful study of the *Fellowship Dissertation* by Consuelo Preti [Preti 2008].
[29] It is clear from the Russell-Moore correspondence that Russell had read the dissertation in November of 1898.

concepts and Russell's theory of terms must not be understated.[30] Importantly, on Russell's theory, the proposition is regarded as basic and the logical nature of the constituent term is determined by the position it occupies within a significant proposition, and the sort of occurrence it has therein.[31] Unlike Moore, Russell distinguishes among terms between those which have the logical nature of *concepts* and those which have the logical nature of *things*. In those propositions which are of the subject-predicate form, there is no term that is a subject or a predicate essentially or in itself, as on a substance/accident ontology, but the position that a term occupies in a judgment and the manner of its occurrence determines its status. For instance, terms which are traditionally viewed as predicates are *concepts*, which may also occupy the subject position in a judgment, terms which occupy the subject position are those *concepts* or *things* that the proposition is about and, among these terms, those which cannot occupy the predicate position are *things*.[32] Since many propositions, particularly mathematical propositions, are not of the subject-predicate form, a logically satisfactory account of the nature and structure of propositions was, for Russell as it was for Frege, intimately connected to the advancement of logic beyond the traditional subject-predicate logic. In the AMR,

[30] The manner of occurrence which a term has in a proposition is central on Russell's approach to analysis and figures crucially in the theory of meaning and denotation which prefigured his 1905 theory of descriptions. In chapter 5, I shall give a more detailed account of the kinds of occurrence which terms have in propositions and the significance this has for logical analysis, particularly for the logical analysis of propositions involving denoting complexes.

[31] As Griffin and Godden point out, "The basic unit is the judgment rather than the term, since a term's place in a judgment will determine how it occurs therein (Griffin 1991, 276)" [Griffin and Godden 2009, 4].

[32] Russell intends his distinction between predicates occurring as subjects and predicates occurring as meanings as such, lacking being (i.e., predicating predicates are not terms) to defeat Bradley's regress argument [A&R, 28]; [AMR, 175]. By PoM, however, Russell has adopted the view that predication is an external relation and regards it as self-contradictory to deny that anything is a term.

however, far from formulating an approach to the analysis of relational propositions, Russell regards relations as reducible to the adjectives of the relata.[33]

In his letter dated Sept 11, 1898, Moore tells Russell: "With regard to the special method of composition [of propositions] I said nothing [in his dissertation]. There would need, I think to be several kinds of ultimate relations between concepts—each, of course, necessary" [RA].[34] In his response of September 13, 1898, Russell replies: "I agree most emphatically with what you say about the several kinds of necessary relations among concepts and I think their discovery is the true business of Logic" [RA]. In January, 1899, Russell gave a paper on the Classification of Relations to the Moral Sciences Club in which he expressed the results of his work on the logical classification of relations and maintained, against Bradley, that relations are external to their terms and not reducible to identity and diversity of content, as he had formerly supposed. Prior to Russell's development of a doctrine of external relations, the analysis of all propositions involving asymmetrical transitive relations was inconceivable and prior to his development of a logic of relations, the analysis of mathematical propositions was crippled. In *My Philosophical Development*, Russell writes that he "first realized the importance of the question of relations when [he] was working on Leibniz" [MPD, 61] from the summer of

[33] It is worth noting that on AMR's theory of terms, predicates are construed as terms of relations and, hence, as having subsistence [AMR, 218]. Russell held a variety of (inconsistent) positions on whether relational predicates are terms in AMR, FIAM, and PoM and I shall devote attention to these passages in Chapter 2.

[34] This is reflected in his later account of the indefinability of 'the good', in *Principia Ethica* [Hylton 1990, 10n2].

1898 to the summer of 1899 and I believe it would be accurate to say that Russell embraced the central theses of Moore's new realism in a piecemeal fashion during this same period.[35] Moore and Russell did not give a clear or complete account of their position at the time it was developed and, even in his book on Leibniz, where Russell does attempt to clarify some of the basic features of his new realist philosophy and his conception of the nature and analysis of propositions, the account of the positive position is far less clear than, and must in places be gleaned from, the account of what is to be rejected.[36] Moore's influence is suspected in Russell's attack on the Kantian theory of knowledge, on which the truth of judgments depends upon conditions for belief.[37] Russell's condemnation of Kant, which began to develop with Russell's rejection of the subjectivity of the *a priori* in his 1897 *Essay on the Foundations of Geometry* (henceforth, EFG) and which had no doubt gathered strength from the arguments

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[35] In MPD, Russell writes "Moore led the way, and I followed closely in his footsteps" [MPD, 42].

[36] Latta thinks as much: "It is a pity that in making so comprehensive a charge Mr. Russell has not given us a more complete account of his own position, for if his contention be just, his relational theory of the proposition must be of incalculable importance to philosophy" [Latta 1901, 527]. This is echoed by Gustav Bergmann in Bergmann 1956, 175. Bergmann remarks: "…Russell's thought, though churning with momentum, was still inchoate at [the time of writing the Leibniz book]". He also aptly remarks that Leibniz's doctrines were closer to the Medieval doctrines than the young Russell or his contemporaries suspected and that Russell attributed to Leibniz his own preoccupations.

[37] Russell clarifies his anti-psychologism in the Leibniz book, taking issue with the conflation of logical questions as to the nature of the proposition and the conditions for its truth with the psychological and subsequent question of the construction and origins of knowledge. He writes: "…The problem we are now concerned with… is not the problem: What are the general conditions of truth? Or, What is the nature of the proposition? It is the entirely subsequent problem, How do we and other people come to know any truth? What is the origin of cognitions as events in time? And this question evidently belongs to psychology…The two questions have been confused…From the strict standpoint of psychology, no distinction can be made between true and false belief, between knowledge and error. As a psychical phenomena, a belief may be distinguished by its content, but not by the truth or falsity of that content. Thus in discussing knowledge, i.e., the belief in a true proposition, we presuppose both truth and belief. The inquiry is thus hybrid, and subsequent both to the philosophical discussion of truth, and the psychological discussion of belief" [PoL, 189].

contained in Moore's fellowship dissertation, pervades the Leibniz book. It is aimed especially at what Russell describes as the view "…constituting a large part of Kant's Copernican revolution, that propositions may acquire truth by being believed" [PoL, 16-17].[38] In PoL, Russell holds that Leibniz's doctrine of innate truths in the *New Essays* is vulnerable to the same criticisms as those which he levels against Kant's doctrine of the subjectivity of the *a priori*, which depends upon what Russell calls "the radically vicious disjunction" that knowledge is either caused by its objects, i.e., by an existent in the case of sense perception, or is uncaused and is to be found already in the mind, as in the case of eternal or *a priori* truths.[39] The view that what is known in perception is an existent and what is known in the case of *a priori* knowledge is a proposition introduces psychological questions about the origins or causes of knowledge into epistemology which could be avoided by the recognition that even in the case of existential judgments, what is known is not the existent that is supposed to be the origin or cause of knowledge, but the fact of existence, i.e., the proposition [PoL, 194]. Though Russell develops the view significantly, Moore's influence is also apparent in his attack on "the Kantian theory of relations", on which a substance-accident ontology premised on a subject-predicate logic requires that relations be useful fictions abstracted from the adjectives of the relata and themselves essentially the work of the mind, versions of which theories are variously

[38] Recall that Russell has a psychologistic reading of Kant (the neo-Kantian idealists' Kant) and that, in this way, his view that Kantian epistemology collapses truth conditions with conditions for belief grounded in objective judgements, especially in the case of empirical judgments, has some plausibility.
[39] Russell's subsequently accepts Couturat's view that all truths, for Leibniz, are analytic in response to material supplied by Couturat in his 1900 work on Leibniz [CPLP, R23.03.1902]. Cf. note 95.

attributed to Leibniz, Lotze, and Bradley. Russell retrospectively gives a clear characterization of what the central theses of the new realist position were in the preface to PoM and I have not discovered any reason to doubt the characterization of Moore's influence given there:

> **[O]n fundamental questions of philosophy, my position, in all its chief features, is derived from Mr. G.E Moore. I have accepted from him the non-existential nature of propositions (except such as happen to assert existence) and their independence of any knowing mind; also the pluralism which regards the world, both that of existents and that of entities, as composed of an infinite number of mutually independent entities with relations between them which are ultimate, and not reducible to adjectives of their terms or of the whole which they compose. Before learning these views from him, I found myself unable to construct any philosophy of arithmetic, whereas their acceptance brought about an immediate liberation from a large number of difficulties which I believe to be otherwise insuperable. The doctrines just mentioned are, in my opinion, quite indispensable to any even tolerably satisfactory philosophy of mathematics [PoM, xviii].**[40]

What remains unclear is how the revolution[41] in Russell's thinking about the nature and constituents of propositions permitted new solutions to formerly insuperable difficulties in mathematics. Russell's embrace of Moore's new realist conception of the nature and constitution of propositions and the ultimate and irreducible nature of relations, developed in both his early mathematical works and in his sustained commentary on the philosophy

[40] The theses characterizing the new realist position are articulated in MTCA I, as follows: "That every presentation and every belief must have an object other than itself and, except in certain cases where mental existents happen to be concerned, extra-mental; that what is commonly called perception has as its object an existential proposition, into which enters as a constituent that whose existence is concerned, and not the idea of this existent; that truth and falsehood apply not to beliefs, but to their objects; and that the object of a thought, even when this object does not exist, has a Being which is in no way dependent upon its being an object of thought: all these are theses which, though generally rejected, can nevertheless be supported by arguments which deserve at least a refutation. Russell notes: "I have been led to accept these theses by Mr. G.E Moore, to whom, throughout the following pages, I am deeply indebted" [MTCA, 432n2].

[41] This is how Russell characterizes it: "There is one major division in my philosophical work: in the years 1899-1900 I adopted the philosophy of logical atomism and the technique of Peano in mathematical logic. This was so great a revolution as to make my previous work, except such as was purely mathematical, irrelevant to everything that I did later. The change in these years was a revolution; subsequent changes have been the nature of an evolution" [MPD, 9].

of Leibniz, constitute the philosophical commitments which would serve as the groundwork for developments in his symbolic logic and the discovery of various techniques by means of which the logical analysis of mathematical propositions could be carried out. It will be the aim of the remaining sections of this chapter to trace these developments.

1.2 TRANSCENDENTAL DEDUCTIONS

In his first published work and in the spirit of 19th century epistemology, Russell pointed out the confusion between the psychologically subjective and the logically *a priori* [Russell 1895, 251].[42] To avoid this confusion as a neo-Hegelian idealist, Russell sought to provide a purely logical test of the *a priori*—the test of whether the experience of the subject-matter of a science would be impossible without the axiom under consideration—and attempted to de-psychologize Kantian arguments,[43] giving them what he thought was a purely logical formulation in EFG.[44] In EFG, Russell attempts to defend the view, also expressed in his 1895 notebook, "Observations on Space and Geometry," that space is known *a priori*. Whereas he had summarily dismissed projective geometry in the earlier work, in the EFG he acknowledges its logical independence from

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[42] For a more detailed treatment, see Griffin 1991, 132.

[43] The neo-Kantians, with a few exceptions, e.g., Herman Cohen and the Marburg school, read Kant as having a decidedly psychologistic notion of the *a priori*. Russell was influenced in this view by Vaihinger, though, according to Griffin, he did read Cohen in March, 1898 [Griffin 1991, 131-4 and, esp, 132 n68].

[44] As Griffin and Godden point out, even as an idealist, Russell rejected the psychologistic views that laws of logic are psychical laws, that thoughts (i.e. ideas) rather than things are the subject matter of arithmetic, and that epistemology could take the form of a "psychology of thought" [Griffin and Godden 2009, 4]. In his introduction to EFG, Russell tells us that if psychology discovers no connection between subjectivity and whatever has been proved a priori in the essay, then the connection between the subjective and the a priori must, in that instance, be abandoned [EFG, 3-4].

metrical geometry and gives a detailed characterization of its main contributions, before condemning the projective definition of distance on the grounds that it ascribes spatial referents to signs which have a mere technical validity.[45] Russell begins EFG with an historical description of the advances of metageometry, inaugurated by the attempt to prove the independence of Euclid's parallel postulate, and quickly moves on to a discussion of metrical geometry and its algebraic treatment of spatial magnitudes. He gives an exposition of Bernhard Riemann's conception of space as a species of the more general conception of a manifold whose elements form a collection of magnitudes, or, more specifically, as a species of a triply extended magnitude whose unique properties must be discovered empirically. He criticizes Riemann[46] both for his neglect of the qualitative aspects of space and the obscurity of his notion of a manifold. Russell writes: "...it is a pity that Riemann, in accordance with the metrical bias of his time, regarded space as primarily a magnitude or assemblage of magnitudes, in which the main problem consists in assigning quantities to the different elements or points, without regard to the qualitative nature of the quantities assigned" [EFG, 15]. In defining space as a species of the more general conception of a numerical manifold, Riemann had, on Russell's view, obfuscated the true nature of spatial magnitudes, which has its basis in a system of

[45] Joan Richards remarks that "[t]he places where Russell broke from the British tradition that identified geometrical signs with spatial referents anticipated the radical break into logicism he was to make a few years later" [Richards 1988, 79].

[46] Russell's criticisms of Riemann are prefigured in his 1895 notebook containing "Observations on Space and Geometry," in which he writes: "Mathematically, Riemann's form is probably as good as any that can be imagined; but philosophically it seems to me very ill fitted to settle what space-conception we require to fit our space-perceptions; and this is the question on which turns the truth to fact of any Geometry, as opposed to mere logical self-consistency," Bertrand Russell, "Observations on Space and 'Geometry" (ms. notebook dated Berlin, June 1895), p. 65-8, cited in Richards 1988.

relations, which is prior to the possibility of regarding it as a system of manifolds [EFG, 16]. This is important for recognizing that well before embracing logicism, Russell had already rejected the arithmetization project in geometry.

Russell contends next with the advances of projective geometry, which dispenses with spatial quantities, employing quantities merely as names for points. A significant portion of the work is devoted to an exposition of Arthur Cayley's reduction of metrical properties (particularly distance[47]) to projective ones, the geometrical use of imaginary numbers, and Felix Klein's extension of Cayley's work to elliptic geometry. On Russell's view, the reduction of metrical to projective properties is merely technical or "apparent", the projective coordinates being purely descriptive, i.e., convenient names for points,[48] and the use of imaginary numbers, despite having logical independence from metrical notions, likewise has a merely technical validity and are without philosophical significance.[49] The projective notion of distance as a function of anharmonic ratio (cross-

[47] Quantities describing the relationship of distance between points change in projection, but the cross-ratio, or relationship between four collinear points, remains invariant in projection. By designating a conic (the Absolute) intersected by any lines in the space at two points at infinity, the distance of any two points on the line can be given as a function of the cross-ratio of these two points and the points at infinity. Cayley writes "…the theory in effect, is that the metrical properties of a figure are not the properties of the figure considered per se apart from everything else, but its properties when considered in connexion with another figure, viz. the conic termed the Absolute" [Cayley 1859, 90].

[48] Again, Russell's criticisms of the mathematically elegant but philosophically impoverished developments of projective geometry are treated briefly in his "Observations on Space and Geometry" (ms. notebook dated Berlin, June 1895), p.50, cited in Richards 1988.

[49] Russell argues that, given a coordinate system, and given a set of quantities which determine a point, a point will be uniquely determined, but it cannot be concluded that to every set of quantities, a point corresponds; the quantities themselves are without spatial significance. Russell writes: "Finally, then, only a knowledge of space, not a knowledge of Algebra, can assure us that any given set of quantities will have a spatial correlate, and in the absence of such a correlate, operations with these quantities have no geometrical import. This is the case with imaginaries in Cayley's sense, and their use in Geometry, great as are its

ratio), though technically possessing the properties of quantitative distances, is strictly qualitative and cannot be used in identifying metrical or quantitative properties.[50] While distance is formally definable in projective terms, real (quantitative) distance is treated by metrical geometry. Russell writes:

> **If A, B, C, be three different points on a line, there must be some difference between the relation of A to B and of A to C, for otherwise, owing to the qualitative identity of all points, B and C could not be distinguished. But such a difference involves a relation, between A and B, which is independent of other points on the line…Before we can distinguish the two fixed points, therefore, from which the projective definition starts, we must already suppose some relation, between any two points on our line, in which they are independent of other points; and this relation is distance in the ordinary sense. When we have measured this quantitative relation by the ordinary methods of metrical Geometry, we can proceed to decide what base-points must be chosen, on our line, in order that the projective function discussed above may have the same value as ordinary distance. But… distance, in the ordinary sense, remains a relation between two points, not between four; and it is the failure to perceive that the projective sense differs from, and cannot supersede, the ordinary sense, which has given rise to the views of Klein and Poincaré. The question is not one of convention, but of the irreducible metrical properties of space [EFG, 35-6].**

The projective definition of distance is formally, but not philosophically valid, since the metrical notion of distance as an independent and unique relation between two points is irreducible to the projective construction.[51] As Russell adopts logicism, he arrives at the

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technical advantages, and rigid as is its technical validity, is wholly destitute of philosophical importance" [EFG, 46].

[50] Russell writes: "[T]he arbitrary and conventional nature of distance as maintained by Poincaré and Klein, arises from the fact that the two fixed points, required to determine our distance in the projective sense, may be arbitrarily chosen, and although, when our choice is once made, any two points have a definite distance yet, according as we make that choice, distance will become a different function of the two variable points. The ambiguity thus introduced is unavoidable on projective principles" [EFG, 35].

[51] Interestingly, the philosophical validity of metrical distance is connected to the relational theory of space. Positions are defined by relations alone and, on Russell's view in EFG, this requires unique relations of distance between any two positions, but such relations, unique to the pair, cannot be inferred from qualities, since points are all alike. Russell writes: "…suppose three positions A, B, C were necessary, and gave rise to the relation abc between the three. Then there would remain no means of defining the different pairs BC, CA, AB, since the only relation defining them would be one common to all three pairs … [F]resh points could not affect the internal relations of our triad, which relations, if they can give definiteness at all, must give it without the aid of external reference. Two positions must, therefore, if definition is to be possible, have some relation which they by themselves suffice to define" [EFG, 143-4]. This is the metrical notion of

view that projective geometry is concerned with distance and belongs to pure mathematics, while metrical geometry is concerned with magnitudes of divisibility and is not a part of pure mathematics at all, which has its origins in his view, in EFG, that the logically subsequent science of metrical geometry must be invoked for the application of quantity to space, i.e., for the measure of real distance.

Russell's defence of the *a priori* nature of space in the EFG is advanced by means of transcendental deductions, which establish the axioms which make possible the experienced subject matter of geometry: the form of externality. In projective geometry, the subject matter is any possible form of externality, some form of which is necessary to experience, and, in metrical geometry, it is the form of externality of more than one dimension insofar as it is capable of (spatial) measurement [Papers 2, xvi].[52] Russell points out that projective geometry, which contends with qualitatively equivalent straight lines and points, is a purely qualitative *a priori* science, presupposed in any quantitative

distance. Metrical geometry likewise depends on the relativity of position: "all parts of space are qualitatively similar, and cannot, therefore, be distinguished by any intrinsic property. Hence positions in space, if our axiom be true, must be wholly defined by external relations, i.e. position is not an intrinsic, but a relative, property... If there could be such a thing as absolute position, in short, metrical Geometry would be impossible. This relativity of position is the fundamental postulate of all Geometry, to which each of the necessary metrical axioms leads, and from which, conversely, each of these axioms can be deduced" [EFG, 60]. Cf. note 162.

[52] It is worth pointing out that, in EFG, Russell claims that metric geometries which entail constant curvature must be established by empirical measurement—a position which he defends against Couturat's objections in EAE.

comparison, for which qualitative similarity is required.[53] Interestingly, Russell addresses the circularity of geometrical definition, i.e., that any definition of points must be carried out by means of the straight line, and that any definition of the straight line must be carried out by means of points [EFG, 127]. Russell concludes that, in pure geometry, we cannot escape this circle: since space is constituted by nothing but relations, "…if we take any spatial figure, and seek for the terms between which it is a relation, we are compelled…to seek these terms within space,…but we are doomed, since everything purely spatial is a mere relation, to find our terms melting away as we grasp them" [EFG, 128]. Though quantitative comparison presupposes the qualitative identity of points on the same line, points can be distinguished from one another only quantitatively, by their relations. The straight line, however, is merely a relation between two of its intrinsically identical points, so that a straight line must be distinguished by the points through which it passes. The antinomy of the point is inescapable: spatial relations require terms, but points are merely the terms of spatial relations and, being distinguished by mere relations, have no intrinsic differences. The analytic component of Russell's treatment of projective and metrical geometry thus yields the axiom of pure relativity, i.e., that all parts of space are intrinsically alike, discernible only by their relations.

Russell employs transcendental arguments to show that certain geometric axioms are necessary to any form of externality. Spatial measurement and hence all metrical

[53] Two points or straight lines having the same anharmonic ratio are, on Russell's account, "qualitatively equivalent" [EFG, 123]. By this he means only that the equivalence is not determined by quantitative comparison as in metrical geometry.

geometry requires congruence, but congruence requires free mobility, i.e., the movement of figures from one region of space to another region of space, which is problematic if the figure can be described only in virtue of its spatial relations to other figures. In metrical geometry, the relativity of space is essential for the axiom of constant curvature (or free mobility)—the requirement that spatial figures may be moved freely in space, upon which congruence of spatial figures depends. However, something further is required to provide points with intrinsic properties by which they might be differentiated, so that the movement of a figure from one part of space to another can be meaningfully described. The form of externality as a condition for the possibility of experience, depends, on Russell's view in the EFG, on the assumption that all knowledge requires the recognition of "'diversity in relation' or, if we prefer it, 'identity in difference'", and matter is introduced to supply simultaneous diversity.[54]

In characterizing Moore's anti-subjectivist critique of Kant, Baldwin points out that Moore's chief objection to transcendental arguments is that such arguments can show only what necessarily follows from the hypothesis that we possess empirical knowledge, and since there is nothing to prevent empirical judgments from being false, what is entailed has a precarious sort of necessity [Baldwin 1993, 11]. These criticisms are

[54] Russell writes: "For so long as we leave matter out of account, one position is perfectly indistinguishable from another, and a science of the relations of positions is impossible…Again, if Congruence is ever to be used there must be motion: but a purely geometrical point, being defined solely by its spatial attributes, cannot be supposed to move without a contradiction in terms. What moves, therefore, must be matter. Hence, in order that motion may afford a test of equality, we must have some matter which is known to be unaffected throughout the motion" [EFG, 77].

essentially those leveled against Russell's purely logical transcendental arguments in Moore's 1899 review of EFG. In his review, Moore criticizes Russell's use of transcendental arguments on the grounds that they establish conditions for the possibility of knowledge concerning some branch of experience and not conditions for the truth of judgments concerning it and, in this regard, the deductions are insufficiently anti-psychologistic. In EFG, Russell clearly does not hold that the propositions of mathematics have an immediate certainty by virtue of operating on determinate contents, but he does hold that the propositions of mathematics are synthetic. Though Russell was aware, even in EFG, that modern logicians rejected the Kantian distinction between synthetic and analytic judgments [EFG, 59],[55] he nevertheless held that synthetic judgments "…combine a subject and a predicate which cannot, in any purely logical way, be shewn to have any connection, and yet these judgments have apodeictic certainty", maintaining that Kant had proven "…with every precaution, that without them, experience would be impossible" [EFG, 59]. Moore finds this to be a dubious sort of necessity, dependent upon universal features of human psychology or operations of the mind. If these psychological features or operations of the mind are contingent matters of fact, then necessary propositions cannot be deduced from them and if they are explained by further *a priori* and necessary truths, then it is not directly from the constitution or operations of

[55]Russell cites Bradley, *Logic*, Bk III, Pt 1, Ch1 and Bosanquet's *Logic*, Bk 1, ch1, contending that judgments are regarded in modern logic as both synthetic, in that they combine parts into wholes, and analytic, in that they analyze wholes into parts.

the mind that synthetic *a priori* propositions are deduced.[56] In his 1898 Fellowship Dissertation and in NJ, Moore contends that transcendental arguments fail as deductions from possible experience, but succeed in showing that space, time, and the categories are involved in particular existential propositions, that is, that geometry, arithmetic, substance, and causality are involved in ordinary empirical judgments, which, he thinks "…is of greater value than a deduction from the possibility of experience would have been" [NJ, 192]. The value of Kant's so-called deductions, then, is that they attempt to give an analysis of the sorts of concepts constituting various existential propositions. The trouble is that, while the application of the categories allows for the objective validity of propositions, so that they can be used in inferences, it is, indeed, for Kant, a reference to existents (objects of intuition) that gives propositions the title of 'knowledge'—that gives them objective reality. On Moore's view, the supposition that the object of judgment is not the (existential) proposition, but that existent which the proposition is about, has the intolerable consequence of making truth dependent upon a correspondence between what is asserted in a judgment and the object which the judgment is about. This supposition, which is implicit in Bradley's claim that a judgment involves "…a reference to something beyond [itself]" and a reference always to something actual [PL, 42] is rejected by Russell in his 1900 book on Leibniz along with the existential theory of propositions,

[56] This would, in Moore's terms, "presume to deduce a necessity from a mere fact, namely that our mind is so and so constituted, and this on Kant's own principles, effectually excludes the propositions deduced from any claim to be absolutely necessary" [Baldwin and Preti 2011, 151]. The constitution of the mind supplies no logical basis for the deduction of a priori necessary propositions.

which, Russell thinks, has the consequence of making truth dependent upon knowledge [PoL, 214].

In EFG, Russell has taken steps in the direction of this view. Asserting Bradley's thesis that all necessary truth is hypothetical, Russell points out that the question which concerns him is what properties the form of externality must possess if externality, that is, interrelated diversity, is to be experienced—the conception of the form of externality, as such, is independent of actual space, and has no existential import [EFG, 62 and 135-6]. The conditions of the experience of the form of externality are stated so that *if* there be experienced externality, *then* there must be a form of externality having such and such properties [EFG, 136]. The motivation is to account for inconsistent geometries by claiming that geometric statements do not assert the existence of the various spaces they define. We shall see that, as Russell begins to work out the axioms of geometry in a more rigorous fashion, he abandons the Kantian formulation, adopting the view that geometric statements assert that if certain axioms hold, then the geometric concepts defined by such axioms possess certain formal properties, not that entities such as those defined actually exist. In EFG, Russell happens to believe that Kant's argument that the form of externality is necessary to the experience of interrelated diversity presupposed in sense perception establishes that there is such a form of externality, but he recognizes the logical independence of the hypothetical propositions established by his transcendental arguments. The question of whether Russell's arguments are indeed vulnerable to

Moore's criticisms cannot be taken up here, but it can be pointed out that they were largely irrelevant to Russell's philosophy by the time they appeared in print. While Russell retained his transcendental arguments in AMR, they began to give way to an unequivocal anti-Kantianism and were rejected altogether in his 1899 work, "Fundamental Ideas and Axioms", before Moore's review was published. Since Russell construes the logicist project that results from his embrace of analysis as a refutation of Kant [PoM, 4][57] it is worth considering these developments.

In considering Moore's claim from NJ that the value of Kant's transcendental arguments consists in the fact that they exhibit the concepts involved in particular existential judgments, Nicholas Griffin writes:

> **[T]he analysis of propositions, their fundamental constituents and the necessary propositions which are 'involved' in them, seem to be passed off as the true form of transcendental arguments. Moore understates the radicalism of his break with Kant, but if such arguments were to be counted as transcendental, then Russell's arguments from the analysis of propositions to the calculus of symbolic logic are also transcendental…Arguments showing that certain concepts and propositions were involved in the analysis of complexes of terms might be regarded as a new, non-psychologistic form of transcendental argument, or alternatively, as a type of argument which was not transcendental at all, but analytical (in the very literal sense that Moore and Russell came to use when talking of their new philosophy) [Griffin 1991, 306].**

Russell's embrace of analysis and its consequences can be charted by the Kantian doctrines he disburdened himself of along the way—-replacing the subjectivity of the *a priori* with a purely logical criterion, the axioms requisite to the possibility of

[57] Also: "The primary aim of *Principia Mathematica* was to show that all pure mathematics follows from purely logical premises and uses only concepts definable in logical terms. This was, of course, an antithesis to the doctrines of Kant, and initially I thought of the work as a parenthesis in the refutation of [Kant]" [MPD, 57].

mathematics with the axioms which make existing mathematics true, and intuitions in mathematical reasoning with strictly logical analyses. In EFG, Russell insists on the separation of the *a priori* and the subjective, on the grounds that results as to the *a priori* must be gleaned from the logical analysis of knowledge and ought not to be "placed at the mercy of empirical psychology." "How serious this danger is," Russell writes, "the controversy as to Kant's pure intuition sufficiently shows" [EFG, 3]. After coming to doubt the transcendental arguments he provided in EFG, but before dispensing with them altogether, Russell's primary concern was to answer this very question of "what it means to have an *a priori* intuition" —a question which, he conveys in his response to Couturat of May 12, 1898, is "perhaps the most difficult in philosophy".[58] In June of 1898, Russell writes to Couturat, who had reviewed EFG, that he has changed his views significantly since EFG was written, but that he nevertheless hopes to defend the view that the axioms unique to Euclid are empirical. In giving this defence in his November 1898 paper, "Les Axiomes propres à Euclid, sont-ils empiriques?", Russell regards the propositions of mathematics as synthetic *a priori* on the grounds that, wherever they are relational,[59] they presuppose the possibility of a diversity of logical subjects, which, Russell believes, requires a material diversity which can only be given in intuition. Russell's position, in EAE, is the following:

[58] In this same letter, Russell tells Couturat that he regards the question as necessary to defending his view from EFG that axioms peculiar to Euclid are empirical. He adds also that he intends to answer the question "…such an intuition, supposing that it exists, can have only some of the properties of space" [CPLP, R12.05.1898].

[59] It is worth noting that identity is not, at this time, taken by Russell to be a genuine relation, in that it does not involve a diversity of terms.

> **Certain mathematical propositions, for instance that if A=B then B=A, or that A>B then B<A, or the axioms concerning order, seem to be necessary and synthetic. The collection of all propositions of this kind, and the proof that they are synthetic, obviously cannot be given here…But…all these judgments depend upon a diversity of logical subjects: they are not restricted to affirming a necessary connection of the contents; they affirm that, if A has an adjective, B must have another, or other more complicated assertions of the same type. In brief, they all depend upon relations which imply material diversity, i.e., a plurality of existent beings. If, then, these judgments are truly necessary, the possibility of several beings is also necessary; and this condition seems satisfied…by space and time. But we cannot say for this reason that space and time are a priori; we can only declare that some form of externality, sufficient for the a priori judgments of Mathematics, is a priori [EAE, 334].**

On Russell's account, the *a priority* of the intuition presupposed in (relational) mathematical propositions consists exclusively in its supplying the possibility of material diversity which such propositions require.[60] The necessity of the fundamental propositions of mathematics consists in the fact that they are presupposed in the methods or "reasoning" of a science according to which empirical knowledge is possible, but strictly, they are incapable of proof. Russell gives the following account:

> **We begin with the necessity of certain fundamental propositions. For this necessity we do not provide more positive proof than for the blue colour of the sky. We can show that some proposition is presupposed in the set of procedures used by science, and that the methods by which an experimental proof is obtained would be impossible without this proposition…But if we are to continue believing in our proposition, and, still more, if we are to believe it its necessity we are obliged…to excuse ourselves from every attempt to prove it. This apparently arbitrary property characterizes, I believe, the necessity of mathematical axioms [EAE, 334][61]**

Russell did not easily abandon his project of giving purely logical transcendental deductions to ground the truths of mathematics and, on June 3, 1898, Russell writes to Couturat that the book he has been working on, what would become AMR, could be titled

[60] Russell gives the argument that the axioms of parallels and of three dimensions can only be called a priori if an undue psychological element is preserved; the a priori is defined, not with respect to our knowledge, but only with respect to truth and necessity [EAE, 338].
[61] See also EFG, 4-5.

"How is pure mathematics possible" and that the results would be, for the most part, "purely Kantian" [CPLP, R03.06.1898]. The revolution in Russell's thinking occurred in the months which ensued.

Russell had read Whitehead's *Universal Algebra* in March 1898, while working on his paper "On Quantity and Allied Conceptions". Whereas mathematics had been regarded as the study of quantity, Whitehead's book had offered an algebraic treatment of symbolic logic, not based on the concept of quantity. In his *Universal Algebra*, Whitehead points out that, "historically, mathematics has…been confined to the theories of Number, of Quantity, (strictly so-called) and of the Space of common experience…" [Whitehead 1898, viii]. In more recent mathematics, a wider concept of quantity was introduced, as the complex quantity of ordinary algebra[62] of which quantity in the strict sense is merely a part. Newly invented algebras, Whitehead points out, are "…not essentially concerned with number or quantity; and this bold extension beyond the traditional domain of pure quantity forms their peculiar interest" [Whitehead 1898, viii]. After conveying to Couturat, on May 12, that he intends to modify the theory outlined in "The Relations of Number and Quantity" (1897) to connect number and quantity via the idea of relation,

[62] "Ordinary algebra, in its modern developments is studied as being a large body of propositions, inter-related by deductive reasoning, and based upon conventional definitions which are generalizations of fundamental conceptions…" [Whitehead 1898, viii].

Russell abandons the attempt to reconcile "On Quantity and Allied Conceptions" with Whitehead's work[63]. Finally, on July 18th, Russell tells Couturat:

> **I don't believe I shall make many allusions to the article on quantity,[64] since the questions the article concerns are too fundamental to be discussed in passing. Moreover, I will need a whole book to give an exposition and proof of what I have said on the subject. I propose, in this book, the same goal that you spoke of in the second part of your article,[65] that is to say, the discovery of the fundamental ideas of Mathematics, and the necessary judgments (axioms) that we must accept in reasoning about these ideas. I also have it in mind that order and quantity must be put on the same level as number, if not in a philosophy of mathematics, in any case in a philosophy of space and time [CPLP, R18.07.1898].[66]**

Whitehead's influence is significant.[67] The aim of Whitehead's work was "…to exhibit the new algebras, in their detail, as being useful engines for the deduction of propositions; and in their several subordination to dominant ideas, as being representative symbolisms of fundamental conceptions" [Whitehead 1898, viii]. Importantly, for Whitehead, mathematics is constituted by "the development of all types of formal, necessary, deductive reasoning" [Whitehead 1898, vi]. It is formal in that it is not concerned with the meaning or content of propositions, but with the rules of inference; it is necessary in that mathematical axioms are necessary, though their empirical or philosophical justification forms no part of this necessity; and they are deductive in that they are based

[63] Sections IV and V of Russell's paper "On Quantity and Allied Conceptions," seem to be initial attempts to make the work compatible with the insights he had recently gleaned from Whitehead, but Russell abandoned the work.

[64] Russell means that he will not refer to his own article "On the Relations of Number and Quantity, (1897)" in his article "Are Euclid's Axioms Empirical?" for the *Revue de métaphysique et de morale*.

[65] The article in question is Couturat 1898.

[66] It has been supposed that order replaced quantity as fundamental category of mathematics [Griffin 1991, 356-7], but it seems that quantity was placed on a par with the other crucial concepts of mathematics.

[67] July 20th, 1898, Russell tells Couturat that he hopes that Couturat will write his review of Whitehead's *Universal Algebra* and adds that he believes it to be a work of very great importance and one by which he had recently been inspired [CPLP, R20.07.1898].

on definitions that need only to be internally consistent, though the definitions must stand in relation to whatever ideas are contained in the subject-matter of the system in question. In AMR, whose title alone exhibits Whitehead's influence, Russell seeks to exhibit the *a priori* foundations of pure mathematics, its fundamental—irreducible, indefinable, and unanalyzable—concepts and the basic, non-demonstrable propositions[68]—axioms or rules of inference, which assert necessary connections between concepts stated in the form of implications. In AMR, the fundamental, indefinable concepts of mathematics are no longer to be confined to number, quantity, and the space of our perceptions, but include addition and the manifold; number introduced in arithmetic; the concept of order, introduced in the theory of the ordinal numbers; relations of equality, and greater and less, in the theory of quantity; the extensive continuum in the theory of extensive quantity; the concept of dimensions; and the concept of a thing. In many respects, AMR is a transitional work—it is influenced, on the mathematical side, by Whitehead's developments in symbolic logic,[69] the mathematical account of extensions (manifolds), and the distinction of signs and, on the philosophical side, by Moore's new realist thesis that propositions, to be the sort of entities to which logical truth may be ascribed and to be used in inference, must be composed of concepts standing in several necessary ultimate relations whose analysis consists in a decomposition of the whole into its constituent parts. The techniques required for a purely logical analysis of the (oftentimes relational)

[68] Called judgments in the AMR, though Moore had already adopted the term 'proposition'.
[69] Whitehead dealt at length with Boolean algebra—the algebra of symbolic logic in the first book of the *Universal Algebra*, which Russell deals with in the first Book of AMR.

propositions of mathematics, however, were not available to Russell in 1898-1899 and, despite advances in symbolic logic informed by Whitehead's Universal Algebra, the work retained many of the commitments of Russell's neo-Hegelian program, most significantly, his doctrine of internal relations.

In FIAM (1899), Russell entirely rejects transcendental arguments and regards intuition as no more necessary for mathematical certainty in an account of space than in arithmetic [FIAM, 270]. In that work, he attempts to give logical analyses of mathematical propositions and logical structure and rigor to mathematical proofs, departing from axioms which allowed existing mathematics to be true without appeal to intuitions. In the 1899-1900 and 1900-1901 drafts of PoM, Russell is no longer asking his earlier psychologistic (Kantian) question— "how is pure mathematics possible?", but rather "what axioms allow mathematics to be true?" Whereas Russell had formerly sought to give a purely logical deduction of the propositions necessary to the empirical methods and (experienced) subject-matter of the science of geometry, he later sought to uncover the fundamental propositions of mathematics, stated in the form of implications, by which the propositions of mathematics could be shown to be true, privileging a logical over an epistemic criterion for necessity. Russell writes:

> **There was, until very lately, a special difficulty with the principles of mathematics. It seemed plain that mathematics consists of deductions, and yet the orthodox account of deductions were largely or wholly inapplicable to existing mathematics…In this fact lay the strength of the Kantian view, which asserted that mathematical reasoning is not strictly formal, but always uses intuitions, i.e., the a priori knowledge of space and time. Thanks to the progress of Symbolic Logic…this part of the Kantian philosophy is now capable of a final and irrevocable refutation [PoM, 4].**

Nevertheless, it is in something of the Kantian spirit of his former transcendental arguments from EFG that Russell retains the view, from his pre-logicist to his post-logicist program, that legitimacy in the choice of axioms depends upon whether true propositions of mathematics follow from them.[70] In PoM, Russell claims that "…formally, my premises are simply assumed; but the fact that they allow mathematics to be true, which most current philosophies do not, is surely a powerful argument in their favour" [PoM, xviii]. Russell's transition from the broadly Kantian project of attempting, in AMR, to exhibit the axioms or rules of inference which make (true) mathematical judgments possible to the notion that advances in symbolic logic permit an irrevocable refutation of the synthetic *a priori*[71] was cemented by his discovery of Peano's symbolic logic and his articulation of a logic of relations by means of which the propositions of mathematics could be analyzed. Russell's adoption of the doctrine that relations are ultimate entities, external to their terms and irreducible to the intrinsic properties of the related terms was a crucial step in this development. Russell's work, in AMR, on the

[70] Consider, for instance, Russell's subsequent articulation of the regressive method in mathematics: "We tend to believe the premises because we can see that their consequences are true, instead of believing the consequences because we know the premises to be true. But the inferring of premises from consequences is the essence of induction; thus the method in investigating the principles of mathematics is really an inductive method, and is substantially the same as the method of discovering general laws in any other science", from "The Regressive Method of Discovering the Premises of Mathematics", read before the Cambridge Mathematical Club, 09 March 1907, [Lackey 1973, 273f]. For a detailed treatment of the method and its significance, see Irvine 1989.

[71] This is somewhat anachronistic since, while Russell later claims, for instance in the second edition preface to PoL, that the analytic/synthetic distinction is simply that of whether propositions can be deduced from logic or cannot be so deduced [PoL, xvii], he in fact holds, in PoM, that the propositions of mathematics are synthetic. What should be emphasized is that in PoM, Russell believes that the deduction of mathematical propositions can be carried out by means of logic alone without recourse to intuition, but it is logic that is no longer to be considered to be "strictly formal"—being "...just as synthetic as other kinds of truth" [PoM, 457]. I shall address the latter point, and the related question of whether logicism involves "strictly formal" methods of deduction in Chapter 3, but for now I wish to emphasize the former.

distinction of signs in the account of quantity, number, order and series, in the absence of a doctrine of external relations, left intact the apparently inexorable contradiction of relativity which Russell believed to pervade the whole of mathematics.

1.3 INTERNAL RELATIONS AND THE CONTRADICTION OF RELATIVITY

The doctrine of internal relations which Russell held in AMR is not the doctrine of internal relations he criticizes Bradley for holding. Russell, as we shall see, quite mistakenly held that Leibniz regarded all relations, including those constituting space and time, as reducible to the states of substances and, what was equivalent for him, regarded all relations as reducible to the predicates belonging to the substances as logical subject which were, by themselves, "destitute of meaning". In the doctrine he ascribed to Leibniz, Russell saw a great affinity to Bradley's doctrine of relations. In the Leibniz book, he writes: "Mr. Bradley, in attempting to reduce all judgment to predication about Reality, is led to the same view concerning his ultimate subject. Reality, for him, is not an idea, and is therefore, one must suppose, meaningless" [PoL, 59n12].[72] It is worth considering Bradley's doctrine briefly. Bradley construes reality as a unified whole of experience, whose aspects can be distinguished only by the abstraction of individual terms and relations between them, but which cannot be resolved into individual terms and relations [PL, 2nd Ed., Ch. II, additional note 50]. To ascribe properties to individual things is, on Bradley's account, either to invoke the unintelligible notion of a thing without properties to which properties are ascribed or to suppose that the individual is an aggregate of

[72] Russell cites PL.

properties, which leaves unanswered the question of how these properties are related.[73] In his famous regress argument against the reality of relations, Bradley maintains that if the relations supposed to unify the terms of a judgment were real, i.e., had independent existence, then further relations would be required to relate these relations to their terms and so on *ad infinitum*. Bradley writes:

> **Let us abstain from making the relation an attribute of the related, and let us make it more or less independent. 'There is a relation C, in which A and B stand; and it appears with both of them.'…The relation C has been admitted different from A and B, and no longer is predicated of them [...] If so, it would appear to be another relation, D, in which C, on the one side, and, on the other side, A and B, stand. But such a makeshift leads at once to the infinite process [A&R, 16–18].[74]**

In denying the reality of external relations, Bradley did not, however, hold that relations are reducible to the internal properties of the relata. In response to criticism, Bradley maintains that external relations assert the independent existence of relata and, hence that external relations in their very nature obfuscate the way in which they are part of a greater totality. Internal relations, i.e., those grounded in the intrinsic properties of the relata, make the requisite difference to that which they relate. Nevertheless, they too fail to be self-consistent in that, as relations, they require the independence of the objects they relate, while their internality requires that these objects are constituted by their relations to other objects. Insofar as they point to a greater totality beyond themselves, they are, in that sense alone, preferable to merely external relations [Bradley 1914, 227-8

[73] Bradley claims that after having read his chapter III on 'Relation and Quality' the reader will "…have little need to spend his time on those which succeed it. He will have seen that our experience, where relational, is not true; and he will have condemned, almost without a hearing, the great mass of phenomena", i.e., space, time, motion, change, activity, causality, etc. [A&R, 29].

[74] See also Bradley's regress argument from Chapter III of A&R.

and 239-40].[75] What, for Bradley, generates an infinite regress and thereby supplies the grounds for denying the independent reality of both terms and relations is, for Russell the makings of a reductio argument against the view that relations modify their terms. In his 1901 draft of PoM, Russell levels the following criticism at what he takes to be the monistic view of relations:

> **[B]oth subject and predicate are simply what they are—neither is modified by its relation to the other. To be modified by the relation could only be to have some other predicate, and hence we should be led into an endless regress. In short, no relation ever modifies either of its terms. For if it holds between *A* and *B*, then it is between *A* and *B* that it holds, and to say that it modifies *A* and *B* is to say that it really holds between different terms *C* and *D*. To say that two terms which are related would be different if they were not related, is to say something perfectly barren; for if they were different, they would be other, and it would not be the terms in question but a different pair, that would be unrelated. The notion that a term can be modified arises from neglect to observe the eternal self-identity of all terms and all logical concepts, which alone form the constituents of propositions [Russell 1901c, 189].[76]**

Bradley, who holds not only that relations are dependent on their terms, but also that terms are dependent upon or constituted by their relations, recognizes that a term would be different if it did not stand in the relations it did.[77] Indeed, this is the grounds for Bradley's rejection of internal relations in Russell's sense (i.e., reducible relations) as well as external relations. On Bradley's view, a related term provides the foundation for a relation in one sense, while being, in another sense, constituted by it and it is this double-aspect of related terms which produces the infinite regress of relational complexes.

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[75] See also A&R, 506 ff., 519-20] and Bradley 1935, 628-676.

[76] Russell claims against the monists, citing Moore's 1901 paper "Identity" [I], that "…another mark which belongs to terms is numerical identity with themselves and numerical diversity from all other terms" [Russell 1901c, 189].

[77] "[R]elations must depend upon terms, just as much as terms upon relations" [A&R, 26].

As a neo-Hegelian, Russell rejected Bradley's monism and the doctrine of internal relations with which he initially takes issue is that on which all apparently relational judgments assign an adjective to the Absolute, the one true subject in which all properties inhere.[78] A metaphysical statement of this view is found in Bradley's *Appearance and Reality,* where he claims that "[i]n every judgment the genuine subject is reality, which goes beyond the predicate and [is that] of which the predicate is an adjective" [A&R, 148]. The logical statement of this view is found in his *Principles of Logic,* where he claims that "Our 'S is P' affirms really that Reality is such that S is P" [PL vol. II, 630]. That this is the view which Russell attributed to Bradley is clear from marginalia, presumed to be written in January, 1898, in Russell's copy of Bradley's *Logic*. Beside Bradley's claim that a judgment does not always have two ideas, Russell remarks: "On your theory, there <u>are</u> two ideas, the wandering adjective & Reality." He notes, along with this marginal comment, that "Reality", for Bradley, is not an idea [Chalmers and Griffin 1997, 55][79] and, underlining Bradley's remark that "an idea is adjectival", Russell writes: "always?" In his marginal comments on Bradley's *Principles of Logic,* Russell puzzles over contentions regarding the impossibility of supplying unique reference by means of adjectives which are universal and over the distinction between identity of content and

[78] In PoM, Russell retrospectively writes: "This doctrine [that every proposition ascribes a predicate to a subject] develops by internal logical necessity into the theory of Mr. Bradley's *Logic*, that all words stand for ideas having what he calls meanings, and that in every judgment there is a something, the true subject of the judgment, which is not an idea and does not have meaning" [PoM, 47]. He cites PL, 58-60 which is precisely the passage he comments on in 1898.

[79] The complete marginal comment is: "On your theory there <u>are</u> two ideas. The wandering adjective & Reality. [but cf. pp. 49-50, where it appears that Reality is not an idea]".

numerical identity, but in January, 1898, Russell is not in a position to offer solutions to these difficulties. By way of criticism, he simply remarks that "[t]he point to be argued is, whether all ideas are purely adjectival, or whether all identity is merely identity of content; but this point is not argued…"[80] On my account, the attempt to develop his new-realist position on precisely these issues which troubled him in reading Bradley's *Principles of Logic,* finally compelled Russell to jettison his own doctrine of internal relations and the "contradiction of relativity" which it occasioned.

We have seen already the manner in which EFG ends in the antinomy of the spatial point, as a result of the fact that, on Russell's relational theory of space, the points which are the terms of spatial relations are qualitatively indiscriminable.[81] The unique internal relation between two positions (i.e., distance) cannot be inferred from the positions it relates, since positions are determined by relations [EFG, 144]. Points are the contradictory outcome of "hypostatizing the form of externality", where, "philosophically, the relations alone are valid" [EFG, 138]. The antinomy of the point is produced by the relativity of position, that is, by the fact that positions are determined by

[80] Russell adds, "see page 156 [of PL]". This comment appears alongside this passage from PL, 59-60: "If there is an idea conveyed by the name, whenever it is used, then it surely means something or, in the language which pleases you, it must be "connotative." But if, on the other hand, it conveys no idea, it would appear to be some kind of interjection. If you say that, like "this" and "here", it is merely the ideal equivalent of pointing, then at once it assuredly has a meaning, but unfortunately that meaning is a vague universal" [Chalmers and Griffin 1997, 60].

[81] It is worth noting that even in EFG Russell regards quantities as relations. However, whereas the quantitative comparison of such magnitudes as colour and pitch depends upon the intrinsic differences of adjectives between the elements concerned, spatial relations can neither be reduced to nor inferred from adjectives, since points are all alike. In metrical geometry, wherein quantitative comparison requires division into parts, spatial relations must be hypostatized.

mere relations and there are no conceptions by which to distinguish points or the parts of space. The contradiction of relativity in AMR has a further precursor in the antinomy of quantity in Russell's 1897 article "On the Relation of Number and Quantity". Judgments of quantity are comparative, which requires that the quantity in question be homogeneous, i.e., that it be qualitatively similar to the elements into which it can be divided and to those quantities with which it is compared. Moreover, quantity is a relative notion and any quantity is distinguishable from others in virtue of its relations to other quantities. In comparing two quantities which, to effect the comparison, must be qualitatively similar, we find that we have a conception of difference (i.e., numerical distinctness), but no difference of conception, since there is no intrinsic difference in the concepts applicable to each quantity. When Russell wrote his March, 1898 article "On Quantity and Allied Conceptions", Russell's distinction between "a conception of difference" and "a difference of conception" has grown more nuanced. Given a manifold of elements, all of the elements which have in common the assumed intrinsic property 'the quantity A', and all of the elements which have in common the assumed intrinsic property 'the quantity B' will form two submanifolds of quantitatively equal elements which are quantitatively different from each other. However, the quantitative properties A and B, which are required to account for the inequality of their respective elements, supply no intrinsic difference, but differ merely in virtue of being the quantities of their respective elements. Russell explicitly maintains that the asymmetrical relations involved in these quantitative judgments "...cannot be analyzed into a relation of adjectives, but confer adjectives with an external reference" [OQAC, 123]. In the AMR, the antinomy of quantity is generalized to "the contradiction of relativity," which holds in all of the sciences. The doctrine of internal relations which Russell espoused as a neo-Hegelian differs from Bradley's doctrine of relations in that it requires that relations are, in some sense, grounded in the qualities of the relata. Importantly, relations involving a diversity of content, i.e. those grounded in different qualities in the relata—asymmetrical relations fundamental to every branch of mathematics and necessary to the concepts of spatial points, instants, quantity, and number—produce the contradiction of relativity, which Russell describes as "…the contradiction of a difference between two terms, without a difference in the conceptions applicable to them" [AMR, 166].

In AMR, Russell maintains that, while symmetrical relations like equality and simultaneity, which confer the same adjective on both terms,[82] can be analyzed into those adjectives in the related terms which ground the relation, asymmetrical relations, which confer differing adjectives on its terms are not, in this sense, reducible to "relations of adjectives" [AMR, 224]. Though differing adjectives can be inferred from an asymmetrical relation, e.g., the adjectives cause and effect, right and left, greater and less, positive and negative, the related terms are, on Russell's view, "…differentiated by the relation, not by any discoverable inherent properties in which they differ" [AMR, 224], which, on the doctrine of internal relations, produces a difference without a point of

[82] See Whitehead's "On the Nature of A Calculus": "[T]he equivalence of distinct things implies a certain defined purpose in view…Then within this limited field no distinction of property exists between the two things" [Whitehead 1898, 5].

difference. In all such cases, there are to be found "adjectives of the relation" which cannot be reduced to "relations of adjectives". Russell's theory of relations, then, admits ineliminable relational properties, but relations themselves, on the doctrine of internal relations, presuppose adjectives or adjectives of the relation and are themselves unreal. The "difference in the conceptions" applicable to two terms, where it is not reducible to a difference in adjectives, but requires adjectives of the relation, is treated by the distinction of signs.[83] Whitehead's work in the *Universal Algebra* influences Russell's theory of the sign and, insofar as Russell's aim in the AMR was to provide a philosophical basis for pure mathematics in answering the question, how is pure mathematics possible?, the original purport of the work is changed significantly by making the appeal to intuition irrelevant to the difference of sense indicated by the distinction of signs. Russell writes:

> **The possibility of two senses, of the difference, emphasized by Kant, between right and left handed screws….of the distinction between eastward and westward, before and after—-the possibility of all such differences appears to me to imply a special idea, the idea embodied in distinction of sign. This idea, in its general form, seems to be applicable to all asymmetrical [transitive asymmetrical] relations of the type involving the contradiction of relativity. It was explained, in Chapter II of the present Book, how two terms A, B, become, by means of such a relation Aβ and Bα. The difference between α and β is, I think, a difference whose meaning cannot be explained in terms of other conceptions, but is expressed, in Mathematics, by means of sign. This idea is one which appears to involve, more evidently than any of the preceding ideas, an appeal to intuition, and this is, in the Prolegomena, the main purpose which Kant makes it serve. But if, as would appear to be the case, the idea is involved in asymmetrical relations of the above type, it involves no more appeal to intuition than such relations do** [AMR, 226–7].

[83] In the typescript material for AMR, Russell points out the important difference between *ratio*, which is a purely conceptual relation (expressing a relation between adjectives?) and *difference*, which is a relation expressing a difference between terms according to the special characteristics of the terms between which it holds [AMR, 231]. Note also changes in his conception of relations in Russell's treatment of anharmonic ratio in Russell 1898.

In the typescript material for AMR, Russell insists that the account he provides of the distinction of signs is applicable, not only to quantities, but to all asymmetrical, transitive relations. In mathematics, the difference of adjectives, e.g., A's being greater than B and B's being less than A, is indicated by a difference of sign in this way: two terms A and B standing in relation R become, in reference to R, Aβ and Bα, where β expresses A's difference from B, e.g., A is greater than B, and α expresses B's difference from A, e.g., B is less than A. Analysis reveals "adjectives of the relation" or the difference in sense expressed by the distinction of signs to be involved in asymmetrical relations and no appeal to intuition is required for the explanation of such differences. The fact that the difference indicated by the distinction of signs cannot be further analyzed would seem to be a step in the direction of a doctrine of external relations, but given the stranglehold of the doctrine of internal relations, it is instead evidence that the signs cannot indicate a real difference. It was the role of Whitehead's mathematical calculus to employ signs substitutively and yet, with the distinction of sign, it was not clear what the signs were intended to signify, i.e., what their correlates were in true propositions.[84] Russell concludes that what the distinction of signs shows is that the contradiction of relativity cannot be eschewed:

[84] Whitehead had described the requirements of a mathematical calculus in the following terms: "The signs of a Mathematical Calculus are substitutive signs. …In order that reasoning may be conducted by means of substitutive signs, it is necessary that rules be given for the manipulation of the signs. The rules should be such that the final state of the signs after a series of operations according to rule denotes, when the signs are interpreted in terms of the things for which they are substituted, a proposition true for the things represented by the signs" [Whitehead 1898, 4].

> **Thus we have [in the case of all asymmetrical transitive relations] a difference without a point of difference or, in the old formula, a conception of difference without a difference of conception. This contradiction belongs, therefore to all relations of our fourth type; and relations of this type pervade almost the whole of Mathematics, since they are involved in number, in order, in quantity, and in space and time. The fundamental importance of this contradiction to Mathematics is thus at once proved and accounted for [AMR, 225–6].**

Russell's doctrine of internal relations was intractable and the corresponding search for the enigmatic "points of difference" was not easily abandoned. Though the change was radical once accomplished, the radical change was prefigured by a series of transitional steps which, in retrospect, exhibit Russell parting gradually with the doctrine between the summer of 1898 and the winter of 1899. The radical nature of the change is exemplified by Russell's revision of his *modus ponens* above to a *modus tollens* argument, so that where the original argument concludes that the contradiction of relativity is accounted for by the nature of transitive asymmetrical relations, the revised passage included in the 1899-1900 draft of PoM concludes that "[w]e cannot hope…so long as we adhere to the view that no relation can be "purely external", to obtain anything like a satisfactory philosophy of mathematics" [Russell 1899–1900, 90]. The fact that there was a shift is evident, but precisely what Russell means will require explanation. Since Russell attributes the doctrine of internal relations to Leibniz and Bradley, it might be thought that overturning their doctrine(s) was what enabled Russell to dismantle the contradiction of relativity. In what follows, I hope to show that Russell rejects both Bradley's and Leibniz's view that no relation is purely external before overturning his own doctrine of internal and that, even once he has dispensed with his own doctrine of internal relations, more gradual developments were required for his admission of the ultimate nature of relations differing in sense and irreducible to adjectives of the relation. The gradual and not straightforward changes which preceded these various revisions will be addressed in the following chapter.

CHAPTER 2: RELATIONS IN ANALYSIS

2.1 EXTERNAL RELATIONS AND THE PRIMITIVE DIVERSITY OF LOGICAL SUBJECTS: THE IMPORT OF RUSSELL'S WORK ON LEIBNIZ

In his paper "The Classification of Relations," (henceforth, CoR), read to the Moral Sciences Club in January, 1899, Russell articulates the doctrine of external relations which was required for the dissolution of the contradiction of relativity and which was supposed to clear the way for a satisfactory philosophy of mathematics. It can be reasonably assumed that the conception of types of relations articulated in CoR was conceived by Russell as an extension of the project of discovering the several kinds of necessary relations between concepts constituting propositions of various types, which, as he indicated to Moore in September, 1898, was the business of logic. In CoR, he writes:

> **I could have wished, had I been able, to give a more systematic enumeration of relations. If I possessed, as Kant believed himself to possess, a complete list of the forms of propositions, my task would be easy; for to every form of proposition some relation must correspond, and no relation can be without a corresponding form of proposition. For the present, I would urge the importance of the problem, and the desirability of completing the list. Such a list would be a real alphabet of Logic, and could hardly fail to have far-reaching consequences in metaphysics** [CoR, 138–46].

In COR, relations are given the following fairly modern classification: i. symmetrical relations, e.g. equality, simultaneity, identity of content, which are relations such that, if ArB, then BrA, and if ArB and BrC, then ArC; ii. reciprocal relations, e.g., inequality, spatial or temporal separation, diversity of content, which are relations such that, if ArB, then BrA, but if ArB and BrC, it does not follow that ArC; iii. transitive relations, e.g., whole and part, before and after, greater and less, cause and effect, which are relations such that, if ArB and BrC, then ArC, but if ArB, it is false that BrA, and iv. one-sided relations, e.g., the relation of predication and occupation of a time or place, which have none of the above properties [CoR, 138-9].[85] In contrast to Bradley's thesis that all relations are reducible to identity and diversity of content, Russell insists that (strict) identity is not a relation, since it has only one term, and that diversity is a relation and not analyzable into a pair of predicates of the related terms. Indeed, he maintains that no relation is analyzable into a pair of predicates of the related terms. Russell is explicit that where "Mr. Bradley has argued much and hotly against the view that relations are ever purely "external"" [CoR, 143], his own view is that all relations are, in the terminology he ascribes to Bradley,[86] "purely external". The argument Russell gives for the externality of the relation of predication echoes Moore's claim in his dissertation and NJ that "...no one concept [is] either more or less an adjective than any other" [NJ, 192]. Russell points out that the fact that the predicate term in one proposition may equally be taken as the subject term in another both exhibits the independence of the two terms, neither of which is more substantive than the other, and also that what is asserted is a relation between two independent terms and not the ascription of a meaning to the logical subject.[87] In his

[85] The so-called "symmetrical relations" are transitive relations. I do not quantify over the terms, since Russell does not do so in CoR.

[86] It seems that Bradley only adopted this terminology himself in response to criticism, e.g., in Bradley 1914.

[87] Russell writes: "A little consideration will show that the predicate is no more dependent on the subject than the subject on the predicate. Instead of saying 'the chair is red', we may say 'red is predicable of the chair'. The two propositions seem identical in meaning, but the second, by making 'red' the subject, brings out more clearly than the first, that what is asserted is a relation" [CoR, 141]. He had formerly held the following view: "[t]he peculiarity of predicates is, that they are meanings. Now although it is impossible to speak of meanings without making them subjects ..., yet meanings as such are the antithesis of

dissertation and NJ, Moore argues against Bradley's view that the (logical) idea is adjectival, that "[a] concept is not in any intelligible sense an adjective, as if there were something substantive, more ultimate than it…" [NJ, 193]. Once it has been admitted, for instance on epistemological grounds, that the concept or term is a logically ultimate and independently subsisting entity and itself the most general concept of metaphysics, there is no longer a metaphysical basis for the supposition that the true form of propositions is subject and predicate, but, as we have seen from Russell's early work, this alone is insufficient for the analysis of relational propositions. Russell writes:

> **When it is considered that almost all systematic Metaphysics, hitherto, has used either Substance or the Absolute, and that either, when taken as the fundamental concept of Metaphysics, implies the preeminence of subject and predicate among forms of propositions, it becomes evident how far-reaching and profound is the dependence of Metaphysics upon Logic, and how much must be reformed if a more complex doctrine of relations be admitted** [CoR, 138–46].[88]

It was in his rather involved work on Leibniz[89] in anticipation of a series of lectures at Trinity College, which he prepared at the time of writing CoR and delivered in January and February of 1899, that Russell began to give serious consideration to the metaphysical and logical doctrines which would have to be abandoned for a logical

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subjects…When I say 'Socrates is human'…I am, in a word, not asserting a relation between two subjects" [AMR, 174].

[88] See also PoM, 47.

[89] Russell's main source on Leibniz was Gerhardt's two seven-volume works, *Philosophischen Schriften* and *Leibnizens Mathematische Schriften*. Russell records reading the *Philosophischen Schriften* in February 1899 in "What Shall I Read?". He found the correspondence with Arnauld and the *Discourse on Metaphysics* to be particularly illuminating [PoL, xiii-xiv]. For secondary sources, Russell made use of Erdmann's commentary [PoL, p. xiii]. What is clear from Russell's marginal comments in the Gerhardt volumes is that Russell undertook the study of Leibniz meticulously, cross referencing passages and paying special attention to dating.

doctrine of relations to be formulated. Since the significance of the work on Leibniz to Russell's development has not received sufficient attention, I shall consider the work in some detail in what follows.

In his article, "The early Russell on the metaphysics of substance in Leibniz and Bradley," T. Allan Hillman treats Russell's work on Leibniz as a case study that corroborates his position that, contrary to the received view[90] on which Russell had merely adopted Moore's anti-idealist views, having "…had very little himself to offer before 1903", Russell had independent reasons for rejecting idealism.[91] Hillman points out that Griffin dismisses the Leibniz book summarily in *Russell's Idealist Apprenticeship*[92] and that Peter Hylton—who maintains, in *Russell, Idealism, and the Emergence of Analytic Philosophy,* that where Russell was concerned in PoL with the reducibility of relations, Moore was concerned with their mind-independence [Hylton 1990, 155] — fails to recognize Russell's independent contributions to the new realist metaphysics. Hillman writes:

[90] The reasons for this view, to the extent to which anybody holds it, are the absence of any record of the conversations in which Moore is supposed to have influenced Russell in the winter of 1898 [Papers 3, 260], the lack of a clear statement of the 'new realist philosophy' at the time it was developed, and Russell's own claims that he was influenced by Moore in purely philosophical or metaphysical matters. However, the careful study of copious pre-1903 manuscripts has certainly shown that Russell made significant contributions of his own to overturning idealism in his early mathematical and logical works.

[91] Hillman writes: "Judging from the literature, one may…come away with the suspicion that commentators have endorsed the sentiment that [Russell] had very little himself to offer before 1903…That Russell's metaphysical outlook reflected the influence of Moore at this time is beyond doubt; but little follows from this fact all by itself. One might hope to show that Russell had his own reasons—perhaps even arguments independent of Moore—for rejecting idealism" [Hillman 2008, 246].

[92] Though Griffin once claimed that the Leibniz book offered "...no very reliable guide to the development of his thought as he abandoned neo-Hegelianism" [Griffin 1991, 343], he has since written a detailed paper arguing that Russell clarified his views on relations in working on the Leibniz book. See: Nicholas Griffin, "Russell and Leibniz on the Classification of Propositions," forthcoming.

> **Both commentators, it seems to me, ignore the fact that in *POL* Russell is making a positive contribution not merely to the logical question according to which propositions either are or are not reducible to relations of subject and predicate, but also to a full-stop endorsement of realism against Bradley and others. These are, to my mind, separate issues, and the *POL* clearly demonstrates that Russell's interests at this time were as much metaphysical as they were logico-linguistic in nature [Hillman 2008, 24n6].**

On my view, the metaphysically motivated arguments given in the PoL against monadism, monism, and the doctrine of substance generally are extensions of Moore's criticism of the idealist view that the logical idea is an adjective, which, in Moore's work, supports the new realist thesis that the concept or term is the only substantive and no concept is more or less substantive than any other. What is unique to Russell is his sustained demonstration that the faulty monadist and monist metaphysics and the psychologistic doctrine of relations result from the *logical* doctrine that all true propositions ascribe a predicate to a subject, which Hylton rightly emphasizes. Insofar as Russell made an independent *metaphysical* contribution—and this is one which Hillman does not draw attention to in considering Russell's arguments against substance in PoL—it was by showing that the notion of bare numerical difference is required to escape the commitment to the incoherent monist assertion that all propositions ascribe a predicate to the Absolute. The most significant consequence of Russell's work on Leibniz was perhaps the changes it brought about in his own doctrine of relations. As Griffin argues in subsequent work, one of the changes brought about by the study of Leibniz may well have been his rejection of his own quasi-Leibnizian doctrine of internal relations, which I shall characterize as the view that relations "presuppose" corresponding intrinsic adjectives, in favor of the view, which he mysteriously attributes to Moore, that relations are ultimate, intensional, and irreducible to relational adjectives.[93] The rejection of the internal doctrine of relations dissolves the contradiction of relativity, but the doctrine attributed to Moore is essential to further developments in the analysis of relational propositions in mathematics. In the light of these considerations, it seems to me that Russell is both correct in his judgment that "On fundamental questions of philosophy, [his] position, in all its chief features, is derived from Mr. G.E Moore" [PoM, xviii], and in his judgment that he "first realized the importance of the question of relations when [he] was working on Leibniz" [MPD, 48]. I hope to show that the developments mentioned above are all deeply connected in Russell's philosophy. By rejecting the "necessity version" of the Principle of the Identity of Indiscernibles (henceforth, PII) i.e., that, necessarily, no *two* things are *qualitatively* indiscernible which, on Russell's view, holds only if the subject-predicate doctrine holds, Russell follows out Moore's criticism of the Bradleian view that the logical idea is adjectival to its conclusion. In so doing, he abandons the fundamental assumption underlying his own doctrine of internal relations on which adjectives of the relation presuppose corresponding intrinsic adjectives, which, as we have seen, produces a contradiction in the case of all asymmetrical relations, where no such adjectives are to be found. However, it is only subsequently that Russell adopts

[93] Russell consistently attributes this view to Moore [PoM, xviii], citing NJ. Just prior to articulating the "intensional doctrine of relations", he also attributes to Moore the view that relational propositions are more ultimate than subject-predicate or class propositions, also citing NJ [PoM, 24]. Though Moore states that relations are ultimate in his letter to Russell of September 11, 1898, and though the view that relations which differ in sense are differing relations is compatible with the central tenets of his new realist philosophy articulated in NJ, I do not know of anywhere where he advocates for these views explicitly, but perhaps my account of the connection between the rejection of the notion that the logical idea is an adjective and the changes in Russell's views concerning relations will make a start on resolving this interpretive difficulty.

the position that asymmetrical transitive relations marked by a difference in sense are ultimate relations, irreducible to adjectives of the relation, which exact analysis reveals to be constituents of propositions. I shall elaborate these contentions in what follows, but it will be useful to first address the most significant among those views which Russell expresses in the Leibniz book.

In the PoL, Russell famously remarks: "That all sound philosophy should begin with the analysis of propositions is a truth too evident, perhaps, to demand a proof" [PoL, 9]. On Russell's account, Leibniz's philosophy, which was "almost entirely derived from his logic" [PoL, v] began with the analysis of propositions,[94] but since Leibniz was committed to the doctrine that in every meaningful proposition a predicate is ascribed to a subject in which it is contained either explicitly or by analysis, all analysis was analysis into the true subject-predicate form of the proposition and all (*a priori*)[95] propositions were construed as analytic. Russell's main concern, in the Leibniz book, is with

[94] Russell writes: "That Leibniz's philosophy began with such an analysis, is less evident, but seems to be no less true" [PoL, 9].
[95] Leibniz's distinction is between necessary and contingent truths. Russell subsequently accepted that all propositions, including existential ones, were analytic in Leibniz's philosophy. Russell writes to Couturat: "Your citations have convinced me on the subject of the principle of sufficient reason, and existential judgments. I cited several texts in my book that are hardly capable of any other interpretation, but I could not suppose that we could take analytic judgments as contingent. For this reason, it is the citation that you give (RMM p.11, note) beginning "Ita arcanum aliquod" that finally persuaded me of the correctness of your theory" [CPLP, R23.03.02]. This is a response to Couturat's letter, in which he tells Russell to watch for the January *Revue de métaphysique et de morale*, where he publishes the most important of the "fragments inédits" of Leibniz and where he discusses Russell's interpretation [CPLP, C12.01.02]. The citation which confirms that all propositions are analytic, even if contingent, in Leibniz is: "I may have thus explained something that has perplexed me for a long time and that was not intelligible to me, to know how a predicate could be in a subject and how nevertheless the proposition could not be necessary. But knowledge of geometry and the analysis of the infinite let me see the light, and made it intelligible to me how notions can be resolved to infinity," which is a translation of the Latin passage given in Couturat 1903, 18. Cf. note 39.

dismantling the metaphysical consequences which result from the subject-predicate doctrine, that is, the doctrine that every true proposition ascribes a predicate to a subject. It is this logical doctrine, Russell thinks, which gives rise to Leibniz's doctrine that all that is real is the states of monads, supposed to be adjectives, and the individual substances which are supposed to underlie them.[96] The simultaneous perceptions of monads are needed to give psychological reality to the unity of aggregates, and monadic states are needed to *ground* relations and all that is constituted by relations, most importantly, space and time. Russell regards Leibniz as holding essentially the same view as Lotze held, on which "…relations and aggregates have only a mental truth; the true proposition is one ascribing a predicate to God and to all others who perceive the relation" [PoL, 16].[97] It is the simultaneous perception of monads, which Russell believes is akin to Kant's "unity of apperception", which synthesizes the plurality of monads and "…a collection, as such, acquires only a precarious and derived reality from simultaneous perception [and] the truth in the judgment of plurality is reduced to a judgment as to the state of every monad which perceives the plurality" [PoL, 136].[98] In all relational propositions, the true judgment concerns the states, that is, the adjectives, of monads in which relations are grounded and the relational propositions are themselves strictly meaningless. It is thus that in subscribing to the subject-predicate doctrine, Leibniz is compelled to uphold "the Kantian theory that relations, though veritable, are the work of

[96] "The ground for assuming substances—and this is a very important point—is purely and solely logical" [PoL, 49].
[97] Russell cites Lotze's *Metaphysic*, S 109.
[98] Moreover, in Leibniz's philosophy, number, strictly, cannot be predicated of an aggregate.

the mind" [PoL, 16].[99] Relations "outside the subjects" are dependent upon the psychological fact of thinking two or more things together, that is, upon *concogitabilitas* in Leibniz's terminology.[100] For the objective reality of relations, like that of eternal truths, *esse est percipi*, and God is supposed to perceive the relations both among and between individual monads and their states.[101] Since the independent subsistence of relations is denied on the subject-predicate doctrine, this view relies on the incoherent notion that eternal truths and relational propositions are the internal objects of God's understanding,[102] which, in truth, is an admission that God has knowledge of what is itself meaningless.

On Russell's account, monadic states and thus the monads in which they are supposed to inhere are introduced into the theories of space and time so that spatial and

[99] See also Papers 3, 260 for Moore's influence. Though I cannot address the matter here, it is worth pointing out that, in order to resolve the contradiction of relativity in the AMR, Russell turns to the psychology of monads to provide *grounds* for relations. Relations thus become the product of mental states or "the work of the mind", which is precisely the doctrine he charges Leibniz, Kant, and the neo-Hegelians with espousing.

[100] Leibniz writes: "A relation is the concogitabilitas of two things" [LHIV, vii, C35]. Leibniz also writes: "If as soon as a plurality of things is called into existence, then we understand that one thing only exists, those are said to be the parts, this the whole (Si pluribus positis, eo ipso unum aliquod poni immediate intelligatur, illa dicuntur *partes*, hoc *totum*.). There is no need that the parts exist at the same time or in the same place: it is sufficient that they all *are considered* at the same time…" [A VI, 4: 627], cited in Mugnai 2010, 6-7.

[101] Russell writes "…relations derive their reality from the supreme reason [which]…sees not only individual monads and their various states, but also the relations between monads, and in this consists the reality of relations. Thus in the case of relations, and of eternal truths generally, *esse est percipi*. But the perception must be God's perception and this, after all, has an object, though an internal one…"[PoL, 210] [GII 438].

[102] Russell complains that the notion that a priori truths are internal objects of our or God's understanding, "…has been encouraged by the Kantian notion that a priori truths are in some way the work of the mind, and has been exalted by Hegelianism into a first principle." Russell aptly points out that "the word *object* [suggests] what the word *internal* is intended to deny, that truths are something different from the knowledge of them" [PoL, 214] [GVI 614].

temporal relations can be reduced to pairs of attributes.[103] However, Leibniz's view that relations are both products of the understanding and, at the same time, have a derived reality in virtue of being grounded in the states of simple substances[104] introduces fundamental confusion into the theories of space and time. Russell objects that space and time must have an objective ground in the differing points of view of individual monads between which relations hold, so that there is some common object of perception [PoL, 151-2]. Regarding Leibniz's space, Russell writes:

> **[T]his ought to have been obvious to him, from the fact that there are not as many spaces as monads, but one space, and even one only for all possible worlds. The congeries of relations and places which constitutes space is not only in the perceptions of the monads, but must be actually something which is perceived in all those perceptions [PoL, 148].[105]**

Regarding Leibniz's theory of time, Russell writes: "…the relations, being between monads, not between various perceptions of one monad, would be irreducible relations, not pairs of adjectives of monads. In the case of simultaneity, this is peculiarly obvious and seems indeed to be presupposed in the idea of perception" [PoL, 153]. Russell's criticisms of Leibniz's doctrines of space and time derive from his belief that, so long as the relations constituting the space and time orders were reducible to monadic states—the eternal predicates of the underlying individual— relations could not be asserted to hold

[103] According to Russell: "relations must always be reduced to attributes of the related terms [and]….to effect this reduction of spatial relations, monads and their perceptions must be introduced" [PoL, 142].
[104] Leibniz maintains that "…although relations are from the understanding, they are not groundless or unreal. For the primitive understanding is the origin of things; and indeed the reality of all things, simple substances excepted, consists only in the foundation of the perceptions of phenomena in simple substances" [GII, 347].
[105] Russell also writes: "It would thus appear that Leibniz, more or less unconsciously, had two theories of space and time, the one subjective, giving merely relations among the perceptions of each monad, the other objective, giving to the relations among perceptions that counterpart, in the objects of perception, which is one and the same for all monads and even for all possible worlds" [PoL, 151].

between the states of differing monads.[106] In his insistence that the resolution into notions be distinguished from division into parts, Leibniz had made a start on a tenable doctrine of relations by recognizing their indivisibility, but, on Russell's account, his belief in the merely mental status of relations committed him to a "complete denial of the continuous" [PoL, 129].[107]

Whatever misunderstandings attach to Russell's interpretation, he did not attribute the subject-predicate doctrine to Leibniz without evidence. Leibniz subscribed to the doctrine that in every true proposition, not only those which assert explicit identities, the predicate is, in some sense, contained in the concept of the subject.[108] In his July 14, 1686

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[106] Here it might be pointed out that Russell ought to have distinguished the notion that the complete concept of the monad contains all its predicates from the notion that the monad contains all its predicates and that the complete concept of the monad includes the relation of its states (predicates) to the states (predicates) of other monads.

[107] Russell expressed an awareness of Leibniz's actual views, for instance writing that, according to Leibniz, "[t]he labyrinth of the continuum…comes from looking for actual parts in the order of possibles, and indeterminate parts in the aggregate of actuals" [PoL, 130]. However, his misrecognition of the entailments of Leibniz's doctrine that relations are ideal leads him to his thesis that Leibniz denies the continuous. He writes, for instance: "Distances may be greater or less, but cannot be divided into parts, since they are relations. This consequence is not drawn by Leibniz, indeed it is expressly denied, but...he says, what suffices for me, that in space and time there are no divisions but such as are made by the mind" [PoL, 132]. Interestingly, in Russell's review of Meinong's *Ueber die Bedeutung*, Russell held that Meinong was correct for regarding distance as an indivisible relation [Russell 1899b]. The review is supposed to have been written as early as September, 1898 [Papers 2, 147].

[108] In a passage from the *Discourse on Metaphysics*, also reproduced in the Gerhardt volume, Leibniz writes "Now it is certain that every true predication has some basis in the nature of things and, when a proposition is not an identity, that is to say, when the predicate is not expressly contained in the subject, it must be included in it virtually. This is what the philosophers call *in esse*, when they say that the predicate is in the subject. So the subject term must always include the predicate term in such a way that anyone who understands perfectly the concept of the subject will also know that the predicate pertains to it. [AG, 41]; [GII, iv, 433]. Russell concludes that an individual subject, for Leibniz, is the mere sum of its predicates. Russell did not (think it intelligible to) distinguish the concept of the individual from the individual. Adams argues that Leibniz held both that the predicate is contained in the complete concept of the individual and in the substance itself [Adams 1994, 71, 79]. Mugnai develops a convincing argument that Leibniz construed the doctrine of the inherence of a predicate in a subject, the doctrine of *inesse*, by analogy to space, e.g., "…just as what is shut up somewhere or is in some whole, is supported by it and goes where it goes, so

letter from his correspondence with Arnauld, which was reproduced in the Gerhardt volume and which Russell references in PoL, Leibniz articulates his containment principle: "In every affirmative true proposition, necessary or contingent, universal or singular, the concept of the predicate is included in that of the subject, *praedicatum inest subjecto*" [GII, 56].[109] Moreover, he held that all extrinsic denominations, including relations (and arguably relational properties or tropes), have (intrinsic) properties for their foundations[110] which are themselves included in the complete concept of the individual. In a passage which Russell marks, "very important" in the second of the Gerhardt volumes,[111] Leibniz writes:

> **[T]he concept of an individual substance includes all its events and all its denominations, even those which are commonly called extrinsic…For there must always be some foundation for the connection between the terms of a proposition, and this must be found in their concepts… This is my great principle…of which one of the corollaries is the common axiom that nothing happens without a reason, that one can always provide [the reason] why the thing has gone this way rather than otherwise [GII, 56].**[112]

accidents are thought of similarly as in the subject- *sunt in subjecto, inhaerent subject*" [A VI, 6, 277-8], cited in Mugnai 2010, 5.

[109] Mugnai argues convincingly that Leibniz reduced the relation of containment of a predicate in a subject to the part/whole relation, e.g., "…an accident is not in any other place or time different from where the subject is, nor is a part in anything other than the whole" [LH IV, 7B, 3, Bl. 56v], cited in Mugnai 2010, 7. Russell rejects the quasi-Leibnizian part/whole theory of predication to which Moore subscribes.

[110] "There is no denomination so extrinsic so as not to have an intrinsic one for its foundation" [GII, 240] [PoL, 242]. See also [PoL, 205]. It is not entirely clear whether, for Leibniz, relational accidents are founded on intrinsic accidents and both properties and relations are included in the complete concept of the individual or relations are founded on relational accidents themselves included in the complete concept of the individual. I cannot develop a position on this interesting question within the scope of the present work. See Cover and Hawthorne 1999 and Plaisted 2002.

[111] Russell's marginalia: 'v[ery] imp[ortan]t' and indexes 'S.R' for sufficient reason. [GII, 56]; [L, ii, 517].

[112] Also, in his letter to Arnauld in early June, 1686, which Russell indexes 'S.R' for 'Sufficient Reason' in his marginalia in Gerhardt's second volume, Leibniz maintains that "…nothing is without a reason, or …every truth is proved a priori, drawn from the concept of the terms, though it is not always in our power to arrive at this analysis" [GII, 62].

In a well-known passage from his 5th letter to Clarke, which Russell considers in PoL,[113] Leibniz offered a formulation of the three ways in which a ratio between two lines L and M may be construed. There is the ratio of the greater L to the lesser M, the ratio of the lesser M to the greater L, and the purely abstract ratio or relation between L and M, indifferent to which is the subject and which the predicate. Leibniz holds that the lines L and M cannot, together, be the subject of the accident, since then there will be "...an accident in two subjects, with one leg in one, and the other in the other," which is, on Leibniz's philosophy, "…always contrary to the notion of accidents".[114] In his letter to Des Bosses, 21 April, 1714, Leibniz stated his position on relational accidents: "…paternity in David is one thing, filiation in Solomon another, but…the relation common to both is a merely mental thing of which the modifications of singulars are the foundations" [GII, 486], [PoL, 243].[115] Leibniz held, then, both that relations are extrinsic denominations having intrinsic ones for their foundations, that all properties of an individual—its intrinsic properties and its relational properties or tropes—are contained in its complete concept, and that relations (and arguably relational properties),

[113] From GIII. 401, 266-7. See also PoM, 222.

[114] Leibniz writes: "...It cannot be said that both of them, L and M together, are the subject of such an accident; for if so, we should have an accident in two subjects, with one leg in one, and the other in the other; which is contrary to the notion of accidents. Therefore we must say that this relation in this third way of considering it, is indeed out of the subjects; but being neither a substance, nor an accident, it must be a mere ideal thing, the consideration of which is nevertheless useful" [GVII, 401]; [L, ii, 1147] as quoted by Russell [PoL, 15]. Russell quotes the passage again in PoM, 222. This scholastic doctrine is also included in Leibniz to De Volder, December 31st, 1700, beside which, in Gerhardt's third volume, Russell writes: "cf. passage on ratio in Fifth Letter to Clarke". The scholastic doctrine of accidents is captured in Aquinas's claim that "*unum accidens non potest in diversis subiecis esse*." Thomas, In quatuor libros Sententiarum, II, d. 42, q.1, ar.1.

[115] This might be better translated "...whose basis is the modifications of the individuals." See also, NE, II, xii, 3; II, xxx, 4.

well-founded on the intrinsic accidents of individual substances, are the grounds for the abstract and ideal relation that arises from thinking the two terms together. Though Russell may have failed to appreciate the complexities of Leibniz's position on relational accidents, he cites the above passages and gives an essentially accurate presentation of Leibniz's scholastic doctrine of relations, correctly attributing to Leibniz the scholastic doctrine that the same accident cannot inhere in two subjects.[116]

On Hylton's account, Russell is concerned with the subject-predicate doctrine because it has the consequence that relations cannot be construed as ultimate and extra-mental. He writes:

> **Moore's usual attitude is that it is uncontroversial that propositions contain relations, and the controversial point is whether propositions, and therefore also relations that they contain, are objective, non-mental entities. Russell, then, takes the subject-predicate view of propositions to be philosophically crucial because he identifies this view with the doctrine that relations are not real, objective non-mental entities [Hylton 1992, 155].**

It seems clear that Moore regarded propositions as being constituted by various kinds of ultimate and necessary connections, but he also held that there was nothing more substantive than the concepts (i.e., concepts and relations) constituting propositions whose objective reality supervenes on the self-subsistence of the concept—a view which Hylton later seems to endorse in his commentary on the part/whole relation in Moore's early work. While Moore rejects any view on which propositions and hence relations are mental entities, the non-mental status of concepts and relations is not dependent on the independent subsistence of propositions, but the dependence runs the other way. Just as

[116] See especially Russell's account of the relation of similarity, PoL, 141. See also POL 63, 67, 73.

the concept is the totality of its properties (also concepts), the proposition is the totality of its concepts and relations. Moore writes: "When… I say 'This rose is red'…[w]hat I am asserting is a specific connexion of certain concepts forming the total concept rose with the concepts 'this' and 'now' and 'red'; and the judgment is true if such a connexion is existent" [NJ, 179]. Moore is committed to a part/whole theory of predication and to the view that the relation of whole to part is internal.[117] Though the specific connections between terms are in some sense ultimate and intensional, it is difficult to see how, given these commitments, Moore could countenance a conception of difference without a difference of conception[118]—on his account, the relations asserted to hold between concepts in a proposition hold between these and no others, so that it is impossible to speak of *the same concepts*, standing in different relations, for such concepts would, by virtue of standing in different relations, be different concepts. Moreover, if concepts are themselves complex and capable of analysis, and the part-whole relation is internal, then the complex concept has internal relations to its parts, which might be regarded as its intrinsic denominations. Prior to his study of Leibniz, Russell, for his part, held that relational propositions are irreducible to subject-predicate ones for the reason that the properties of such propositions are not preserved in the reduction. As we have seen from

[117] Arguably, by accepting that things (concepts) are the sum total of their predicates (also concepts), Moore has accepted a refined version of the containment principle.
[118] Thomas Baldwin points out that, while Moore rejects metaphysical holism in NJ, and though it indicates the atomist pluralism he would advance soon after in his articles for J. Baldwin's *Dictionary of Philosophy and Psychology*, it is a mistaken exaggeration of Moore's position to attribute to him the view that all relations are external. He writes: "Even in the article 'Relative and Absolute' (in Baldwin's dictionary) in which Moore first explicitly attacked the concept of an organic whole and denied that all relations are internal, he maintained that having the parts it does have is internal to a whole's existence" [Baldwin 1993, 25].

his anti-psychological thesis in AMR that asymmetrical relations confer differing adjectives which are captured by the purely mathematical distinction of signs whose meaning cannot be further analyzed, Russell's commitment to the *anti-psychological* conception of the *proposition* and its analysis did not commit him to the doctrine of external *relations*. While there is no doubt that Russell believed the subject-predicate doctrine to be responsible for the commitment, on the part of all who subscribed to it, to the view that relations have a merely mental status, it is for good reason that Russell maintains that his study of Leibniz caused him to appreciate the importance of relations and not that his anti-psychological doctrine of relations caused him to realize the untenability of the subject-predicate theory of propositions.

It is also worth pointing out, in this connection, that having established relations as concepts, external and implying no corresponding intrinsic adjectives, Russell is still concerned with how relations belong to their terms. Consider the passage from COR in which he states the difficulty involved in terms of the Bradley regress, which, we shall see, turns out to be the Leibniz regress as well:

> **I must confess that the above theory raises a very difficult question. When two terms have a relation, is the relation related to each? To answer affirmatively would lead at once to a regress; to answer negatively leaves it inexplicable how the relation can in any way belong to the terms…[119] To solve this difficulty—if it indeed be soluble—would, I conceive, be the**

[119] On Russell's view, the equivalent problem for those who subscribe to the subject-predicate doctrine is the inability to answer the question: "When a subject has a predicate, is the predicability of the predicate a new predicate of the subject?" However, this is not the sort of regress that concerns either Leibniz or Bradley.

> **most valuable contribution which a modern philosopher could make to philosophy [CoR, 146].**[120]

The regress is, for Leibniz, taken as grounds for the mental status of relations. In December, 1676, Leibniz considered the status of relations taken independently of their terms and outside the subjects and concluded that the reality of relations gives rise to an infinite regress and that, *for this reason*, relations must be mere intelligible things. He wrote:

> **Suppose, for example, that there is a relation between *a* and *b*, and call it *c*; then, consider a new relation between *a* and *c*: call it *d*, and so forth to the infinite. It seems that we do not have to say that all these relations are a kind of true and real ideas. Perhaps they are only mere intelligible things, which may be produced, i.e. that are or will be produced [cited in Mugnai 2010, 1].**[121]

Leibniz's conclusion concerns the ontological status of the relations produced by the regress, but it would seem that the relations which result from thinking of two terms together, and so forth, would be groundless as well as abstract and it would be difficult to say how such relations belong to their terms. For Russell, the non-mental status of relations is simply guaranteed by the fact that they are not part of the content of ideas or any part of mental facts. Interestingly, in a note that Russell is supposed to have written around December, 1899 "Do Differences Differ" [Papers 3, 553], Russell maintains that when two terms A and B differ, 'the difference of A and B' is unanalyzable and fails to relate, but is "related to differences as Point to points [and] the relation of a specific difference to its terms is no part of the meaning of "A and B differ", though it is logically

[120] On my view, the problem of whether differences differ is a different problem than the unity problem, or the problem of how a relation relates its terms—a view I shall develop subsequently.
[121] For a detailed account of this passage, see Mugnai 2010.

implied by this proposition" [Papers 3, 557]. Russell concludes that every relation is unique to the pair of terms it relates and has a unique relation of relating them [Papers 3, 557]. Russell's subsequent treatment of this problem in PoM, in answer to the question of whether "differences differ" reveals that, at the level of the *concept (relation)*, "the difference of a and b" is indistinguishable from bare difference and that it is in the *proposition* "a differs from b" that the relation implies a relation to the terms, *ad infinitum* [PoM, 50-52].[122] The regress is not a problem for logical analysis, since it is non-vicious, that is, an infinity of terms do not constitute the *meaning* of the proposition. The trouble for Russell's view in the published version of PoM is that the twofold occurrence of relations, as relating their terms and as ultimate concepts in relational judgments, rests on the arguably psychologistic notion that the meaning of the proposition and its analysis is concerned with relations as concepts and relating relations are introduced into propositions only by the psychological fact of *assertion*. One wonders whether, by making the relation as *concept* extra-mental, psychologism in not reintroduced into the relational *proposition*. This question, though it is worth addressing, will be set aside so that I can return to a consideration of the results of the Leibniz book.

Leibniz's subject-predicate doctrine issues from his principle, conveyed in the 1686 letter to Arnauld, that in every true affirmative proposition, the predicate is

[122] While Leibniz was a nominalist about relations, his commitment to the contingency of the PII seems to entail the relations still are and ought to be, when outside the subjects, irreducible to relations or relational accidents, which is easy to see if the relation is that of bare difference, as it is in Russell's version of the regress argument.

contained in the complete concept of the individual subject.[123] Insofar as what interests us is not the subject-predicate doctrine that Leibniz actually held, but the one which Russell believed him to hold and himself rejected, it seems to me that this is rightly identified as the one found in the correspondence with Arnauld. Following Moore's pronouncement in NJ that "...a thing becomes intelligible first when it is analyzed into its constituent concepts [NJ, 182],"[124] Russell accepted Leibniz's doctrine that the analysis of concepts consists in their decomposition into simple constituents, these simple concepts being *indefinable* (for instance, the indefinables of mathematics). However, Russell rejected Leibniz's view that true propositions are demonstrated by the resolution of concepts by which implicit identities are converted into explicit ones.[125] That is, Russell rejected Leibniz's *concept containment theory of truth*,[126] on which the truths of mathematics are analytic, i.e., resolvable into identities. Leibniz writes:

> **The predicate or consequent, therefore, is always in the subject or antecedent, and this constitutes the nature of truth in general, or, the connection between the terms of a proposition, as Aristotle also has observed. In identities this connection and inclusion of the**

[123] In this, I follow Griffin, who writes that it is "…patently clear from Leibniz's statement of the containment principle" in the 14 July letter to Arnauld [GII, 56] "that it applies to all true (affirmative) propositions" and that "An immediate consequence of this is that..if there are any propositions not of the subject-predicate form then none of them is true" [Griffin, forthcoming].

[124] At least this is the view until AOG, when Russell remarks that it is a dogma that a thing cannot be understood unless it is defined.

[125] Leibniz writes: "In short, there are simple ideas of which no definition can be given; there are also axioms and postulates, in a word, primary principles, which cannot be proved, and indeed have no need of proof; and these are identical propositions, whose opposite involves an express contradiction" [GVI, 612]; [PoL, 22]. On Russell's account, propositions ordinarily taken to be analytic, e.g., 'the equilateral rectangle is a rectangle' or 'the round square is round' presuppose synthetic propositions asserting the compatibility or incompatibility of the subjects and it is to these propositions and not the ideas that the law of contradiction applies [PoL, 23-4].

[126] This is the theory that in every true statement the concept of the predicate is contained in the concept of the subject [Rodriguez-Pereyra 2010, 50].

> **predicate in the subject is express, whereas in all other truths it is implicit and must be shown through the analysis of notions, in which *a priori* demonstration consists [GVI, 608].**[127]

Russell's rejection of this view, however, can hardly be supposed in itself to constitute a contribution to the advancement of analytic philosophy, for it is an objection given on the thoroughly Kantian grounds that there must be recourse to intuition if the primitive truths of mathematics are not to be tautologous. Though Russell maintained that all sound philosophy begins with the analysis of propositions [PoL, 8][128] he still subscribes, in the Leibniz book, to his pre-logicist view[129] that the *a priori* propositions of mathematics are synthetic.[130] Russell's objection to the formalist view on which, for any *a priori* science, the mere analysis of concepts would produce the identities which constitute the axioms of that science, results from his concern that establishing the fundamental concepts or primitive axioms of a science becomes an aphilosophical enterprise when these are simply established by analysis into identities. Russell writes: "The problems of

[127] "Primae Veritates"; C 518-19, translation from Morris and Parkinson 1995, 87-8.
[128] In his July 1901 paper , "Necessity", Moore argues that 'analytic' propositions, if this is taken to mean those whose contraries are self-contradictory, involve a synthesis of the proposition in which the terms and the relation are asserted and the one in which this connection is denied. If analyticity is taken to mean the containment of the predicate in the subject, either the subject is identical with the predicate and there is no proposition, or what is asserted is a relation between predicates (compatibility), which is certainly synthetic [N, 295].
[129] Though Russell held, subsequently to his adoption of logicism, that mathematics (and even logic) was synthetic, he meant by this only that it was not derived from non-contradiction alone. In the Preface to the 1937 edition of the Leibniz book, Russell renounces the claim that the propositions of pure mathematics are synthetic, attributing his 1900 view to his ignorance of mathematical logic and Cantor's theory of infinite numbers. "The important distinction," he writes, "is between propositions deducible from logic and propositions not so deducible; the former may advantageously be defined as 'analytic', the latter as 'synthetic'" [PoL, xvii]. By this standard, which is the most useful for appreciating the turn in his thinking after he had come to appreciate Cantor, he did view mathematics as 'analytic' and not 'synthetic', in the relevant senses, after the adoption of logicism.
[130] Russell writes: "[T]he propositions of Arithmetic, as Kant discovered, are one and all synthetic. In the case of Geometry, which Leibniz also regards as analytic, the opposite view is even more evidently correct" [PoL, 25]. He arrives at this view in his 1905 paper, "Necessity and Possibility".

philosophy should be anterior to deduction. An idea which can be defined, or a proposition which can be proved, is of only subordinate philosophical interest. The emphasis should be laid on the indefinable and indemonstrable, and here no method is available save intuition" [PoL, 201-2].[131] During the course of the following year, Russell would refine his conception of the requirements of the adequate analysis of concepts and dispense with intuition, but this development was subsequent to his acceptance of the doctrine of external *relations* and, as we shall see, depends upon it.

On Griffin's account, it is the scholastic doctrine, which was shared by the neo-Hegelians and which was an unstated assumption in Russell's own doctrine of internal relations, that it is absurd that there be an accident in two subjects, which was brought to light by Russell's study of Leibniz. It was, on Griffin's view, the dismissal of this assumption that permitted Russell to reject his own doctrine of internal relations [Griffin forthcoming, 69]. Russell's interpretation of Leibniz's doctrine that relations are well-founded in intrinsic accidents and his doctrine that no accident can inhere in two subjects is reflected in his assessment of Leibniz's conception of relations of similarity. In assessing Leibniz's construction of sameness of place, Russell recognizes that the relation of similarity is merely mental in that for two terms to be in precisely the same place

[131] Leibniz includes the following, in his January 14, 1688 letter to Arnauld: "Some day, if I find leisure I hope to write out my meditations upon the general characteristic or method of universal calculus, which should be of service in the other sciences as well as in mathematics. I have already made some successful attempts. I have definitions, axioms, and very remarkable theorems and problems in regard to coincidence, determination (or de unico), similitude, relation in general, power or cause, and substance, and everywhere I advance with symbols in a precise and strict manner as in algebra."

would, on the doctrine that all relations are grounded in intrinsic accidents, require their possession of a common property, which, on that doctrine, is absurd.[132] In Russell's own philosophy, e.g., in the AMR and earlier works, and, prior to his study of Leibniz, common properties are inferred from equivalence relations.[133] True judgments of quantity, for instance, assert relations of quantitative comparison and quantity is thus an "assumed intrinsic property" [EAE, 328], common to equal quantities. The trouble is when relations require "assumed intrinsic properties" which are not common, as in the case of all asymmetrical relations. The problem is not that an accident cannot have a leg in two terms, since this is precisely the nature of Russell's irreducible relational properties which give rise to the distinct senses only in virtue of their reference to one another, but rather that such correlative relations are supposed, insofar as they assert a difference, to "presuppose a point of difference", though there is nothing apart from their mutual relations in which these differences consist.

Griffin goes on to cite Leibniz's remark to Arnauld that "…there is no denomination so extrinsic that it does not have an intrinsic denomination for its

[132] Leibniz's view is this: "[I]t may not be amiss to consider the difference between place and the relation of situation which is in the body that fills up the place. For the place of A and B is the same, whereas the relation of A to fixed bodies is not precisely and individually the same as the relation which B (that comes into its place) will have to the same fixed bodies; but these relations agree only. For two different subjects, such as A and B, cannot have precisely the same individual affection, since it is impossible that the same individual accident should be in two subjects or pass from one subject to another. But the mind, not content with an agreement, looks for an identity, for something that should be truly the same, and conceives it as being extrinsic to the subjects, and this is what we call place and space. But this can only be an ideal thing, containing a certain order, in which the mind conceives the application of relations" [Leibniz and Clark 2000, 54].

[133] "Such relations as equality or simultaneity, which confer the same adjective on both terms, do not seem to involve this difficulty [as relations conferring adjectives of relations]" [AMR, 224]. See also Russell 1899–1900, 91.

foundation" [Griffin forthcoming, 69] [134] as the first explicit statement of the doctrine of internal relations which, once brought to Russell's attention, was rejected. Russell certainly emphasized this passage, but it is not obvious what Leibniz was assuming in this remark or that Russell held the same assumption.[135] If this assumes Leibniz's containment theory of truth, on which the subject concept is supposed to include all of its predicates so that there are grounds for the connection between terms in a true proposition, then it can be pointed out that Russell rejects this Leibnizian version of the containment principle prior to his study of Leibniz. As early as *AMR*, Russell admits irreducibly relational propositions [AMR, 169], but this does not, in itself, dissolve the impediments to the logical analysis of relational propositions or prevent his doctrine of internal relations from producing the contradiction of relativity. Griffin acknowledges this earlier in the paper when he points out that, well before undertaking his study of Leibniz, Russell held only that concepts contained all of their monadic predicates and denied the reducibility of the whole class of relational propositions whose properties could not be preserved in the reduction into subject-predicate form. Griffin is not attributing to Russell a rejection of the containment principle and hence a rejection of the doctrine of internal relations, for he rightly acknowledges that Russell subscribed to the latter well after having rejected the former.

[134] The passage is reproduced in GII, 240 and PoL, 242.
[135] Russell did not give an explicit statement of his doctrine of internal relations while he held it, but in order to claim that it was this remark in particular that motivated Russell to change his doctrine, it is worth trying to clarify both Leibniz's remark and the assumptions which were and were not involved in Russell's doctrine at the time of working on the Leibniz lectures.

Insofar as the remark to Arnauld is a reiteration of the view that relations presuppose corresponding adjectives, other difficulties arise. The main point of contention in Leibniz scholarship concerns the status of relational accidents in his philosophy, particularly whether he was a realist or a nominalist concerning them, both of which positions are compatible with the scholastic doctrine of relations and properties. One interpretation is that relational accidents, for Leibniz, are the extrinsic denominations, irreducible to intrinsic ones, that are themselves contained in the complete concept of the individual and on which relations are well-founded. Another interpretation is that relational accidents are contained in the complete concept of the individual in reduced form and, in this sense, are just adjectives inferred from relations and themselves have intrinsic accidents for their foundations. The latter formulation is arguably akin to the assumption, which produces the contradiction of relativity on Russell's doctrine of internal relations, that adjectives of the relation which assert a difference presuppose a point of difference. It seems to me that the essential difficulty is exhibited in trying to reconcile Leibniz's claim that extrinsic denominations are contained in the individual with his claim that extrinsic denominations have intrinsic ones for their foundations, when one has failed, as Russell has, to distinguish the concept of the individual from the individual. If relations are ideal and relational accidents are well-founded on intrinsic accidents, it is difficult to know what could be meant by saying that relational accidents are well-founded in, but not reducible to, the intrinsic accidents of the *individual,* except, perhaps, for an ideal distinction of signs. The more plausible reading is that, while relational accidents are well-founded in the intrinsic accidents of individuals, these extrinsic denominations are not contained in the individual, but are contained in its complete concept. However, Russell, whose new realist philosophy required that an individual be a concept, did not have recourse to, or at least would likely have refused to countenance such a distinction.

Supposing that Russell believed that Leibniz's contention that every relation has a foundation is equivalent to the claim that wherever relations are asserted, corresponding predicates are presupposed, then dispensing with this Leibnizian assumption would have dismantled his doctrine of internal relations. It is clearly Russell's view in AMR, completed in July, 1898, that "[w]e cannot use the difference between [the distinct signs] α and β to supply the point of difference, for both α and β state a difference and therefore *presuppose* a point of difference" [AMR, 225]. As we have seen in considering Russell's own doctrine of internal relation—*the very doctrine that relations "presuppose" corresponding predicates*— the contradiction of relativity was preserved whether relations of difference were analyzed into relations of adjectives, adjectives with an external reference, or adjectives of the relation, not further analyzable.[136] Russell's internal relations do not depend upon the assumption that relations or relational adjectives are *reducible* to intrinsic adjectives or can be analyzed into pairs of adjectives.

[136] Recall that Russell is not a scholastic and has no problem with the possession of a common property required for equivalence relations, though he does revise his conception of the possession of a common property by members of equivalence classes as he refines his notion of classes.

Russell holds that relations imply, and are equivalent to, the (relational) adjectives in each of the terms, i.e., the "adjectives of the relation." Where these relational adjectives, inferred from the relation, have no essential reference to one another, the adjectives are the ground for the relation. Where these relational adjectives can only be expressed by reference to one another and the related terms are differentiated solely by these relational adjectives, e.g., A's "being an instant earlier than B" and B's "being an instant later than A", or A's "being a cause of B" and B's "being an effect of A", these are called "adjectives of relations" [AMR, 224]. Concerning "adjectives of the relation", Russell expressly states that these adjectives of relations are "existent" [AMR, 228] and supply differences of sense which cannot be further analyzed. It also seems, though, that these particularized relations fail to meet the Bradleian requirement for internal relations articulated by Russell in CoR, that "…a relation must make a difference to the related terms, and that the difference must be marked by a predicate which the terms would not otherwise possess" [CoR, 142]. Despite the fact that it is the adjectives of the relation alone that mark a difference between terms which those terms would not otherwise possess, they fail to make a difference to their terms and it is for this reason that the "points of difference" apart from them must be "presupposed". If anything can be said to be the foundation of relations of difference, it is the so-called "adjectives of the relation" into which such relations are analyzed and to which, if anything, they are reducible.[137]

[137] The view that all relations are external cannot depend on their being irreducible to relational adjectives, since Russell doesn't hold this view until he distinguishes "relations differing in sense" in "On the Notion of Order" in July, 1900.

However, since adjectives of relations are dependent on the relations from which they are inferred, rather than on the terms they differentiate, it would seem that the foundations of the relations are the relations themselves.

Difference of sense, expressed by adjectives of the relation, ought to have led Russell to suppose that the adjectives of the relation are not reducible to intrinsic adjectives and in no way involve intrinsic adjectives in their analysis, or, in that sense, "imply" intrinsic adjectives. Intrinsic adjectives are not "implied" in the way relational adjectives are implied by relations or imply each other, but are "presupposed" as corresponding differences in the terms, the way relational judgments "presuppose" differing terms, or a diversity of logical subjects. It would seem that intrinsic adjectives are no more the foundations of relations of difference than is the diversity of logical subjects. However, so long as Russell assumes that corresponding predicates are "presupposed", he has retained an assumption analogous to Leibniz's assumption that extrinsic denominations have intrinsic ones for their foundations. Perhaps the notion that terms are "differentiated *by* the relation" [AMR, 224] and not by any intrinsic adjectives bears some analogy to Russell's separation of numerical from conceptual diversity. After all, adjectives of relations compelled him to admit diversity not based on intrinsic properties.

In the CoR, Russell maintains that all relations are purely external, that is, do not imply corresponding intrinsic adjectives. Importantly, it is the material diversity of terms in reciprocal relations and, by extension, the status of subject and predicate concepts as distinct terms in relations of predication, that supply the model for the externality of relations. On my view, the arguments from diversity are important. It is by considering the connection between Leibniz's PII and his containment principle, and in rejecting the PII as a consequence of Moore's view that the logical idea is not (ever) an adjective, that Russell develops an argument against the view that differences presuppose a point of difference. It is clear that Russell believed there to be a non-severable connection between the PII and the containment doctrine.[138] Russell offers the following Leibnizian argument in favour of the Identity of Indiscernibles: if A and B differ, the relation of difference must be a difference from B, which entails a corresponding predicate in A, but since B does not differ from itself, it cannot have this predicate and, hence, for A and B to differ

[138] Russell indexes the following passage, from Leibniz to Count Ernst von Hessen-Rheinfels. (September) 1687, "I of I" for "Identity of Indiscernibles": "Can it be denied that everything, whether genus, species or individual has a complete concept according to which God conceives of it (he who conceives of everything perfectly), a concept which involves or embraces all that can be said of the thing? And can it be denied that God is able to have such an individual conception of Adam or of Alexander that it shall embrace all the attributes, affections, accidents and, in general, all the predicates of this subject? And finally since St. Thomas could maintain that every separate intelligence differed in kind from every other, what evil will there be in saying the same of every person and in conceiving individuals as final species, provided that the species shall not be understood physically but metaphysically or mathematically; for, in physics when a thing engenders something similar to it, they are said to be of the same kind, but in metaphysics or in geometry we say that things differ in kind when they have any difference in the concept which suffices to describe them, so that two ellipses in one of which the major and minor axes are in the ratio of two to one and in the other in the ratio of three to one, differ in kind. Two ellipses which differ only in magnitude or proportionately, and where, in their description, there is no difference of ratio in the axes, have no specific difference or difference in kind, for it must be remembered that complete beings cannot differ merely because of differences in size." Russell also makes use of *Discourse on Metaphysics*, IX: "There follow from these considerations several noticeable paradoxes; among others that it is not true that two substances may be exactly alike and differ only numerically, solo numero, and that what St. Thomas says on this point regarding angels and intelligences (quod ibi omne individuum sit species infima) is true of all substances, provided that the specific difference is understood as Geometers understand it in the case of figures" [AG, 42]. See also the July 14, 1686 letter to Arnauld.

requires a difference in predicates [PoL, 68].[139] On Leibniz's account, the PII is required by the containment principle, which is itself required by the Principle of Sufficient Reason. Leibniz writes: "there cannot be in nature two individual things which differ in number alone. *For it must be possible to give a reason why they are diverse, which must be sought from some difference in them*" [G VI. 608]. Bradley's monist assertion concerning the sufficient reason for all relations is reminiscent of Leibniz's claim [A&R, 517]. The apparent externality of relations is not evidence for their independent subsistence, but rather is evidence that they are merely abstractions from a non-relational unity which is real, for purely external relations have no grounds in the nature of the Absolute. Consider, for instance, the following passage from *Appearance and Reality*:

> **Somewhere there must be a reason why this and that appear together. And this reason and reality must reside in the whole from which terms and relations are abstractions, a whole in which their internal connection must lie, and out of which from the background appear those fresh results which never could have come from the premises [A&R, 517].[140]**

The notion that relations must have some grounds, i.e., must presuppose corresponding non-relational predicates or qualitative points of difference, is the notion that Russell rejects—the basis for his own doctrine of internal relations. On my view, Russell rejects the notion that all relations have some grounds in, or presuppose corresponding

[139]Russell writes: "…Suppose A and B were two indiscernible substances. Then A would differ from B exactly as B would differ from A. They would, as Leibniz once remarks regarding atoms, be different though without a difference" [NE 309]; [G. v, 268]; [POL, 58].
[140]Consider, also, Bradley's account of spatial terms and relations: "The terms and the relations between them are themselves mere abstractions from a more concrete qualitative unity. Neither the things in space nor their space, nor both together, can be taken as substantial. They are abstractions depending on a more concrete whole which they fail to express. And their apparent externality is itself a sign that we have in them appearance and not ultimate reality" [A&R, 577].

predicates, not because there is insufficient reason for the assumption, but because there is an argument against it. Though I am not sure precisely when he formulated the argument, the rejection of this assumption is a consequence of his consideration of the requirements of "diversity" in his arguments against substance and seems to me to be deeply connected to his and Moore's anti-Bradleian theory of meaning.

Hillman maintains that Russell's arguments against substance—both monads and the Absolute—have a metaphysical import which can be separated from the logical question of the reducibility of relations and relational propositions and which are not to be found in the philosophy of Moore. On my view, Hillman is mistaken. The arguments Russell gives against substance and the Absolute are, in all chief philosophical features, similar to Moore's objection, in NJ, to the view that logical ideas or meanings are adjectival and are ascribable to substances more ultimate than them, though Russell puts this in terms of the supposition that adjectives are of the logical nature of predicates, as Hylton stresses. What Hylton doesn't stress is that Moore was concerned with something apart from the extra-mental status of relations. One ought not to underestimate the significance of Moore's remark that "…we must, if we are to be consistent, describe what appears to be most substantive as no more than a collection of such supposed adjectives: and thus, in the end, the concept turns out to be the only substantive or subject, and no one concept either more or less an adjective than any other" [NJ, 93].[141] Where Russell

[141] Hillman appropriately cites this passage, but does so merely to show that Moore's theses were 'lurking in the background' of Russell's arguments, which, in my opinion, does not go far enough.

had wondered, in reading Bradley's *Principles of Logic* in January, 1898, whether logical meanings are *always* and *purely* adjectival, or whether all identity is merely identity of content, he has, by the time of delivering the Leibniz lectures,[142] accepted the new realist thesis that there is nothing more ultimate in the proposition than the concept or term. Where Russell makes a positive contribution—and this is what, at the same time, accounts for both the metaphysical claims that outstrip those in NJ and for the significance of his realization of the importance of relations in working on Leibniz—is in his extension of the thesis that the logical idea is not an adjective to the rejection of the PII, to which Moore still subscribes.[143] In the Leibniz book, Russell rejects the PII on the grounds that numerical diversity is logically prior to a difference in predicates. It seems that on this fundamental point, Moore was to follow in Russell's footsteps.[144] In his 1901 paper on "Identity", Moore points out equivocations in the notion of "identity in difference" employed in the claims, for instance, that there are individuals or that the world is an organic unity [I, 103]. Moore wonders, in connection with the question as to

[142] Russell's theory of terms was well worked out in AMR, completed in mid-1898.
[143] This extension of the thesis shared with Moore is non-trivial and, in some sense, has its origins in the uniquely Russellian emphasis on a term's manner of occurrence. On his theory of terms in AMR, it is their capacity for occurring as logical subjects which allows for the difference between terms to be expressed. He writes, for instance: "...there is a certain unique kind of difference between subjects, dependent on their being subjects. Redness differs from blueness, 2 differs from 3, one subject differs from another...This manner of differing would be inexpressible if we refused to regard such terms as subjects; numeration, which depends upon just this kind of difference, would be impossible" [AMR, 168].
[144] Between 1901 and 1902, Moore changes his position from the view that "…material diversity of things, which is generally taken as a starting point is only derived" [NJ, 182] to the view that particulars may differ merely numerically [Moore 1901–5, 402]. See Baldwin 1993 for a more detailed account. In I, Moore holds not only that the PII does not apply to particulars, but also that almost every universal—quality or relation—has instances to which it does not apply. Predicates are sortal, allowing for identity and individuation of what possesses it [Baldwin 1993, 49]. In PE, Moore claims that "[properties] are, in fact, rather parts of which the object is made up than mere predicates which attach to it…[They] give the object all the substance it has" [PE, 41].

whether the common properties of terms are concepts or tropes, whether numerical difference (as to subjects) is distinct from conceptual difference (as to predicates). Moore's own *reductio* argument against the claim that there is only conceptual difference is that if two things possessing a common predicate do not differ merely numerically, then each of the two will be analyzed into (i) the point of difference, (ii) the common predicate and (iii) the relation to the common predicate, and since ii and iii are identical in each, each of the two will be nothing apart from their points of difference, the points of difference alone will differ from each other, and these points of difference will be necessary to any further assertion of differences constituted by mutual relations [I, 103]. Moore writes:

> **We fancy that the uniqueness of a thing ought in every case to be capable of being expressed in some predicate...But the fact is that every predicate that we can assign does also belong to some other thing...and that the only thing that gives absolute uniqueness to any proposition is the subject [I, 120].**

In these views, Moore seems to be indebted to Russell.[145] Contrary to what Russell maintains in the Leibniz book, Moore claims that insofar as it is deduced from the Principle of Sufficient Reason, Leibniz's PII cannot be a necessary truth and consistency in fact requires that the difference between numerical and conceptual difference be acknowledged [I, 107].

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[145] See, for instance, CoR. See also Russell 1900, which Russell read to the Aristotelian Society in February, 1900 and the bulk of which appeared in the PoL.

Now it is not clear that Leibniz ought to or did in fact subscribe to the necessity-version of the PII, i.e., that, necessarily, no two things are qualitatively indiscernible,[146] but it is fortuitous that Russell thought this had to be Leibniz's view,[147] for it was his objection to this version of the PII that allowed Russell to escape Absolute Idealism and its notion of "Reality". In Russell's own philosophy, the diversity of logical subjects was supposed to be supplied by space and time, but this was not a possibility in Leibniz's philosophy for the reason that any reference to space and time was reducible to monadic states, i.e., on Russell's interpretation, adjectives in the relata. Consider the conclusion of Russell's argument against substance:

> **The substance must be numerically determinate before predication, but only predicates give numerical determination. Either a substance is wholly meaningless, and in that case cannot**

[146] Leibniz states that he accepts only the contingent version of the PII in his 5th paper to Clarke and this is of fundamental importance for interpreting his views. His principle is not that the PII is contingent, but that the PII is necessary concerning contingent things—i.e., applies to individuals, which are different from other concepts expressed in eternal truths. Leibniz's view is that "it is not possible for two individuals to exist entirely alike or differing solo numero" [July 14, 1686 letter to Arnauld]. See this letter also for the meaning of 'individual' and its relation to contingent truths. See *Discourse on Metaphysics*, IX, for Thomistic origins of the PII.

[147] Russell believes Leibniz's assertion of the contingent version of the PII in the 5th paper to Clarke results from the fact that the Principle of Sufficient Reason, from which the non-existence of two indiscernibles is derived, establishes contingent truths [PoL, 66]. Russell clearly characterizes the PII as the necessity version stated in GII.131 and, where he comes across evidence of the contingency version, dismisses it as inconsistent, e.g., in his marginalia on the correspondence with Arnauld, he writes:: "This seems inconsistent with the I of I" in reference to the following passage: "Also the notion of the sphere in general is incomplete or abstract, that is to say one there considers only the essence of the sphere in general or in theory without regard to particular circumstances, and consequently it involves absolutely nothing that is required for the existence of a certain sphere; but the notion of the sphere that Archimedes had put on his tomb is realized [accomplie] and must involve everything which pertains to the subject of this form. This is why in individual or practical considerations, which turn on singular things [quae versantur circa singularia], besides the form of the sphere, there enter in the matter of which it is made, the place, the time, and the other circumstances, which by a continual chaining finally envelop everything in the universe that follows from it, if one could pursue everything that these notions involve" [GII, 39]. See also, *Monadology*, IX: "Each Monad, indeed, must be different from every other. For there are never in nature two beings which are exactly alike, and in which it is not possible to find a difference either internal or based on an intrinsic property" [G VI, 608].

> **be distinguished from any other: or a substance is merely all or some of the qualities which are supposed to be its predicates [PoL, 70].**

As we have seen from Chapter III of Bradley's Appearance and Reality, the argument Russell gives against substance is not a new realist argument for the theory of terms or concepts any more than it is a monistic argument against the independent subsistence of individual terms and relations. It is only by rejecting the assumption that subject and predicate is the canonical form of propositions that Russell's argument becomes a *reductio*. On Russell's account, Leibniz succeeds in showing "that if subject and predicate be the canonical form of propositions, there cannot be two indiscernible substances …", which is a crucial premise of the relational theories of space and time,[148] but since the ascription of predicates to *differing* substances requires their logically prior *numerical diversity*, Russell thinks "… the difficulty is to prevent proving that there cannot be two substances at all" [PoL, 68]. If the true judgment asserting the numerical diversity of substances is one in which predicates are ascribed and there is no bare numerical difference which does not reduce to a difference in predicates, then there can be no two indiscernible substances, but, by the same token, if all differences are (or involve or imply) differences as to predicates, a judgment asserting a plurality of individual substances is incoherent. This renders intelligible Russell's subsequent claim that the doctrine of subject and predicate develops "by internal logical necessity" into Bradley's

[148] Recall that Russell also expressly holds the corollary view that relations are supposed to hold between states of substances, for the reason that the states are supposed to be of the logical nature of predicates (are adjectives) requiring substances in which to inhere.

view that every proposition assigns an adjective to the Absolute [PoM, 47].[149] Russell writes that "In the belief that propositions must, in the last analysis, have a subject and a predicate, Leibniz does not differ from his predecessors or from his successors. Any philosophy which uses either substance or the Absolute will be found, on inspection, to depend upon this belief" [PoL, 18]. That Russell's criticisms of Leibniz are derived from Moore is clear from Russell's claim that the assumption underlying Leibniz's doctrine of substance as well as his theories of space and time is the assumption that there are adjectives or, in Russell's logical terminology, predicates. Russell writes: "The ground for assuming substances—-and this is a very important point—is purely and solely logical. What Science deals with are states of substances… and they are assumed to be states of substances, because they are held to be of the logical nature of predicates, and thus to demand subjects of which they may be predicated" [PoL, 58].[150] Russell maintains that the criticisms of Leibniz's monadism are "… applicable also to Lotze, and generally to all theories which advocate a plurality of things" [PoL, 138][151] pointing out the radical inconsistency between the assertion that "there is a plurality of things" which asserts bare diversity without assigning a predicate to a subject (and cannot be reduced to propositions of the subject-predicate form), with the doctrine that all propositions assign a predicate to

[149] Hillman remarks that "Moore's account of concepts corresponds nicely with Russell's discussion of predicates. For instance, Russell observes that "it is only the predicates which give a *meaning* to [a substance]" [Hillman 2008, 60, my italics]. On the old Bradleian formulation, the logical idea or adjective gives a meaning to that which is substantive. On Moore's view, the adjective is a concept no less ultimate than any other. Russell emphasizes the logical status of the adjective as predicate-concept.
[150] It is also assumed that some terms can only be subjects and these are identified according to whether they cannot be attributed to any other term [PoL, 50], i.e., are not of the logical nature of predicates.
[151] Russell acknowledges that Lotze does not ultimately hold that there are a plurality of things, [PoL, 138n].

a subject (and relational propositions are reducible to subject-predicate ones).[152] While Bradley's monism results from following the consequences of the view that all diversity is diversity as to meanings/adjectives, he does not follow them far enough. Just as the doctrine of substance is inferred from the notion that there are adjectives or properties, it is overturned by a consideration of adjectives. The self-identity of the Absolute must also, on the subject-predicate doctrine, consist in the identity of predicates and not numerical self-identity, and the Absolute, being thus indiscernible from the sum of its predicates, cannot coherently be asserted to exist apart from its predicates.[153] Put otherwise, both the monadist notion of substance and the monist notion of the Ultimate Substance are particular cases of the dependence of substance upon attributes or upon the notion that propositions contain meanings, which are adjectival—a logical doctrine which culminates in the view that the subjects are the sum of their predicates and all diversity is diversity as to predicates.

The notion that relations must be construed as being, in the terminology Russell attributes to Bradley [CoR, 143], "purely external" to their terms develops by internal

[152] In this connection, Russell cites A&R, 29-80. In his 1901 paper, "Are Space and Time Absolute or Relative?" he levels a similar criticism at the monistic dogma shared by Bradley and Lotze that all propositions attribute a predicate to the Absolute, which is the only subject. Russell points out that the proposition "the Absolute has predicates" both presupposes diversity in presupposing that predicates exist apart from the Absolute and is itself an irreducibly relational proposition asserting a purely external connection between a predicate and the Absolute.

[153] Russell writes: "As against many substances, we may urge, with Mr. Bradley, that all diversity must be diversity of meanings; as against one substance, we may urge that the same is true of identity. And this holds equally against the supposed self-identity of Mr. Bradley's reality" [PoL, 70]. Russell held as early as March, 1898 that numerical identity is something apart from qualitative identity. In PoM, Russell claims that the assertion "there are predicates", which is not of the subject-predicate form, is logically prior to (presupposed in) the assertion "the Absolute has predicates" and that even "the Absolute has predicates" asserts a diversity of terms.

logical necessity from Bradley's view that *all diversity is diversity of meaning.* Since all meanings, adjectives, or logical ideas are concepts, all diversity that is *presupposed* in relations of difference is a diversity of concepts—the material diversity of contents.[154] In the CoR, Russell writes:

> **The assertion of diversity involves taking our concepts as terms, and not merely adjectivally. Hence, diversity is always diversity of terms, i.e. numerical diversity. It is always of the kind which is involved in saying there are two terms. But the terms which are diverse are always concepts, and the diversity is therefore diversity between meanings [CoR, 143].**

In the Leibniz book, Russell writes:

> **The view that a subject and a predicate are to be found in every proposition is a very ancient and respectable doctrine; it has, moreover, by no means lost its hold on philosophy, since Mr Bradley's logic consists almost wholly of the contention that every proposition ascribes a predicate to Reality, as the only ultimate subject… The plainest instances of propositions not so reducible are the propositions which employ mathematical ideas. All assertions of numbers, as e.g., "There are three men," *essentially assert a plurality of subjects,* though they may also give a predicate to each of the subjects. Such propositions cannot be regarded as a mere sum of subject-predicate propositions, since the number only results from the singleness of the propositions, and would be absent if three propositions, asserting each the presence of one man, were juxtaposed [PoL, 13–14].[155]**

On the theory of number in AMR, "There are three men" was an existential judgment, predicating 3 of "men there" [AMR, 197].[156] In the CoR, Russell holds that the same kind of diversity is found between existents and between concepts and that all diversity is always a diversity of terms, i.e., the kind of diversity involved in saying "there are two terms." It is the relation of diversity, which does not presuppose and cannot be analyzed

[154] Moreover, Russell approaches the view that propositions asserting identities are not propositions, identities are not relations, and there are no subject-predicate propositions, strictly speaking, but only ones which assert a relation between subject and predicate, taken as terms.
[155] Russell cites PL, 49, 50, 66. See also Russell 1900, 517.
[156] Numbers were asserted of manifolds, i.e., of extensions of concepts and the non-arbitrary meaning of number consisted solely in the notion of ratio, expressed, wherever number was predicated of a collection, in the relation of each of the parts to the whole.

into predicates, but is presupposed in predication, which is the model for the externality and irreducibility of all other relations. Russell had formerly held that "The peculiarity of predicates is that they are meanings. Now although it is impossible to speak of meanings without making them subjects ...meanings as such are the antithesis of subjects, are destitute of being, and incapable of plurality. When I say 'Socrates is human',… I am, in a word, not asserting a relation between two subjects" [AMR, 174]. However, by extension of the notion that all diversity is diversity in meaning and that diversity of meaning is a precondition for the assertion of relations, the antithesis is entirely dissolved and even judgments of subject and predicate themselves exhibit external relations of different types, e.g., in "This is red," and "Red is a colour."[157]

However, relations of order, that is, the asymmetrical, transitive relations which are crucial to mathematics present an interesting case for analysis. In EAE, published November, 1898, but written in August, Russell took propositions involving relations of identity and difference in mathematics, e.g., if A=B then B=A or if A>B then B<A to *presuppose* a diversity of logical subjects, in involving the assertion that if A had an adjective, B must have another and, at the time, believed the required diversity was

[157] It is worth noting the view Russell expressed in his December 1, 1898 letter to Moore: "…I have read your dissertation—it appears to me to be on the level of the best philosophy I know. When I see you, I should like to discuss some difficulties which occur in working out your theory of Logic. I believe that props. are distinguished from concepts, not by their complexity only, but by always containing one specific concept, i.e.—the copula 'is'. That is, there must be, between the concepts of a prop., one special type of relation, not merely some relation. 'The wise man' is not a prop[osition] as Leibniz says. Moreover, you need the distinction of subject and predicate in all existential prop[ositions] e..g., existence is a predicate not a subject. 'Existence is a concept' is not existential. You will have to say that 'is' denotes an unsymmetrical relation. This will allow concepts which only have predicates & never are predicates—i.e., things—& it will make everything except the very foundation perfectly orthodox" [RA].

supplied in intuition.[158] In the same paper, Russell remarks that the contradiction of relativity does not involve Leibniz's identity of indiscernibles, since the intrinsically identical entities, e.g., points, do not have a purely material diversity, but differ according to special relations which hold uniquely between them, e.g., the unique distances which hold between pairs of points on a line [EAE, 328n5].[159] It seems that Russell believed, on the one hand, that mutual relations supplied the relevant point of difference and, on the other hand, that they presupposed a point of difference. Hence, propositions asserting (relations of) difference presuppose the material diversity of logical subjects, but since they are asymmetrical relations and not reciprocal ones, this cannot exhaust their differences and, so far as they have mutual relations which constitute their differences beyond numerical difference, these presuppose a "point of difference". By the time of writing the CoR in January, 1899, Russell rejects this assumption, I believe, by rejecting PII. However, having rejected this assumption, he does not seem to know quite how to account for the analysis of asymmetrical, transitive relations, upon which order

[158] Russell writes: "Certain mathematical propositions, for instance that if A=B then B=A, or that if A>B then B<A, or the axioms concerning order, seem to be necessary and synthetic… [A]ll these judgments depend upon a diversity of logical subjects: they are not restricted to affirming a necessary connection of the contents; they affirm that, if A has an adjective, B must have another, or other more complicated assertions of the same type. In brief, they all depend upon relations which imply material diversity, i.e., a plurality of existent beings. If, then, these judgments are truly necessary, the possibility of several beings is also necessary; and this condition seems satisfied…by space and time" [EAE, 334].

[159] Since EAE was begun in June and completed in August [Papers 2, 323-4], he perhaps makes this remark before he has had occasion to consider Leibniz's reasons for holding the PII or its connection to the Principle of Sufficient Reason, prior to reading the correspondence with Arnauld. By this point, Russell may not yet have begun reading Duncan's *The Philosophical Works*, in August, 1898, and Latta's *Monadology and Other Writings*, in October, 1898. In the preface to the PoL, he remarks that during the time he prepared his January/February lectures on Leibniz, he was in the dark about Leibniz's reasons for holding PII and as to what he meant by the PSR before reading the correspondence with Arnauld [PoL, xxi], which he records having read in February in 'What Shall I Read?'.

depends.[160] For his account of such relations, Russell relies on considerations not exhausted by those involved in establishing that the reciprocal relation of diversity does not presuppose a point of difference. Having rejected the view that relations which assert a difference presuppose a point of difference, Russell holds, in CoR, that if asymmetrical, transitive relations were to presuppose pairs of predicates, these would need either to be constituted by those predicates' mutual relations to other predicates or else asymmetrical, transitive relations would have to hold between the implied predicate and the other predicate to which it would otherwise have a relation of mere diversity, so that either way, relations are inevitable. For instance, if the relation involved in "A is before B" were thought to imply 'A's position in time' and 'B's position in time', either the positions of each would consist in all of the mutual relations of before and after between positions or else irreducible asymmetrical relations would need to hold between positions, to establish a difference of order [CoR, 144].[161] The former is Russell's earlier view and the latter is the view which Russell accepted in adopting absolute theories of space and time.

On Russell's initial account of absolute position, there are two kinds of series: independent series in which relations between terms determine their order, and series by correlation, in which each term of the series is correlated with a definite term in an

[160] Russell cites Leibniz's *New Essays* for the distinction between relations of comparison (equality, similarity) and relations of concurrence (order, whole and part, cause and effect), but does not think the distinction sufficiently clear [CoR, 138].
[161] Also, Russell writes in "Is Position in Time Absolute or Relative," "If we hold—as do most modern logicians—that all relations are reducible to identity or diversity of content, we must condemn both theories, and wholly deny the possibility of order" [Russell 1900b, 225].

independent series, so that a collection without intrinsic order can be put into correspondence with an ordered collection, giving relations of order between terms [Russell 1899c]. Absolute theories of space and time are required because Russell, who held in AMR that position is an "assumed intrinsic property" [AMR, 222] presupposed by equivalence relations, could not conceive of how asymmetrical relations could hold between positions by virtue of the mere adjectives of relations by which they were distinguished. In abandoning the doctrine of internal relations, he jettisoned the notion that positions were constituted by mere mutual relations.[162] Asymmetrical relations hold between events in time or bits of matter in space, not by virtue of the nature of the related terms, but by correlation with the asymmetrical relations holding between absolute points and instants. Temporal relations must hold between temporal positions and not between events, as Leibniz holds,[163] so that the order relations between events which have no intrinsic order can be determined by their relation to the series of temporal positions. Simultaneity can be analyzed as having the same relation to one and the same temporal position or, what is the same, by the common property in virtue of which events bear the same relation to the same absolute position, and relations of earlier and later may hold between events by correlation with the order of moments in absolute time. On his view, spatial position, likewise, cannot be determined by the mutual distances of bodies, as

[162] This is not to say that there is any incompatibility between a doctrine of internal relations and an absolute theory of space or time. However, where in EFG the relativity of position necessitated internal relations between points for the definition of position, Russell's abandonment of internal relations grounded in intrinsic qualities seems to have led him to conclude that points cannot be distinguished by relations. Cf. note 51.

[163] In attributing this view to Leibniz, Russell cites GII, 183.

Leibniz holds, since this presupposes some specific difference in the terms, but this difference cannot belong to bits of matter, for two pieces of matter can be said to successively occupy or fail to occupy the same position without loss of identity.[164] In the 1899-1900 draft of PoM, Russell maintains that asymmetrical relations "are of absolutely vital importance for a sound philosophy of mathematics, and it is they that best exhibit the inadequacy of the traditional logic, according to which every proposition is at bottom one assigning a predicate to a subject…[T]he problem of relative or absolute position turns on this point, and as we have already seen that the problem of relative or absolute magnitude is a particular case of the problem concerning position" [Russell 1899-1900, 144].[165] The relativity of position, on Russell's view, entails the contradiction of relativity, i.e., "…a specific difference without any point in which the different terms differ" [Russell 1899-1900, 144], which can be avoided only by admitting that asymmetrical, transitive relations hold between simple terms which have a primitive identity with themselves and an immediate diversity from others. Thus, in the theories of space and time, absolute

[164] Russell conveys to Couturat (May 9, 1899) that he believes that Poincaré's argument from Dynamics for absolute position is sound. On his subsequent account, what Russell calls 'the relativity of position' makes motion impossible: if bits of matter have the distances they do in virtue of their own natures, then any change in its distances to other terms effectuates a change in its nature, so that matter must be perpetually annihilated and substituted for by another piece of matter with a different nature and thus different distances to other terms. Matter, then, must have some property in virtue of which it is related to a set of terms, i.e., a set of spatial positions, at differing moments in time. Russell writes: "If this be not admitted, motion becomes wholly arbitrary; for if A now and B then are not identical at all, there is no reason for saying A has moved to B, but we might equally well say that C or D had become B. such a view would be absurd, and would ignore the most elementary analysis" [Russell 1899–1900, 145].

[165] In the 1899-1900 draft of PoM, the absolute theories of space, time, and magnitude are intended to avoid the doctrine of internal relations involved in the 'relative' theories. In EAE, magnitudes which themselves have no quantitative properties become quantities by quantitative comparison, producing the contradiction of relativity in the same way that it is produced when spatial positions consist in the mutual distances of bodies.

points and instants supply the so-called common properties[166] according to which events or bits of matter could be said to belong to equivalence classes[167] and hence, to be at a time or occupy a position. The notion that symmetrical relations are constituted by the relation of two or more terms to a common property is a modification of the traditional doctrine, which Russell recognizes explicitly [PoM, 166]. It is in the analysis of asymmetrical relations that the traditional doctrine is utterly overturned. Once the immediate diversity of terms is recognized, and asymmetrical, transitive relations are seen to hold between these simple terms, neither implying intrinsic differences nor modifying their terms, the difficulties involved in the theories of space and time "vanish like smoke."[168] Russell attributes to Moore both his appreciation of the difficulties in the relational doctrines of space and time[169] and his recognition of the "eternal self-identity"

[166] Russell points out that these 'common properties' are in fact terms to which other terms may have the same relation [Russell 1899–1900, 94]. See also Russell 1901b, 293.

[167] Prior to embracing the principle of abstraction in 1901, Russell borrowed from Whitehead the notion that classes resulted from a property common to the terms of that class. In his *Universal Algebra*, Whitehead defines the notion of a manifold as "a collection of terms having the kind of unity and relation which is found associated with a common predicate" (common property) [Whitehead 1898, 16].

[168] In PoM, he states his position clearly: "[the difficulties found] in the nature of space…seem to have been derived almost exclusively from general logic. With a subject-predicate theory of judgment, space necessarily appears to involve contradictions, but once the irreducible nature of relational propositions is admitted, all the supposed difficulties vanish like smoke" [PoM, 455 and 455n]. In his 1901 paper "Is Position in Time and Space Absolute or Relative?", Russell considers Lotze's argument, which he credits Leibniz with having introduced, that points are exactly alike, that is, on Leibniz's Identity of Indiscernibles, have no differing predicates, though their mutual distances must differ and each relation which holds between a pair of points must be particular to that pair. On Russell's view, this argument depends upon the subject-predicate doctrine, which obscures the manner in which two simple terms differ immediately, i.e., numerically, which does not depend upon a difference in predicates, but is prior to it [Russell 1901, 313].

[169] In "Is Position in Space and Time Absolute or Relative?" Russell writes, "The logical opinions which follow are in the main due to Mr. G.E. Moore to whom I owe also my first perception of the difficulties in the relational theory of space and time" [Russell 1901, 272 n8].

and primitive diversity of terms, which precludes the notion that relations must modify, or make a difference to their terms [Russell 1901, 279].[170]

Interestingly, in his 1899 paper "Axioms of Geometry" (henceforth, AOG), which was designed as a response to Poincaré's reply to EFG [Poincaré 1899], Russell maintains that while, in mathematics, an object is defined by its relations to some other known relation of terms,[171] a philosophical definition is concerned with *conceptual analysis*, which cannot consist in its relations to other terms.[172] Hence, a term's relations are not included in its meaning, nor, therefore, in its analysis. When Russell, who holds that any term or concept has a primitive identity with itself and a primitive diversity from other terms, decides that the meaning of a concept must not involve its relations, he must mean that it does not involve all of its mutual relations, which, on his earlier view, were supposed to constitute the differences of sense marked by the distinction of sign. In "Is

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[170] Russell refers reader to NJ [Russell 1901, 279n9]. See also the May 1901 draft of PoM [Russell 1901c, 189n2] where Russell refers reader to Moore's article on "Identity" [I].

[171] Russell writes: "We will correctly admit that a term cannot be usefully employed until it signifies something. Its signification can be complex or simple. In other words, either it is composed of other significations, or it is itself one of these ultimate elements that constitutes the other significations. In the first case, we define the term philosophically in enumerating its simple elements. But when it is itself simple, no philosophical definition is possible. The term can still have a particular relation with some other term, and can thus have a mathematical definition. But it cannot signify this relation, and consequently the mathematical definition becomes a theorem" [AOG, 410]. See also the French text, « Sur les axioms de la geometrie » in Papers 2, 446-450.

[172] Russell objects to Poincaré's claim that where an object possesses two properties, A and B, if it can be defined by A, completely independently of B, then B will not constitute a definition, but a theorem. If the properties A and B are elements of the concept of the object, they will both be necessary to the definition, and if they are relations to other concepts, neither will be necessary and they will both be theorems. On Russell's view, mathematics holds the prejudice that a term cannot be understood unless it is defined, but if the sense of a term A is understood as a function of B, and the sense of B as a function of C etc, there will need to be some terms which are indefinable or else a vicious circle will be introduced into the definition of terms, e.g., in the proposition 'the straight line is determined by two points,' the terms 'point' and 'straight line' are supposed already to be understood and it is between indefinable terms that relations are supposed to hold.

Position in Time Absolute or Relative?", given as a lecture to the Moral Sciences Club on May 6, 1900, Russell takes issue with the PII and asserts the following:

> **The difference between A and B in virtue of which they are two, must be prior to any difference of relation or adjective, since differences in these respects presuppose two differing terms. Thus red and blue, or 2 and 3, or identity and diversity, or any other pair of simple terms, are primarily distinguished in and for themselves, and are only subsequently found to have different relations. Similarly, the moments of time, being simple, are not susceptible of an analysis which shall reveal differences, but are themselves simply and immediately different, in the same kind of way as red and blue are different. After the moments have been distinguished, they can be seen to have different relations to the qualities exiting in them; but this cannot be the ground of the distinction, since the distinction between simple terms, being itself the ultimate ground of other distinctions, can never have any ground whatever [Russell 1900b, 232].**[173]

The relative diversity constituted by mere mutual relations implies immediate diversity, and to construe the diversity of two things as defined by (analyzed into) their mutual relations, involves a regress.[174] However, it remains to clarify the status of irreducible relations of order involving the distinction of signs—greater and less, before and after, right and left—and the manner in which they are to be analyzed. In the case of all those relations characterized by the distinction of signs, Russell neither maintains that these relations are ultimate or irreducible to adjectives of the relation nor that these are revealed, in decompositional analysis, to be constituent concepts of the irreducibly relational propositions asserting transitive asymmetrical relations. This would require his unequivocal adoption of the notion that the distinct senses of relations of order were in

[173] Russell condemns the doctrine of the 'floating adjective' on the grounds that if what is predicated is anything at all, then predication is a relation. This is taken as the basis for admitting relations of other types, as in CoR. In "The Notion of Order and Absolute Position," 1901, Russell tells us that "Complex terms, it is true, have differences which may be found by analysis... [but] the source of every mediate difference must be found in immediate differences" [Russell 1901d, 255].
[174] Reported in *Revue de métaphysique et de morale*, 8 (1900): 561-3, from Russell's lecture at the Paris Congress 1900 [Papers 3, 236].

fact different relations. The substantive aim of this chapter was to shed new light on how to construe the role which Russell's work on Leibniz played in arriving at his account of external relations and it was argued that the arguments which Russell leveled against Leibniz's PII in favour of the primitive diversity of logical subjects provided the basis for his first articulation of external relations and extended his new logic beyond the central tenets of Moore's new realist position, though it was the logical culmination of Moore's rejection of the Bradleian thesis that meanings are adjectival. On the doctrine of relations which Russell attributes to Moore, relations are not merely external and irreducible to adjectives of the relata, but are also intensional—a view which Russell arrived at subsequently to his view that relations are external and which informed his logic of relations. It remains to account for how Russell's commitment to this doctrine came about, which will prove important to understanding the conception of the analysis of the propositions of mathematics to which Russell subscribed as he arrived at logicism.[175]

2.2 THE INTENSIONAL DOCTRINE OF RELATIONS AND THE PHILOSOPHICAL APPROACH TO ANALYSIS

Working on Leibniz may have encouraged Russell to dispense with the view that relations of difference presuppose corresponding predicates, but it did not suffice for his adoption of the view that relations differing in sense were differing relations revealed by analysis to be the ultimate constituents of propositions. It was this doctrine which would

[175] Without an understanding of the origins of these developments, it might be difficult to understand the intensional logic of relations at work in the definition of mathematical concepts or Russell's early attempt to dissolve the Contradiction by means of relations taken in intension.

sweep away all traces of the contradiction of relativity. Since this intensional doctrine of relations is crucial to the logic of relations by which his early logicism is achieved in the various branches of mathematics, it is worth being clear about what that doctrine was. In AOG, which attempts a rigorous axiomatization of the foundations of geometry independent of Kantian intuition, Russell attempts an analysis of fundamental spatial concepts, beginning with the point.[176] Points are, on Russell's account, the simple and indefinable terms of relations which characterize geometrical figures that do not have proper parts (e.g., the straight line and plane), and are related to the simple and indefinable concept of the class 'point' "…to which particular points are related like red and blue are related to the concept of colour" [AOG, 412], i.e., by the relations of membership in a class. Importantly, there are, between points, fundamental and indefinable relations of distance and direction.[177] Distance is an indivisible relation without distinction of signs[178], i.e., the distance of A to B is the distance of B to A, while (distance with) direction, which requires the preservation of order between terms, is a relation which differs as to sense, i.e., the direction of A to B differs from the direction of B to A.[179] The order of points requires the complex notion of distance in a direction,

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[176] In FIAM Russell adopts the anti-Kantian position that space does not require intuition any more than arithmetic, which he plans to defend in Part IV; Chapter IV [Papers 2, 261].
[177] Russell writes: "[Direction] is the same as the projective straight line. It is not direction in the Euclidean sense, where two lines can have the same direction, but in the only sense which does not entail the parallel axiom, which is to say that where two distinct straight lines never have the same direction" [AOG, 413].
[178] Recall that Russell commended Meinong, in his 1899 review, for emphasizing the indivisibility of relations in Russell 1899b.
[179] Together, distance and direction supply the notion of distance on a straight line. Russell writes: "The projective straight line is the relation of the given direction; the metrical straight line is the class of points of

where distance is taken positively in one sense and negatively in the other [AOG, 433]. In FIAM, Russell holds that relational propositions that require the preservation of order between terms, e.g., 'A is greater than B,' in which A and B cannot be interchanged [FIAM, 295] , involve single concepts, e.g., A's excess over B (A-B), whose meaning cannot be preserved in analysis, but consists in its relation to the class-concept 'excess' [FIAM, 295]. In his "Note on Order", Russell adopts the view that a relation, considered apart from its terms is, in the analysis, something distinct from the relation insofar as it proceeds from one term to the other term[180] and from the other to the one for its converse, e.g., the distance between A and B is different from the distance from A's relation of greater than to B or B's relation of lesser than to A, and the same is true for any relation R, distinct from its senses R_1 and R_2.[181] In the 1899-1900 draft of PoM, Russell puts forth the view that asymmetrical relations, crucial to mathematics, may be considered abstractly, and independently of the two senses, expressed by the distinction of signs, which correspond to the differing adjectives which they confer upon the related terms.[182] The distinct senses expressed by signs make it clear that relations cannot be reducible to the intrinsic nature of the relata. However, it is not clear whether propositions asserting

which any two have this relation between them. Thus the points that form a straight line are distinguished from other points, not by their intrinsic nature, but by their mutual relations" [AOG, 414].

[180] The analogy is to a vector, which gives a relation in a direction.

[181] Russell 1898, 353–5.

[182] In connection with Leibniz's famous example, this amounts to the view that the abstract relation between L and M is a signless quantity or relation measurable in terms of distance or divisibility, while L's property of being greater than M and M's property of being less than L are the senses, expressed by signs, by which the relation confers properties on L and M respectively.

asymmetrical relations contain one relation with two senses or two distinct relations.[183] It is this question which Russell would address in his paper on "The Notion of Order," written just prior to his discovery of Peano.

Order, which had rightly come to predominate over quantity, rested on the hitherto obscure notion of relations differing in sense. In "The Notion of Order", completed in July, 1900, Russell concludes that the correlated pair of asymmetrical transitive relations differing in sense, e.g., A is greater than B and B is less than A, are different relations, attributing this view that the two senses of the relation expressed by the distinction of sign are two distinct relations to Moore and referring the reader to NJ. We have seen that the contradiction of relativity held despite the admission of adjectives of the relation, not further analyzable. It is only in accepting that relations involving a difference in sense are distinct relations that the last vestige of the contradiction of relativity goes out of his philosophy. Concerning the inadequacy of Leibniz's account of relations and, evidently, his own earlier account of asymmetrical transitive relations differing in sense, Russell writes:

> **[T]he inadequacy of this account is evident when we consider that α involves reference to *A*, and β involves reference to *B*. If, as in Leibniz's instance, *A* and *B* are two magnitudes of which *A* is the greater, then α is "less than *A*", and β is "greater than *B*". But these are not simply adjectives of their terms: they are analyzable, respectively, into *less* and *A*, *greater* and *B*. Thus the abstract relations *less* and *greater* remain necessary, and instead of having, in α and β, mere adjectives of *B* and *A*, we have in each relations to *A* and *B* respectively. Thus the relational form of proposition must be admitted as ultimate: *greater* and *less* must be regarded as two distinct relations, of which it is significant and true to say that, if one holds between *A* and *B*, then the other holds between *B* and *A*. *A* is not intrinsically greater,**

[183] Or, as Russell puts it, "distinct relations with a relation of difference to sense" [Russell 1901b, 300].

> **nor *B* intrinsically less...Thus unless relations be admitted as ultimate, we arrive at a definite contradiction...In short, the [A and B] have no difference of adjective, but only the immediate difference which consists in the fact that they are diverse terms[184]....We have thus a difference between *A* and *B*, namely that expressed by [the adjectives of the relation] α or β, but we have no corresponding point of difference. We cannot use the difference between α and β to supply the point of difference, for both state a difference, and therefore, *on the traditional logic*, presuppose a point of difference. We must, in fact, have a difference between *A* and *B*, without any corresponding point in which they differ... Only when relations are accepted as ultimate, and allowed to be what is called "external", does this cease to be a contradiction [Russell 1901b, 299].[185]**

In accordance with the requirement of part and whole analysis that the proposition contain its constituents and not involve, in its meaning, whatever is not among its constituents, Russell points out that A's relation of greater than to B involves "greater than" and "B" and so cannot be reduced to a relational adjective, while the proposition "A is greater than B' does not have "less" as a constituent and so must be a different proposition from "B is less than A," which does not have "greater" as a constituent and in which B's relation of "less than A" to B contains "less" and "A" and so a relation with a distinct sense and not a relational adjective. Russell has thus become clear in his position that those relations which differ in sense are what he calls differing "correlative relations"[186] and that an exact analysis of such relational propositions requires that the

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[184] In the complete passage, Russell writes: "This argument may be put generally as follows. Let two terms A and B have an asymmetrical relation R, which is to be expressed (if possible) by the adjectives β and α, where β has a reference to B, and α to A. Neither α nor β can be expressed without this reference, and they differ in content, not only by referring to A and B respectively, but also by having different senses. A and B, considered without reference to R, have no difference of content corresponding to α and β, though either α or β alone may be considered as expressing a difference between A and B. In fact, α gives to B the adjective of differing from A in a certain manner, while β expresses the same difference with A as starting point. We have thus a difference between A and B, namely that expressed by α or β, but we have no corresponding point of difference" [Russell 1901b, 299].

[185] See also Russell 1901b, 299 n12, where Russell notes that he was led to abandon the old theory of relations, to which he still subscribed in RNQ, by Moore, referring the reader to NJ.

[186] Russell writes: "We thus see that the difference of sense, or, speaking generally, of sign, is a fundamental and unanalyzable logical fact, which is the source of order and series. Some, if not all, relations, other than

difference be preserved in analysis. Though analysis did not progress very much once the ultimate status of relations differing in sense was admitted, it would significantly inform the uses to which Russell would put the new symbolic logic after discovering it from Peano and the logic of relations with which he would supplement the Peanistic logic.

It is worth remarking briefly on the short-lived significance attached to the notion of order in Russell's philosophy of mathematics immediately prior to his discovery of Peano. In "On the Notion of Order", Russell points out the difference between stretch and distance, construing the former as all of the intermediary terms interpolated between two fixed ones, and the latter as the magnitude of a relation between two terms in a series in which the relations are magnitudes, that is, if two relations differing in sense denoted by R_1 and R_2 are magnitudes, then AR_1B and BR_1C not only implies AR_1C, but also that the distance of A to C is greater than that from A to B or B to C and the same is the case for R_2. For instance, if R_1 is earlier than B and R_2 is later than A, then R_1 will be less earlier in AR_1B and BR_1C than R_1 in AR_1C, and R_2 will be less later in AR_2 B and BR_2C than in AR_2C. In the theory of magnitudes, order is produced by relations of greater and less—and this applies to all distances which are magnitudes and hence to the relations between

diversity, are such that, if one of them holds between A and B, then it is not the same relation, but a correlative one, that holds between B and A. It is plain that, if A is related to B, B is also related to A; hence, if B does not have to A the same relation that A has to B, then B must have a relation to A which is correlative to that of A to B. The difference between these correlative relations is the difference of sense" [Russell 1901b, 299].

terms in most series.[187] In his 1899 review of Meinong's *Über die Bedeutung des Weber'schen Gesetze,* Russell remarks that the most important insight generated by Meinong's work on Weber's law is that "… the dissimilarity of two measurable quantities [*Grossen*][188] of the same kind may be regarded as measured by the difference of the logarithms of these quantities" [Russell 1899b, 251].[189] In time, distance, which is the magnitude of priority or posteriority between instants generates order, though the instants between two instants A and B can be measured by stretch. In space, distance, which does not provide distinct senses does not confer order, and direction is required, i.e., if points AB and BC have the same sense (e.g., to the left of) or if BA and BC have opposite senses (e.g., to the left of, to the right of respectively) then B is between A and C. In projective geometry, the order cannot be assumed by direction, but must be proved subsequently to the construction of a relation between four points, the anharmonic ratio

[187] Russell's theory of quantity just prior to his adoption of logicism shows, as does his earlier theory of quantity, that he is not interested in the arithmetization project. This is consistent with Gandon's view [Gandon 2008, 9].

[188] It is curious to find Russell translating 'Grossen' as 'quantity', not only because this is inconsistent with his translation of 'Grossen' as 'magnitude' throughout PoM, but more importantly, because the correspondence with Couturat shows that as early as May, 1898 Russell had accepted Couturat's remark of May 4th, 1898: "...le mot quanitity doit se traduire par grandeur, et au contraire magnitude par quantité."

[189] Russell, in his 1899 review, rejects the notion that magnitudes of certain kinds, e.g., sounds, pitches, pleasures, are capable of numerical measurement and does not countenance any comparison between magnitudes of differing types, but has a certain affinity for Meinong's view that certain kinds of relation, namely, relations of dissimilarity are indivisible magnitudes that can be measured in terms of their correlation with divisible magnitudes or, more precisely, that distances can be measured in terms of the logarithms of their correlated stretches. The divisibilityof infinite wholes is not measured by cardinal numbers but derived from relations. In the same way that Russell rejected Riemann's notion of space as a numerical manifold, he rejected the reduction of magnitude theory to arithmetic. Unlike Burali-Forti, however, who regards Pieri's definition of operations on magnitudes satisfying the Peano axioms as reason to think that numbers and arithmetic are derivative from the theory of magnitudes, Russell regards the two branches of mathematics as separate, before and after his embrace of logicism. For additional commentary, see Gandon 2008.

(cross-ratio). Russell notices that distance, however, seems extraneous, since the order of points supplied by direction is definite and all measurement can be carried out in terms of stretch.[190] In the theory of whole and part, the relation of the whole to its simple parts is its magnitude of divisibility, which is akin to distance. In the theory of number, ratio is the intensive magnitude according to which a relation holds between two integers and since one ratio is greater or less than another, they are akin to distances. Since equal ratios are correlated with equal fractions, ratios can be measured by logarithms of the corresponding fractions. Russell, at this time, has begun to regard order, rather than quantity, as the fundamental category of mathematics and it is essential to Russell's early arithmetization project that number not be abstracted from quantity, but defined in terms of ratio or order within series, quantity being derivative. The numbers, themselves, are indefinable entities and this early programme of the arithmetization of mathematics is not what lies at the foundation of the logicist project adopted subsequently to the 1899-1900 draft of PoM.[191] However, in both the theory of whole and part and the theory of number, there are few innovations introduced by the notion of order alone. Though the notion of relations with sense would remain important, the connection between order and number

[190] In PoM, Russell's view was that quantity was not properly part of pure mathematics. As for metrical geometry, which he formerly held to be an extension of projective geometry, introducing quantity above the qualitative comparisons of projective geometry, he now holds that it is not necessary that distances be magnitudes, but only that they form a series with certain properties [PoM, 408-9], such that the relations between every pair of points are numerically measurable. Indeed, the notion of distance in metrical geometry could be dispensed with in favor of the magnitude of divisibility of the corresponding stretch, which suffices for measurement, but magnitude of divisibility is not a logical concept.
[191] I shall address this in greater detail in a subsequent section.

was to change significantly with the embrace of symbolic logic and the introduction of Russell's logic of relations.

Although the most significant consequence of Russell's embrace of Peano's symbolic logic was its contribution to the "true logical calculus" which would revolutionize Russell's conception of number, allowing him to venture a definition in his 1901 "On the Logic of Relations", it is the views on whole and part which would be impacted first, initiating the gradual decline of the part/whole theory which would be accelerated by the appearance of a version of the contradiction intimately connected with it. Russell's logic of relations and its consequences for his logicist project will be the topic of the next section, but in order to appreciate the commitments underlying Russell's logicist project, it will be important to continue to trace those developments which form the basis of Russell's commitments concerning the connection of logic and mathematics. Since logical analysis prior to the discovery of Peano was essentially part/whole analysis, it is worth considering briefly. In PoM, Russell writes:

> **For the comprehension of analysis, it is necessary to investigate the notion of whole and part, a notion which has been wrapped in obscurity—though not without certain more or less valid logical reasons—by the writers who may be roughly called Hegelian [PoM, 137].**

Part/whole analysis was initially adopted by Russell and Moore as part of the break with the neo-Hegelian thesis that the universe was an organic unity and that any conceptual divisions introduced did not correspond to real divisions.[192] Moore regards the internal

[192] Moore illustrates his position on organic unities, which typify the internal relation of part to whole, by pointing out that an attached arm has the properties it has in virtue of the relation it has to the other parts

relation of whole to part as admissible, since the complex whole consists in the parts and their arrangements.[193] Though the part/whole relation is internal, it is of the part itself that we assert that it belongs to the whole, and not of the part together with the predicate of belonging to the whole. He writes:

> **When we think of the part *itself*, we mean just *that which* we assert, in this case to *have* the predicate that it is part of the whole; and the mere assertion that *it* is a part of the whole involves that it should itself be distinct from that which we assert of it. Otherwise we contradict ourselves since we assert that, not *it*, but something else—namely it together with that which we assert of it—has the predicate which we assert of it [PE, 33].**

Russell held that wholes could be uniquely analyzed into simple parts and that real divisions corresponded to these conceptual divisions.[194] In PoM, Russell writes:

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which make up the body and which the arm, severed, would not have. He writes: "...those properties which are possessed by the living, and not by the dead, arm, do not exist in a changed from in the latter: they simply do not exist there at all. By a casual necessity their existence depends on their having that relation to the other parts of the body which we express by saying that they form a part of it. Yet, most certainly, if they ever did not form part of the body, they would be exactly what they are when they do" [PE, 34-5].

[193] Moore writes, for instance: "...we might think just as clearly and correctly about a horse, if we thought of all its parts and their arrangement instead of thinking the whole" [PE, 8]. The supposition that the relation of having the parts it does is internal to the existence of a whole causes the problem that propositions include existent particulars as parts though propositions themselves do not exist, which introduces difficulties into the view that a proposition is akin to a state of affairs into which constituents enter directly. According to Baldwin, Moore holds this view from 1899-1900, including in his 1900 paper, "Necessity" [Baldwin 1993, 48].

[194] The part/whole analysis gives rise to two antinomies which ought to be mentioned, though one is quickly dissolved and the other not very important to Russell. In the FIAM, Russell had not yet been persuaded of Cantor's theory of infinities and believed that the notion of totality, applied to number, gave rise to the contradictions of infinite number or of "the number of numbers." On Russell's account, "this arises most simply from applying the idea of a totality to numbers. There is, and is not, a number of numbers. This and causality are the only antinomies known to me" [Papers 2, 267]. By the time he writes his 1899-1900 draft of PoM, Russell has understood Cantor's conception of infinity. In FIAM, part/whole analysis gives rise to the antinomy of causality, which leads Russell to conclude that "...given a causal pair of terms, we cannot be sure they have any relation at all" [FIAM, 295n]. In his book on Leibniz, Russell has no satisfactory solution to the antinomy of causality, which he believes no theory of dynamics can escape. He writes: "If we do not admit…particular causes, every part of matter, and therefore all matter, is incapable of causal action and Dynamics….becomes impossible. But…a sum of motions, or forces, or vectors generally, is a sum in a quite peculiar sense—-its constituents are not parts of it…Thus no one of the constituent causes

> **A distinction is made, in support of organic unities, between conceptual analysis and real division into parts. What is really indivisible, we are told, may be conceptually analyzable. This distinction, if the conceptual analysis be regarded as subjective, seems to me wholly inadmissible. All complexity is conceptual in the sense that it is due to a whole capable of logical analysis, but is real in the sense that it has no dependence upon the mind, but only upon the nature of the object. Where the mind can distinguish elements, there must *be* different elements to distinguish [PoM, 466–7].**[195]

Russell began to overcome obscurity of part/whole analysis before the break from idealism was complete. The analysis into part and whole was, in the AMR, connected with Boole's logical calculus, which Russell adopted from Whitehead. As Russell tells Couturat on July 18, 1898, "[w]hole and part…form the category on which the logical calculus rests" [CPLP, R18.07.1898].[196] The part/whole relation, importantly, does not presuppose arithmetic [Russell 1899–1900, 35–8].[197] Wholes may be divided in different ways into mutually exclusive parts and the sum of the parts (addition), when the whole is

ever really produces its effect: the only effect is the one compounded, in this special sense, of the effects which would have resulted if the causes had acted independently" [PoL, 116]. He reiterates the difficulty in the 1899 -1900 draft of PoM, considering the case wherein some causal effect results solely from the effects of the parts, but the whole is a new term and the compounded effects from which it results are not properly parts of it. He writes: "I will illustrate this difficulty by the case of gravitating particles. Let there be three particles, A, B, C. We say that B and C both cause accelerations in A…The effect which they produce as a whole can only be discovered by supposing each to produce a separate effect: if this were not supposed, it would be impossible to obtain the two accelerations whose resultant is the actual acceleration. Thus we seem to reach an antinomy: the whole has no effect except what results from the effects of the parts, but the effects of the parts are non-existent" [Russell 1899–1900, 169].

[195] For a similar characterization of idealism, see also Russell 1899–1900, 39, 96.

[196] Also: "This Algebra may, therefore be regarded as the Algebra specially applicable to whole and part and as strictly coordinate with Arithmetic, which it nowhere presupposes. Its essence is contained in the fact that, given any two units, there is always one definite unit which is the whole composed of them, or, as we may say, their common whole, and also there is (if we include 0) one definite unit which is their common part". Russell adds that 0 is the common part of two wholes which have no common part, and the Universe is the whole of which everything is a part [Russell 1899–1900, 35–8].

[197] See also Whitehead 1898, Bk. I, Ch. I: "…[T]he laws of Algebra, though suggested by Arithmetic, do not depend on it. They depend entirely on the convention by which it is stated that certain modes of grouping the symbols are to be considered as identical. …the laws regulating the manipulation of the algebraic symbols are identical with those of Arithmetic. …If the laws be identical, the theorems of the one science can only give results conditioned by the laws which also hold good for the other science; and therefore these results, when interpretable, are true."

divided one way, can be equated with the sum of the parts divided another way. Every whole is also the common part of differing sets of wholes and the common part of one set of wholes (multiplication) may be equated with the common part of another set [Russell 1899–1900, 38]. In FIAM, Russell retains the view that symbolic logic is the calculus of whole and part and the work is concerned chiefly with the part/whole analysis of propositions, where the relation of whole and part is the indefinable relation of inclusion [FIAM, 266]. Material implication is also to be analyzed in terms of one thing's being a part of another,[198] that is, the inclusion of the consequent in the antecedent. The indefinable part/whole relation[199] is, in the 1899-1900 draft of PoM, to be distinguished from implication [Russell 1899–1900, 35–8], though it involves logical priority,[200] which is defined in terms of the indefinable relation of implication: proposition p is logically prior to proposition q if proposition q implies proposition p but proposition p does not imply proposition q.[201] If the relation of part to whole involved only implication and not logical priority, then to establish the asymmetrical relation of part to whole would require

[198] As in PoL, propositions ordinarily thought to be analytic involve the part/whole relation by which unity, akin to the unity involved in numeration or the assertion of a whole, is conferred on the constituents of the subject [PoL, 26].
[199] Russell distinguishes wholes which are aggregates, the wholes common to mathematics specified by enumeration of their parts, and those which are unities, e.g., propositions, which are not specified by the enumeration of their parts, e.g., 'A' 'greater than', and 'B'.
[200] Moore attempts to clarify the notion of Logical Priority in his July, 1900 paper, "Necessity". Propositions which are presupposed by, involved in, or imply others are 'logically prior' and, in this sense, necessary. The necessity of a proposition increases to the degree it is involved in other propositions (presupposed in them) and the necessity of connections (of implication) between propositions consists not in the fact that the connection necessarily holds, but in the fact that, if it holds, the truth of what is implied does follow necessarily from the truth of what implies it [N, 303].
[201] In PoM, Russell writes "a proposition is "more necessary" if it is logically prior to more propositions. See also PoM, 454, where Russell refers to Moore's N. Logical priority means "one proposition is presupposed, or implied, or involved in another" [PoM, 300].

a relation between two wholes, 'q implies p' and 'p does not imply q', where neither is a part of the other. Nevertheless, it was not this nuance, but the very notion that the logical calculus is based on the relation of inclusion between part and whole which would be disrupted by Russell's discovery of Peano's symbolism, which, according to Peano, was "capable of representing all the ideas of logic, so that by introducing symbols to represent the ideas of the other sciences, we may express every theory symbolically" [SW, 190]. For Russell, the discovery of the works of Peano and his school was, as we have noted, tantamount to the discovery of "the true logical calculus" and the immediate changes in Russell's views are reflected in marginal comments which Russell added to the 1899-1900 Draft. Peano's distinction between membership (ε) and (universally quantified) material implication made it clear to Russell that the part/whole relation of inclusion is distinct from implication,[202] and that whole is distinct from class and is not involved in the logical calculus. Peano's impact is summarized by Russell in his January, 1901 letter to Couturat:

> **For that which concerns the value of [Peano's] symbolism, I am not entirely in agreement with you. I find it, to the contrary, excellent from a symbolic point of view, and I find that it is in the first instance Peano's symbolism that permitted the Italians to produce such good works on mathematical logic. I now employ, in all problems of this sort, entirely this algorithm, which I completed with *an algebra of relations* different than Peano's and Shröder's. I found (1) that logical analysis is facilitated enormously; (2) that the paralogisms become much more rare; (3) that formulas and demonstrations become a thousand times easier to understand. When I read Cantor, for example, I always translate him into Peanistic formulas, even though, before the Congress, I had not read a word from this school. And**

[202] The membership relation is that of individual to class, x is a, and inclusion can be defined by means of implication, a is contained in b where if x is a, then x is b. The containment of the individual in the class is the Cantorian composition of the class, not the Boolean part/whole relation. This relation isn't transitive, as the relation of class to class is (Russell's illustration is, ironically, that 2 is a number, number is a class, 2 is not a class) and if x is in a or b, it must be in either a or b.

> **from the point of view of formal logic, I find that there has been far too much insistence on equations, which have no real importance, and it has been wrong to misrecognize the distinction between ε and $\supset$ —an indispensible distinction, by my lights, for the theory of the infinite, and even to all that is called mathematics. I even succeeded in making new discoveries in the field of pure mathematics, which I never succeeded in doing by the old methods. For these reasons, I find in the symbolism of Peano an immense superiority to all his precursors [CPLP, R17.01.1901].**

In a survey of recent Italian work in logic, which largely reflects the comments to Couturat, Russell credited Peano with "…the revival, or at least the realization, of Leibniz's great idea, that, if symbolic logic does really contain the essence of deductive reasoning, then all correct deduction must be capable of exhibition as a calculation by rules" [RIW, 353]. The project was at least superficially similar to Frege's project, begun in his *Begriffsschift*, which Frege remarked, in his polemical essay against Boole, was intended to offer a "fresh approach to the Leibnizian idea of a *lingua characteristica*," but which was not supposed to be a rational calculus confined to pure logic like Boole's [PW, 12-13]. Frege had recognized in his earliest works that the part/whole relation was pseudo-logical and incapable of determining where the division into parts is complete, by contrast with the logical relation of membership, where the elements of a class are uniquely determined. Frege held that a whole thought is constituted by the senses which make it up, but he was not committed to the view that breaking down a thought into its constituent senses yielded exact analyses. On his early view, a whole thought can be differently divided [PW, 192],[203] and it has been argued that the fact that these different divisions result in new concepts was the basis for his notion, prior to the introduction of

[203] For a discussion of this issue see Levine 2002.

the Sinn/Bedeutung distinction, that mathematics contains informative analytic truths.[204] However, what is more important is that, in Frege's logic, the subject-predicate doctrine was overturned, not by the admission of ultimate and intensional relations as constituents in propositions, but by the extraction of function and argument from judgeable contents, which provides an approach to analysis that permits concept formation without being confined to subject-predicate logic. For Russell, the nature of terms depends on the kind of occurrence they have in true or false propositions—the occurrence of a concept differing in an indefinable way from the occurrence of subjects or terms of relations [Russell 1899-1900, 190], and, as we have seen, relations have a twofold occurrence in propositions as relating relations and as concepts whose relating capacity cannot be preserved in analysis. However, relations differing in sense are differing relations and, though a relation with one sense implies the relation with the opposite sense as its converse, it is the ultimate relation which must be preserved in analysis. For Frege, the judgeable contents are prior to concepts formed out of them and relations are not ultimate or self-subsistent entities. Functions, like all concepts that require objects falling under them, are incomplete or unsaturated, and require, for their completion, that their argument-places be filled in by objects, which alone are complete or self-subsistent.[205]

[204] On Frege's account: "the more fruitful type of definition is a matter of drawing boundary lines that were not previously given at all. What we shall be able to infer from it, cannot be inspected in advance; here, we are not simply taking out of the box again what we have just put into it. The conclusions we draw from it extend our knowledge, and ought therefore, on Kant's view, to be regarded as synthetic; and yet they can be proved by purely logical means, and are thus analytic" [FA, 100-101].

[205] In the *Grundlagen*, Frege writes that "if, from a judgeable content which deals with an object a and an object b we subtract a and b, we obtain as remainder a relation-concept, which is, accordingly, incomplete at two points" [FA, 82].

Functions can never occur without argument places, never have a twofold occurrence in propositions, and can never occur as objects.[206]

By the time he articulates his logicist thesis, the philosophical conception of analysis as decomposition had already committed Russell to the notion of philosophically exact analyses, to the intensional doctrine of relations and classes, correspondingly, which would significantly inform the manner in which he construed the logical analysis of mathematical propositions. We have seen already that, along with this decompositional conception of analysis, Russell inherited from Moore the notion that the composition and decomposition of propositions hinges on the ultimate relations they express, so that an adequate method of analysis is one which preserves the ultimate and necessary types of relations holding between concepts in a proposition. Relations are included among the ultimate constituents of propositions and, functioning like vectors between relata and referents, they have ultimate senses, opposite to those of their converse relations.[207] The

[206] On Frege's view, Peano and Russell do not understand the nature of functions: "I distinguish function-letters from object-letters, using the former to indicate only functions and the latter to indicate only objects, in conformity with my sharp differentiation between functions and objects, with which Mr. Peano is unacquainted" [CP, 248]. In the letter published in the *Rivista di Matematica*, Frege complains of an instance of Peano having used a function letter without an argument place and insists that "this is to misunderstand the essence of a function, which consists in its need for completion. One particular consequence of this is that every function sign must always carry with it one or more places which are to be taken by argument signs; and these argument places – not the argument signs themselves – are a necessary component part of the function sign." As we shall see, the notion that a function could be asserted of itself and that function could be "non-assertability of self" involves a contradiction, so that the function (assertion) in Russell's propositional functions cannot be separated from the variable (term), but rather must "live in propositions of the form φx and cannot survive analysis".

[207] More precisely, the relational symbolism supplants Russell's earlier attempt to articulate difference of sense in the terms of the vector calculus, adopted from Whitehead's *Universal Algebra*. On his earlier view, distances are at once asymmetrical relations differing in sense with opposite signs, having the nature of vectors, and abstract relations without distinction of sign, having the nature of a scalar [Russell 1899, 387].

result of this is the view that equivalence in extension does not suffice for identity. In essence, Russell has adopted an intensional view of relations[208]—a doctrine which, as we shall see, both informs his logic of relations[209] and subsequently gives rise to some of the failures in analysis that lead him to refine his logical apparatus. Russell would object, from the outset, to Peano's failure to distinguish relations in extension as classes of ordered pairs, from relations in intension, and to his failure to distinguish the class in extension from the intensional class-concept—two notions which Moore warns against confounding in his 1900 paper on Identity [I, 125][210]—that is, the predicate-concept that is supposed to determine the terms forming the class, since class-concepts may be extensionally equivalent, but not identical. We have also seen that the analysis which

The former sort, he regards as relations of direction having sense [AOG, 413]. As Russell refines his notion of order so that relations differing in sense are differing relations, he no longer regards relations differing in sense as "of the nature of vectors", but this notion is nevertheless useful in appreciating the mathematical origins of the later view.

[208] In PoM, Russell contends that he embraces the "intensional view of relations" inherited from his friend, G.E Moore, and though it is not entirely clear what Moore's doctrine of relations was or where it was stated, the view which Russell had in mind would seem to be the view that relations differing in sense are differing relations, i.e., the anti-Heglian view that conceptual differences are real differences to be captured in decompositional analysis. Relations are taken in intension iff two distinct relations are co-extensive. More formally, relations R and S are extensional iff $(Vx,y)((xRy \Leftrightarrow xSy) \Rightarrow R = S)$; otherwise they are intensional.

[209] In a retrospective letter to Jourdain dated April 15th, 1910, Russell writes that his paper, "On the Logic of Relations", and the logicist definition of number were carried out according to the intensional doctrine of relations. In fact, in that same letter, he gives a synopsis of how his early work developed: "Until I got hold of Peano, it had never struck me that Symbolic Logic would be any use for the Principles of Mathematics, because I knew the Boolean stuff and found it useless…I had already discovered that relations with assigned formal properties…are the essential thing in mathematics, and Moore's philosophy led me to wish to make relations explicit, instead of using only ε and $\subset$. This hangs together with my attack on subject-predicate logic in my book on Leibniz…I read Schröder on Relations…in September, 1900, and found his methods hopeless, but Peano gave just what I wanted. Oddly enough, I was largely guided by the belief that relations must be taken in intension, which I have since abandoned, though I have not abandoned the notations…" [Grattan-Guinness 1977, 134].

[210] Russell's formulation of the point is clearer, since he avoids talking of concepts having instances, pointing out as early as his preparatory notes for FIAM that the instances of a concept are the concepts having an extensional relation to the (class) concept in question [FIAM, 276-7].

Russell adopted prior to his discovery of Peano was the analysis of part and whole. An analysis of the content of a judgment, i.e., by means of its decomposition into constituent concepts, and the analysis of complex constituents into their simple parts may introduce philosophical precision into analysis, but it was difficult to see how it was supposed to comport with the aims of mathematical reasoning or the analysis of the propositions of existing mathematics.[211] However, with its origins in Boole's propositional calculus, the new logical calculus is, in the first instance, the logic of propositions (implication/identity). Though implication comes to be distinguished from the part/whole relation, when Russell commits himself to logicism,[212] he comes to regard pure mathematics as being defined as "the class of all propositions of the form 'a implies b', where a and b are propositions, each containing at least one variable, and containing no constants except logical constants or such as can be defined in terms of logical constants."[213] To be clear about Russell's brand of logicism, it will be helpful to clarify

[211] Of course, it is a central difficulty of Russell's problematic early theory of denoting that the logical subject of propositions containing denoting phrases does not occur in the proposition, while the denoting concept occurs only as meaning and cannot be denoted. A full explanation of the connection between decompositional analysis and the analysis of mathematical propositions will require an account of how Russell conceived the role of a theory of denoting, i.e., of the philosophical commitments underlying his approach to mathematical definition, which will be taken up in chapter 5. For the moment, it will suffice to establish that both the view that propositions are logically basic and the decompositional approach to analysis are retained as Peano's logic replaces the earlier part/whole logic.

[212] The first informal articulation is in the popular paper, "Recent Work in Mathematics". Interestingly, Leibniz had held that "As to *eternal truths*, it is to be noted that at bottom they are all conditional, and say in effect; Granted such a thing, such another thing is. For instance, when I say 'Every figure which has three sides will also have three angles', I say nothing but this, that supposing there is a figure with three sides, this-same figure will have three angles" (Langley 1916, book IV, chapter 11, section 14); these are of the form if x is a, then $\boldsymbol{\varphi}$, where the antecedent asserts a condition which restricts the variable in the consequent, which is precisely Russell's notion of implication.

[213] This is the first (clear) articulation of logicism, given in the May, 1901 draft of PoM : "Pure mathematics is the class of all propositions of the form "a implies b", where a and b are propositions, each containing at

what motivated it and the manner in which it developed, which will be the undertaking of the following chapter.

least one variable, and containing no constants except logical constants or such as can be defined in terms of logical constants. And logical constants are classes or relations whose extension either includes everything, or at least has as many terms as if it included everything" [Russell 1901c, 185].

CHAPTER 3: LOGICISM AND THE ANALYSIS OF MATHEMATICAL PROPOSITIONS

3.1 LOGICISM AND EXISTING MATHEMATICS

According to Peter Hylton, "Russell was both a metaphysician and a working logician. The two are completely intertwined in his work: metaphysics was to provide the basis for logic; logic and logicism were to provide the basis for arguments for the metaphysics" [Hylton 1990, 9].[214] On this account, Russell's metaphysical commitments were adopted along with Moore's new logic and significantly informed the logic which grew out of Russell's break with idealism. Russell's logic and his metaphysics are indeed intertwined in the "philosophical approach to analysis" which arose out of his initial anti-Hegelian commitment to the part/whole approach to analysis. The approach, as we have seen, involves the decomposition of propositions into constituent concepts and complex concepts into indefinable simple constituents, where conceptual differences indicate real differences which the logic must preserve.[215] Though he dispenses with the part/whole approach to analysis as he adopts symbolic logic and formulates logicism, the new logic informs Russell's view that logical analysis has philosophical as well as technical

[214] In *Propositions, Functions, and Analysis*, Hylton claims, in considering Moore's influence, both that "...logic [for Russell] has metaphysical implications, which must be correct if logic is true" [65] and that "...the metaphysics was independent of and prior to the logic" [71]. The latter statement, on my view, has some truth in connection with part/whole logic, but becomes less true as it is supplanted by Peanistic logic.

[215] For more on the connection between exact or unique analyses and the part/whole approach to analysis, see Levine 2002, 202. Levine calls these exact analyses 'ultimate analyses'. Though this applies to the analysis of the constituent concepts of a proposition, which, if complex, should be further analyzed to reveal simple terms, it does not apply to what might be called equivalent propositional contents, as we have seen, for instance, in the case of two relations differing in sense, where exact analysis requires that they be kept distinct where an ultimate analysis would reduce both relations to some more ultimate term.

requirements, so that, on the decompositional approach, analyses must be philosophically exact (i.e., must preserve sense) as well as preserving the relevant formal features of the analysandum in the analysans.[216] After discovering Peano, Russell's analyses (or nominal definitions) seem intended merely to preserve the formal properties required of the "entities" under consideration and not their meanings, but in his embrace of nominal definitions and even on the earliest articulation of the principle of abstraction, Russell remains concerned that the mathematical definitions of concepts are philosophically unsatisfactory.[217] Though Russell was concerned with the ontological consequences of his logicist definitions, i.e., of his abstraction principle, the aim was to achieve logical precision in definitions and, correspondingly, metaphysical precision about what is defined, for instance, in distinguishing carefully between the definition of any w-series (Dedekind's definition) and numbers (the Frege-Russell definition). It is only by invoking classes to serve as the guarantors of "purely logical objects" secured by explicit definitions, that the mathematical entities concerned are no longer regarded as problematically incapable of philosophical definition and it is only by his problematic

[216] This is not inconsistent with his earlier work. For instance, in AMR, Russell distinguishes types of relations and types of propositions according to their formal properties and the rules applicable to them and continues to maintain, in PoM, that the introduction of particular notions by logical rules of inference is the basis for the classification of relations or types of propositions [PoM, 11]. As we have seen, formal (i.e., universally quantified) implication is distinguished from membership for the reason that differing rules of inference are applicable in each case.

[217] We shall see, in Chapter 3, that Russell initially holds that all mathematical definition is philosophically inexact: though a mathematical definition specifies the relation possessed uniquely by the object defined to a specified concept, it does not give the (philosophical) *meaning* of the term. Subsequently, classes are invoked to supply the logical objects defined, but in the light of the Contradiction, Russell struggles again with his conception of mathematical definition and, as we shall see in Chapter 5, briefly replaces class abstract notation with functional notation before arriving finally at the theory of descriptions by which classes and functions could be eliminated.

inclusion of set-theoretic notions within logic, that these definitions are strictly logical. At the same time, philosophical problems given rise to by analysis like the problem of the unity of the proposition, namely, that the whole cannot be specified by the enumeration of its constituent parts, were revealed to be problems particular to propositions, not classes and, though it was philosophically significant, mathematics merely required classes determined by class-concepts and predicates. Initially, relations in intension and the distinction between the class and the (intensional) class-concept introduced in accordance with the part/whole approach to analysis acquired from Moore and Boole, are preserved in PoM so as not to introduce logical confusion into statements of implication: relations in intension are to be identified with class-concepts, which are to be distinguished from the corresponding classes in extension. However, in the light of the logical demands imposed by the contradiction, the limitations of analyses into simple constituents become increasingly clear and Russell comes to treat classes and relations in extension, within an intensional logic of propositions and propositional functions. Even after classes are abandoned altogether, Russell adheres to this philosophically-motivated decompositional approach to analysis, on which logically exact analyses carve up reality with exactness, preserving the intensional dimension of meaning within the logical analyses themselves.

The connection Hylton perceives to exist between Russell's logic and his metaphysics may be the sort which holds between his logic and then-existing mathematics. For Kant, conditions for the construction of knowledge reveal the synthetic *a priori* status of mathematical propositions in which their truth consists. Russell begins PoM by telling us that symbolic logic studies inference, which is deductive and relies on the relation of implication, [218] which asserts, in both arithmetic and geometry, that whatever has such and such properties also has such and such properties, indifferently to whether the entities in question exist. Logical deduction, formerly regarded as tautologous unless it was supplemented by intuition, is in itself informative. Though its axioms are formally assumed, the fact that they allow existing mathematics to be true—and not approximately so, as the Hegelians would have it, or true of the objects of intuition, as the Kantians would have it—is, as Russell puts it, a "powerful argument in their favour".[219] Russell's logicist definitions dispense with entities inferred from collections and identify such entities with the classes of classes or relations having the properties required for the propositions about them to be true—as we shall see, Russell's inclusion of the apparatus of set theory within logic inclined him to identify logical objects with classes of classes, supplying existence theorems in set-theoretic terms to show that there are such classes as those defined. The motivation for Russell's logicism was to establish, where traditional

[218] "I may as well say at once that I do not distinguish between inference and deduction" [POM, 11n1]. Also, 1906 paper "The Theory of Implication", "In order that one proposition may be inferred from another, it is necessary that the two should have that relation which makes one a consequence the other. When a proposition q is the consequence of a proposition p, we say that p implies q. Thus deduction depends upon the relation of implication" [Russell 1906, 159].

[219] Russell writes: "Formally, my premises are simply assumed; but the fact that they allow mathematics to be true, which most current philosophies do not, is surely a powerful argument in their favour" [PoM, xviii]. In PM, he writes: "[T]he chief reason in favour of any theory on the principles of mathematics must always be inductive, i.e. it must lie in the fact that the theory in question enables us to deduce ordinary mathematics. In mathematics, the greatest degree of self-evidence is usually not to be found quite at the beginning, but at some later point; hence the early deductions, until they reach this point, give reasons rather for believing the premises because true consequence follow from them, than for believing the consequences because they follow from the premises" [Whitehead and Russell 1910, v]. These claims are echoed in Russell 1973, 194 and in PM, 37.

logic had failed as a result of its reliance on syllogistic argument forms and analyses into subject and predicate, to establish the truths of existing mathematics without regard for the constitution of the mind or (psychological) conditions for the construction of knowledge. Gandon has argued, in particular, that Russell's was a topic-specific logicism, on which the integrity of the body of knowledge constituting various branches of mathematics is to be preserved in the reductions to logic. It might be said that mathematics—the extant body of knowledge comprising its various branches—was to provide the basis for logicism, while "logic" and "logicism"—the formal requirements of the propositional calculus and predicate calculus with polyadic quantification supplied by the logic of relations, together with the content supplied by individuals, classes and relations between them, sufficing for informative deductions—provided the arguments establishing pure mathematics. I shall consider this possibility in greater detail in what follows.

Russell defines pure mathematics as "the class of all propositions of the form 'p implies q' where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants".[220] It has been proposed, originally by Putnam, that it is necessary to

[220] Pure mathematics is defined as "...as the class of propositions asserting formal implications and containing no constants except logical constants" [PoM, 106]. He adds: "And logical constants are: Implication, the relation of a term to a class of which it is a member, the notion of such that, the notion of a relation, and such further notions as are involved in formal implication, which we have found to be the following: propositional function, class, denoting, and any or every term." In the Preface to the 2nd edition, Russell writes: "This brings me to the definition of mathematics which forms the first sentence of the "Principles". In this definition various changes are necessary. To begin with, the form "p implies q" is only

distinguish Russell's "strong" or "categorical logicism" in *Principia Mathematica* (henceforth, PM) from his earlier "weak" or "conditional logicism" in PoM. On Putnam's account, Russell comes to reject what he calls "if-thenism" (i.e., "conditional logicism") in favour of what he calls "logicism", i.e., the "standard logicism" on which explicit definitions permitting the applications of numbers are privileged.[221] Coffa adopts the distinction, but characterizes "standard" or "categorical logicism" as the view that all theorems of mathematics can be stated in terms of logical concepts and proved by logical axioms and rules of inference; and characterizes "conditional logicism" or "if-thenism", following Putnam, as the view that all propositions of pure mathematics are conditionals whose antecedents are the axioms of a branch of mathematics and whose consequents are the theorems provable by logic. "Conditional logicism"/"if-thenism" is supposed, by Coffa, to be Russell's view in connection with geometry in PoM and integral to his refutation of Kant, whereas "strong" or "standard logicism" is supposed to be the view

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one of many logical forms that mathematical propositions may take. I was originally led to emphasize this form by the consideration of Geometry. It was clear that Euclidean and non-Euclidean systems alike must be included in pure mathematics, and must not be regarded as mutually inconsistent; we must therefore only assert that the axioms imply the propositions, not that the axioms are true and therefore the propositions are true. Such instances led me to lay undue stress on implication, which is only one among truth-functions and no more important than the others…" [PoM, vii].

[221] Though it seems to me that Putnam is correct to point out the importance of so-called 'standard logicism' to ordinary applications of arithmetic, I cannot assess his characterization of standard logicism in PM here. The important point, and the one relevant to Coffa's interpretation, is Putnam's claim that, before developing explicit definitions, Russell held that mathematics consists of if-then statements, i.e., : "If there is any structure which satisfies such-and-such axioms … then that structure satisfies such-and-such further statements …" [Putnam 1975, 20]. George Boolos's contention that it is, rather, in PM that Russell abandons logicism and adopts if-thenism in adopting the axiom of infinity has some plausibility [Boolos 1998, 255-274], though his argument is not decisive. Cf. note 222.

that Russell simultaneously held for arithmetic in PoM.[222] The idea is that, while pluralism in geometry requires that it be reduced to the conditional form—i.e., if such and such axioms hold, then such and theorems are implied—there is nothing in arithmetic to rival Peano's axioms, and, without inconsistent systems to reconcile, there is no need for arithmetic propositions to be stated as conditionals with axioms as antecedents and theorems as consequents.[223] There is supposed to be a textual basis for this in Russell's own remark that it was the fact that Euclidean and non-Euclidean systems (both internally consistent) are to be included in *pure mathematics*, that led him to presume that implications are the true form of mathematical propositions. Ian Proops—who agrees with the thesis that Russell subscribed to different brands of logicism for arithmetic and for geometry—tries to save Russell from the trivialization of logicism which results from the notion that anything that can be axiomatized can be logicized by reminding us that the

[222] While Putnam contrasts explicit definitions with the conditional form of logicism and believes the former supplanted the latter in PM, Coffa believes that Russell fully embraced the categorical version of logicism in PM, but accepted the standard logicism for arithmetic, and the conditional version for geometry in PoM. It seems to me that Putnam's attribution of 'if-thenism' and its attendant formalism to Russell [Putnam 1975, 251-2] is intended to be charitable, since it is supposed to do the work done by a model-theoretic account, i.e., if such and such a system of axioms holds, then such and such a mathematical proposition is true in that system', but to attribute such a view to Russell would be to misrecognize the universality of Russell's logic of propositions. Nevertheless, as Boolos points out, the need for important axioms (e.g., the axiom of infinity and the axiom of reducibility) not stated in the system in PM may commit Russell to a kind of 'if-thenism'. In his recent book, Landini defends the view that Boolos is mistaken in his claim that, in adopting the axiom of infinity in PM, Russell adopts if-thenism and rejects logicism [Landini 2011, 98-103]. I shall confine my discussion to standard and conditional logicism in PoM. Cf. note 221.

[223] Coffa writes: "roughly speaking, those mathematical theories for which there appeared to be no alternatives (i.e. arithmetic) were to be reduced to logic in the standard sense; those for which there were colegitimate alternatives (e.g. geometry) were to be reduced to logic only in the conditional sense" [Coffa 1981, 252].

concepts (geometric as well as arithmetic) are supposed to be derived from logical concepts.[224] I shall address these various attributions all together in what follows.

The first claim in Coffa's interpretation that I wish to address is the question of whether Russell subscribed to "standard logicism" for arithmetic and "conditional logicism"/"if-thenism" for geometry. Evidence for the conditional view of geometry is, on Coffa's account [Coffa 1981, 247–263], that there are two inconsistent theories of metrical geometry *which were supposed to be logicized*. However, as Gandon and Byrd point out, this was not the case. In EFG, as we have seen, Russell had held that metrical geometry, which presupposes projective geometry as the science of purely "qualitative" comparisons,[225] extends it by introducing quantity, its chief merit consisting in its establishing distance as a relation between two points, rather than the "merely technical" quadrilateral construction. In his 1899 "Notes on Geometry", Russell holds that projective geometry is "not essentially concerned with order or series", that the quadrilateral construction cannot give order between points *per se*, and that distance needs to be introduced to give order between two points on a line [Russell 1899, 379]. In AOG, anharmonic ratio is derived from the quadrilateral construction, whose uniqueness is proved from certain axioms, but showing that any four points on the straight line have an

[224] Ian Proops accepts the distinction between conditional and categorical logicism, but disputes Coffa's charge of 'if-thenism'. He thinks Russell avoids the consequence that any body of knowledge could be 'logicized' in the conditional sense, by the requirement that the concepts of mathematics must be definable in logical terms, which precludes musical and geographical concepts, and so forth [Proops 2006].

[225] Russell's treatment of projective geometry and its axioms left much to be desired, as Poincaré would point out in his review, to which Russell responded with a rigorous axiomatization in AOG.

anharmonic ratio requires that *all points* can be obtained from the quadrilateral construction and that there is no finite gap on the straight line, which requires the introduction of metrical notions [AOG, 405–6]. In PoM, however, Russell's view is that quantity is not properly part of pure mathematics [PoM, 158]. While it is not necessary that distances be magnitudes,[226] but only that they form a series having the properties required for the numerical measurement of the relations between every pair of points [PoM, 408], Russell concludes that, for convenience, the notion of distance in metrical geometry can be dispensed with in favor of the magnitude of divisibility of the corresponding stretch, which suffices for measurement, but he clearly holds that magnitude of divisibility is not a logical concept. So, while metrical geometry, as a theory of distance that is no longer dependent on the introduction of quantity but merely on the introduction of metrical properties in purely projective terms, i.e., on distance as a function of an anharmonic ratio (cross-ratio), is purely logical, metrical geometry, conceived independently of projective geometry as a theory of magnitudes of divisibility is not part of pure mathematics and is not to be logicized.[227] Interestingly, Gandon regards this as a strong argument against the trivialization of logicism that is supposed to result from Russell's "if-thenism":

[226] Russell writes: "It may well be asked, however, why we should desire to define a function of two variable points possessing these properties. If the mathematician replies that his only object is amusement, his procedure will be logically irreproachable, but extremely frivolous…[T]he (projective) theory of distance, unless we regard it as purely frivolous, does not dispense with the need of (the theory of magnitude of divisibility). What it does show…is that, if stretches are numerically measurable, then they are measured by a constant multiple of the logarithm of (a certain) anharmonic ratio" [PoM, 425].

[227] My reading, though perhaps not the consequences I draw from it, is similar to Byrd's. See Byrd 1999.

> **Contrary to what Coffa's …argument presupposes, it is not the case that just because Russell had the technical means to annex a given field to logic that he believed he should therefore do so. The derivation could threaten the place that a body of knowledge had in the scientific architecture, and if this was the case, then the logicist had to renounce what appeared to be a mere formal trick [Gandon 2008b].**

When he adopts logicism, then, the quantitative theory of metrical distance does not belong to pure mathematics because the concept of magnitude of divisibility is non-logical. The fact that Russell preserves the theory of metrical geometry conceived as a non-logical theory of magnitudes of divisibility at all is evidence that Russell's logicization project preserves the internal structure of the body of knowledge belonging to the various branches of mathematics, which fits nicely into Gandon's broader thesis that Russell's brand of logicism is topic-specific [Gandon 2008b]. Since the topic-specific nature of the reductions constitutes evidence against the view that Russell held differing versions of logicism for arithmetic and for geometry, I shall briefly consider some of the reasons which have been given in favour of the view that Russell not only determined which branches of mathematics were to be logicized on topic-specific grounds, but advocated topic-specific approaches to logicist reductions in projective geometry and in arithmetic.[228]

[228] If the preservation of the internal structure of a body of mathematical knowledge decides whether a given mathematical topic is to be logicized, then it would seem that there are epistemological aims which outstrip the logicization project. This gives rise to the question of whether and in what cases a topic is to be mathematically characterized and whether and in what cases it is to be logically characterized. The question of how a topic is to be characterized presents difficulties for Gandon's view, but his "topic-specificity thesis" is informative and its disadvantages are ones worth resolving, though I cannot do so here. I will merely suggest that the topic-specificity of logicism is not required by pure mathematics, but is required by the ordinary applications of mathematics. Cf note 268.

I shall attempt to briefly state Gandon's basis for insisting on the topic-specificity of Russell's logicization of projective geometry. Russell's chief concern after EFG, as indicated in his 1899 reply to Poincaré, AOG, was to "prove the uniqueness of von Staudt's quadrilateral construction" from which projective geometry was to be deduced [Sur les axioms de la geometrie, 684–707]. In PoM, however, Russell outlines two very different theories of projective geometry. Projective geometry is characterized first, as a theory of ordinal relations, following Pasch, in which the indefinables are 'point' and the relation of 'between', with plane, line, and incidence between lines defined in terms of these, and, second, as a theory of incidence relations, following Pieri. Gandon argues convincingly that the latter is the culminating achievement of Russell's attempt to deduce projective geometry from von Staudt's quadrilateral construction, relying solely on incidence relations. Importantly, in contemporary projective geometry, it had been shown that the theorems of projective geometry could not all be proved by incidence axioms alone without axioms of order.[229] However, as Gandon points out, Pieri had showed that the projective segment, an ordinal notion, could be defined in terms of harmonic conjugation which made use of the quadrilateral construction and, hence, only of

[229] Gandon points out, for instance, Klein had shown that the 'fundamental theorem' of projective geometry— that a projective transformation between two ranges of points is uniquely determined when three points of one and the corresponding three points of the other are given— could not be proved by means of von Staudt's quadrilateral construction, that is, by incidence axioms alone [Gandon, 2009, 43]. Gandon also points out the ultimate status afforded to ordinal notions, not only in Kant, for whom they were to be filled in by intuitions, but also for Hilbert, for whom order was axiomatized. The point, I think, is important. It must have impressed Russell to discover, not only that deductions from geometric axioms did not need to be supplemented by intuitions, but that ordinal relations weren't ultimate in pure projective geometry.

incidence relations for the intersection of lines in a plane.[230] As Russell puts it: "...Pieri has shown how, by means of certain axioms, this relation of four terms may be used to divide the straight line into the two segments with respect to any two of its points, and to generate an order of all the points on a line."[231] Separation and projective order on a line can be defined, then, in terms of incidence relations without the need for ordinal notions. In considering the historical details of Gandon's account, which I have merely given in rough outline above, it would seem that, far from being the outcome of a topic-neutral reduction, the theory of projective geometry adopted from Pieri was both an outgrowth of, and a significant contribution to contemporary developments in geometry.

Russell is thought to have considered the theory of number given in PoM, developed according to Peano's axioms, supplemented by his own logic of relations, as the standard model of arithmetic, and the explicit definitions introduced are supposed to constitute the standard form of logicism. Between October 1900 and May 1901, Russell arrives at a logicist definition of cardinal numbers as common properties (classes) of similar classes—a definition which will be explored at length in Chapter 4. Peano's implicit (axiomatic) definition of number did not identify the objects satisfying the Peano axioms and Russell's own attempt to give definitions by abstraction gave rise to the

[230] Russell writes: "If four points x, y, x′, y′ be given, it may or may not happen that there exist two points a, b such that $xH_{ab}\,y$ and $x'H_{ab}\,y'$. The possibility of finding such points a, b constitutes a certain relation of x, y to x′, y′. . . Pieri has shown how, by means of certain axioms, this relation of four terms may be used to divide the straight line into the two segments with respect to any two of its points, and to generate an order of all the points on a line" [PoM, 385]. Here, $xH_{ab}\,y$ states that x, y are harmonic conjugates with respect to points a, b.

[231] The further axioms referred to here are set forth in Gandon 2009, 48.

uniqueness problem—indefinitely many classes possess the defining property—but by taking the number to be the class of all such classes, the problem is avoided. In defining the cardinals in LOR, Russell gave his abstraction principle as follows:

> **[A]ll relations which are transitive, symmetrical, and non-null can be analyzed as products of a many-one relation and its converse, and the demonstration gives a way in which we are able to do this, without proving that there are not other ways of doing it. [This proposition] is presupposed in the definitions by abstraction, and it shows that in general these definitions do not give a single individual, but a class, since the class of relations S is not in general an element. For each relation S of this class, and for all terms x of R, there is an individual that the definition by abstraction indicates; but the other relations S of that class do not in general give the same individual. …Meanwhile, we can always take the class ...as the individual indicated by the definition by abstraction [LOR, 320].**

The abstraction principle is also employed in the definition of the ordinals. Russell criticizes Dedekind for postulating ordinal numbers where really what he has defined are numbers having order, so that it cannot be held that what all progressions—infinite, well-ordered series— have in common is the ordinals, but only that the same rules apply to them as to ordinals [PoM, 248-9].[232] In other words, Dedekind's implicit (axiomatic) definitions define any progression, not the numbers. Russell writes:

> **It is impossible that the ordinals should be, as Dedekind suggests, nothing but the terms of such relations as constitute progressions. If they are to be anything at all, they must be intrinsically something; they must differ from other entities as points from instants, or colors from sounds... Dedekind does not show us what it is that all progressions have in common, nor give any reason for supposing it to be the ordinal numbers, except that all progressions obey the same laws as ordinals do, which would prove equally that any assigned progression is what all progressions have in common... His demonstrations nowhere–not even when he comes to cardinals–involve any property distinguishing numbers from other progressions [PoM, 249].**

[232] Russell also tells us that "it is impossible that the ordinals should be, as Dedekind suggests, nothing but the terms of such relations as constitute a progression. If they are anything at all, they must be intrinsically something; they must differ from other entities as points from instants, or colours from sounds" [PoM, 249] The point, however, is not that implicit definitions are not valid, but merely that they defined progressions and not the ordinal numbers. Russell's point is that if the ordinal numbers exist, then they must be identified with a certain kind of concept and it is not enough that they satisfy the properties of progressions.

An order, Russell tells us, is not a property of a given set of terms[233] but of a serial relation whose field is the given set of terms, and, in the light of the principle of abstraction, we may define the ordinal number of a serial relation, R, as a class of well-ordered relations similar (i.e., order isomorphic) to R.[234] Hence, the cardinal numbers may be defined without any recourse to the properties of progressions and, likewise, the ordinal numbers can be defined in the manner stated above.

Russell's explicit definition of the reals in mathematical Analysis is an interesting case. In an effort to give a rigorous foundation to the real number system, Dedekind had introduced the property of ordered systems that they can be "cut" into two classes, which together exhaust the elements of the system, where every element in the first precedes every element in the second, and the system is continuous if every element of the system gives rise to such a cut. The reals were shown to be a continuous system comprised by the rationals and irrationals corresponding to such cuts, the latter arising wherever a cut in the rationals was not produced by a rational.[235] Though the reals are uniquely correlated with

[233] Russell lucidly points out that to say that we can consider a given set of terms in any order we like is really to say we can consider any serial relation whose field is the given set.
[234] In "General theory of Well-Ordered Series", Russell says it would be more usual to regard ordinal number as the common property of a class of similar serial relations, but that there is no such property apart from the class and the relation of similarity. A term is nth in respect to a given serial relation when, in respect to that serial relation, it has n-1 predecessors.
[235] I have followed Grattan-Guinness's account in Grattan-Guinness 1980, 222-3. Importantly, no notion of quantity was required for continuity and the calculus could be arithmetized. Russell, in rejecting the arithmetization project, did not follow Burali-Forti who believed number and arithmetic could be derived from quantity.

the cuts, they are not identified with them in Dedekind's philosophy.[236] For Russell, real numbers are defined in a manner similar to Dedekind's, though less intuitively, in terms of "segments of the rationals," which form a compact (dense) series, that is, (sub)classes of rationals akin to the lower bounds in Dedekind's cuts.[237] Imagining a division into a left-hand side of the cut (L) and a right-hand side of the cut (R), the reals correspond to the greatest lower bound of R, which, given that R has no least element, is in L.[238] Irrationals are segments of the rationals without a limit, that is, as (sub)classes of rationals determined by being less than any given one, i.e., a (sub)class of rationals less than the greatest lower bound which is an element in the class).[239] It is the properties of progressions, not of numbers, that is of crucial importance in the theory of segments, i.e., what is significant is that numbers form a progression from which a compact series may be obtained [PoM, 241]. For this reason, the Peano and Dedekind axiomatic (implicit) definitions of progressions suffice, though they do not define numbers.[240] Russell also gives an explicit definition of rationals as classes of what Cantor calls "coherent" classes of rationals. Cantor's definition of the coherence of two infinite classes of rationals, u, v

[236] Grattan-Guinness notes that Dedekind's philosophical commitment to the view that the reals ought not to be identified with the cuts, but are 'created' out of them, resembles Riemann's view that if actual space is discontinuous, continuous space can be created by creating new point-individuals [Grattan-Guinness 2000, 87].
[237] For additional comments, see Byrd 1994, 62.
[238] Another equally good definition is in terms of the least upper bound of L, which, since L has no greatest element, is in R. Russell had the option of identifying the reals with either the least-upper bound or the greatest lower bound.
[239] Less awkwardly, a segment of the rationals is the subclass of rationals identified with 'the class of rationals x such that x is less than y, where y is a rational of the class'. Interestingly, this definition was present in printer's copy of November, 1900 according to Byrd, though the explicit definitions of cardinal numbers were not [Byrd 1994, 57].
[240] That is, the axioms define the progressions by defining the triplet, first term, successor, and elements, constituting the meaning of any progression.

stipulates that "*u* and *v* are coherent if *u* and *v* have no maximum, for every element of *u*, there is a greater element of *v*, and conversely, for every element of *v*, there is a greater element of *u*."[241] According to the abstraction principle, the equivalence relation (of coherence) requires a common property to which the coherent sets of rationals have a relation. Russell supposes segments, which have all the properties of reals, to be these common properties. Interestingly, in chapter xxxiv of his November, 1900 additions to Part V of PoM, he remarks that this leaves doubt as to what the reals are, concluding that the reals, distinct from segments, should not be posited, the segments having all the properties required. In chapter xxxiii, however, he identifies the reals with the segments that are the common properties of equivalence classes of coherent classes of rationals.[242] The latter is the sort of explicit definition that is supposed to characterize Russell's "standard logicism".

Russell explicitly claims that numbers defined as classes of classes are essential to any *assertion of number*, but that this definition is irrelevant to numbers as they are employed in arithmetic and analysis, where what is significant is that numbers form a progression [PoM, 241]. As Byrd points out, the number terms used in assertions are an application of arithmetic as the general theory of progressions, not the exclusive or

[241] Byrd gives this definition in Byrd 1994, 62. It would seem also that the the two classes *u* and *v* should be "bounded above"., i.e., that there is some element of the class such that any other element is either less than or equal to it, so that a segment cannot be identified with the whole of the rational number set.
[242] For more detail, see Byrd 1994, 62.

standard model of arithmetic [Byrd 1999, 53].[243] Though the definition of numbers in terms of classes allows for definitions in mathematics to proceed by means of number (in line with the arithmetization program), Russell does not take this approach and even holds explicitly that the properties of progressions and of most series in general are independent of number[244]—indeed, even Russell's preference of the term 'progression' over 'denumerable series' emphasizes their independence from number [RIW, 359]. Once its role in applied number statements is clarified and it is understood that Russell simply regarded arithmetic as the theory of *progression*s, the fact that Russell's theory of numbers does not seem to reflect the proofs within contemporary arithmetic practice[245] is not grounds for identifying Russell's logicism with a content-neutral method of reduction. It seems that, as Gandon would have it, Russell's logicist project is not carried out indifferently to the preservation of the internal structure of the existing body of knowledge belonging to the branch of mathematics in question.

Metrical geometry *can* be logicized and projective geometry worked out in terms of order or incidence axioms, but Russell recommends a non-logical approach to metrical

[243] Gandon points out that "the definitions of both the order type η of the rational numbers, and the order type θ of the real numbers, are founded on progressions" and "the entire doctrine of continuity is then independent of cardinal arithmetic" [Gandon 2008, 15].
[244] This is Gandon's thesis: clearly, Russell holds that Kantian intuition be dispensed with (i.e., that the continuum and real numbers can be explicated without spatial concepts and that space can be freed of Kantian antinomies), and extends the arithmetization program of introducing logical rigor into proofs, but as Gandon argues, the stronger requirement that mathematics be reduced to the theory of numbers does not describe Russell's logicist program, on which the reduction of various branches of mathematics to logic is not carried out by means of a reduction of mathematics to arithmetic. In this light, Russell's definition of the cardinal and ordinal numbers is not just an extension of the Weierstrass/Cantor/Dedekind project of constructing the real numbers from arithmetic [Gandon 2008, 3].
[245] Gandon contrasts arithmetic and geometry on this point [Gandon 2009, 48, 59].

geometry and privileges Pieri's approach in projective geometry in the spirit of contemporary mathematical practice. Explicit definitions of number can be given as alternatives to the definitions given by Cantor/Weierstrass/Dedekind, and yet order need not be accounted for in terms of number, i.e., Russell rejects the arithmetization program and gives a relatively marginal, albeit important role to number theory. In both geometry and arithmetic, then, the strong evidence for Gandon's topic-specificity thesis[246] challenges both the claim that Russell subscribed to the conditional version of logicism as a formal method of reduction for non-Euclidean geometries and to "standard" or "categorical logicism", and its explicit definitions, as supplying the standard model of arithmetic and, by extension, the means of arithemetizing mathematics. Gandon introduces the topic-specificity of Russell's logicist reductions (i.e., in geometry) to spare Russell's PoM version of logicism from the consequences of the "if-thenist"/"conditional logicist" position that he is supposed to have held in PoM. I wish to maintain, however, that if Russell subscribed to "if-thenism" as it is characterized by Coffa, in preparing the material for PoM, he did so only in the period that marked the transition to logicism, around the fall of 1900 and did subscribe to "if-thenism" and logicism concurrently in his early work. The "if-thenism"[247] that prefigures logicism is nevertheless informative in

[246] Gandon concludes that the reductions were carried out according to relational types …. This view may be arrived at by considering which reductions Russell in fact privileged, but I'm not sure what this means for his logicism in general, other than a restatement of the topic-specificity thesis.

[247] It should be pointed out that I am concerned only with the 'if-thenist' position that Putnam attributes to PoM and that Coffa attributes to geometry in PoM. Since Putnam tried to resurrect 'if-thenism' in "The Thesis that Mathematics is Logic," Musgrave uses the term 'if-thenism' to characterize a later logical empiricist position adopted after the breakdown of logicism [Musgrave 1977]. I am not concerned with this position.

characterizing the position which Russell did hold immediately prior to logicism and, in particular, the formalist approach to the logicization of mathematics that he might have espoused had his adoption of Peano's formal implications not coincided with his acceptance of Cantor's set-theory. This brings me to my second point.

3.2 MORE LOGICISM: THE FORM OF IMPLICATIONS AND THE ROLE OF EXPLICIT DEFINITIONS

The second point I wish to address is Coffa's claim that, while "categorical logicism" requires that the concepts of mathematics be definable in logical terms, "conditional logicism" requires only that the propositions of mathematics are conditionals whose antecedents are axioms and whose consequents are mathematical theorems, provable by logic. In a brief review of Coffa's account of "conditional logicism"/"if-thenism", Griffin writes:

> **Where Coffa goes wrong, I believe, is in claiming that these conditionals had axioms as their antecedents and theorems as their consequents. Rather the propositions of pure mathematics were, for Russell, formal (i.e. quantified) conditionals the consequents of which asserted some condition of *every* value of an untyped variable ranging absolutely without restriction over the domain of terms, while the antecedent imposed some categorical condition on the variable, thereby ensuring that the whole proposition remained true (by failure of antecedent, if necessary) for every value of the variable [Griffin 1982, 77].**

Griffin finds it strange that Coffa acknowledges Peano's influence and correctly attributes to him a "conditional interpretation of mathematics" on which the antecedents determine the range of variables in the corresponding consequents, without recognizing that this was precisely Russell's view. On my account, both Griffin and Coffa are correct, but for different periods in Russell's development: the form of the hypothetical statements of mathematics which Griffin attributes to Russell is the one which he had gradually arrived at by May, 1901, while the "implication of theorems by axioms" characterization of the hypothetical statements of geometry which Coffa attributes to Russell applies to the position he arrives at in the fall, 1900 draft. Arguably, this position had not altogether disappeared by the time of writing his January, 1901 paper "Recent Italian Work"[248], where Russell emphasizes the fact that geometry asserted implications, i.e., that certain propositions were implied by certain axioms, and did not assert the axiom or (therefore) the proposition and, hence, asserted nothing as to the nature of actual space or the points in it. Certainly in fall, 1900, Russell had not clearly conceptualized the nature of the variable,[249] and while he had adopted the view that mathematical propositions can take the form of formal (universally quantified) implications, he still held that the genuine propositions between which implications hold contain indefinable mathematical concepts. This "if-thenist" position is indeed closely connected to Russell's initial use of Peano's symbolic logic and is integral to his refutation of Kant, but it was one which Russell subsequently abandoned with the formulation of logicism. Around the fall of 1900, Russell privileges mathematical over philosophical definition and briefly adopts a kind of formalism with respect to the structures defined. However, by May, 1901 he has, as we

[248] The paper was finished in early winter, 1901[Papers 3, 350].
[249] Byrd notes that the additions involving variables were likely made in June, 1901. One such addition is: "In Universal Algebra, our symbols of operation, such as + and x, are variables, the hypothesis of any one Algebra being that these symbols obey certain prescribed rules" [PoM, 377]. Another, more fundamental, is that in which Russell claims that we can "dispense altogether with indefinables", replacing non-logical constants in the axioms with variables, so that "the axioms then become parts of a definition, and we have neither indefinables nor axioms" [PoM, 397]. For the consequences Byrd draws from these passages, see Byrd 1999, 47–8.

shall see, not only decisively arrived at, and even improved upon, the view of formal implication which Coffa attributes to Peano (on which the antecedents in the universally quantified implications determine the range of variables in the corresponding consequents), but has abandoned formalism. That is, he has arrived at the view that the concepts of mathematics are definable in logical terms by virtue of the naive comprehension principle on which there are genuine classes determined by the properties asserted in the implications. To establish these claims, I shall trace these developments.

In regarding the hypothetical statements of mathematics as implications from axioms to theorems, Coffa follows Putnam, who attributes to Russell the following characterization of the "if-thenist" position in PoM:[250] "...if there is any structure [of a certain kind] which satisfies such and such axioms (e.g., the axioms of group theory), then that structure satisfies such and such further statements (some theorems of group theory or other)" [Putnam 1975, 20]. Putnam notes that the existence of any such structure need not be asserted, and the derivation of consequences from axioms determines the properties of all such structures. While this is not Russell's view in the published text of PoM,[251] he may have come near to such a view in the period

[250] Ironically, Putman attributes Russell's "if-thenism" correctly to the period "before he espoused 'logicism", but he means the period prior to logicism of PM [Putnam 1975, 20].
[251] We shall see that definition by axioms does not suffice, since Russell takes it to be crucial that existence theorems are supplied by explicit definitions in arithmetic. In his Preface to the 2nd edition to PoM, Russell criticizes Hilbert's formalism, on which we are concerned, not with what the numbers are, but with asserting axioms of them which permit the deduction of arithmetical propositions. Russell condemns formalism for its negligence of the practical uses of arithmetic, which requires existence theorems to supply the logical objects that the numbers are, where formalism produces limitless systems of non-contradictory axioms, all supposed to define some set of objects [PoM, vi].

immediately preceding logicism, and Putnam's use of group theory to illustrate "if-thenism" may, in fact, be especially apt. In the October, 1900 draft of LoR, written prior to Russell's articulation of logicism and prior to the logicist definition of number,[252] Russell included a section on group theory treated by the logic of relations, but the treatment of group theory disappears in the published paper. In a paper intended to explain its disappearance, Griffin points out the connection which existed between group theory and geometry.[253] At the time he discovered Peano, Russell had been assimilating Klein's treatment in group-theoretic terms of the preservation of the characteristic properties of the various types of geometry under corresponding transformation groups.[254] As we have seen in EFG, the invariance of anharmonic ratio (cross-ratio) in projective transformation was acknowledged and, apart from the desire that distance be a relation between two points, not four, Russell readily admits both projective geometries and metrical geometries as viable theories of actual space, but neglects to employ group

[252] It is in LoR that Russell lays the groundwork for his logicist project, giving the familiar definition of the cardinals, cardinal and ordinal addition, and a construction of the reals, all by means of Peanistic logic supplemented by his newly invented logic of relations; "standard logicism" originates in LoR.
[253] Griffin cites the following passage: "The field of the group may be arranged according to the values of this invariant, and the relations of the group merely permute terms which give the same value of the invariant. A group may be defined by the above relation S. The field of the group consists of all terms having the relation S to some term. When a term in the field of the group is given, there is only one term to which it has the relation S; but there are in general many other terms having this same relation S to the same term. Thus for instance the group of collineations leaves anharmonic ratios [cross-ratios] unchanged, and there is a collineation which relates any two ranges having the same anharmonic ratio. Here S is the relation of a range, pencil or sheaf to its anharmonic ratio. Similarly the group of motions leaves magnitudes unchanged; here S is the relation of a figure to its magnitude, and SŠ is metrical equality" [Papers 3, 595] cited in Griffin forthcoming b, 6.
[254] Griffin writes: "The truth is…that the work he had been doing immediately before the Paris Congress of 1900, especially the work on geometry, made it quite natural for him to think of group theory as an appropriate target for his new logic of relations" [Griffin forthcoming b, 1].

theoretic methods in his essay.[255] In his critical commentary on the *Essay*, Poincaré urged that group-theory would introduce precision into the EFG's treatment of geometry,[256] but in his 1899 response, AOG, Russell maintains that "the language of the theory of groups cannot help us to a philosophical account of the foundations of geometry" [AOG, 412], by which he meant that it was not amenable to conceptual analysis or, to put it simply, to knowing what the terms *mean*. In AOG, Russell maintains that "A mathematical definition consists of any relation to some specified concept which is possessed only by the object or objects defined" adding, however, that the term in question "…cannot *mean* this relation, and that the mathematical definition thus becomes a theorem, which is true or false" [AOG, 410]. Russell initially rejects group theory for the reason that it cannot give us the philosophical meaning of the concepts of geometry. It is not long after the discovery of Peano that Russell takes a different view of the importance of philosophical definition.

In the fall, 1900 draft of Part VI of PoM, written around the same time that he applied the logic of relations to group theory, Russell privileges a mathematical definition

[255] In EFG, Russell does not make use of Klein's group-theoretic contributions, i.e., transformation groups are not employed. Lie's contributions recognized in detail, but, in a residually Kantian spirit, Russell determined the geometries resulting from group-theory to be abstract, not true to actual space, which is homogenous, i.e., has constant curvature [Griffin forthcoming b, 11].

[256] Griffin points out the trouble with the Lie's/Klein's definition of a group, which Russell did not recognize: "In the case of finite groups the existence of a inverse could always be proven from the closure condition, but in infinite groups it had to be separately assumed as part of the definition. Lie ([1889], p. 558) had noticed this defect and corrected it; and Klein (reprinting his [1873]) in 1893 had made note of it, too. Lie stated the associativity law at about the same time (Lie [1888], p. 553). Russell's account of groups in EFG, which looks at first sight to be hopelessly inept, was in fact just out-of-date" [Griffin forthcoming b, 10].

of points, concerning himself only with whether the concept defined has the requisite formal properties:

> **It [projective space] is defined like all mathematical entities solely by the formal nature of the relations between its constituents, not by what those constituents are in themselves. Thus we shall see that the points in a projective space may each be an infinite class of straight lines in a non-projective space. So long as the points have a requisite type of mutual relations, the definition is satisfied [cited in Byrd 1999, 46].**

The significance of philosophical definition has waned significantly and, in a footnote in his January, 1901 paper "Recent Italian Work", Russell makes the following remark:

> **It should be observed that, in Mathematics, a term is considered to be defined when it is the only term having an assigned relation to one or more known terms. This is not the sense in which the word *definition* is usually used in philosophy; but it seems doubtful whether the philosophical use is capable of any precise meaning, and if it can be made precise, it would seem that, in the resulting sense, *all* ideas are indefinable [RIW, 360n15].[257]**

In an addition to the manuscript of Part 6 of PoM, likely added in December, 1900, Russell makes the striking remark that "…a definition is no part of mathematics at all, and does not make a statement concerning the entities dealt with by mathematics, but is simply and solely a statement of a symbolic abbreviation; it is a proposition concerning

[257] The philosophical notion of definition Russell has in mind is that of the analysis of the meaning of terms, where the meaning of the fundamental terms cannot be given [AOG, 412]. Moore, for whom definition consisted in the analysis of a whole into its parts, continued to treat simple concepts as indefinable in PE. Wholes have parts in common and it is the ultimate differences between simple parts which are responsible for exhibiting the peculiarity of the whole in definition. For Russell, the uniqueness of a term may be supplied in definition by its peculiar connection with a complex of known terms. Thus "yellow", for Moore, is a simple and indefinable concept [PE, Ch1, Sect. 10], but for Russell it might be defined as "the colour evoked by light rays of 570-590 nm", i.e., by a denoting complex. Here, the difference of Russell's theory of terms from Moore's theory of concepts is significant: the manner of occurrence of "yellow" in "yellow is a colour" is as logical subject, while the manner of occurrence of "the colour evoked by light rays of 570-590 nm" in "yellow is the colour evoked.." is as concept/meaning. I shall address the associated difficulties with denoting in connection with mathematical definition in chapter 5.

the symbols, not concerning what is symbolized" [PoM, 429].[258] In privileging mathematical over philosophical definition, Russell is concerned only that the geometric concepts defined have the requisite formal properties, and not with fixing the meaning of the terms defined.[259]

Recall that Coffa holds that "conditional logicism"/"if-thenism" and "categorical logicism" coexist in PoM, but believes that Russell subscribed to the latter for arithmetic and the former for geometry, in connection with the need to account for the inconsistent

[258] The date cannot be established for certain, since the section of the manuscript to which the passage belongs is lost [Grattan-Guinness 2000, 304].

[259] This position is similar to Pasch's and to Hilbert's view. Pasch, for instance, claims the following: "If geometry is to be truly deductive, the process of inference must be independent in all its parts from the meaning of the geometric concepts, just as it must be independent from the diagrams. All that need be considered are the relations between the geometric concepts, recorded in the statements and definitions. In the course of deduction it is both permitted and useful to bear in mind the meaning of the geometric concepts that occur in it, but it is not at all necessary. Indeed, when it actually becomes necessary, this shows that there is a gap in the proof, and—if the gap cannot be eliminated by modifying the argument—that the premises are too weak to support it" [Pasch 1882, 98]. In a letter to Frege on December 29, 1899, Hilbert writes the following: "Every theory is only a scaffolding or schema of concepts together with their necessary mutual relations, and the basic elements can be conceived in any way you wish. If I take for my points any system of things, for example, the system love, law, chimney-sweep…and I just assume all my axioms as relations between these things, my theorems—for example, the theorem of Pythagoras—also hold of these things. … This feature of theories can never be a shortcoming and is in any case inevitable" [PMC, 40]. On my view, Russell abandons formalism for the reason that explicit definitions are required for practical applications of arithmetic and to supply existence theorems within the various branches of mathematics (existence theorems which are not supplied by the definition by axioms which suffice for mathematical purposes) [PoM, vi]. It is also clear that, in the period immediately prior to logicism, Russell had determined that the quadrilateral construction could be carried out without presupposing metrical notions (for instance, in "Geometry, non-Euclidean, composed in December, 1899 and revised in August 1900 [Papers 3, 470; 487] in and was working his way toward the view that metrical geometry was to be excluded from pure mathematics, while projective geometry was to be logicized. Gandon has suggested to me that the difference between formalism and logicism is not as great as I imply, for the reason that nominal definitions can be given wherever a formal characterization is given (by means of axioms). If this is so, it would seem that topic-specificity is not a requirement of pure mathematics, which could proceed formally, which is consistent with my thesis that existence theorems are introduced to account for applications which are strictly outside arithmetic.

axiom systems of geometries in the refutation of Kant by means of symbolic logic.[260] In PoM, in connection with Geometry, which he later admits to have inspired him to emphasize implication as the true form of mathematical statements, Russell writes:

> **Geometry has become...a branch of pure mathematics, that is to say, a subject in which the assertions are that such and such consequences follow from such and such premises, not that entities such as the premises describe actually exist. That is to say, if Euclid's axioms be called A, and P be any proposition implied by A ... then the geometer would only assert that A implies P, leaving A and P themselves doubtful [PoM, 373].**

Importantly, this passage is added to Part 6 of the manuscript PoM in January, 1901 [Grattan-Guinness 2000, 303]. In EFG, Russell had appreciated that the different properties of various conics in projective space gave rise to different metrics, which had led him to emphasize the conditional nature of geometrical statements, though in a more transcendental than formal sense.[261] However, what is perhaps more important, is that, as we have just seen, Russell had been content, in December of 1900, to embrace a certain formalism regarding the "entities" of interest in the hypothetical statements of geometry. The passage is also immediately prefigured by that which informed the topic-specificity of Russell's logicist reductions in geometry: Russell's appreciation of Pieri's work in projective geometry is conveyed in the December, 1900 draft of Part VI, in which he also expresses his non-logical account of distance and angle in metrical geometry in terms of stretch. The treatment of distance and angle given in the October, 1900 draft of LoR but

[260] Frege, who might be said to have adopted standard logicism in arithmetic, did so much earlier than Russell and the state of the art in late 19th century geometry would seem to have contributed to his view that geometry could not be logicized, whereas Russell, who considered the inconsistent axiom systems of geometry on the earliest formulation of logicism believed geometry could be logicized.
[261] As Gandon puts it: "...in the new perspective [advanced by Klein], the alleged incompatibility between the different kinds of metric (hyperbolic, elliptic, Euclidean) was reduced to the differences between the properties of various kinds of projective conics" [Gandon 2008b, 6].

absent from the published paper is, along with group theory, on its way out. Despite all of these advances, Russell has not, by January, 1901, fleshed out a logicist position concerning them. Though there is no reason to suppose that Russell misspoke when he claimed that it was the fact that Euclidean and non-Euclidean geometries belong equally to pure mathematics that led him to emphasize implications as the true form of mathematical propositions, this position predates logicism.

A crucial component of the refutation of Kant, and one not exhausted by the attempts at rigorization in the derivation of theorems from axioms is, in the logicist project, defining geometrical concepts *in logical terms*.[262] On Coffa's account this is not a part of "conditional logicism"/"if-thenism", but it seems to have been fundamental to Russell's logicist project from its first articulation. Ian Proops, who concurs that Russell subscribed to "conditional-logicism" for geometry and "standard logicism" for arithmetic, wishes to save Russell from the trivialization of logicism involved in "if-thenism" by stressing that the conditionals contain concepts which must be logicized. Russell maintains that the aim of logicism is "to show that all pure mathematics follows from purely logical premises and uses only concepts definable in logical terms" [MPD, 57]. My claim is that, if he subscribed to "if-thenism" at all, he did so at a time during which he did not hold that all of the concepts of mathematics could be derived from logic,

[262]This applies also to other concepts: for instance, Russell's notion that segments of rationals have all the required formal properties of the reals establishes the anti-Kantian position that the real number system is independent of spatiotemporal notions, though he only later becoming concerned with establishing their uniqueness and existence by means of explicit definition.

though he jettisoned this view as he arrived at his logicism between January and May, 1901.[263] Coffa's claim is that in the logicization of various geometries, Russell was concerned only that the derivations be logical. As part of the rigorization project and the attempts at gapless proofs in the derivation of geometrical theorems, Russell was certainly concerned to dispense with Kantian intuition, as Coffa emphasizes, but the step of dispensing with indefinables in mathematics was integral to logicism and crucial to the refutation of Kant.[264] In the May, 1901 draft, of Part 1 Russell takes a step in the direction of the logicist account of mathematical propositions involving only logical constants and variables whose values form a class, emphasizing the new meaning thereby attached to *a priority* of mathematics:

> **Thus pure mathematics must contain no indefinables except logical constants, and consequently no premises, or indemonstrable propositions, but such as are concerned exclusively with logical constants and with variables whose possible values form a class which is a logical constant. It is precisely this that distinguishes pure from applied mathematics….Thus, for example Euclidean geometry, considered as the study of all possible spaces of a certain type, is a branch of pure mathematics; but considered as the study of actual space, it belongs to applied mathematics...It may be observed that the connection of mathematics with logic, according to the above account is exceedingly close. The fact that all mathematical constants are logical constants, and that all the premises of mathematics are concerned with these, gives, I believe, the precise statement of what philosophers have meant in asserting that mathematics is a priori [Russell 1901c, 187].**

Byrd points out that it is in making revisions to the text, certainly later than fall, 1900 and likely around June, 1901, that Russell inserts a new leaf which would constitute Section

[263] This was the period during which Russell both adopted explicit definitions and replaced non-logical constants with variables so that mathematical concepts could be logicized.

[264] This is not to say that the logicist definition of mathematical concepts could not be carried out without explicit definitions. As we have seen, implicit definitions are sufficient in all but securing existence theorems required for practical applications of number. What is significant here is that Russell rejects formalism as he adopts logicism and dispenses with the indefinables of mathematics, adopting the form of conditionals which Coffa attributes to Peano and contrasts with Russell's approach.

378 of PoM, in which he claims that it is possible to eliminate indefinables altogether by replacing non-logical constants in the axioms with variables, the axioms in the antecedents becoming "parts of a definition" [PoM, 397]. Byrd writes:

> **The proposal is to take the axioms, replace the non-logical constants in them by variables and to regard the result as the definition of a certain kind of structure: "The axioms then become parts of a definition, and we have neither indefinables nor axioms" *(PoM,* p. 397). On this view, the propositions of pure mathematics are generalized implications, whose quantifiers range over logical entities, such as classes and relations. The antecedents may be regarded as defining a class of logically characterizable structures [Byrd 1999, 47–8].**

The so-called "if-thenism" predates logicism and coincides with Russell's endorsement of mathematical definition in Fall, 1900, while the so-called "categorical logicism", which, on Coffa's account, requires that mathematical concepts are definable in logical terms, coincides with Russell's use of the variable in place of the non-logical constants of mathematics in the May, 1901 draft of PoM.

Though he had immediately recognized the importance of the variable in connection with the notion of "any",[265] it was not until the May 1901 draft of PoM, after his initial formulation of logicism, that Russell introduces the variable, ranging over everything in the universe (and, as we have seen, variables replace primitive terms and the axioms become definitions in the hypothetical statements of geometry). In a "Note on

265 Immediately after his return from the Paris Congress, before he had even finished reading all of the works of Peano and his school which would inform the new symbolic logic adopted in PoM, Russell became concerned about the notion of 'any', writing to Moore on August, 16 1900: "Have you ever considered the meaning of any? I find it to be the fundamental problem of mathematical philosophy. E.g., "Any number is less by one than another number." Here any number cannot be a new concept, distinct from the particular numbers, for only these fulfill the above proposition. But can any number be an infinite disjunction? And if so, what is the ground for the proposition? The problem is the general one as to what is meant by any member of a defined class. I have tried many theories without success" [RA].

All and Formal Implication", likely written around May, 1901, in preparation of Part 1 of the Principles, Russell writes:

> **It seems *all* must be taken as an indefinable: for a formal implication is the assertion of all implications of a certain class, so that xεa. ⊃ₓ. xεb cannot be taken to define all, though it may define "a is part of b".**
>
> **A formal implication may perhaps be derived from a relation of assertions, as e.g.….εa. ⊃. …εb, but we shall still need formal implication as well as the relation of assertions.**
>
> **Observe that fallacies may arise if φ(x) is a proposition for some values of x but not for others. It may be doubted whether aεx is a proposition if x is not a class. It is not impossible that the contradiction may be soluble in this way [Russell 1901–2, 566].**[266]

In re-writing Part 1 and outlines for Part 1 between May, 1901 and April, 1902, Russell cannot decide whether to title it "the variable" or "the indefinables of mathematics", preferring the former in the May, 1901 draft, the latter in an intermediary draft, reverting to "the variable" in the May, 1902 draft, and finally settling on "the indefinables of mathematics" in PoM. In the April, 1902 outline, Russell does not intend to make changes to the definition of pure mathematics, but he does continue to puzzle about the nature of formal implication and assertion [Papers 3, 211-12] and, in the April, 1902 outline of Part 1, arrives at the notion of classes defined by propositional functions, arriving at the form of hypotheticals which Griffin attributes to him and which characterize his logicism of PoM.[267] In the use of propositional functions, central to the version of logicism he embraced in PoM on which propositional functions have the role

[266] The contradiction is the paradox of predication identified in Paper 2, May 1901 draft of PoM.
[267] In May, 1902, the explicit definitions of the cardinals, ordinals, and relation numbers are added [Byrd 1994, 57].

of defining a class of structures of a certain kind, Russell differs from Peano,[268] but in all other important respects, his conditionals take the form which Coffa attributes to Peano's conditionals. On the final version, Russell's conditionals are formal (quantified) implications in which the antecedents contain variables ranging over everything and the consequents assert a propositional function of the same variable ("for all x, if x is an *a*, then φx"). By clarifying such notions, Russell has arrived at the version of logicism on which pure mathematics is construed as the class of propositions of the form 'p implies q' where p and q are propositions containing one or more of the same variables and involving only logical constants. It seems that Russell's original attempt at deciding upon the true form of mathematical statements in the light of non-Euclidean geometry may well have emphasized the fact that it is the implication between the axioms and the propositions of mathematics, and not the axioms or the propositions which are asserted (or whose truth-value is concerned, as Coffa states it),[269] and that this is so without regard for whether the entities exist —thus distinguishing branches of pure mathematics from applied mathematics. It seems also that Russell refined his conception of the form of mathematical propositions along with his refined notion of implication, so that the

[268] Russell ultimately adopts the notion that classes are defined by propositional functions, though propositional functions are introduced quite late. In the May, 1901 draft, Russell construes 'x is a man', symbolized by f(x) as a complex proposition conjoining as many propositions as there are terms in the class of terms such that f(x). The chapter titled 'Assertion' in the April, 1902 outline is called 'Propositional Functions' in PoM and it would seem that propositional functions were added to Chapter 2 in early May, 1902, a few weeks prior to submitting PoM to Cambridge University Press for publication [Blackwell 1985]. For a detailed discussion, see Beaney 2009. Russell's notes for his Lectures on Logic at Cambridge in October, 1901, however, make use of propositional functions [Papers 3, 383].

[269] The importance of this earlier conception of implication is, obviously, that the various geometries do not consist in the assertion of inconsistent primitive propositions.

propositions of both arithmetic and geometry are formal implications of the sort which Coffa attributes to Peano, namely, formal (i.e. quantified) implications whose antecedents impose a categorical condition on the unrestricted variable, and whose consequents assert, by means of a propositional function, a condition of every value of the variable: "for all x, if x is an *a*, then φx".

Even if "if-thenism" is a position which prefigures logicism and logicism, on its earliest articulation, is shown to involve the logical definition of mathematical concepts, it nevertheless remains to reconcile this form of logicism with the so-called "standard logicism" which depends on explicit definitions. We have seen that the definitions of geometric concepts, as Russell construes them in the fall, 1900 draft, merely specify the formal properties required of a certain class of structures, and do not assert their existence or even fix their philosophical meaning so that, for instance, various spaces are said to be defined where the classes of terms in question are such that the terms have the required type of mutual relations. On this articulation of Russell's prioritization of mathematical over philosophical definitions of geometric concepts, which seems to commit him to a kind of formalism, group theory would be perfectly acceptable for the foundations of geometry or, at least, there is nothing, on the face of it, which prevents a group-theoretic basis for the logicization of geometry. In AOG, groups had a logical definition of a permutation group and in the October, 1900 draft of LoR, groups are treated in terms of relations of 1:1 correspondence, via the logic of relations. What has changed, then, for

Russell in the period between the October draft of LoR and “Recent Italian Work”, so that groups fall by the way just as soon as they become promising for logicist reductions in geometry?[270] This question may be answered by answering the question of why Russell abandoned formalism with the adoption of logicism, i.e., on the published version of LoR, the logicist requirements of the definition of number are not exhausted by those of implicit definition but require, apart from this, that the definitions secure the existence of the objects defined and, as we shall see, Russell’s appeal to explicit definitions for existence theorems in mathematics is crucial to his logicist project.[271] Consider the view which Russell espoused concerning the reals and the cardinals at the same time that he

[270] It should be noted that groups do not disappear altogether, for they are invoked in Russell’s account of distances in PoM. In LoR, Russell defines a quantitatively comparable class of distances, that is, a kind of distance as a series in which there is a term between any two, and it is also a group, such that if any two terms belong to the field of this group, there is a relation of the group which holds between them. Distance is here taken to be distinctive of series, which was the usual view, but what is more important is, we are told, that the group is such that the “relation of the group” holds between any two of its terms and it is in virtue of this operation, analogous to a special kind of addition. Gandon informs us that the group-structure is itself defined as: “a set K of bijective relations having the same field such that, firstly, if P belongs to K, the converse P belongs to K, and such that, secondly, if P and R belong to K, the relative product PR belongs to K” [Gandon 2008, 20]. In PoM, the group operation is accomplished by first transforming additive operations constituting the group into relations. Russell writes: “It sometimes happens that two quantities, which are not capable of addition proper, have a relation, which has itself a one-one relation to a quantity of the same kind as those between which it holds. Supposing a, b, c to be such quantities, we have, in the case supposed, some proposition aBc, where B is a relation which uniquely determines and is uniquely determined by some quantity b of the same kind as that to which a and c belong. Thus for example two ratios have a relation, which we may call their difference, which is itself wholly determined by another ratio, namely the difference, in the arithmetical sense, of the two given ratios. If α, β, γ be terms in a series in which there is distance, the distances $\alpha\beta$, $\alpha\gamma$ have a relation which is measured by (though not identical with) the distance $\beta\gamma$. In all such cases, by an extension of addition, we may put $a + b = c$ in place of aBc. Wherever a set of quantities have relations of this kind, if further aBc implies bAc, so that $a + b = b + a$, we shall be able to proceed as if we had ordinary addition, and shall be able in consequence to introduce numerical measurement” [PoM, 180]. It is the special operation of additivity, substituting for ordinary addition, upon sets transformed into relations having this feature that allow for the measure of distances.

[271] I would like to leave open the possibility that the rejection of formalism was incidental to the logicization of pure mathematics. It may be that the logicization of pure mathematics can be carried out by implicit definitions, and it is only applied considerations that necessitate explicit definitions giving existence theorems (and only considerations from within applied mathematics that place restrictions on logicization, i.e., topic-specific requirements may not be ‘logicist’ requirements).

privileged mathematical over philosophical definition in geometry. Although Russell's grounds for the identification of numbers with the common properties indicated by equivalence relations between classes arose out of the primacy of mathematical definition, which supplies the requisite formal properties of the "entities" with which philosophical definition is concerned, the axiom of abstraction is employed so that the entities that are the numbers are supplied in the definitions. On the early axiom of abstraction, equivalence relations between classes—the relation of coherence between classes of rationals in the case of the reals, and the relations of similarity (equinumerosity) between classes in the case of the cardinals—indicate common properties with which the numbers can be identified. In the case of the cardinals, Russell's view in November 1900 is that the inferred common properties "…make it plain that there are such entities"[272] and though he initially holds that it is not philosophically correct to identify the numbers with common properties in the case of the definition of the reals in the November, 1900 draft of Part V of PoM, the philosophical point is disregarded in a subsequent section of the same draft of the manuscript in favour of the view that the common properties indicate the existence of the reals.[273]

A crucial development around the time of the October, 1900 draft of LoR, where the definition of transfinite numbers was presented, was that Russell had finally abandoned the view that Cantor's set theory was riddled with paradoxes. In the 1899-

[272] Folio 97, November 1900, see Byrd 1994, 59.
[273] It is perhaps worth pointing out that when Russell recognizes the uniqueness problem, he is concerned that the numbers are indefinable—a concern which does not trouble him in geometry.

1900 draft of PoM, Russell's objections to Cantor on the basis of the paradox of the "number of (finite) numbers" had dissolved, but he went on wrestling with the philosophical problems associated with infinity. It is only in his January, 1901 paper, "Recent Work on the Principles of Mathematics", that Russell unequivocally praises Cantor for solving all the problems of infinity. Indeed, he is so convinced that Cantor's work is free of paradox, that he dismisses the paradox of the largest cardinal in the Winter of 1900-1901, believing it to be the result of a "very subtle fallacy" in Cantor's diagonal argument [RW, 375]. [274] Though the paradox of the largest cardinal leads to the contradiction of classes, Russell not only initially fails to appreciate the significance of the paradox, but, as we shall see in the following section, he only arrives circuitously at the contradiction of classes which vitiates the logicization of arithmetic, by a consideration of predicates not predicable of themselves. So, set theory and the problematic comprehension principle are ushered into logic to provide the basis for the theory of number and, in an important sense, the formalism which seems to have briefly accompanied Russell's conception of mathematical definition yielded to explicit

[274] Russell conveys the problem to Couturat on December 8th, 1900: "I have discovered a mistake in Cantor, who maintains that there is no largest cardinal number. But the number of classes is the largest number....[Cantor's proof] in effect amounts to showing that, if υ is a class whose number is α, the number of classes included in u (which is 2^{α}) is larger than α. The proof presupposes that there are classes included in u which are not individuals (members of) υ; but if υ=Class, that is false: every class of classes is a class" [CPLP, R08.12.1900].

definition. And, with the embrace of Cantorian set theory, groups fall by the way,[275] precisely as Russell begins to privilege mathematical over philosophical definition.

It is decidedly Russell's view in the final version of PoM that definitions of classes (or single members of a unit class) are given where a propositional function is asserted which specifies the defining property of the class. Where the object is defined by means of a transitive symmetrical (equivalence) relation, the principle of abstraction, by which Russell defined numbers as classes of classes, guarantees a class of classes as the logical object defined. It is the explicit definitions adopted in the identification of numbers with classes[276] which are essential to what Putnam calls Russell's "standard logicism." In arithmetic, applications like ordinary number statements favour explicit definitions and, we might add, the identifying of numbers as logical objects with classes of equinumerous classes makes it easy to express what is involved in counting.[277]

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[275] Russell may have been persuaded, in part, by Whitehead: the abstract of Whitehead's paper on group theory, given to the Royal Society in February, 1899 indicates that Whitehead held that groups are "a special type of set" and sets are fundamental, so that it may have been Whitehead's insistence on the primacy of sets over groups that led Russell to continue to hope that Cantorian set theory could eschew the paradoxes of infinity [Griffin forthcoming b, 23].

[276] Byrd points out that the explicit definitions are only introduced in the June, 1901 version of Part II of PoM. It is around this time that Russell identifies the numbers with classes of similar classes in LoR.

[277] Russell makes this case for the 'correct' definition of number as late as the *Introduction to Mathematical Philosophy*, where he maintains that we want our numbers to be such as can be used for counting common objects, and this requires that our numbers should have a definite meaning, not merely that they should have certain formal properties." He points out that "number", "0", and "successor" might be regarded, not as indefinable primitives with fixed meanings, but rather as variable terms, but concludes that, if this were so, then "it does not enable us to know whether there are any sets of terms verifying Peano's axioms" [IMP, 10]. Russell makes a case against the formalists in the 1938 Introduction to the 2nd edition of PoM, characterizing the position as follows: "As presented by Hilbert, for example in the sphere of number, it consists in leaving the integers undefined, but asserting concerning them such axioms as shall make possible the deduction of the usual arithmetical propositions. That is to say, we do not assign any meaning to our symbols 0,1,2…except that they are to have certain properties enumerated in the axioms….accordingly the symbols 0,1,2….do not represent one definite series, but any progression

Though, evidently, this incentive is not at work in geometry, the explicit definitions do have a role to play in geometry.

The role of the explicit definitions given by arithmetic, apart from supplying the logical objects involved in ordinary assertions of number and eliminating complication accruing to formalism in the account of counting (for instance, in a formal account of counting in terms of bijection of sets), is to supply "existence theorems" in the various branches of mathematics [PoM, 497]. Conditionals in Russell's logicism define classes of structures of certain types, but existence theorems given via the apparatus of set theory show that there are such classes satisfying the axioms, that is, there is some class defined. In short, Russell's logicism on the final version of PoM precludes formalism.[278] Insofar as Russell rejects formalism in PoM and identifies logical objects with the classes of (logically specified) structures of certain kinds, "conditional logicism" does not merely coexist with "standard logicism". Even if Russell's formal implications are correctly characterized as statements of the form 'if x is an *a*, then φx', explicit definitions remain a necessary supplement to the implicit definitions by means of axioms.

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whatever. The formalists have forgotten that numbers are needed, not only for doing sums, but for counting. Such propositions as 'there were 12 Apostles'…cannot be interpreted in their system. For the symbol "0" may be taken to mean any finite integer, without thereby making any of Hilbert's axioms false; and thus every number-symbol becomes infinitely ambiguous" [PoM, vi].

[278] We have seen already that this is not restricted to geometry: initially reluctant to identify the reals with segments of the rationals, content that the segments have the formal properties required (and hence all that is required for a refutation of the Kantian notion that intuitive notions must be introduced into the concept of continuity), Russell later committed himself to the view that the abstraction principle secured the reals as logical objects.

On the published version of PoM, definition in logical terms—a crucial component of the logicist project— involved the illicit theory of classes: the concepts of mathematics can be defined in terms of logical concepts, where these definitions are definitions of classes determined by propositional functions, and where existence theorems are supplied to show that there are such classes as those defined. Russell writes:

> **A definition is always…the definition of a class: this is a necessary result of the plain fact that a definition can only be effected by assigning a property of the object or objects to be defined, i.e., by stating a propositional function which they are to satisfy…And wherever the principle of abstraction is employed, i.e., where the object to be defined is obtained from a transitive symmetrical relation, some class of classes will always be the object required [PoM, 497].**

The following synopsis may be given of Russell's articulation of the important existence theorems derived from arithmetic in the concluding pages of PoM: The existence of zero is derived from the null-class, 1 from the unit class whose only member is the null-class, and so on for all finite numbers by the successor relation, aleph null, the least infinite cardinal, from the class of all finite cardinals, and ω, the least infinite ordinal, from the series of finite cardinals in order of magnitude.[279] The order type, η, of the dense, well-ordered infinite denumerable series is given from the definition of the rationals and their order of magnitude. The existence of the reals is given from the segments of rationals, the reals and the order type of the reals, θ. From the definition of the complex numbers,

[279] In PoM, ω can be defined as the class of serial relations such that, if u is a class contained in the field of one of them, then 'u has a successor' implies and is implied by 'u has aleph null terms or a finite number of terms', and the series of ordinals of the first and second classes in order of magnitude is of this type, so that a1 can be proved and defined as the number of terms in a series whose generating relation is of the type ω1, and so on for a1 and ω 2, up to aω and ωω, where ωω is the type of generating relation of a series such that, if u is a class contained in the series, to say that u has successors is equivalent to saying that u is finite or has, for an appropriate value of n, an terms. This process gives us a one-one correlation of ordinals with cardinals.

which bear "an essential reference to the plurality of dimensions", the class of Euclidean spaces of n dimensions is proved [PoM, 379] and the class of projective spaces is also given [PoM, 413]. Russell's criticisms of Dedekind, Weierstrass, Cantor, and Peano for the absence of explicit definitions seem to reflect a concern that the definitions specify the properties that certain mathematical entities must have without deciding the matter of whether such entities exist. This special task is carried out by Russell's "standard logicism".[280] Russell's logicism is first articulated with full awareness of the crucial insight of "if-thenism" that the propositions of mathematics do not assert that certain entities exist, but that if something is such and such an entity, then it will be such that 'so and so', i.e., if x is an *a*, then φx, where *a* might be a number or a point. However, Russell ascribes this form of conditionals to the propositions of mathematics precisely for the reason that it is crucial to the logicist project that the concepts of mathematics be definable in logical terms: by replacing the indefinables with variables and permitting propositional functions to transform axioms into definitions, Russell has departed from the "if-thenist" position on which the conditional status of mathematical propositions is

[280] According to the definition of 'existence' adopted from Peano, i.e., the class α is non-empty, symbolized by Ǝα, [Grattan-Guinness 2000, 300]. Russell appears to have held that it was necessary, in defining classes in mathematics, to show that they were not null. In PoM, Russell indeed construes existence in this way., writing: "The existence-theorems of mathematics – *i.e* the proofs that the various classes defined are not null – are almost all obtained from Arithmetic" [PoM, 497]. When Russell says that existence theorems are required to show that the relevant classes are not null, he adds that he means this in his "strict sense," [PoM, 372] presumably, in the sense that for all x, x is an a is not always false [PoM, 21]. Apart from the notion of existence involved in Russell's account of existence theorems, the notion of existence has generally been a source of confusion in attempts to interpret PoM, which is reflected in attempts within the literature to make sense of his changing conception of the null-class, his theory of terms, and of his conception of classes generally. I shall address the issues of the null class in connection with the theory of denoting in a subsequent section.

constituted by the logical derivation of theorems from axioms. Russell did supplement this brand of logicism with the untenable "standard logicism" which trades on the misbegotten naïve comprehension principle on which explicit definitions supply the existence theorems guaranteeing that there are such classes as those defined, which rests on a fundamental confusion not only about the requirements of mathematics, but also about what belongs properly to logic. Clearly, the trouble is caused by "the puzzling notion of the class" involved,[281] a notion which makes the logic employed in his refutation of Kant more informative than Russell had hoped.[282] The fact that "standard logicism" has a marginal application, even in arithmetic, where the implicit definitions given by Peano, Cantor, and Dedekind suffice,[283] combined with the fact that logicism without explicit definitions is not merely "if-thenism", is significant in interpreting the logicist project of PoM. It is also helpful for understanding the broader context in which Russell's explicit definitions were carried out.[284]

3.3 THE LOGIC OF RUSSELL'S LOGICISM AND THE CONTRADICTION

[281] This is Russell's expression at PoM, 497.
[282] In hindsight, Russell appears to have seen the humour in this, remarking in the introduction to the 2nd edition of PoM: "Henri Poincaré, who considered mathematical logic to be no help in discovery, and therefore sterile, rejoiced in the contradiction: 'La logistique n'est plus sterile; elle engendre la contradiction!'" [PoM, xii].
[283] The idea, stated generally, is that any axiomatic system produces definitions merely in terms of the relational structures exhibited by the axioms.
[284] It seems clear enough that Russell believed it necessary to supplement implicit definitions with existence theorems supplied by explicit definitions, though the extent to which this should be regarded as central to his logicism is less clear. An advantage of Gandon's topic-specificity thesis is that on such an account, the role of explicit definitions can be excluded from logicism and the fact that, on Russell's account in PoM, implicit definitions sufficed for mathematical purposes can be accurately presented, while allowing for restrictions to be imposed, on topic specific grounds, on what is to be logicized (preventing if-thenism). The disadvantage is that the conditions for logicization follow no general formula.

A great deal of work has been done to capture the similarities and the differences between Frege's and Russell's logicist projects. The most obvious difference is that Frege's logicist project was confined to Arithmetic. Arguably, this had to do with advances in the axiomatization of projective geometry which allowed for the reductions to be carried out by means of relations without appeal to spatiotemporal notions.[285] As we have seen, Russell's project of logicizing the various branches of mathematics, including geometry and the theory of magnitudes, is aimed at preserving the truth of existing mathematics and ought not to be understood as the arithmetization programme [Gandon 2008]. While various attempts have been made to characterize the different features of the logic to which arithmetic propositions were supposed to be reduced on Frege's and Russell's respective logicist programs, there is a point on which they are supposed, for good reason, to agree: the definition of the cardinal numbers, which, we have seen, Russell had regarded as indefinable before his discovery of Peano and the development of his own logic of relations. In considering some of the crucial developments in Russell's logicist project in the preceding section, I pointed out both that Russell's notion of the formal implications constituting mathematics did not precisely resemble Peano's and that, even though it was at work in Cantor's paradox of the greatest cardinal, the contradiction of classes which threatened the logicization of arithmetic was initially articulated in terms of the paradox of predicates not predicable of themselves. These two important points are connected by the introduction of propositional functions and are both addressed in

[285] In this, Russell differed from Frege, though Frege was also aware of these advances in Projective geometry. See Wilson 1992.

Russell's philosophical treatment of the indefinables of mathematics in Part 1 of PoM. Since they also figure prominently both in Russell's and Frege's different conceptualizations of the problem confronting the logicization of arithmetic and in their proposed solutions, I shall present them briefly before examining the so-called Frege-Russell definition of number to see whether the points of divergence are not so significant as to make it impossible to say that Russell and Frege were in agreement as to the logicization of arithmetic.

After defining pure mathematics as implications involving variables and logical constants, the notion of relation, and the notion of "x *such that* φx" (where the values of x are a class satisfying the propositional function), Russell contends with the fundamentals of symbolic logic, by which he means the true symbolic logic developed by Peano and his school and supplemented by his own (intensional) logic of relations. Whereas Boolean algebra, with its emphasis on equations, had regarded "=" as standing for either the co-extensionality of classes or the equivalence of propositions and in general regarded the letters in symbolic expressions as standing for either classes or propositions with emphasis on the parallelism between inclusion and implication,[286] Russell wishes not only to follow Peano in strictly separating the two notions, but to introduce further logical precision into the distinction. Importantly, in this connection, Russell points out that the true logical distinction is between the relation of class inclusion and the relation of

[286] Of course, this approach influenced Russell through Whitehead's *Universal Algebra* where the calculus was concerned precisely with equivalence statements and this was embodied by the substitutivity of signs, where, under some limitations, no distinction of property prevented the substitution [Whitehead 1898, 5].

implication between *genuine propositions*. Propositions of the form “x is a man” are not genuine propositions, since they are neither true nor false, but contain real variables whose differing values produce differing propositions. Propositions of the form “(for all x) x is a man implies that x is mortal” are genuine propositions, since the whole implication is either true or false independent of the values of the variable, which is only apparent, that is, such propositions assert a relation which holds for all values of x. In the proposition “p implies q”, what is asserted is the relation of implication and the propositions p (x is a man) and q (x is mortal) are merely unasserted complexes under consideration and, in order for the proposition q to have the logical property of assertion, the proposition p must not merely be considered, but asserted, along with the assertion of the implication. The trouble with this view, of course, is that if p and q are unasserted propositions in “p implies q”, it must be these very propositions which are capable of being true and not new propositions possessing the logical property of being asserted. Russell’s concern is that what is asserted in a formal implication—φx implies ψx—will be understood to be an assertion concerning the meaning of *the symbol* (i.e., a definition of x). Russell alleges that this is what Peano has in mind—for instance, he holds that in ‘x is a man implies x is a mortal’, ‘x’ designates the class of men. The meaning of the *symbol* is being interpreted in such a way that the assertion of the consequent ‘x is mortal’ depends on whether x is a man, where what is supposed to be asserted is the formal implication, ‘for any x, x is a man implies that x is mortal’, in which the variability of x is unrestricted.[287] Peano, according to Russell, nullifies the purpose of formal implication [PoM, 37]. We can see Russell working towards a clarification of these notions in his April, 1902 outline of Part I of PoM, where he writes, concerning Chapter III 'Implication and Formal Implication, "...Meaning of $\varphi x \supset \psi x$. Notion of *all* terms essential?" and, concerning Chapter VII "Assertion": "...Formal implication again: Is $\varphi x \supset \psi x$ an assertion about x? Difficulties in so analyzing a proposition" [Russell 1902b, 212]. The fundamental confusion is clear in Peano's notion that 'the x's such that x is an a are the class a', which trades on the confusion between the class of x's (such that x is an a) and the class-concept 'a'. It is of the utmost importance, in the assertion of formal implication, that the unrestricted variable be preserved, that is, that in 'x is a man, implies that x is mortal', x should not mean 'the class of men', but 'for all values of x', so that x is varied for the proposition 'x is a man implies x is a mortal' as a whole[288] and wherever

[287] This hinges on the recognition that formal implication is not reducible to the relation of inclusion between classes. Russell's way of making sense of the unrestricted variable is to say that when we assert the implication, what we are really asserting is that every member of a class of material implications is true. Ordinarily, this means the class of all propositions in which an assertion made of a subject(s) is affirmed to imply another assertion concerning the same subject(s), e.g., Socrates is a philosopher is affirmed to imply Socrates is human is affirmed to imply Socrates is mortal. Where the subject is replaced by a variable, it might appear that what is involved in an implication is the relation of inclusion between classes. Russell holds that this error arises from regarding the assertion as giving the meaning of the variable symbol, reducing formal implication to the relation of inclusion between classes. If an implication merely asserts a relation of class inclusion, e.g., "x is a man implies x is a mortal" merely states the inclusion of "all men' in 'all mortals", then the relation between the assertions for any x with unrestricted variability is nullified [PoM, 36-7].

[288] Russell considers whether what seems basic and indefinable in propositional functions might be identified with assertions plus 'every term' about which it is made or 'every proposition' containing it. Once a proposition is decomposed into its constituent terms (or into its subject-term and assertion), its original unity is destroyed: We have the relation or assertion as term, but not the relation or assertion as it relates the terms or asserts something of a term.. Recall that there is a single propositional function for the whole corresponding formal implication; there, what is asserted is the truth of the propositional function "x is a man implies x is a mortal" for all x. To analyze this into a relation asserted to hold between the

a constant replaces x the resulting proposition, where true, implies the proposition 'x is mortal' for that value of x. As we have seen, mathematics involves propositions of the form of formal implications, where the conditionals are quantified, 'for all x' and, to avoid Peano's confusion of the class and the class-concept, the classes are *defined by propositional functions.* What is involved, then, is a single propositional function, indicated by the class concept, where 'χ is a υ' is a propositional function iff υ is a class-concept, whatever the value of χ. Every propositional function which is not null defines a *class*, denoted by 'x's such that φx', where the corresponding *class-concept* is the singular 'x such that φx'. All values, then, for which 'x such that φx (is true)' form a class[289]—the class of x's such that φx—and Russell is led to say that "any propositional function in which a fixed assertion is made of a variable term is to be regarded as giving rise to a class of values satisfying it" [PoM, 77].

An important feature of the true symbolic logic is, then, that it does not confound class-propositions with subject-predicate ones. In the Boolean logic, this conflation results in the notion that the inclusion relation was essentially that of being part of a

function and the variable (subject)– "x is a man implies x is mortal" is represented by φx, where "y is identical to φx" is equivalent to "y has the relation R to x"—introduces more complexity than in the original propositional function. Propositions containing two independent variables and a constant relation, xRy, cannot be analyzed into the assertion R concerning x and y, since the directionality of the relation is not reproduced when R is taken to be the assertion. It is not analyzable into the assertion …Ry, since this makes y a constant, which needs to be varied by introducing to …Ry, "for different values of y", but this fails to capture the independent variability of x and y in the propositional function xRy When the terms are fixed and the concept (relation) is variable, as in propositions which satisfy the propositional function aRb, the unrestricted variability of the relation requires that the propositional function be "R is a relation implies aRb". This propositional function (R is a relation implies aRb) defines the class of relations holding between a and b, where the propositional function is satisfied by only some values of R [PoM, 85-6].

[289] I shall address the paradox this gives rise to in a subsequent section.

manifold or collection unified by possession of a common predicate, and to ensure that it is not at work in the new logic, Russell requires that the account of inclusion and formal implication be carried out independently of analysis into subject and assertion (e.g., 'Socrates' and 'is a man') and, further, that propositional functions be introduced to preserve the distinction between the class and the class-concept, primarily in order to avoid confusions in the extensional treatment of classes. We have seen that a fixed assertion made of a variable term, indicated by the notion of 'such that', gives rise to a class and that formal implication is the assertion of a proposition involving universal quantification of individuals 'such that', i.e., over propositional functions. A predicate-concept does not suffice for determining a class and the relation of inclusion between predicate-concepts does not suffice for formal implication. The conflation of class propositions with subject-predicate ones also undergirded the assumption that class propositions are more ultimate than relational propositions, so that relations, which are not given a formal treatment in Peano's logic, were treated, by Schröder and Peirce, as classes of couples.[290] Russell begins instead with an intensional logic of relations to supplement Peano's logic. In his retrospective account in a letter to Philip Jourdain, dated

[290] Russell's notes on Schröder's *Vorlesungen über die Alge-bra der Logik, Der Operationskreis des Logikkalkuls* (1877), and "Sur une extension de l'idée d'ordre" (1901) are dated 1901 [Anellis 1990/1991]. Russell's notes on Charles Peirce's "On the Algebra of Logic" (1880) and "On the Algebra of Logic: A Contribution to the Philosophy of Notation" (1885) are dated from 1900-1901 [Anellis 2004/2005, 75]. Peirce, who claimed to have found PoM "superficial to nauseating", objected to Russell's criticisms. According to Irvine Anellis, "The gist of Peirce's marginalia to his copy of the Principles was that Russell's difficulties with Peirce's and Schröder's ostensible lack of proper distinctions was rooted in Russell's own failure to distinguish material implication and truth-functional implication (conditionality), and in Russell's erroneous attempt to treat classes, in function-theoretic terms, as individual entities" [Anellis 2004/2005, 78].

April 15, 1910, Russell wrote: "During September, 1900 I invented my Logic of Relations; early in October I wrote the article which appeared in RdM VII2-3…Oddly enough, I was largely guided by the belief that relations must be taken in intension, which I have since abandoned, though I have not abandoned the notations which it led me to adopt" [Grattan-Guinness 1977, 132–4].[291] In "The Logic of Relations with Some Applications to the Theory of Series", Russell does not define 'relation' from 'class' and 'ordered pair', but introduces it as a primitive, and preserves the intensional doctrine throughout.[292] We have seen that the intensional doctrine arises from the analysis, influenced by Moore, of relations differing in sense in propositions. In Russell's doctrine of relations, the primitive proposition is required that where a relation holds between two terms, that relation is ultimate and does not hold between any other two terms, which, he tells us, is analogous to the view that any term is the only member of some class.[293] Russell's intensional view of relations leads him to treat co-extensive relations as distinct

[291] This refers to the October, 1900 draft of LOR. In the published paper, "The General Theory of Relations" takes the place of "The General Propositions of Logic" for the first chapter and then Russell proceeds to cardinal number, leaving out groups. In "The Logic of Relations with Some Applications to the Theory of Series", Russell writes that "the logic of relations…must serve as a foundation for mathematics, since it is always types of relations which are considered in symbolic reasoning" [LOR, 314].

[292] The notion of relation requires the axioms: If R is a relation, then so are its converse and its complement; if R and S are relations, then so is their relative product; if K is a class of relations, then its union and intersection are relations; for any x and y, there is a relation holding only between x and y; membership, identity of individuals, and similarity are relations [LOR, 311]. In PoM, Russell abandons axioms that similarity, class membership, the identity of individuals (classes), and the union of classes of relations are relations.

[293] A term is not to be identified with the class whose only term it is—Russell attributes this view to Frege's influence, but it is contained already in his criticisms of Peano, before he read Frege and revised PoM in 1902.

and merely logically equivalent, but not identical.[294] He characterizes co-extension in terms of the implication relation between the equivalent relations, such that R and R′ have the same extension when xRy implies and is implied by xR′y for all values of x and y. Given the extension of a relation, it is possible (even though relations are taken in intension) to define a relation that is specified uniquely when the extension is specified. The formal identity of the two co-extensive relations is explained by the identity of the classes of relations equivalent to each of the co-extensive relations respectively. When the extension is determinate, he tells us, we can identify two co-extensive relations (R, R′) by replacing one relation R with the logical sum (what Russell also calls a 'class') of the relations equivalent to R which, in virtue of the logical equivalence of the two relations R and R′, will be identical to the logical sum of the relations equivalent to R′.[295] The purpose of the construction is technical but it enables us to identify a relation given some extension by means of identical classes. Having noted these features of Russell's logic, it is possible to examine the first articulation of the contradiction, which, as we shall see, necessitates revisions in Russell's intensional notion of classes and relations.

The first statement of the paradox of predication and Russell's first insight into the need for rejecting Peano's naïve comprehension principle, i.e., the principle that every

[294] Recall that relations R and S are extensional iff (Vx,y)((xRy ⇔ xSy) ⇒ R = S); otherwise they are intensional, cf note 208.
[295] In fact, Russell offers two ways of executing the formal identity: we may produce a relation from either the logical product or the logical sum of all the relations having the same extension. The logical product and logical sum of two relations is a relation. The logical product R^S is (Vx,y)(xRy & xSy), the logical sum RvS is (Vx,y)(xRy v xSy).

definable collection of terms forms a class defined by a common predicate, is found in the May, 1901 draft of PoM. In the 1901 draft of PoM, the difficulty with predicates not predicable of themselves led him to reject the notion that every definable collection of terms forms a class defined by a common property. Concerning those predicates which are not predicable of themselves, Russell writes:

> **These are the referents (and also the relata) in a certain complex relation, namely the combination of non-predicability with identity. But there is no predicate which attaches to all of them and to no other terms. For this predicate will either be predicable or not predicable of itself. If it is predicable of itself, it is one of those referents by relation to which it was defined, and therefore, in virtue of their definition, it is not predicable of itself. Conversely, if it is not predicable of itself, then again it is one of the said referents, of all of which (by hypothesis) it is predicable, and therefore again it is predicable of itself. This is a contradiction which shows that all the referents considered have no common predicate and therefore do not form a class.…It follows that not every definable collection of terms forms a class defined by a common property [Russell 1901c, 195].**

Those predicates not predicable of themselves form a determinate collection of referents of the relation of non-predicability of self, but there is no predicate which is common to the members of the collection by which they may be said to form a class. The conclusion is reiterated in the chapter on Relations in PoM: "This is a contradiction which shows that all the referents considered have no exclusive common predicate and, therefore, if defining predicates are essential to classes, do not form a class" [PoM, 97].[296]

Interestingly, in *PoM* (prior to revisions added late in 1902), Russell thinks the important consequence of the contradiction is that it is not clear that there is always a

[296] The contradiction is reiterated in the published version in the chapter on Classes, in which Russell writes: "It is natural to suppose that [the predicates not predicable of themselves] form a class having a defining predicate. But if so, let us examine whether this defining predicate belongs to the class or not. If it belongs to the class, it is not predicable of itself, for that is the characteristic property of the class. But if it is not predicable of itself, then it does not belong to the class whose defining predicate it is" [PoM, 80].

defining predicate for a class determined by "being an x such that φx". In Appendix B on the doctrine of types, Russell gives an independent reason for rejecting this principle and adopting an extensional view of classes, which has to do, again, with predicates: "There are, we know, more classes than individuals; but predicates are individuals. Consequently, not all classes have defining predicates. This result, which is also deducible from the Contradiction, shows how necessary it is to distinguish classes from predicates, and to adhere to the extensional view of classes" [PoM, 526].[297] It is the need to recognize that not every proposition containing only one real variable asserts a predicate or class-concept, that predicates and class-concepts must be distinguished from propositional functions, and that not every propositional function which defines a class indicates a corresponding predicate or class-concept that are the lessons of the contradiction:

> **It must be held, I think, that every propositional function which is not null defines a class, which is denoted by '*x*'s such that φ*x*.' such that it will always entail the concept of a class and corresponding class-concept will be the singular 'x such that φx'. … But it may be doubted—indeed the contradiction with which I ended the preceding chapter gives reason for doubting—whether there is always a defining predicate of such classes [PoM, 88].**

The same is the case for class concepts not members of their own extensions:

> **We shall maintain, on account of the contradiction there is not always a class-concept for a given propositional function φ*x*, i.e. that there is not always, for every φ, some class-concept *a* such that *x* ε *a* is equivalent to φ*x* for all values of x [PoM, 514].**

[297] Russell also points out the need for the extensional view of relations on the same grounds, i.e., that there are more classes of couples than couples and hence individuals, but every relation as verb is an individual, so not every class of couples is the extension of some relation (verb).

As we have seen, the analysis of propositions into propositional functions which have propositions for their values is an alternative to the analysis of a whole proposition into its simple constituent parts by analysis into subject and assertion, and prevents the confusion of class propositions with subject-predicate ones.[298] We have seen that Peano's confusion of the class and the class-concept in 'x's such that x is a are the class a' and that rather the class of x's such that φx must consist in all those values of x which satisfy the propositional function. Initially, Russell holds that there is only a difficulty in the notion that "any propositional function in which a fixed assertion is made of a variable term is to be regarded as giving rise to a class of values satisfying it" if the assertion is a predicate or class-concept separable from the function which is supposed to define the class of terms. Importantly, propositional functions contain their arguments as constituents just as propositions contain their terms as constituents and propositional functions must not be regarded as entities separate from their variables, but "live in propositions of the form φx and cannot survive analysis" [Russell 1904b, 86]. That is to say, propositional functions are not separable into function (assertion) and variable (term) and so are not constituents of propositions, because this would engender a contradiction. If we regard the φ in φx as separable, so that we can predicate it of itself or assert it of itself, φ is φ or φ has φ, then we can also deny it ~(φ is φ) or ~(φ has φ) . Where the predicate is non-predicability of self, or the assertion is "non-assertability of self", this results in a contradiction. However, it is important to stress that the lesson is that propositional functions are *not akin to*

[298] Russell's distinction between assertions and propositional functions is given in PoM, 39-40, 83ff.

predicates or class concepts and, for that reason, *do not give rise to a contradiction*, so that there is no difficulty in the notion that the propositional function always determines some class, but only with the notion that every class has a corresponding class-concept or is defined by a common predicate. In other words, the contradiction does not arise for propositional functions, properly understood. Again, in the chapter on Relations, Russell draws the consequence from the contradiction that the notion that all terms having a fixed relation to a given term form a class defined by a common predicate results from the analysis of aRb into subject a and assertion Rb, where Rb is a predicate. However, when xRy is considered, it is not clear that a *predicate* is implied by being a term of which Ry, for some value of y, can be asserted, though the doctrine of propositional functions requires that such terms form a class [PoM, 98].

There are, it turns out, propositional functions of the sort that do seem to give rise to the contradiction. Certain propositional functions, which he calls "quadratic forms", differ from ordinary propositional functions in which the φ and the x in φx are constant or varied without reference to one another, in that, in their case, the x is a function of the φ, so that it is varied where the φ is varied, that is, in such cases, φ is asserted of x in the sense of being asserted of the class of terms satisfying φ. Initially, Russell attempts to solve this problem by proposing that such a propositional function guarantees only a collection of terms, but not a "class as one".[299] Russell is persuaded by the contradiction

[299] In his July 10, 1902 letter to Frege, Russell writes: "I believe I can therefore say without contradiction that certain classes (namely those defined by quadratic forms) [those defined by propositional functions of

to adopt an extensional view of classes and relations and to restrict the range of significance of the propositional functions. The propositional functions are hierarchized according to their ranges of significance of propositional functions and thus corresponding to types—the class of x's such that φx is a proposition. In Appendix B "On the Doctrine of Types", Russell writes:

> **The doctrine of types is here put forward tentatively, as affording a possible solution of the contradiction; but it requires, in all probability, to be transformed into some subtler shape before it can answer all difficulties...Every propositional function φ(x)—so it is contended—has, in addition to its range of truth, a range of significance, i.e., a range within which x must lie if φ(x) is to be a proposition at all, whether true or false. This is the first point in the theory of types; the second point is that ranges of significance form types, i.e., if x belongs to the range of significance of φ(x), then there is a class of objects, the type of x, all of which must also belong to the range of significance of φ(x), however φ may be varied [PoM, 523].**

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the above type] are mere manifolds and do not form wholes at all" [PMC, 137]. The propositional function is satisfied by the terms of the class (the class as many), but not by the class itself (the class as one). In PoM, this is articulated first in terms of class concepts: "Let R be a relation, and consider the class *w* of terms which do not have the relation R to themselves. Then it is impossible that there should be any term *a* to which all of them and no other terms have the relation *R*. For, if there were such a term, the propositional function 'x does not have the relation R to y' would be equivalent to x has the relation R to *a*'….When in place of R, we put ε—the relation of a term to a class-concept which can be asserted of it—we get the above contradiction [PoM, 102]" Then in terms of the class as a single term satisfying the propositional functions: "Every propositional function which is not null…defines a class, and every class can certainly be defined by a propositional function. Thus to say that a class as one is not a member of itself as many is to say that the class as one does not satisfy the function by which itself as many is defined….If any propositional function were satisfied by every class having the above property, it would therefore necessarily be one satisfied also by the class *w* of all such classes considered as a single term. Hence, the class *w* does not itself belong to the class *w*, and therefore there must be some propositional function satisfied by the terms of *w* but not by *w* itself….[W]e must suppose, either that there is no such entity as *w*, or that there is no propositional function satisfied by its terms and by no others" [PoM, 103]. What Russell initially thinks the Contradiction shows is that it is not always the case that the class as many, determined by a propositional function, requires a class as one which also must satisfy the propositional function: "Perhaps the best way to state the suggested solution is to say that, if a collection of terms can only be defined by a variable propositional function, then, though a class as many may be admitted, a class as one must be denied. When so stated, it appears that propositional functions may be varied, provided the resulting collection is never itself made into the subject in the original propositional function. In such cases there is only a class as many, not a class as one. We took it as axiomatic that the class as one is to be found wherever there is a class as many; but [b]y denying it...the whole difficulty will be overcome" [PoM, 104].

In granting primacy to propositional functions, Russell privileges structures over entities and, though the change is by no means immediate, he is led to dispense with classes and, ultimately, propositions. However, before attempting to solve the contradiction, Russell had first to recognize its significance.[300]

In a letter to Frege, June 16, 1902, Russell conveyed the contradiction in terms that seemed inapplicable to Frege's philosophy, but which would undermine the foundational Basic Law V of his arithmetic. Russell wrote:

> **Let w be the predicate of being a predicate which cannot be predicated of itself. Can *w* be predicated of itself? From either answer follows its contradictory. We must therefore conclude that *w* is not a predicate. Likewise, there is no class (as a whole) of those classes which, as wholes, are not members of themselves. From this I conclude that under certain circumstances a definable set does not form a whole [PMC, 130].**

While Frege does not have a difficulty on the intensional version of the contradiction, that is, with the paradox of predication, given that, in his philosophy, a concept cannot be predicated of itself ,[301] he has nevertheless to contend with the extensional version of the paradox of the class of classes not members of themselves, since his Basic Law V—that the course-of-values of the function (concept) F is identical with the course-of-values of

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[300] To the extent that he did not immediately apprehend the significance of the paradox, Russell was in good company: Burlali-Forti had, in his 1897 paper "On Well-Ordered Classes", articulated the paradox of the largest ordinal without recognizing it as such, and Cantor had, in letters to Dedekind written in the summer of 1899, taken the paradox of the largest cardinal as evidence for the need to distinguish "inconsistent multiplicities" from genuine unities (sets) [Griffin 2004, 351].

[301] That is, a function (a predicate in extension) is never an object and a first-level function (which a predicate is) must have an object for its argument, and never a function. In his June 22, 1902 letter to Russell, Frege writes: "…the expression 'A predicate is predicated of itself' does not seem exact to me. A predicate is as a rule a first-level function which requires an object as argument and which cannot therefore have itself as argument (subject). Therefore I would rather say : 'A concept is predicated of its own extension' [PMC, 132-33].

the function (concept) G if and only if F and G are co-extensional—assumes that every concept has an extension. Frege uses Basic Law V to prove Hume's principle—that the number of F's is identical to the number of G's iff F and G are equinumerous. This strategy may not be strictly necessary,[302] though it is doubtful whether bypassing this axiom and preserving a *Fregean* logicism may be accomplished at the same time—a question which I most certainly shall not attempt to tackle. In an appendix added to his *Grundgesetze der Arithmetik*, Frege wondered:

> **Is it always permissible to speak of the extension of a concept, of a class? And if not, how do we recognize the exceptional cases? Can we always infer from the extension of one concept's coinciding with that of a second, that every object which falls under the first concept also falls under the second? [Irvine 1999, 1].**

Frege's comprehension principle was insupportable and, though Basic Law V may not be needed, Frege did not see a way around it for introducing the equivalent of the set-theoretic apparatus out of which arithmetic is built, where the many-one relation between equinumerous concepts and numbers supplied by Hume's Principle is backed up by the one-one relation between concepts and extensions supplied by Basic Law V. For Russell, who does not distinguish concept and object in the first place, the problem which initially appears only to concern the quasi-logical predicates or class-concepts, is at the heart of the theory of classes and, hence, is inherent to his logic (of classes and relations), so that the existence of classes would have to be jettisoned from the logic. However, to preserve the universality of logic—wherein the propositions of logic are wholly general and

[302] Parsons 1965, and Hale and Wright 2001.

variables involved in them range over everything in the universe, and are not restricted to a universe of discourse—and to preserve the logicization of mathematics, Russell develops a logic in which (intensional) propositional functions, which we have seen are implicit in the logic of propositions, are logically basic and classes are incomplete symbols appearing in sentences expressing propositions about propositional functions.[303] Russell's explicit definition of number may be illustrative of the conception of logic underlying the attempt to logicize mathematics and, here, the comparison with Frege, who is supposed to have shared the so-called Frege-Russell definition of number, will be informative.

[303] In the Introduction to PM, Russell and Whitehead write: "[A] function can be apprehended without its being necessary to apprehend its values severally and individually…What is necessary is not that the values should be given individually and extensionally, but that the totality of the values should be given intensionally, so that, concerning any assigned object, it is at least theoretically determinate whether or not the said object is a value of the function" [PM, 40]. Propositional functions are intensional and type-stratified. The fact that propositional functions are type-stratified is not supposed to be a problem for logicism, since mathematics is concerned with extensions (of propositional functions) and a propositional functions of the lowest order co-extensive with a propositional function of any order is given on assumption of the Axiom of Reducibility.

CHAPTER 4: LOGIC AND ANALYSIS IN RUSSELL'S DEFINITION OF NUMBER

4.1 RUSSELL'S AND FREGE'S LOGICIST DEFINITIONS OF NUMBER

It is generally agreed that in defining the cardinals as classes of equinumerous classes in 1901, Russell had independently discovered Frege's definition of the cardinals.[304] The claim to independent discovery is true enough,[305] but the claim that what was discovered was Frege's definition may require some qualification. The extent to which Russell`s conception of the cardinals should be viewed as akin to Frege's is a matter of historical importance, insofar as points of divergence between Frege's and Russell's definitions of the cardinals illuminate more fundamental differences in their logicist projects on the very point on which they are supposed to agree, namely, the logicization of arithmetic. It has been argued that while Frege simply accepted that numbers as logical objects are correlated with value-ranges (classes), i.e., correlated with concepts whose extensions we apprehend,[306] Russell was concerned with the metaphysical status of abstracta resulting from definition by abstraction. James Levine writes:

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[304] Russell puts it this way in Chapter II of his *Introduction to Mathematical Philosophy*.

[305] There is, however, a connection: in March, 1901 Russell read a paper in which Peano rejected the definition of the number of any class, a, as the class of classes similar to a on the grounds that numbers have different properties than these classes of classes. This is the 'same definition' given in Frege's *Grundgesetze*, which Peano reviewed in 1895 [Papers 2, xxvii].

[306] For Frege, the concept of number is not a class *per se*, but is essentially a second-level concept that a first level concept falls within and which, on Frege's view, can nevertheless be correlated with an object, i.e., with a range of values.

> **Frege, unlike Russell, does not introduce such definitions in order to address fundamental questions regarding the metaphysical status of abstracta or our knowledge of them, [hence] Frege, unlike Russell (in *PoM*), is in a position to hold that with regard to those fundamental questions, classes are no different from other abstracta [Levine 2007, 71].[307]**

There is some truth in an account of this sort. It invites us to consider the important manner in which Russell, in PoM, favoured "exact analyses" intended to exhibit the basic constituents of the universe, while Frege settled on the view that value-ranges (classes) were, ontologically, on a par with all other logical objects which could only be apprehended as extensions of concepts (ranges of values of functions).[308] Nevertheless, on my view, it also tacitly invites us to view the points of divergence between Russell's and Frege's conceptions of abstracta as "philosophical" or "metaphysical" concerns, separate from the logical issues that Russell thought were introduced into a purely formal definition of the numbers as classes within his logic of relations and propositional functions.[309] I wish to reject interpretations on which the central difference between the Fregean definition of the cardinals and Russell's early attempts at an analogous definition is supposed to be philosophical, even to primarily concern the metaphysical implications of abstraction principles. On such interpretations, Russell did not depart significantly from the Fregean definition of number in PoM, but simply clarified the definition by

[307] Here, Levine uses 'classes' to mean Frege's 'value-ranges'.
[308] In his letter to Russell, July 28th, 1902, Frege writes: "…the question is, How do we apprehend logical objects? And I have found no other answer to it than this, We apprehend them as extensions of concepts, or more generally, as ranges of values of functions" [PMC, 141]. It is not clear how Frege ought to be interpreted concerning the existence of value-ranges, which seems to depend on recognizing Axiom V as a law of logic and not, in the first instance, on 'how we apprehend' logical objects.
[309] Hylton also characterizes Russell as building his logic on a pre-existing metaphysics and Frege as having a metaphysics that fell out of the logic. For a more informative formulation of the position, see Hylton 2005, 71.

addressing philosophical considerations on the metaphysical status of abstracta.[310] The result of this reading, I think, is that the importance of the difference between Frege's function-argument analyses and Russell's analyses into relations and propositional functions for their respective definitions of number never becomes entirely clear.

While the definition of the cardinals that Russell articulates in 1901 is very similar to Frege's definition and has some of the same advantages, an understanding of the difference between these definitions depends essentially on how the relation of "being the cardinal number of a class (or concept)" is defined and on the notion of 'class' involved. This, I shall suggest, can be appreciated only by recognizing that Russell's version of the definition emerges from his intensional logic of relations and propositional functions. It is important to recognize that the status of classes underwent a series of changes as Russell attempted to work out a logical solution to the Contradiction, but the changing ontological status of classes resulted from Russell's requirement that a solution to the Contradiction be carried out within an intensional logic of relations and, later, propositional functions.[311] For Frege, number statements certainly have an intensional dimension insofar as the meanings of number statements have the two aspects of *Sinn* and *Bedeutung*

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[310] Arguably Frege is unconcerned with such philosophical considerations as what sort of entity a value-range is, since he does not face the Russellian problem that in making the value-range a (subject) term, the contradiction is reintroduced [PoM, 516-18]. Frege's sense/reference distinction, together with his commitment to functionality as primitive, permits him to avoid regarding value-ranges as having this sort of occurrence.

[311] Classes were regarded as entities in "On the Logic of Relations," as both extensions and (intensional) class-concepts in the drafts of PoM, as defined by propositional functions, and subsequently, as mere notation, but Russell was prepared to afford classes whatever status was compatible with the logic required for a solution to the Contradiction.

which must not be collapsed, and the *logic* to which arithmetic notions are reduced is the logic of (intensional) functions. On Frege's logic of function and argument, first-level functions have arguments and themselves fall within second-level functions and, in this distinct way, are their arguments.[312] In the drafts of PoM, classes in extension are defined by means of intensional propositional functions[313] and by the 1903 version of PoM or shortly thereafter, classes are defined in an intensional logic in which propositions (and, briefly, propositional functions) are fundamental, classes and relations being subsidiary. My aim will be to consider whether further inspection of Russell's views in PoM, as well as in the 1902-1905 letters from his correspondence with Couturat, exhibit the logical motivations for Russell's adoption of his unique definition of numbers as classes. The remainder of this chapter will be concerned with outlining the development of Russell's views from PoM to the first articulation of the substitutional theory, concerning the irreducible intensional aspect of relations, classes and propositional functions underlying the definition of number, in support of my contention that Russell's logicist definition of number differs from Frege's in non-negligible respects. Chapter 5 will be concerned with exhibiting the substantive character of these differences in connection with Russell's philosophical conception of logical analysis.

[312] See Frege's CO.

[313] Russell initially held that "Any propositional function in which a fixed assertion is made of a variable term is to be regarded as giving rise to a class of values satisfying it" [PoM, 79]. On Russell's early view 'χ is a υ' is a propositional function iff υ is a class-concept, whatever the value of χ and if υ is a term and not a class-concept, then there will be no proposition of the above form. However, Russell recognizes, even in 1902, that this gives rise to the contradiction -see correspondence of 1902 in PMC- and addresses this problem explicitly in PoM [PoM, 88 and 103-5]. He has no effective solution, however, until the zig-zag theory of 1904.

In order to advance my view that it is problematic to assume that Russell embraces the Fregean definition of the cardinals, it will be important to briefly consider Frege's groundbreaking contribution to the extensional definition of the cardinals, both in his use of the context principle and in his explicit definition of number by means of the extensions of concepts. It is worth stating at the outset that the chief mathematical aim of Frege's *Grundlagen* is to give a purely logical definition of number, namely, one which can be used in proofs of mathematical truths which are not self-evident:[314] the Peano axioms can be proved from Frege's definition, that every number has a successor and two numbers cannot have the same successor, which together implies the infinity of the finite cardinals. It will be useful to begin with an articulation of the contextual definition of number—a definition which Frege puts forth in the *Grundlagen der Arithmetik* as insufficient in itself for a definition of the cardinals. The definition is advanced by appeal to Hume's principle (Hp),[315] which can be reformulated as follows:

> ***[Hp] The number of Fs is equal to the number of Gs if there is a one-one correspondence between the Fs and the Gs.***

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[314] The Fregean conception of logical generality is not characterized by an indifference to objects or the particular features of objects, for this would render fruitless the logicization of basic arithmetic notions like the numbers, whose unique properties must be preserved in their logical definition. He rejects the traditional notion that general logic 'abstracts from all contents' or is purely formal, as Kant supposed.

[315] Cantor had made use of this principle, and it has come to be referred to as the 'Cantor-Hume' principle. It is worth noting that both Frege and Russell approach the Hume principle differently than Cantor who makes use of this principle to define the cardinals in terms of the ordinals, which Frege defines instead by the ancestral relation. So, for present purposes, I shall refer to it as the Hume Principle, to emphasize the fact that Frege employed it in a context in which the cardinals were not to be defined in terms of the ordinals (or those numbers corresponding to the order types of well-ordered sets). In contemporary arithmetic, the Cantor approach is taken as axiomatic.

Frege's contextual definition (CD) of 'the number of *F*s' can be stated as follows:

> ***[CD] The number of the concept of F is identical to the number of the concept of G if and only if the concept of F and the concept of G are equinumerous, where "equinumerous" means that there is a one-one correspondence between the concept of F (i.e., value-range)and the concept of G (i.e., value-range).***

From this principle[316] we may glean the general truth that any number is the result of a one-one correspondence between concepts, but the principle seems to tell us only what it is for concepts to have the "same number" and not what it is for any particular number to belong to these concepts. Frege is clear that this cannot suffice for a logical definition of the cardinals.

Defining number within the context of arithmetic theory may, perhaps, be carried out contextually, but defining it for objects in the domain of the conceptual will require a one-one correlation between *concepts* in virtue of their demarcation of the objects falling under them. To arrive at any particular number will require that number attach to concepts which, being sortal in nature, demarcate the definitely many objects falling under them. For Frege, the number is attached to the concept (or value-range), which is

[316] Demopolous interprets Frege's context principle as establishing that Hume's principle provides a criterion of identity whose instances form a class of statements which allow us to recognize, by means of a recognition of the relation of identity at work in number statements, where the number of two distinct concepts is "the same". Demopolous words this differently, claiming that the instances of Hume's criterion of identity form a class of statements associated with numbers in virtue of which we can "say when the same number has been "given to us" in two different ways, as the number of one or another concept." [Demopolous 1998, 482]. Russell's version of this principle, since numbers are not applied to concepts in his philosophy, would be something like the number of a set α= the number of a set β.

not a mere aggregate or collection of the objects falling under it, but is itself an abstract object. Number can, on Frege's account, be defined in terms of the equinumerosity of concepts, where the equinumerosity of concepts is itself a second-order concept that is correlated with an equivalence relation between the extensions of first-order concepts. The following definition of equivalent extensions (EE Def) is given:

> ***[EE def] The extension of the concept F is identical to the extension of the concept G if and only if all and only the objects that fall under F fall under G.*** [317]

That is, an object is a member of the extension of a concept if and only if it falls under that concept and if two extensions have the same members, they are identical. Frege rejects the Contextual Definition of number for the reason that, like all definitions by abstraction, it does not secure the reference of the numbers, but guarantees only the "sameness of number".[318] In the *Grundlagen*, he attempts to define objects within the domain of the conceptual by introducing the concept of equinumerosity. It is assumed that in defining the cardinals as classes of equinumerous classes, Russell has essentially

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[317] I have decided not to address Frege's subsequently formulated Axiom V, i.e., the axiom that the "value-range" of the function $f(x)$ is the same as the "value-range" of the function $g(x)$ if and only if $\forall x[f(x) = g(x)]$ for the reason that it was adopted to prove the Hume Principle and the logical difficulties to which it gives rise are not the subject of this chapter.

[318] This is Frege's famous "Julius Caesar problem". The problem is that contextual definition tells us what it is to be the same number, i.e., when #F=#G, which suffices when we know already that the x in "for some G, x=#G" is a number and is not some other object that is not a number, for instance, Julius Caesar. Hence, Frege tells us, the contextual definition by means of Hume's principle does not permit us to "decide by means of our definitions whether any concept has the number Julius Caesar belonging to it" [FA, §55]. This is the problem faced by any contextual definition giving an identity condition, e.g., "The direction of line a=direction of line b iff line a is parallel to line b" does not permit us to "decide…whether England is the same as the direction of the Earth's axis" [FA, §66]. Hume's principle gives identity conditions for any number, #F=n, but not for any object, #F=x.

adopted the Fregean definition of numbers as set-theoretic objects defined by their extensions. Once his Sinn/Bedeutung distinction is introduced, however, Frege has to contend with both the sense and the reference of concept/value-range expressions, and Frege's correlation of number with what Russell calls the 'class-concepts' gives rise to problems of reference. Importantly, the status of classes as logical objects is not a metaphysical concern for Russell, but a logical one, in that Frege's identification of classes with concepts/value-ranges is a symptom of his commitment to a logic that cannot escape the Contradiction. Russell recognizes that he must contend with the extensional view of classes from within an intensional logic in a way that obviates both the predicate version and the class and, subsequently, the function version of the paradox.

To make the case for the divergence of views, Russell's independent discovery of a Fregean nominal definition of the cardinals in 1901, prior to having read the *Grundlagen*, must be addressed. As we have seen, Russell came even earlier than this to share a rejection of Peano's notion of relations as ordered couples, where classes of relations are classes of ordered couples. Russell's rejection of an extensional definition of this sort shows that his aims are plainly similar to Frege's. Russell's definition, like Frege's, is supposed to be an advance upon definitions by abstraction and avoids the problem introduced by defining number by means of Hume's Principle, which, as both Frege and Russell recognize, suffices only for establishing the "sameness of number" and does not provide a definition of the numbers.[319] However, Russell's definition is carried out within a logic that separates propositions (intensional "entities") from truth-values. Russell is explicit in saying that the primitive truths of the logic of classes are not mere alternatives, as Couturat believes, to the primitive truths of the logic of propositions. If the logic of propositions is more basic than that of classes, there is an immediate sense in which Russell diverges from Frege's view that value-ranges are "logical objects". To establish the interesting differences and to distinguish these from the uninteresting ones, it will be helpful to consider Russell's nominal definition more closely.

It is clear that Russell had adopted a nominal definition of number as early as February, 1901, in "On the Logic of Relations,"—a paper for Peano in which he treated cardinal numbers in terms of the similarity between two classes *u* and *v*.[320] Russell writes: "[i]f we wish to define a cardinal number by abstraction, we can only define it as a class of classes, of which each has a one-one correspondence with the class 'cardinal number'

[319] I shall not defend any position with respect to the neo-logicist attempt to define the numbers by means of Axiom V alone, though it may be that to do so is incompatible with the aims of Frege's logicist project or with his epistemological concern that numbers be apprehended as courses-of-values. Cf note 309. It is worth pointing out, however, that while, in FA, Frege attempts to avoid the Julius Caesar problem by defining numbers in terms of the extensions of concepts, he did so only for lack of alternatives. See Frege's July 28, 1902 letter to Russell [PMC, 139-42]. Moreover, the identity statement employed in the definition by extensions suffers the same problem and we cannot determine where any x is to be identified with the extension of a concept, i.e., "we can neither decide, so far, whether an object is a course-of-values that is not given us as such" [GG, §10,]. It has been supposed that Frege circumvented the problem by restricting his quantifiers to extensions, but this seems at odds with the universality of logic and is, on Wehmeier's interpretation of §10 of GG, mistaken. See Wehmeier, K., 1999, 'Consistent Fragments of *Grundgesetze* and the Existence of Non-Logical Objects', *Synthese*, 121: 309–328.

[320] The nominal definition also appeared in a paper written with Whitehead's "On Cardinal Numbers" in 1902, defining 0 as the class of the null class, 1 as the class of all unit classes, with the defining expressions formulated to avoid the vicious circle, and with the class Nc (class of cardinals) defined as the class of classes of classes [Russell and Whitehead 1902].

and to which belong every class that has a correspondence” [LOR, 321].[321] Russell’s version of the Fregean definition of the cardinals is a development of the definition given in “On the Logic of Relations”. Russell there defines the relation of similarity:

$$\mathbf{*1\ 1}\ \boldsymbol{u, v} \in \mathbf{Cls}\ .\supset .:\ \boldsymbol{u}\ \mathbf{sim}\ \boldsymbol{v}\ .=.\ \exists\, \mathbf{1{\to}1} \cap \mathbf{R} \ni (\boldsymbol{u} \supset \rho\,.\,\rho \boldsymbol{u} = \boldsymbol{v})\ \mathbf{Df\ [LOR, 320].}$$

This says that if *u* and *v* are classes, then they are similar (i.e., equinumerous) if and only if there is a one-to-one relation R such that the range of R restricted to the class *u* is *v*.[322] In the “General Theory of Well-Ordered Series”, published in 1902, but written in the summer of 1901, Nc,’*u* the cardinal number of a class *u*, is defined as well as the relation of *being the cardinal number of* Nc, from which it is derived:

$$\mathbf{7.\,1}\ \boldsymbol{u} \in \mathbf{Cls}\ .\supset.\ \mathbf{Nc}\text{’}\boldsymbol{u} = \mathbf{Cls} \cap \boldsymbol{v} \ni (\boldsymbol{u}\ \mathbf{sim}\ \boldsymbol{v})\ \mathbf{Df}$$

$$\mathbf{.11\ Nc} = \mathbf{Cls}\text{’}\mathbf{Cls} \cap \boldsymbol{w} \ni \{\exists\, \mathbf{Cls} \cap \boldsymbol{u} \ni (\boldsymbol{v} \in \boldsymbol{w}\ .=.\ \boldsymbol{u}\ \mathbf{sim}\ \boldsymbol{v})\}\ \mathbf{Df}$$

Nc is the relation which *u* bears to *w* when *w* is the class of classes *v* similar to *u*, so Nc’*u* is the class of classes *v* which are similar to *u*. This is the accepted Russellian version of the “Frege–Russell definition” of cardinal number [Linsky 2006/2007, 165–66]. Linsky’s findings are further confirmation that Russell arrives at his version of the Fregean

[321] Griffin has pointed out the likelihood that this was a late addition to the text.
[322] According to Bernard Linsky, this should be read as saying that if *u* and *v* are classes, then they are similar if and only if there is a one-to-one relation *R* such that *u* is included in the domain of *R* and the range of *R* is the whole of *v* [Linsky 2006/2007, 134]. Citing Papers 3, xiv, Linsky points out that “Gregory Moore reports that Russell used ⊃ for class inclusion as well as implication until March or April 1902, when he started to use ⊂ for class inclusion…” Evidently, the first such ‘⊃’ is an implication; the second means class inclusion.

definition of cardinals by the summer of 1901, before he has read Frege[323]—a view established by Rodriguez-Consuegra's study of the manuscript for Russell's article for Peano's journal [Rodriguez-Consuegra 1991]. In his March, 1902 letter to Couturat, Russell is clear that he is able to provide a purely logical definition of number, announcing that in his course at Cambridge,[324] he gave purely *logical* definitions of number, of the numbers, and of diverse spaces, adding, importantly, that he does not find Peano's definitions by abstraction to be at all necessary, since the *logic of relations* provides the means by which to arrive at nominal definitions in all cases.[325] This is presumably a remark on his own earlier attempt, written and revised by February 1901, to dispense with definitions by abstraction in favour of a definition of cardinal number by the principle of abstraction in "Sur la logique des relations," where the logic of relations needed to carry out constructions of arithmetic notions is liberated from the obsolete view that relations must be treated as ordered couples. Peano's definition by abstraction had defined numbers by giving an equivalence relation between classes, x and y, which gives rise to a function φx, i.e., "being the cardinal number of x", and holds between the classes x and y iff φx=φy. By Russell's principle of abstraction, the numbers can be defined by

[323] While the "General Theory of Well-Ordered Series" was published in the *Revue de mathématiques* in 1902, it was written in the summer of 1901.
[324] Russell gave a course on "the Principles of Mathematics" in 1901-1902 for the Mathematics Tripos at Cambridge.
[325] Interestingly, Russell construes this as a continuation of Leibniz's project, noting that Leibniz came nearer to these ideas than anyone [CPLP, R23.3.1902]. Also, in the preface to the 2nd edition of PoM, Russell writes: "…of the three kinds of definition admitted by Peano—-the nominal definition, the definition by postulates, and the definition by abstraction—I recognize only the nominal: the others, it would seem, are only necessitated by Peano's refusal to regard relations as part of the fundamental apparatus of logic, and by his somewhat undue haste in regarding as an individual what is really a class" [PoM, 107].

the relation of similarity between classes, where any equivalence relation can be stated as the relative product of a many-one relation S and its converse. However, Russell realized that S is not uniquely determined, adding in a marginal comment: "This won't do: there may be many such relations as S. Nc must be indefinable" [Papers 3, xxvii]. In PoM, Russell gives a very clear statement of the fact that his reason for having rejected the definition by abstraction—that is, the definition of number which relies on the many-one relations possessed by similar classes to the common property that is their number—is that such definitions fail to establish that there is only one entity to which similar classes have this relation. Russell writes:

> **Now this definition by abstraction, and generally the process employed in such definitions, suffers from an absolutely fatal formal defect: it does not show that only one object satisfies the definition. Thus instead of obtaining one common property of similar classes, which is the number of the classes in question, we obtain a class of such properties with no means of deciding how many terms this class contains. In order to make this point clear, let us examine what is meant, in the present instance, by a common property. What is meant is, that any class has to a certain entity, its number, a relation which it has to nothing else, but which all similar classes (and no other entities) have to the said number. That is, there is a many-one relation which every class has to its number and to nothing else. Thus, so far as the definition by abstraction can show, any set of entities to each of which some class has a certain many-one relation, and to one and only one of which any given class has this relation, and which are such that all classes similar to a given class have this relation to one and the same entity of the set, appear as the set of numbers, and any entity of this set is the number of some class. If, then, there are many such sets of entities—and it is easy to prove that there are an infinite number of them—-every class will have many numbers, and the definition wholly fails to define the number of a class. This argument is perfectly general, and shows that definition by abstraction is never a logically valid process [PoM, 114-15].**

Between February and July 1901, Russell adds to his definition that for any equivalence relation R, we can take the equivalence class of a term u as "the individual indicated by the definition by abstraction; thus for example the cardinal number of a class u would be the class of classes similar to u". By June, 1901, Russell had completed his part of the joint paper with Whitehead "On Finite and Infinite Cardinal Numbers"[326]. In his correspondence with Frege, Russell recommends that Frege consult the joint paper with Whitehead, published October 1902, for the definitive statement of the definition, adding that he had been ignorant of Frege's independent discovery at the time he wrote it.

It was well before reading the *Grundlagen* in the summer of 1902, then, that Russell has realized that although the principle of abstraction from which it takes its start is unproblematic in itself, the definition by abstraction does not produce the required results. Interestingly, from his notes on the *Grundlagen*, it appears that Russell regards the Fregean contextual definition of number as akin to that carried out by the principle of abstraction, the chief advantage of which is that it does not rely on any primitive notion of counting[327] or what Frege calls "aggregative thought" [FA, iv]. Russell's notes reveal this:

> ***Definition of NC***
>
> **Take e.g. set of parallel lines. What is meant by saying they all have the same *direction*? Can *define* "direction of line *a* " as "all lines parallel to *a* ". Similarly "shape of triangle *ABC* " is "all triangles similar to *ABC* ". *Principle of abstraction.* Two concepts "equinumerous" [similar] when 1→1 between terms under them. Nc'*F* = extension of concept "equinumerous with *F*".**
>
> **Df 0 = Nc'(not equal to identical with itself)**
>
> **Df 1 = Nc'(identical with 0)….**

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[326] If it was not this date, it was sometime between January and June, 1901, but June, 1901 is definitive [Papers 3, 422-3].

[327] Russell regards this, in PoM, as one of the main obstacles to a purely logical definition of number: "Some readers may suppose that a definition of what is meant by saying that the two classes have the same number is wholly unnecessary. The way to find out, they may say, is to count both classes. It is such notions as this which have, until very recently, prevented the exhibition of Arithmetic as a branch of Pure Logic. For the question immediately arises: What is meant by counting? To this question we usually get only some irrelevant psychological answer, as, that counting consists in successive acts of attention" [PoM, 114].

> **Observe with above definition of cardinal numbers NC, no need of *counting* [Linsky 2006/2007, 165–6].**

This shows Russell approving of Frege's advances upon the definition of number, insofar as they had the mutual aim at arriving at a purely logical definition. As Levine points out, however, Russell maintains as late as May 1902 [328] that "for formal purposes, numbers may be taken to be classes of similar classes" [Levine 2007, 64], providing an argument intended to show that numbers are "... philosophically, not formally definable ... [and] these indefinable entities are different from the classes of classes which it is convenient to call [numbers] in mathematics" [Byrd 1987, 69].[329] Levine points out that it was only during his correction of page proofs, after June 1902, that Russell changes this passage to read:[330]

> **Numbers are classes of classes, namely of all classes similar to a given class ... [N]o philosophical argument could overthrow the mathematical theory of cardinal numbers set forth [above] [PoM, 136].**

The text from the printer's copy of Part II of PoM, likely changed in May, 1902, actually reads: "…these indefinable entities are different from the classes of classes which it is convenient to call classes in mathematics." Michael Byrd notes, "sic: "classes" is underlined lightly in pencil and should, I think, be "numbers" here" [Byrd 1987, 69]. If Byrd is correct, the text should read "…classes of classes which it is convenient to call

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[328] Though most of the changes to Part II of PoM to have been made as early as June, 1901, Chapter XV, from which this passage is taken, is an exception. Changes to Chapter XV were made in May, 1902 [Byrd 1987, 63]. This makes sense of remarks concerning the Contradiction and is in keeping with Russell's comment to Jourdain in 1910 that Parts I *and* II were "wholly later, May 1902" [Grattan-Guinness 1977, 133].

[329] Interestingly, this parallels Russell's concerns about the principle of abstraction in connection with the question of whether to identify the reals with segments of the rationals.

[330] See Byrd 1987, 64.

numbers in mathematics." I do not think, however, that the June, 1902 text represents a departure from the view Russell expresses in May, 1902. Consider the preceding text from the May, 1902 alterations to Chapter XV from Part II of PoM:

> **[F]ormal definability results from the assumption made by the symbolism that a definable class can always be taken as a single term. But philosophically numbers are not predicates and not class-concepts; for predicates and class-concepts apply to single terms. But numbers are closely allied to predicates, for they are asserted of classes in the same kind of way in which predicates are asserted of terms: they are concepts occurring otherwise than as terms in propositions which are not in the ordinary sense relational [Byrd 1987, 69].**

Russell, by this point, has adopted the view that not every "definable class" is a single term, i.e., not every propositional function defines some class-as-one and, hence, numbers cannot be identified with classes of classes, but, for the same reason, they cannot be class-concepts or predicates in the ordinary sense. Rather, they must be properties common to equivalent classes and, as such, indefinable. Hence, when Russell writes that numbers must be regarded in mathematics as classes of similar classes, he has not changed his view. What is required is some amendment of the view that, philosophically, numbers are the common properties of classes in extension and themselves indefinable.

The extensional definition of number in terms of classes trades on the identity relation, which has an intensional dimension that must be captured in logical terms if the Contradiction is to be circumvented. While the extensional view of classes is necessary for mathematics, the intensional dimension of the logical connectives on which the logic of classes rests must be accommodated in logic. Identity, for Russell, must be a relation in intension for roughly the same reasons as those suggested by Frege in advancing the sense/reference distinction: for otherwise, there would be no cognitive difference between the notion that a = a and a = b. To speak in terms of the sense/reference distinction, Frege recognizes that we cannot arrive at the objects that the cardinals are merely by virtue of grasping the senses of which number statements are made up. In considering 'The number of the concept F is the number of the concept G iff the concept F is equinumerous with the concept G', we cannot arrive at the reference of 'the number of the concept F' merely in virtue of the sense of 'the concept of F's being equinumerous with the concept G'. Identity, then, is a relation in intension, but this does not, on Frege's view, present problems for the definition of number by means of classes, since number is identified with what Russell calls "the class-concept" (i.e., Russell thinks Frege has an intensional view of classes giving rise to the difficulty of knowing whether two classes *u*, *v* are identical in case they are ranges determined by their corresponding functions [PoM, 512]). Interestingly, Russell reflects simultaneously on the intensional definition of classes and the logicist project in his 1902 notes on the *Grundlagen*:

> **Hope to have made probable that arithmetical laws are analytic and therefore à priori, and arithmetic mere prolongation of logic… .** ***Classes and Concepts.*** **Classes must be defined by** ***intension*****—even enumeration, which is only possible with** ***finite*** **classes, is really giving intension, i.e. identical with** ***a*** **or with** ***b*** **or etc. [Linsky 2006/2007, 166].**

If we understand Russell to mean that the meaning of the thing defined, i.e., the class, can only be given by an intensional philosophical definition and not the extensional definitions required for mathematics, then Russell is simply saying that even enumeration, which identifies the members of the class, must be an intensional definition constituting a philosophical analysis of that class. In PoM, however, Russell points out that it is precisely because relations are taken in intension that numbers must be identified with classes in definition and not with class-concepts or common predicates.

In December, 1903, Russell reflects on the status of the principle of abstraction in response to a letter from Couturat. Couturat writes:

> **I would like a clarification on the principle of abstraction. You say (p.166) that you applied this principle in the definition of the cardinal numbers. Yet in the 2nd part I do not see where you made use of this principle, since you define the cardinal number as a class of classes…You do not need this principle to define, e.g. equivalence classes (similar classes); and this principle could serve you in deducing from a class of equivalent classes the idea of the cardinal numbers that is their common property. It thus furnishes you with the cardinal numbers as singular entities, and not as classes of classes [CPLP, C07.12.1903].**

Russell responds:

> **The essence of the principle, as it is demonstrated, is to replace the hypothetical quality common to all of these objects [classes] with the very class of objects involved by the class involved.[331] …Instead of 'the principle of abstraction', I would have done better to have called it 'the principle that replaces abstraction'. …I do not deny that there is often [a common property of equivalence classes that is the cardinal number], but it is not necessary to introduce it; it would in general be indefinable and the class has all of the qualities we need [CPLP, R10.12.1903].**

The abstraction principle given in LOR states that where an equivalence relation holds between two terms, there is an entity to which the terms have a many-one relation.[332] The

[331] The French is: "l'essentiel du principe, tel qu'il se demontre, est de substituer la class meme des objets dont il est question a la qualite hypothetique commune a tous ces objets."
[332] While the principle of abstraction given in LoR is destroyed by the Contradiction of classes, the existence of number can be given by a symbolic construction of the class of classes. Russell sketches the view in "On the Relation of Sense-Data to Physics". He writes: "so long as the cardinal number is inferred from the collections, not constructed in terms of them, its existence must remain in doubt, unless in virtue of a metaphysical postulate *ad hoc*. By defining the cardinal number of a given collection as the class of all equally numerous collections, we avoid the necessity of this metaphysical postulate, and thereby remove a needless element of doubt from the philosophy of arithmetic. A similar method, as I have shown elsewhere,

class of equivalence classes fills this role. It seems that Russell, in jettisoning the inferred common property of equivalence classes with which the cardinal number could be identified and in embracing the notion that the class (of such classes) has all the properties required, Russell has embraced the Fregean definition of number as classes of equinumerous classes. Russell seems to retain this view in POM, defending it against Peano's definition by abstraction. He writes:

> **The other remedy [to the defect involved in the definition of number by abstraction] … is to define as the number of a class the class of all classes similar to the given class. Membership of this class of classes (considered as a predicate) is a common property of all the similar classes and of no others; moreover every class of the set of similar classes has to the set a relation which it has to nothing else, and which every class has to its own set. Thus the conditions are completely fulfilled by this class of classes, and it has the merit of being determinate when a class is given, and of being different for two classes which are not similar. This, then, is an irreproachable definition of the number of a class in purely logical terms [PoM, 115].**

Recall that on Russell's conception of mathematical definition, an object is defined when its unique relation to a given concept is specified. It might be supposed that, in the definition of number, the relation to a common property of similar classes is specified, but, Russell points out, if *u* and *v* are similar classes, "similar to *u*" and "similar to *v*" are different predicates or class-concepts, but a definition of number requires that it is the same object defined, for which reason it must be the class and not the class-concept or common predicate that should be identified with the number in definition. In his Appendix on Frege, Russell articulates this dilemma in Frege's terms, wondering whether

can be applied to classes themselves, which need not be supposed to have any metaphysical reality, but can be regarded as symbolically constructed fictions" [Russell 1914, 115].

two classes *u*, *v* are identical in case they are ranges determined by their corresponding functions [PoM, 512].[333]

While it is clear that Russell independently arrived at a logical definition of the cardinal numbers by means of the principle of abstraction and independently accepted that being a cardinal number is being the cardinal number of a class, which is akin to the Fregean notion that being a cardinal number is to be the number of some concept, this is insufficient, on my view, for attributing to Russell a Fregean definition of the cardinals. Russell's definition is developed within his intensional logic of relations[334] (where, for instance in the above definition, relations are to be identified with class-concepts/predicates giving differing predicates for "similar to *u*" and "similar to *v*") and diverges significantly from the similar Fregean definition, both in terms of how the relation of being the cardinal number of a class or concept is defined and, more fundamentally, in terms of the notion of "class" involved. For Russell, it will not do simply to regard classes as the extensions of concepts, i.e., value-ranges, as uncomplicated logical objects. The point is not merely that Russell did not remain content with a logical definition that met the formal requirements without a definitive conception of the logical objects defined, but that he rejected Frege's definition on account of the differing notion of the relations and, later, (propositional) functions by which they were

[333] In the face of the contradiction, Russell believed that what was required was a denotation of what symbolically corresponds to the class as one [PoM, 514].
[334] The idea is not the Fregean notion that where designations are the same, modes of presentation may differ, but suggests instead a real difference of logically equivalent relations.

defined. This is not to deny that Russell has metaphysical and even epistemological reasons for rejecting Frege's definition. In characterizing equinumerosity by the extension of concepts, what results is an extensional definition of the cardinals that provides a surrogate for counting which accounts for the predicative applications of number statements involving cardinals, but does not provide any notion of what the cardinals are as objects.[335] That is, we know that a cardinal number must be defined in terms of the extension of some concept or in terms of the class of all n-membered classes and we might say that the cardinal number thus has all the properties we require it to have. However, on the supposition that such definitions do not construct, but rather reveal objects, it remains unclear, to both Frege and to Russell, what the logical object that is the cardinal number is. Russell at first believes that *numbers* must be indefinable entities, but he becomes content to give an extensional definition of the cardinals by means of the criterion for class-membership. He then looks to the *classes* to provide the objects that are the cardinals as entities, and remarks to Frege in a letter dated August 8, 1902, that he lacks "…a direct intuition, a direct insight into what [Frege] call[s] a range of values;" "…logically it is necessary," he writes, "but it remains for me a justified hypothesis" [PMC, 143-44].[336] "The contradiction," he goes on to say, "could be resolved with the

[335] This is not to deny any value to the principle of abstraction (or that which dispenses with abstraction) articulated by Russell in PoM: that every equivalence relation R that is instantiated can be viewed as the relative product of some function S and its converse -S. From this, it can be established that the range of the function S, given some definite extension of S, will have all of the properties possessed by the cardinal numbers, which is the desired result.

[336] They are necessitated by logic by the fact that they fulfill the logical requirements of arithmetic, where the null-class must be admitted, the unit class distinguished from its single member, and relations (identified with co-extensive class-concepts) require that there be some corresponding class-as-one, most

help of the assumption that ranges of values are not objects of the ordinary kind" [PMC, 144]. What I hope to show is that this problem, which at first appears to be a metaphysical or epistemological issue, becomes a strictly logical one in the light of the Contradiction.

4.2 RUSSELL'S REJECTION OF FREGE'S (AMENDED) DEFINITION

In 1902, Russell entertains the idea that an extensional hierarchy might block the Contradiction, but he is clear in his correspondence with Frege that obviating the Contradiction will require some logical characterization of classes apart from the notion that extensions are correlated with value-ranges. In July, 1902, Frege points out the "complete agreement" between his own definition of number and Russell's nominal definition in "On the Logic of Relations", i.e., that the cardinal number of a class *u* would be the class of classes similar to *u* [PMC, F28.07.1902]. Russell's mistake, he thinks, is the failure to recognize that the bearer of a number is not an aggregate or a whole consisting of parts, but a concept with a given extension. He writes:

> **It seems to me that you want to admit only systems [wholes] and not classes. I myself was long reluctant to recognize ranges of values and hence classes; but I saw no other possibility of placing arithmetic on a logical foundation… I have always been aware that there are difficulties connected with this, and your discovery of the contradiction has added to them; but what other way is there? [PMC, F28.07.1902].**

It seems, initially, that Frege has a better grasp on numbers and classes, in the light of the fact that Russell continues to differentiate the class as a whole from the class as an

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notably in the definition of the cardinals, where "similar to *u*" and "similar to *v*" are co-extensive class concepts for which there must be some corresponding class-as one if the same number is to be asserted of similar classes [PoM, 488].

aggregate, thereby missing Frege's point about the nature of classes as logical objects. However, Russell's underlying insight that Frege's notion that classes are apprehended as value-ranges is unavailing for resolving the Contradiction survives Frege's attack on aggregates. In this connection, it is worth briefly clearing up a misreading of Russell's 1902-1903 view of classes. In PoM, Russell *seems* in places to endorse the very notion of a class that Frege wished to reject, that of a collection or aggregate, and to thus misunderstand Frege's view of classes. For instance, in PoM, Russell tells us that "with the strictly extensional view of classes,...a class which has no terms fails to be anything at all: what is merely...a collection of terms cannot subsist while all the terms are removed" [PoM, 74]. Appealing to this passage can be terribly misleading in the attempt to characterize Russell's conception of classes in PoM. Not only has Russell abandoned the notion of aggregates and wholes in his letter to Frege in August, 1902 [PMC, R08.08.1902], but he is also explicit in PoM that the conception of 'class' in the above citation is the customary account of the null-class *which he rejects*. Russell is aware that if the null-class is merely a collection of non-entities, then it is not that it fails to denote any entity, but that it fails altogether to denote. Russell is clear that analysis requires that the denoting concept be treated as a class-concept, not merely in the sense of being a collection of terms, such that if it denotes the null class it denotes nothing at all or denotes a class of non-entities, but instead defined in terms of a propositional function, such that the denoting concept 'a' denotes the null-class when, for all x, 'x is a' is false. At least a formal denotation can be provided, then, if not an exact analysis.

On Russell's view, the attempt to identify numbers with classes apprehended as value-ranges is stultifying to a resolution to the Contradiction in its various forms. Initially, the trouble is that, while number can be defined by the formal requirements for membership in the class, so that they possess all those properties we would expect them to have, the attempt to regard classes themselves as the entities that the cardinals are supposed to be cannot escape the paradox of predication or the paradox of classes. From the time that he adopts his quasi-Fregean notion of classes in 1902, Russell takes seriously the "philosophical indefinability" of classes in a way which Frege does not.[337] It should be pointed out that the Contradiction provoked some anxiety in Frege about the status of 'classes' in 1906 for reasons akin to those underlying Russell's concerns as early as 1902. In the 1906 note in "What may I regard as the result of my work?", Frege is explicit that "… extension of a concept or class is not the primary thing …" and his Correspondence from 1918 indicates that he was still hopeful that the paradox could be resolved by the introduction of some other notion of a class [PW, 184].[338]

Before he has dispensed with the view that classes could be regarded as aggregates [PMC, R08.08.1902], Russell expresses dissatisfaction with Frege's treatment of classes as value-ranges. In his letter to Frege in August, 1902, Russell approaches the resolution of the Contradiction by extending a type-hierarchization of ranges of values to

[337] For a more detailed look at the dispute between Russell and Frege concerning the status of classes, see PMC, esp. Russell's July 24, 1902 letter to Frege [PMC, 138-9] and Frege's July 18, 1902 letter to Russell [PMC, 139-42].
[338] See also Frege 1983, 200.

the theory of relations. He writes: "The contradiction could be resolved with the help of the assumption that ranges of values are not objects of the ordinary kind; i.e., that φ(x) needs to be completed (except in special circumstances) either by an object or by a range of values of ranges of values, etc." Russell extends this to the theory of relations, maintaining that relations between relations must be of a different logical type than relations between objects. "For every function, φ(x)", he writes in his August 8, 1902 letter to Frege, "there would accordingly be not only a range of values but also a range of those values for which φ(x) is decidable or for which it has a sense" [PMC, 145]. In Appendix A of PoM, Russell again points out that the Fregean definition involves the underlying view that statements of cardinality are about concepts and Russell immediately recognizes the problem that crops up in connection with the attempt to identify his classes with Fregean value-ranges. In Appendix A, Russell remarks that "Frege gives exactly the same definition of cardinal numbers as I have given, at least if we identify his *range* with my *class*. But following his intensional theory of classes, he regards the number as a property of the class-concept, not of the class in extension" [PoM, 519]. Just as he did in his letter of 1902, Russell adds that "[i]n view of the contradiction of Chapter X, it is plain that some emendation is required in Frege's principles; but it is hard to believe that it can do more than introduce some general limitation which leaves the details unaffected" [PoM, 519]. Russell is quite sensitive to the conflation of classes and extensions with class-concepts and intensions.[339] For

[339] We shall see that this is important for infinite classes, which cannot be given in extension. In PoM,

Russell, numbers are properties of classes in extension and apply to objects, not concepts, which are intensional. While PoM was in proof, Russell recognized the need for some kind of extensional hierarchy to avoid the paradox, but in the passage quoted above, he is clear that the solution to the Contradiction will need to resolve the extensional versions of the paradox, though he has no conception of a solution that will do more than introduce a general limitation, i.e., one that circumvents the extensional versions of the paradox in an *ad hoc* fashion, preserving the original Frege-Russell definition of number according to the principle of abstraction.

In all of the relevant respects, the definitions are the same if, as Russell himself suggests, we equate Russell's classes with Frege's value-ranges. However, Russell takes issue immediately with the notion that number is a property of the class-concept and not the class in extension and, as he tries to work out the Contradiction, he arrives at views that make it difficult to identify his 'class' with Frege's 'range'. Frege's extensional hierarchy of entities, concepts (or predicates) predicates of predicates, etc. suffices for blocking the Russell version of the Contradiction, since the Sinn/Bedeutung distinction applies where the range is the reference of different senses, though the problem persists in the definition of number [Grattan-Guiness 2000, 305]. For Russell, intensional relations are a part of the analysis of number statements and the intensional view of relations motivates a unique corresponding extensional hierarchy that is first expressed in the

infinite classes are not involved in the meaning of a proposition about them (an infinity of terms is excluded), but are denoted by a concept having a relation of denotation to the class in extension.

distinction between the relation to the class as one and the relation to the class as many, where propositions of different types are briefly introduced to block the Contradiction [PoM, 76].[340] Russell's concern, as he tries to solve the Contradiction, is with the *identity relation* within a logic of propositional functions, which, by 1904, are themselves regarded by Russell as more fundamental than ordinary mathematical functions, classes, or relations.

On Russell's early conception of it—though, as we have seen, not the earliest conception— the Contradiction results from holding both that every class is a term and the axiom that any propositional function containing a single variable is equivalent to the membership of a class defined by the propositional function [PoM, 103]. The result of defining classes by means of propositional functions, i.e., by any propositional function that is not false for all arguments, appears to be the Contradiction presented by Russell in Chapter X of PoM. The definition of classes by means of propositional functions and the problem to which it gives rise is clarified by Russell's remarks in PoM:

> **A propositional function, wherever it is not null, is supposed to define a class, which is denoted by 'x's such that φx', such that it will always entail the concept of a class and corresponding class-concept will be the singular 'x such that φx'. But it may be doubted…whether there is always a defining predicate of such classes. Apart from the [paradox of predication described above]…the problem might appear to be merely verbal: "being an x such that φx" it might be said, may always be taken to be a predicate. But in view of our [paradox], all remarks on this subject must be viewed with caution [PoM, 88].**

[340] In Appendix A of PoM, Russell re-examines the doctrine of classes and takes care to reiterate that not every class-as-many requires a class-as-one or that there is not always a class-concept for a given propositional function. In Appendix B, he introduces his hierarchy of types of class, but a paradox of propositions recurs. Between 1902 and 1904 he attempts various solutions that it is impossible to elaborate here.

There is a problem with the notion that any propositional function φx (or P(x)), where x is a variable, determines a class, whose class-concept will be 'x such that φx' and whose members will be all those 'x's such that φx '. There are classes which are not members of themselves, such that the propositional function is satisfied by the terms of the class (the class as many), but not by the class itself (the class as one) [PoM, 102]. All of this is articulated by Russell in terms of *relations*:

> **Let R be a relation, and consider the class *w* of terms which do not have the relation R to themselves. Then it is impossible that there should be any term a to which all of them and no other terms have the relation *R*. For, if there were such a term, the propositional function 'x does not have the relation R to y' would be equivalent to x has the relation R to a'….When in place of R, we put ε—the relation of a term to a class-concept which can be asserted of it—we get the above contradiction [PoM, 102].**

It is in this context that he goes on to state the contradiction in terms of propositional functions:

> **Every propositional function which is not null, we supposed, defines a class, and every class can certainly be defined by a propositional function. Thus to say that a class as one is not a member of itself as many is to say that the class as one does not satisfy the function by which itself as many is defined….If any propositional function were satisfied by every class having the above property, it would therefore necessarily be one satisfied also by the class w of all such classes considered as a single term. Hence, the class w does not itself belong to the class w, and therefore there must be some propositional function satisfied by the terms of w but not by w itself….[W]e must suppose, either that there is no such entity as w, or that there is no propositional function satisfied by its terms and by no others [PoM, 103].**

What Russell initially thinks the Contradiction shows is that it is not always the case that the class as many requires a class as one. It is the need to preserve the distinction between the class in extension and the logical object (or logical subject in a proposition), and not a belief in aggregates, which motivates Russell's July 10, 1902 letter to Frege, in which he writes: "I believe I can therefore say without contradiction that certain classes (namely those defined by quadratic forms) are mere manifolds and do not form wholes at all" [PMC, 137]. The solution to the Contradiction offered in PoM trades on the difference between a class as many (a collection of terms) and as one, in such a way that the latter can be dispensed with:

> **Perhaps the best way to state the suggested solution is to say that, if a collection of terms can only be defined by a variable propositional function, then, though a class as many may be admitted, a class as one must be denied. When so stated, it appears that propositional functions may be varied, provided the resulting collection is never itself made into the subject in the original propositional function. In such cases there is only a class as many, not a class as one. We took it as axiomatic that the class as one is to be found wherever there is a class as many; but this axiom need not be universally admitted, and appears to have been the source of the contradiction. By denying it, therefore, the whole difficulty will be overcome [PoM, 104].**

This approach trades on a rejection of the view which was the obstacle to Russell's progress, namely, that a propositional function assures us of anything more than a relation between a term to its class as many. Once this is granted, there is room for his view that a class as many does not require a class as one *in the sense of being a subject-term* wherever the class as many is of a different *type* from the terms of the class, even when there is only one term.

Russell continues to articulate the role of the propositional function in terms of its exhibiting the relation involved between any term that may be the value of the variable in the propositional function and the class as many and it is in this context that he first articulates the utility of the type distinction:

> **A class as one, we shall say, is an object of the same *type* as its terms; i.e. any propositional function $\phi(x)$ which is significant when one of the terms is substituted for x is also significant when the class as one is substituted. But the class as one does not always exist, and *the class***

> ***as many is of a different type from the terms of the class*, even when the class has only one term, i.e. there are propositional functions $\phi(u)$ in which *u* may be the class as many, which are meaningless if, for *u*, we substitute one of the terms of the class. And so "*x* is one among *x*'s" is not a proposition at all *if the relation involved is that of a term to its class as many*; and this is the only relation of whose presence a propositional function always assures us. In this view, a class as many may be a logical subject, but in propositions of a different kind from those in which its terms are subjects; of any object other than a single term, the question whether it is one or many will have different answers according to the proposition in which it occurs…It is the distinction of logical types that is the key to the whole mystery [PoM, 105].**

The notion that a propositional function P(x) may be significant when one of the terms of a class is substituted for the variable x, though the class as one may not be substituted, is unproblematic, provided we regard the propositional function as indicating the relation holds only between a term and its class as many, such that the class as one must occur in propositions of a different logical type than the propositions in which the terms of the class as many occur. Everything here is still characterized in terms of relations and Russell views the logic of relations as a logic which ought to supply an intensional definition of classes, being itself more fundamental than the logic of classes.

It is by introducing this distinction of types to contend with the "class as one" that Russell is led to a more promising conception of the "class as many".[341] Initially he regards treating the class as many in purely logical terms as tantamount to treating it purely in terms of his logic of relations, but soon realizes the deficiency in this approach. Russell's attempt to dispense with classes could not be carried out by means of identifying classes with Frege's "value-ranges" and proceeding by means of Fregean functions. The problem of the indefinability of classes, taken seriously by Russell as a

[341] Interestingly, in 1905, Russell's reply to Boutroux shows him still committed to a view of relations as propositional functions of two independent variables, asserting the need to distinguish the class as such from a listing of its members. Cited in Grattan-Guinness 2000, 356.

logical issue, ultimately leads Russell to dispense with classes altogether, which would not have transpired were propositional functions simply Russell's version of functions from objects to truth-values.

In fact, what the results would have been had Russell adopted Fregean functions is easy to know, since the fact that the classes represent problematic entities led Russell to consider such a theory in the summer of 1903, when he substituted the notion of functions for the notion of classes. Russell writes to Couturat in June of 1903 that "At present, I've resolved this contradiction; but the solution consists in relinquishing the notion of *class* or set, in making use exclusively of the notion of *function*. …" and adds that he is "arriving, little by little, at a new simplicity; for example, logic is simplified enormously by doing without classes" [CPLP, R09.06.1903]. When Russell writes this, he regards functions as entities.[342] Russell tells Frege of his attempt to eliminate classes in May of 1903, believing himself to have "… discovered that classes are entirely superfluous" [PMC, 158], but in his December 12, 1904 response to Frege, he writes: "…I have known already for about a year that my attempt to make classes entirely dispensable was a failure" [PMC, 166]. Russell's thoughts are clarified by his April, 1904 letter to Couturat, in which he confirms the persistence of the Contradiction:

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[342] Russell to Jourdain, [R.25.03.1906]. Russell writes: "Then, in May, 1903, I thought I had solved the whole thing by denying classes altogether; I still kept propositional functions, and made φ do duty for $\hat{z}(\varphi z)$. I treated φ as an entity. All went well until I came to consider the function W, where $W(\varphi) . \equiv_{\varphi} . \sim(\varphi)$. This brought back the contradiction, and showed that I had gained nothing by rejecting classes" [R25.03. 1906 in Grattan-Guinness 2000, 78].

> **I am working at the moment on my Vol. II. It is the theory of functions and classes that causes me the greatest difficulties, because of the contradiction. Last summer I believed that one could dispense with classes; but I found that the contradiction returned for functions. At present, I have another method, which seems conducive to the aim; but it will take me some time to know whether it is correct [CPLP, R22.04.1904].** [343]

The theory adopted to dispense with classes refers to Russell's brief adoption of a Fregean functional theory in 1903 to dispense with Peano's class-abstract notation. In the Appendix on Frege in PoM, Russell urges his readers to consult the solution to the contradiction which Frege had included in the *Grundgesetze* [PoM, 522]. Essentially, the solution Frege had proposed was an emendation of his flawed Axiom V, on which two functions (concepts) determine identical value-ranges (i.e., the same class) iff they are co-extensive (i.e., have the same values for their arguments). On the revised treatment, Frege holds that two functions may determine equal classes without having the same value for their arguments (without being equivalent). In May, 1903, Russell became hopeful that the contradiction could be solved and arithmetic carried out without classes by replacing classes and Peano's class abstraction notation with functions and Frege's functional notation.[344] He soon discovered that by treating $\boldsymbol{\varphi}$ as a separable entity, it could be asserted of itself, giving rise to the Contradiction he sought to avoid in PoM by maintaining that the $\boldsymbol{\varphi}$ in $\boldsymbol{\varphi}$x was never a separable entity.[345] Russell explains this development retrospectively in his May, 1906 letter to Jourdain:

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[343] The alternative method mentioned may be the zig-zag theory of 1904.

[344] The class of terms x such that $\boldsymbol{\varphi}$x is replaced by ' x $\boldsymbol{\varphi}$x for the value-range of $\boldsymbol{\varphi}$x (' is the smooth breathing mark, which should occur over the x).

[345] A detailed account would give a far more complex picture of this development, but what is essential is that Russell's May, 1903 attempt to resolve the contradiction by replacing class notation with function

> **Then, in May, 1903, I thought I had solved the whole thing by denying classes altogether; I still kept propositional functions, and made φ do duty for *z*'(φz). I treated φ as an entity. All went well until I came to consider the function W, where $W(\varphi).\equiv_{\varphi}.\sim(\varphi)$. This brought back the contradiction, and showed that I had gained nothing by rejecting classes" [R25.03. 1906 in Grattan-Guinness 2000, 78].**

Russell's response to the Contradiction was to continue to explore the potential of Fregean functions, presumably because he needed to find a way of eliminating the problematic functions which do not determine classes and thought that he might isolate their properties and introduce restrictions to dispense with them, as he had tried to do in PoM by introducing propositional functions with restricted ranges of significance. So, despite the contradiction that arises from treating functions as separable entities, Russell continues to use the Fregean smooth-breathing operator notation, '$x\boldsymbol{\varphi}(x)$,[346] for the value range of the function $\boldsymbol{\varphi}(x)$, restricting the class of functions in the primitive propositions to the class of functional complexes, from which the non-functional $\sim\boldsymbol{\varphi}(\boldsymbol{\varphi})$ was excluded. Frege's range abstraction operator is employed until May, 1904, when it is replaced with "uKl$\boldsymbol{\varphi}$' ^x", which says u is the class determined by the propositional function $\boldsymbol{\varphi}$' ^x [Papers 4, xxv].[347] The Contradiction began to preoccupy Russell again in April, 1904, after attempting a variety of failed solutions and it was particularly those functions which do not determine classes that concerned him. In the summer of 1904,

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notation offered no viable solution to the contradiction, but did inform his approach to functional complexes. The problem with functions is mentioned again by Russell in a retrospective letter to Jourdain, where he writes that it was between April, 1904 and January, 1905 that in the attempt to discern which functions determine classes he gradually discovered that "…to assume a separable φ in φx is just the same, essentially, as to assume a class defined by φx, and that non-predicative functions must not be analyzable into φ and x" [R25.03. 1906 in Grattan-Guinness 2000, 79].

[346] The smooth-breathing mark ' should occur over the x.

[347] The circumflexes ˆ should occur over x's.

Russell wished to find some way of eliminating functions that gave rise to the Contradiction, but had no satisfactory alternative.[348] Russell writes to Couturat again in June of 1904:

> **I am still occupied as always with irreducible functions, that is to say functions that do not determine classes. Such are:**
>
> **$X=f(\varphi).\supset_{\varphi}.\varphi(x)$**
>
> **where f(φ) is a function such that x зφ(x), (x). φ(x), etc….These functions are the source of the contradiction (Chap. X); it is necessary to know how to eliminate them [CPLP, R13.06.1904].**

These irreducible functions are, again, "quadratic forms", i.e., functions whose arguments are functions of a variable assertion (function)[349] and do not, in the language of PoM, determine a "class as one"[350]. In his short-lived embrace of Frege's functional theory, Russell did not immediately abandon Frege's notation in response to the Contradiction generated by the separability of the function, but restricted the class of functions to preclude functions generating the Contradiction.[351] Rather, as we shall see in Chapter 5,

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[348] Later in 1904, Russell arrives at his zig-zag theory of classes and by 1906, he has adopted the 'no-class theory' which treats classes as incomplete symbols which acquire their meaning by reference to intensions, without supposing propositional functions to be entities.
[349] See Russell's definition in Papers 4, 614. It is worth noting that he concludes on the following page that 'quadratic form' is indefinable.
[350] Quadratic forms are of the form $\boldsymbol{\varphi}$ (f ($\boldsymbol{\varphi}$)). At this time, Russell regarded the function as separable into assertion and the variable contained in the function. Klement points out that "When Russell adopted the smooth-breathing abstract notation in May, 1903… the variable which is part of the function is maintained in a different way. In "*f*|*g*", the"*g* " can stand for the entire function consisting of both assertion and variable, since the allowable instances of "*g* " look like, e.g., "'*x* (*x* > 7)", so that we might write '*f* |'*x* (*x* > 7)'. The abstraction notation contains the variable letter '*x*'. Here the argument is more than just the assertion" [Klement 2004, 129 note 25]. The smooth-breathing marks ' should appear over the x's.
[351] On this view, Landini is right to point out that Russell's substitutional theory was aimed at preserving the unrestricted variable, but also that, even more fundamentally, Russell was motivated by a desire to preserve his conception of logical analysis for which denoting complexes, which occur only as meanings, was problematic. I shall address these issues in Chapter 5.

he rejected Frege's approach because, in his attempts to arrive at a theory of denoting, he could not reconcile the assimilation of propositional to mathematical functions [OMD, 342; FUND, 362] with his conception of analysis—the conception which led him to reject Frege's function-argument form of analysis in the first place [PoM, 509].

Importantly, the logic to which arithmetic notions are reduced does not, on the Russellian view, consist indifferently of the logic of classes, propositions, and relations. Russell's view, in PoM and after, is that arithmetic depends upon the theory of classes, but that the propositions of the theory of classes depend upon the logic of propositions. In November of 1903, Russell writes to Couturat: "You seem to believe that the Pp of the logic of classes are an alternative to those of the logic of P. This is not true. It is necessary to establish the logic of P before being able to make deductions; so, when one has thousands of Pp in the logic of classes, one cannot draw a single consequence without the logic of P" [CPLP, R12.11.1903]. The important point of divergence is the Russellian view, not shared by Frege, that logic has its roots in the logical form of propositions, and that the theory of classes rests on logic's capacity for exhibiting these logical forms. On this view, the intensional logic of relations is more basic than the logic of classes and Russell has a profound and nearly intractable commitment to the notion that mathematics must be articulated in terms of it, even if number statements within arithmetic can be analyzed extensionally.

By 1904, it seems that Russell was beginning to seriously consider the intensional view of relations to be an impediment to the analysis of arithmetic statements, and granted a fundamental status to propositional functions. To understand this, it is important to recognize that Russell did not think that Frege's strategy offered a viable solution to the Contradiction and by April 1904, as we shall see in Chapter 5, had indispensible reasons for holding propositional functions to be more fundamental than Frege's mathematical functions. Russell would soon propose a manner in which mathematical functions can be defined by means of the fundamental propositional functions with which symbolic logic is concerned. The trouble was that, on his current theory of denoting, he could not eliminate mathematical functions. In July, 1904, Russell writes to Couturat:

> **About relations, I have come to take the extensional point of view, for the same reasons that determined me to do so in the theory of classes. That is to say, I recognize that what one calls a *relation* in philosophy, and what we must call it, is the analogue of the *predicate*; but that mathematics must employ the analogue of the class [CPLP, R05.07.1904].**

On this view, propositional functions, $\boldsymbol{\varphi}!\,(x, y)$, determine classes of couples with sense. Importantly, it is from within an intensional logic that Russell adopts his extensional view of relations. He then adds:

> **What complicates matters, is that the fundamental relations of our calculus are relations in intension: these are $\supset$, ε, $=$, etc. This is to say, relations are defined by their signification, not by their extension.**

In a manuscript called "Fundamental Notions", written in stages in fragments in mid-1904, Russell makes it clear that single letters (when they do not stand for individuals) stand for classes or relations in extension, but the propositional complexes assert relations in intension (class-concept) which the letters satisfy. For instance, in p⊃q, what is asserted, unlike ordered couples whose relation is completely determined, is that the relation of implication holds between p and q; it "does not assert that p and q are a couple of the sort between which, as a matter of fact, implication holds" [Papers, 112]. Moreover, even if numbers may be defined extensionally by means of classes, the counting of classes, he thinks, requires acknowledgement of the intensional dimension of logical relations. Russell points out:

> **In arithmetic, it is essential to take relations in extension…When we take them in intension, the number of relations filling the given conditions is not determined. But in mathematics, we often need to count classes of relations [CPLP, R05.07.1904].**

In considering the role of explicit definition in Chapter 3, we saw that the counting of classes is problematic if the "class as many" is what is concerned, since this amounts to counting relations whose identity cannot be established by their logical equivalence alone [PoM, 516-17]. In the case of the definition of the cardinals, there is, on the one hand, (i) the notion of a one-one correlation between concepts that does not suffice for defining particular numbers, but must be supplemented by the extensions of concepts (classes) and, on the other hand, (ii) the notion that the identity of equinumerosity of concepts and the "sameness of the number of a concept" is a relation in intension. While classes are entirely extensional and the use of relations that are intensional (i.e., identity) in defining classes makes use only of their extensional aspect, it remains important to Russell that the intensional aspect of such relations be captured in logical terms. However, he appears to think that it is only by acknowledging identity as a relation in intension that the counting of relations can be carried out in purely logical terms.[352] In an important sense, Frege believes that the logic of functions is more fundamental than the logic of relations, while Russell, up until PoM, regards the logic to which arithmetic is reducible as a logic of relations (the *logic* of relations as opposed to the logic of propositions undergirding a construal of equivalent relations as constituents of propositions in terms of classes as sums of equivalent relations). This intensional dimension of classes via the intensional dimension of logical relations (e.g., equivalence, membership, implication, etc.) requires that the logic to which arithmetic statements are reducible is not, for Russell, an undifferentiated logic of classes, propositions, and relations, but tiered, so that there are the classes in virtue of which the coextensive non-identical relations constitutive of propositions may be treated as identical, then the primitive propositions that support this, then the logic of relations needed for the counting of classes, where the counting of classes is prior to their definition. The nominal definition of number would have to be carried out—like the analysis of all arithmetic statements—in terms of (identity) relations, in such a way that, even if the definition itself is extensional, it occurs within the

[352] Interestingly, in a 'Note on Class', that is likely to have been composed prior to May, 1901, Russell writes: "CLASS IS NOT one of the fundamental notions of Logic. Every term without exception is a Cls. Those terms which are not ordinarily so called are all equal to the null-class. The fact is, equality (a=b) does not imply identity (aI'b)...[W]hen we are counting classes, we take as one individual all those such that a=b, i.e.. we substitute for the number of classes the number of classes of classes u such that ∃Cls ⋂aʒ(bεu.=.a=b). Yet not so either; for there may be many such for one collection. The fact is that, when we are counting classes, we must substitute equality for identity in our definitions, as e.g. ∃Cls⋂aʒ(u,vεa,⊃.u=v:∃a) Thus a given collection of classes may be counted in two ways: (I) by their number as individuals, (2) by their number as classes. Thus man, featheress biped, rational animal, as individuals are three; as classes, one" [Russell 1901–2b, 566].

intensional context. This remains the case when Russell adopts the extensional view of relations, defined by propositional functions.

In the definition of number, Russell had come to view functions as more fundamental than relations and, by September, 1904, he has found the method of defining the cardinal numbers and of demonstrating the fundamental theorem without introducing relations. He says**:**

One thus has a much cleaner theoretical division:

Classes————-→-Cardinal numbers

Relations——-→-Relational numbers (ordinals).

Put:

f "u= ý {(∃x). xε𝓊.y= f 'x} Df

Unf$_u$ **'v.=: yεv. ⊃ . ˙x {xεu . y= f 'x} ε1 Df**

Such that:

Nc 'u= ˙v [{∃f). v= f " u Unf$_u$ **'v} Df**

The theory of cardinal numbers is greatly simplified by this method; but I had to redo all that Whitehead and I had done. We will have:

Sim= ű(Nc 'u) Df

The equations y= f 'x take the place of relations Nc→1, and when we have Unf$_u$ **'v, the function f, when the arguments belong to the class u, take the place of a 1→1 relation [CPLP, R22.09.1904].**

The equations of the form y=f ‘x in this definition, which express many-one relations and take the place of RεNc→1, involve ineliminable mathematical (denoting) functions.[353] In his notes on "Fundamental Notions", Russell entertains the idea that denoting functions are fundamental and suggests that relations in extension should be regarded, not quite as classes of couples, but "as correlations of every x with a denoting function of x as a new primitive idea" [Papers 4, 117][354] determined by functions. Here, Russell regards the f ‘x as satisfying the propositional functions ***φ*** ‘ (x, y), which is more fundamental than it, but points out that the variability of ***φ*** is restricted for ***φ*** ‘(x), i.e., the function is denoting. Later, he holds that the restricted variability of the ***φ*** occurs only where the relation is many-one. In all such cases, denoting functions are involved. When, in 1904, Russell announces that he has adopted the extensional view of relations and that arithmetic cannot proceed without eliminating classes and irreducible functions, he does not have the logical devices required to eliminate functions-in-isolation.

In his attempt to arrive at a theory of denoting, Russell settled on the view that propositional functions were the fundamental sort and, I shall argue in the following chapter, it would seem that the attempt to arrive at a theory of denoting compatible with his conception of the logical analysis of propositions was what prevented Russell from a theory on which restrictions on functions were introduced to block the Contradiction

[353] This is opposed to his earlier relational view, on which many-one relations are expressed in relational propositions, e.g., "y is the father of x" whose structure is xRy, permits us to derive ‘the’Ř of x, e.g., "the father of x".
[354] Russell may have adopted this notion of ‘correlation’ from Couturat. See CPLP, R05.07.1904.

arising from functions assertable of themselves. In his postcard to Couturat from October, 1904, Russell misleadingly expresses his contentment at Couturat's having adopted Frege's notion of propositional functions, though the whole struggle of dispensing with classes and relations in favour of propositional functions is pervaded by misgivings about Frege's notion of functions. This is made clear in his letter from April, 1904, where Russell expresses his dissatisfactions with Fregean functions. Russell writes:

> **I prefer to begin with what is most simple, this is to say, with the cardinal numbers, to then advance to more complicated ideas. …Concerning functions, [Frege] does not make any advances on what he has already published. He is preferable, on this subject, to all other authors, but I find that he does nothing but pose the problem where he believes himself to have resolved it. It is just this problem that occupies me at the moment. I believe that I glimpse that this is the crucial problem not only in mathematics, but in the whole of logic. But until now, I know of no theory that I do not know how to refute [CPLP, R04.04.1904].**

Frege offers no solution to the problem of (non-predicative) functions which do not determine classes. The proposed solution merely states that two equivalent functions do not determine the same class when these are quadratic.[355] The crucial problem which must be resolved, rather than merely posed, is, we shall see, the problem of denoting, which will be addressed in the following chapter.

In an important passage in the Couturat correspondence, Russell concludes that propositional functions are more fundamental than either classes or relations and, moreover, are more fundamental than mathematical functions. Russell writes: "—What

[355] This is how Russell puts it in his notes on "Classes", written in the first half of 1903. See Papers 4, 9. Quadratic functions are of the form that, in Frege's proposed solution, are inadmissible: those are functions (concepts) determined by a second-level function having a function (concept) as argument, where two functions determine the same values, but the value falls under one concept, but not another.

there is that is constant in P. xRy, x´Ry´ is simple: it is the concept of R itself. —I am now of the opinion that the idea of *functions* is more fundamental than the idea of *relations;* but it is the *propositional function*, not the mathematical function, that serves as the foundation of the edifice" [CPLP, R30.09.1904]. Couturat enthusiastically responds: "Yes, it is the propositional function that is the foundation of relations, in accordance with Frege's ideas; and I adopted this manner of seeing it in my little book on la Logique mathématique; while the mathematical function is posterior to the idea of relation…" [CPLP, C01.10.1904]. Of course, this cannot be Frege's view according to Russell, since Russell rejects Frege's functional treatment of relations in PoM for the very reason that these "double-functions" are subsidiary to the relations and relational propositions from which they are derived. Russell does not distinguish his views from Frege's, but rather responds that he is "glad that [Couturat] shares Frege's opinion on the Prop Fo".[356] This does not represent Russell's adoption of Frege's notion of functions. Rather, I believe this represents Russell abandoning his own earlier notion that the logic of relations will accommodate a hierarchical ordering of the terms of a class, the class, the class of classes etc and represents a step in the direction of collapsing classes and the (intensional) relations that define them into (intensional) propositional functions. In a letter to Jourdain, Russell recalls:

[356] He adds to this "For the conversion, here is the translation into function: one has, instead of xRy, xεf ‘y. in the case that P. Boutroux contemplates, one has R ε Nc→1; so we can put x=φ ‘y. Put (this is Whitehead's notation): arg φ ‘x= ý (φ ‘y=x) Df. So x= φ ‘y .≡. y ε arg φ ‘x. There is the conversion in terms of functions. For the relative product, if we have x= φ ‘y . y=ψ ‘z, we have x= φ ‘ψ‘z. So, in putting f‘ for φ ‘ψ‘, we have x= f ‘z. If we have x ε φ ‘y. y εψ ‘z, we have xε∪ ‘f ‘z, in putting f ‘ for φ ‘ψ‘."[CPLP, R06.10.1904].

> **[I]n April 1904 I began working at the Contradiction again, and continued at it...till January 1905. I was throughout much occupied by the question of Denoting, which I thought was probably relevant, as it proved to be...The first thing I discovered in 1904 was that the variable denoting function is to be deduced from the variable propositional function, and is not to be taken as indefinable. I tried to do without ℩ as an indefinable, but failed...Most of the year...I worked at different sets of primitive propositions as to what functions determine classes [Grattan-Guinness 1977, 79].**

Without the theory of descriptions which allows for incomplete symbols, Russell lacks a viable alternative to the Fregean theory of denoting and has no way of eliminating mathematical functions. The difficulty with a separable $\boldsymbol{\varphi}$, having unrestricted variability and regarded as entity or logical subject, is that it can be asserted of itself. The problem with restricting the variability of the function is, according to Landini, that it destroys Russell's foundational thesis that logic includes only entity variables, since any entity is capable of occurrence "as one" in a proposition, that is, as logical subject. I shall take up this issue in the following chapter, in the hope of establishing that a Fregean functional theory is antithetical to Russell's conception of the logic of propositions and to his approach to logical analysis in both in the period in which Russell took relations in intension to be fundamental and in the subsequent period in which he regarded propositional functions to be more fundamental than relations, expressly seeking a solution in which propositional functions could serve as the "foundation of the edifice".

In April of 1905, Bôcher writes to Russell:

> **The central point at issue is your 'class as one'. Your attitude towards this term is that of the realist, if I understood you correctly; mine is that of the nominalist. I cannot admit that a class is in itself an entity; it is for me always many entities (your class as many)…If you were to accept my position here…your remarkable paradox would crumble to pieces [Grattan-Guinness 2000, 374].**

Russell does seem to embrace the distinction between a formal or mathematical definition of the class as an entity (the class as many) and a philosophical definition of the class as an entity (the class as one). Russell remarks upon the distinction between mathematical (formal) and individual (philosophical) existence in his March, 1904, letter to Couturat that: "…mathematical existence applies to a class: ...one states Ǝa.=.~{(x).x ~εa} Df… But existence in the philosophical sense is another thing entirely: it applies to an individual" [CPLP R06.03.1904]. Had Russell attempted to resolve the Contradiction by dispensing with classes prior to the theory of descriptions, the most appealing approach to eliminating "the class as one" would have been to identify classes with value-ranges, but, in escaping the paradox of classes in this way, he would have met with the paradox given rise to by the assumption that co-extensive concepts have identical value ranges.[357] Prior to the 1905 theory of descriptions, propositional functions seem to be regarded as complex structured entities containing variables that denote the propositions that result from the filling in the values of variables. If they are entities that denote propositions, then it is possible to ask of the independent entity that is a propositional function satisfied by those propositional functions that do not satisfy themselves, whether it satisfies itself. Despite a reluctance in PoM to regard every propositional function as defining some class [PoM, 103], it is only in 1904 that he realizes that "indefinable functions" must be

[357] "The exact formulation of the paradox in Frege's system uses the notion of the *extension* of a predicate P, which we designate as εP. The extension of a predicate is itself an object. The important axiom V is: (Axiom V) $\varepsilon P = \varepsilon Q \;\equiv\; \forall x[P(x) \equiv Q(x)]$. This axiom asserts that the extension of P is identical to the extension of Q if and only if P and Q are materially equivalent. We can then translate Russell's paradox (*) in Frege's system by defining the predicate: $R(x)$ iff $\exists P[x = \varepsilon P \wedge \neg P(x)]$. It can then been checked, using Axiom V in a crucial way, that $R(\varepsilon R) \equiv \neg R(\varepsilon R)$" [Coquand 2010].

eliminated if the Contradiction is to be resolved. It is only once the role of propositional functions is secured after the 1905 theory of descriptions that mathematical functions can be eliminated and propositional functions can be viewed as the structures that entities share, without the need for denoting functions in isolation.

The unique role of propositional functions might be exhibited by attempting to answer the following question: if the Fregean definition of number can be carried out with the adoption of an extensional hierarchy of types—the need for which was recognized by Russell prior to Appendix B of PoM—and, if such an extensional hierarchy is supplied by Frege's distinction of objects, predicates (concepts), and predicates of predicates, then why did he not adopt the Fregean definition and block the "Russell version" of the paradox?[358] The answer is partially contained in the Appendix B and those texts which prefigure it, where the need for a hierarchy of types is recognized in connection with the need to preserve the distinction between extensionally equivalent propositions. The logic of propositional functions preserves the difference between the aspects of meaning in extensionally equivalent mathematical statements and so gives an exact analysis of the identity relation in mathematics in logical terms. From the time he was first confronted with the Contradiction, Russell was aware of the need to place restrictions on the way in which classes were determined by propositional functions and it is well known that when PoM was in proof, he identified the distinction of logical types as "the key to the whole mystery". Despite his awareness of the need for an extensional hierarchy of types of

[358] That is, the paradox of predication.

classes (or classes occurring in propositions of differing types) and his brief adoption of the view that classes might be dispensed with altogether and relations treated in extension, Russell recognized that the Fregean notion of functions would not suffice for capturing the relations in intension that make up arithmetic statements, where extensional equivalence does not suffice for identity, but where the intensional dimension of the meaning of equivalent things must be captured in logic itself.

Propositional functions, distinct from mathematical functions, gained in significance and, by the time of PM, they capture the intensional aspect of logical connectives that are defined from disjunction and implication.[359] The importance of propositional functions is exhibited in both the substitution theory and type theory, which were under consideration at the same time by Russell and Whitehead. In October, 1905, Russell writes to Couturat, informing him that he will send his article “On Denoting” and clarifies that he is not developing the theory of denoting *functions*, but only of the theory of denoting in general. He writes:

> **[…]For denoting functions, here is the principle. I find that to avoid the contradictions, and to make the starting points of mathematics rigorous, it is absolutely necessary not to employ only one letter, such as φ or f, for a variable that cannot become any entity, but which is really a dependent variable. What one wants to say, e.g.:**
>
> **(φ, f) :φ!f‘x (A)**
>
> **The values of φ and of f in question are not the same as the values of x which are in question in (x).φ!x. And yet, one can always reduce the Ps such as (A) to another form that does not incorporate this other species of variability. The theory of denoting functions does nothing but replace such variability as that possessed by f by the variability possessed by φ: it is a first step. Instead of f ‘x, one takes as a general denoting function ψ 7 ‘x,**

[359] See, for instance, PM, *9.12.

> **or ψ ℩ 'x = ℩ 'y' (ψ! (x,y)) Df**
>
> **E.g, 'the son of x'= 'the y such that x fathered y'. So, instead of (A), one would have:**
>
> **(φ,ψ):φ!ψ ℩ 'x (B)**
>
> **Instead of φ!x, we can put $p^{x}{}_{/a}$, which must signify "the result of the substitution of x by a in p"; if a is not found in p, $p^{x}{}_{/a} = p$.**
>
> **So, instead of 'all values of φ" one will have "all values of p and of a"**
>
> **E.g., one has:**
>
> **$x=y.=.(p,a).p^{x}{}_{/a} \supset p^{y}{}_{/a}$ Df**
>
> **which is nearly Leibniz's Df. [Which is**
>
> **$x=y.=.(p).p \supset p^{y}{}_{/x}$ Df**
>
> **There will thus be only one kind of independent variable. This, properly understood, is a method for the principles: we do not need to drag this across the development of mathematics. I believe anew that the solution to the contradictions is found in affirming that there are neither classes nor relations [CPLP, R23.10.1905].**

In his January 1906 letter to Couturat , Russell writes: "…I am more and more satisfied by the solution to the contradiction that I've found. The essence is that classes, relations, etc., are only a *façon de parler*. The same is true for functions: we can talk about φx or of φ(x,y), but φ by itself is nothing" [CPLP, R17.01.1906]. Russell goes on to offer an account of how to vary a function by substituting for it the proposition and the subject of the proposition, reminding us that we do not define the symbol itself, but the propositions of which it forms a part.[360] The limitations set forth in OD and exhibited in the terms of

[360] Frege's position on this point is not obvious. He appears to be committed to some form of part/whole analysis in regarding the whole thought as constituted by the senses which are its parts and, though it is not clear that this commits him to the view that propositional contents are uniquely analyzable, it is not obvious how this is to be reconciled with function-argument analysis. On Levine's characterization, the issue to be resolved is that of how the sense of a function, F, may be a part of the thought expressed by the sentence 'Fa' when the thought is one of the values of the function, i.e., when it has the sense of a as argument. For this characterization of the trouble and a plausible solution, see Levine 2002.

substitution in the letter to Couturat, are supposed, by Russell to be "just the limitations needed to avoid the contradiction, neither more nor less" [CPLP, R17.01.1906]. In his 1906 reply to Poincaré's paper, "Les mathématiques et la logique",[361] Russell announces the no-classes theory of classes to be the most satisfying solution to the Contradiction and the theory in question is the substitutional theory, whereby the presence of the constituent a within the proposition p, is the basic matrix of substitution, written p/a, where the result of substituting b for a in p to produce proposition q is symbolized: p b/a !q [Grattan-Guinness 2000, 360]. The point is that propositional functions are the guarantor of the no class theory of classes adopted in the substitution theory.[362] As the 1904 letters of the correspondence with Couturat suggest, the reducibility of functions was recognized to be significant well before ramified type theory and even before the substitutional theory. Recall Russell's remark to Couturat in June of 1904: "I am still occupied as always with irreducible functions, that is to say functions that do not determine classes... These functions are the source of the contradiction…[and] it is necessary to know how to exclude them" [CPLP, R13.06.1904]. One might say that it is a logic of intensional

[361] "Les Paradoxes de la Logique", *Revue de métaphysique et de morale*, 627-50. The English version, "On 'Insolubilia' and Their Solution By Symbolic Logic" is reprinted in Lackey 1973.
[362] Likewise, propositional functions are the guarantors of the no-class theory of classes in the ramified type theory of PM. The ramified theory has the same advantage of the substitution theory in that it avoids the assumption of classes as entities by subsuming them under intensional propositional functions, written $\psi!x$. Typical ambiguity allows that "in practice we never need to know the absolute types of our variables, but only their relative types" [PM, *65]. From within the logic of propositional functions the reason for the ramified version of the type theory is clear, though it is not obvious why it was preferred over the substitution theory. Nevertheless, a ramification of the theory of types is necessary for defining numbers as classes of classes, and the axiom of reducibility is needed to preserve the identity relation, since it permits the notion of "sharing all properties" (equivalence) in terms of the order to which the properties belong, with the guarantee that for every case of "all properties" of nth order, there is an equivalent predicative property: A and B can be determined to be identical, for any order n, if they have all nth order properties in common.

propositional functions that a type theory must accommodate, so that propositional functions do not serve the type hierarchy, but the latter serves and accommodates the propositional functions that preserve the intensional dimension of the meaning of equivalence statements in mathematics and the universality of the logic to which they are reduced.

Though, as we shall see in Chapter 5, Russell struggled to account for ineliminable denoting functions from 1903-1904, his 1905 theory of descriptions secures his view that propositional functions are fundamental. Russell then adopts a no class theory of classes to escape the paradox of classes[363] and extensional notions in the theory of classes are supplied within the intensional logic of propositional functions. This theory of classes is carried out in the simple theory of types and it is the logic of propositional functions that is adopted between PoM and PM, where propositional functions become more crucial than classes or relations on both the substitutional and type theory, which are developed at the same time. Propositional functions are crucial to Russell's conception of the logic to which arithmetic concepts and principles are to be reduced. Russell's approach serves many of the same aims as the Fregean approach, but underlines the mathematical importance of retaining an intensional view of propositional functions. Functions are formally equivalent where they take the same truth-values and equivalence can be established for the extensions of functions, the class of its arguments [Grattan-Guinness 2000, 392], where "… an extension (which is the same as a class), is an

[363] The no-class theory is retained in PM [PM, *20].

incomplete symbol, whose use always acquires its meaning through a reference to intension" [PM, 72]. Russell, who initially rejected Frege's notion that the extension of a function (the class of arguments satisfying it) must be apprehended as the value-range or logical object correlated with the extension, does not adopt Frege's proposed solution, but instead ascribes a unique role to intensional propositional functions, eliminating classes as entities altogether. While Russell did not recognize that classes might be treated as incomplete symbols until he formulated his 1905 theory of descriptions, we shall see in the following chapter that the move toward the 1905 theory is crucially connected with Russell's initial reasons for both rejecting Frege's function-argument approach to analysis and privileging propositional functions . While both Frege and Russell recognize that it is important for mathematics that extensional equivalence does not guarantee identity, Russell struggled to include the intension dimension of the meaning of non-identical but extensionally equivalent statements in the logic itself to which identity statements are to be reduced. Arithmetic can be carried out by equivalence relations but identity is a relation in intension and identity statements need to have their intensional dimension captured in logical terms by means of the logical structure captured in propositional functions. This remains true even where Russell comes to recognize that propositions themselves are mere notation and are to be dispensed with along with classes and (relating) relations.[364] It is this motivation to avoid the Contradiction while preserving, in

[364] In May of 1906, Russell writes to Couturat: "…I believe again that my solution to the contradiction is good, but it seems to me that it is necessary to extend it to propositions, that is to say that these, like classes and relations, cannot replace ordinary entities" [CPLP, R15.05.1906].

logical analysis, the intensions by reference to which extensions (classes) acquire their meaning, i.e., propositional functions, that leads Russell to reject the uniquely Fregean definition of number. To carry this out, as we shall see, Russell had to dispense with the problematic denoting concepts which served this function in PoM.

In considering the differing logical apparatus underlying Frege's and Russell's definitions of number, it was pointed out that Russell could not discover any means of avoiding the Contradiction from within his logic of relations, but resisted the Fregean theory of functions, regarding propositional functions as more basic than classes or relations. To be certain, however, that the differences in the logic underlying these mathematical definitions were not merely technical differences, but philosophical differences concerning the nature and aim of logical analysis will require a closer consideration of Russell's reasons for rejecting the Fregean functional account of mathematical definition. We shall see in the following chapter that, between 1903 and 1904, Russell did attempt to develop a Fregean theory of functions to obviate the Contradiction, but, in the light of the philosophical commitments at the heart of his approach to analysis, offered explicit reasons for treating propositional functions as more basic than mathematical (denoting) functions—reasons which were consistent with the motivations for the 1905 theory of descriptions, but which were offered prior to the articulation of any theory by which denoting functions could be eliminated. These developments will be addressed in the following chapter, whose aim it will be to establish the crucial connections between Russell's approach to developing a theory of denoting, necessary to mathematical definition, and his philosophical conception of logical analysis.

CHAPTER 5: TOWARD A NEW THEORY OF DENOTING

5.1 FAMILIAR WOES: *DENOTING* CONCEPTS

The purpose of this chapter is, first, to show that Russell's 1903 theory of denoting exacerbated existing difficulties in his conception of logical analysis and, second, to suggest that it was nevertheless his commitment to preserving his conception of logical analysis that led him to the theory of descriptions by which he was able to dispense with his problematic denoting complexes. It was pointed out in the preceding chapter that the theory of descriptions enabled Russell to treat classes defined by propositional functions as incomplete symbols, thereby obviating the Contradiction which arose from introducing classes as entities into his logicist project. However, from the account Russell gives in OD of his motivations for the theory of descriptions, one does not receive the impression that the theory emerged from Russell's attempt to solve the problems given rise to by his own approach to logical analysis, much less that he envisioned the theory as providing the technical apparatus to carry out logicist definitions.

It is worth remarking at the outset that the importance of "the" and definite descriptions consists in the fact that they are crucial to mathematical definition which is supposed to assert both the uniqueness and existence of the object defined. Recall that, immediately prior to articulating logicism, Russell held that definition is no part of mathematics "…but is simply and solely a statement of a symbolic abbreviation; it is a proposition concerning the symbols, not concerning what is symbolized" [PoM, 429].[365] However, as mathematics was brought within symbolic logic, Russell rejected formalism and adopted explicit definitions, identifying mathematical entities with the class of terms defined. We have seen that, on Russell's conception of mathematical definition, a term is defined when it is the only term having a fixed relation to a given term. We have seen also that definition by abstraction, i.e., the definition of number which relies on the many-one relations possessed by similar classes to the common property that is their number, fails to establish that there is only one entity to which similar classes have this relation. On the principle of abstraction given in LoR, there is at least one entity to which similar classes have a relation and the class is taken for this entity, in such a way that the definition of number gives the class, not the class-concept (predicate) in intension that is common to the terms. In the light of the Contradiction, Russell can no longer hold that the class as one is the entity defined and this undermines mathematical definition. Rather, in the light of the Contradiction, classes must be taken in extension, so that, where it appears that an analysis into subject and assertion produces a predicate or class-concept or relation to the class-as-one defining a collection of terms, there is really only a collection of terms determined by a propositional function. Russell recognizes the need for ranges (i.e., classes as one) in order to admit the null-class and the unit class, distinct from its single member, and to establish the identity of the number of a class on the basis of relations of similarity to distinct classes (co-extensive class-concepts), all of which are

[365] The date cannot be established for certain, since the section of the manuscript to which the passage belongs is lost [Grattan-Guinness 2000, 304].

crucial for arithmetic. Hence, Russell initially concludes that what is required is an extensional account of classes on which equivalent propositional functions determine some "class as one",[366] but, as Russell puts it, "...we cannot get any way of denoting what symbolically should correspond to the class-as-one" [PoM, 514]. The question Russell faced was this: if mathematical definition is supposed to assert the uniqueness and existence of the object defined, how should this be possible without class-abstract notation?

In the previous chapter, we noted Russell's brief adoption of a Fregean "functional theory", in 1903, on which Peano's class-abstract notation was replaced with Frege's function-abstract notation. Taking the function to be separable from the variable, Russell quickly arrived at the function version of the Contradiction, given rise to by the fact that the separable function could be asserted of itself, allowing for $\sim(\boldsymbol{\varphi})$, but concluded that it was arbitrary to deny a separable function in the case of $\boldsymbol{\varphi}(\boldsymbol{\varphi})$. He continued to work from within the functional theory, attempting to specify the properties of non-predicative functions, i.e., those which do not determine classes, and to eliminate them by introducing restrictions into his primitive propositions. The question of denoting occupied Russell from April, 1904 to January 1905—a question which, he tells Jourdain in March, 1906, he "...thought was probably relevant [to the Contradiction], which it

[366] In Appendix A on Frege in PoM, for instance, Russell points out that if u and v are distinct but similar classes, the relation "similar to u" will differ from the relation "similar to v" and, in the absence of an extensional notion of the "class as one", it cannot be asserted that the number of u is the number of v [PoM, 514].

turned out to be" [Grattan-Guinness 1977, 79]. The essential difficulty Russell faced was that, no matter how ill-fated he perceived his attempts to work within the functional theory to be, he could not eliminate mathematical functions, i.e., denoting complexes. In the subsequent section of this chapter, we shall see that Russell, unlike Frege, construed mathematical functions as denoting complexes having a certain structure on the basis of his commitments concerning logical analysis, laying crucial groundwork for the theory of descriptions. We shall see that, in April 1904, Russell came to regard propositional functions as the fundamental sort and took steps in the direction of the view that mathematical functions/denoting complexes do not have meaning in isolation. In the present section of this chapter, I shall point out the manner in which denoting complexes exacerbated old problems in Russell's conception of logical analysis—difficulties given rise to by taking the "adjective", "relating relation", or that which Russell calls "propositional complex" as logical subject of a proposition. I shall then explain how mathematical functions—the problematic sort of denoting complexes, which contain variables—are derived from so-called "propositional concepts".

To understand how denoting complexes exacerbated old problems in Russell's approach to analysis, it will be useful to say a word about Russell's reasons for introducing denoting concepts.[367] In PoM, "class-concepts" (predicates) were taken to

[367] For a full discussion of the 1903 theory of denoting, see Makin 2000. On my view, what Makin establishes is that the motivations for the new theory of denoting were not ontological, that the new theory of descriptions is not primarily a device for resolving issues in the philosophy of language, and that its chief virtue was not to resolve the puzzles presented in OD-- the 1903 theory was explicitly employed by Russell

determine classes. Recall Russell's concern that Peano's conception of formal implication gives the *meaning* of the variable, so that in "$x\varepsilon a \supset_x x\varepsilon b$", the x appearing in the consequent of the implication, which should have an unrestricted variability, means "the x's such that x is an a" or, "any a", so that the whole implication merely states that "any a is b". Class-concepts and intensions were to be distinguished from classes and extensions, for the reason that the same extensions/classes of terms may be denoted by philosophically distinct class-concepts. Russell conceives of the distinction between the concept and its denotation as akin to Frege's distinction between *Sinn* and *Bedeutung*, which serves the same function [PoM, 476].[368] Class-concepts and intensions are also to be distinguished from classes and extensions for the purpose of preventing propositions which are about an infinite complexity of terms (infinite classes) from involving an infinite complexity of terms in their meaning, thereby allowing propositions to be formulated about infinite classes. A denoting concept has a special relation of denotation to the object the proposition is about, and it denotes when it occurs as constituent of a proposition which is about its denotation, e.g., in "Every finite number is even or odd", the denoting concept "every finite number" logically denotes the particular numbers that the proposition is about, without having an infinite complexity of terms enter into the

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to deal with these puzzles. Makin also recognizes that analysis in mathematics is the analysis of propositions and that the theory of denoting employed in mathematics is the same theory of denoting which is employed in the analysis of the propositions of ordinary language.

[368] In OD, Russell goes as far as to say that the theory of PoM is "very nearly the same" as Frege's theory of *Sinn* and *Bedeutung* [OD, 415n], though we shall see that the differences are crucial. The similarities chiefly consist in the fact that denoting concepts have the two sides of meaning and denotation, akin to Frege's *Sinn* and *Bedeutung* and in the fact that, in the case of denotationless denoting phrases, what is denoted is the null class, defined in Fregean terms as, the class defined "x is a" is false for all x.

proposition as constituents. As Gideon Makin puts it, "[d]enoting is the relation which obtains between the class-concept and the class itself, and it is essentially the same as the 'determining' involved in saying that a concept determines a class" [Makin 2000, 15]. In "Recent Work on the Philosophy of Leibniz," which he finished in March of 1903 [Papers 4, 535], Russell glosses the problem of denoting:

> **M. Couturat sums up his account by saying that Leibniz possessed almost all the principles of Boole and Schröder … but he failed to constitute symbolic logic because it cannot be based upon the vague idea of intension. There is, no doubt, a certain broad truth in this statement: the Logical Calculus undoubtedly requires a point of view more akin to that of extension than to that of intension. But it would seem that the truth lies somewhere between the two, in a theory not yet developed. This results from the consideration of infinite classes. Take e.g. the proposition "Every prime is an integer." It is impossible to interpret such a proposition as stating the results of an enumeration, which would be the standard point of pure extension. And yet it is essentially concerned with the terms that are primes, not, as the intensional view would have us believe, with the concept prime. There appears to be here a logical problem, as yet unsolved [Papers 4, 548-9].**

In PoM, Russell maintained that symbolic logic has its lair in the position intermediate between extensions and intensions [PoM, 66]. "Every prime" is what Russell calls a denoting concept, which is an intension by which the class in extension is denoted. On Russell's 1903 theory of denoting, these are concepts which, when they occur in propositions, have an inherent logical relation[369] of denotation to some term or terms which the proposition is about, but which do not occur in it [PoM, 53]. For instance, in the proposition "every prime is an integer", the proposition is about every prime number

[369] In "On Meaning and Denotation" Russell writes: "The logically important matter is the relation between what is expressed and what is designated. For when one name both designates and expresses, this is not arbitrary, but is due to a relation between the objects designated and expressed. This relation is what I shall call denoting. Thus it is the meaning, not the name, which denotes the denotation; and denoting is a fact which concerns logic, not the theory of language or of naming" [OMD, 317–8].

and not about the complex denoting concept "every prime"[370] by which every prime number is denoted, though it is the concept itself that is involved in the proposition, since every prime number cannot enter into the proposition as its constituents. The trouble is that if we state that what is concerned is the extension, we must either take the extensional view and involve infinite complexity in the proposition, or else we must take the intensional view and say that what is involved is the extension by means of the intension denoting it, but here we have recourse only to the intension, to the meaning of the concept constituted by "every" and "prime". We may say that this is not a problem if the meaning denotes, as it should. However, how then are we to distinguish the concept "every prime" from its denotation? We might use some technical device like inverted commas and say that "every prime is an integer", but "'every prime' is a denoting concept". The difficulty, however, is the very fact that the denoting concept denotes. Hence, if we have truly taken *it* and not something else to be the logical subject of the proposition " 'every prime' is a denoting concept", then, in involving the meaning, we have involved the denotation and our proposition states that every particular prime number is a denoting concept. Stated in the terms of OD, "…we cannot succeed in *both* preserving the connection between meaning and denotation *and* preventing them from being one and the same" [OD, 421]. While Russell may not have articulated the problem in quite this way in 1903, it would not be surprising to find him concerned with the logical difficulties arising from the attempt to denote meanings without either invoking

[370] Since it is these complex denoting concepts that are of interest, and not the terms 'any', 'every', 'all', 'some', 'a', and 'the', which are of interest, I shall refer henceforth, to these as denoting complexes.

their denotations or transforming them into some other term by taking them as the logical subject of the proposition, for this has analogues in old difficulties which his conception of analysis faced just as soon as it had been articulated.

Recall that Russell's conception of logical analysis developed in reaction to Bradley's contention that the logical form of all judgment is "Reality is such that S is P", where an adjective is referred to Reality as the true logical subject. We have seen that Russell not only rejects Bradley's idea that the logical idea is an adjective, but presses it to the conclusion that it is contradictory to deny that anything is a logical subject and that, between such logical subjects, there is a primitive diversity. [371] We have also seen, in connection with Russell's work on Leibniz, that he further adopts the primitive diversity of logical subjects as the model for his external view of relations, so that in the relational proposition "A differs from B", precisely the same abstract relation "difference" enters as a constituent into the proposition as that which enters into the proposition "C differs from B". There are, however, two problems with external relations and these will be significant

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[371] In articulating his theory of terms, Russell writes: "Attempts have sometimes been made to restrict the logical subject to certain classes of ideas. It may be held that the subject must be a thing or, with Mr. Bradley, that it must be Reality as a whole. Such views I entirely reject. Every possible idea, everything that can be thought of, or represented by a word, may be a logical subject. If I say "2 is numerical", "number is categorical", "before is relative to after", I make judgments which have a subject and a predicate, and express a meaning which no form with a different subject can accurately represent. And thus every predicate may be made a logical subject. I may say: "one is predicable of any subject", and thus make one a subject. Moreover there is a certain unique kind of difference between subjects, dependent upon their being subjects…This manner of differing would be inexpressible if we refused to regard such terms as subjects; numeration, which depends upon just this kind of difference, would be impossible…The kind of difference, which belongs to different subjects as such, is to be distinguished both from material diversity and from diversity of content" [AMR, 168].

to understanding the problems faced by denoting concepts. For the moment, I shall simply state the problematic theses whose significance I shall subsequently explain:

i. On the theory of terms, a term's manner of occurrence in a proposition determines what sort of term it is, but the same term which occurs as concept must be capable of occurring as logical subject without change of meaning. For instance, the same relation of difference must enter into "A differs from B" as that which enters into "Difference is a relation".

ii. The whole proposition must be constituted by its constituents. For instance, if the constituents of a proposition are "A", "B" and "Difference", then the whole proposition is to be constituted by them and nothing else.

On Russell's conception of analysis, then, it is contradictory to deny that anything is a logical subject and the constituents of a proposition must (re)constitute the whole. As we shall see, it is difficult to see how this conception of analysis can be preserved where propositional concepts or denoting complexes occur in propositions.

On Russell's early conception of analysis, predicates and relations have a twofold type of occurrence in propositions in that they may occur either as concepts (indicated by adjectives or verbs) or as subject-terms, without change of meaning. The question arises of whether it is conceptually and numerically the same term which occurs as subject-term as that which occurs as adjective or verb. In the case of predicates, a difficulty arises from the fact that, while a predicate occurring as adjective or concept clearly differs from a predicate occurring as subject-term, it is impossible to state a difference between the term as adjective or concept (e.g., "this is one") and the term as subject term (e.g., "1 is a number") without formulating a proposition in which the term as concept is turned into a subject-term [PoM, 46].[372] Though there appears to be a difference between the predicate occurring as adjective and the predicate occurring as logical subject, it is impossible to state this difference without contradiction. In the case of relations, Russell claims that to avoid this same contradiction, it is necessary to hold that the relation occurring as logical subject, indicated by the verbal noun, e.g., "'difference' is a relation", is precisely the same relation as that which holds as "relating relation" indicated by the verb, e.g., "A differs from B". However, there is a special difficulty in the case of relations: while it is the relation occurring as "relating relation", indicated by the verb, e.g., "difference" as it actually relates A and B in "A differs from B", that is the source of propositional unity,[373] this unity is destroyed in formulating any proposition about the asserted relation, where it is taken as logical subject. The failure of analysis, however, is not resolved by any other approach to analysis and, in fact, the alternative approaches give rise to graver difficulties. We have seen that Bradley held that capturing the relation that rightly relates its terms in analysis leads to a regress of relations between the relation and the terms *ad infinitum*. The attempt to specify the relations "difference" has to A and B not only fails to reconstitute the unity of the proposition, but, if these secondary relations are exhibited

[372] Russell regards this as a serious instance of the contradiction of denying that anything is a logical subject. In what can only be regarded, in hindsight, as an amusing passage, Russell worries that this same contradiction might arise for the "class as many" invoked to solve the contradiction of classes, but dismisses the concern on the grounds that assertions may be made of more than one term, as in "A and B are two" [PoM, 76-7].

[373] Whereas Russell held, in PoM, that relations embody the unity of the proposition and held that analysis into function and argument, by removing a term from a propositional concept, destroys its unity, he ascribed this role to functions supplying the mode of combination in propositions in 1904 "On Functions."

in the analysis of propositions, then an infinite complexity of terms is shown to be involved in the meaning of relational propositions. Invoking particularized relations to analyze the proposition into A, B, and the particularized relation "A's difference from B" will not do, since particularized relations of difference are instantiations of the abstract relation "difference" and, as such, must have in common some relation to 'difference' that is not particularized. As with his argument for external relations based on bare diversity, Russell extends his argument to all relations, concluding that particularized relations never occur in relational propositions as instantiations of abstract relations.[374] The failure of analysis, then, consists in the inapplicability of the two doctrines of analysis mentioned above. First, the fact that the adjective or relation taken as logical subject is something distinct from the adjective occurring as concept or the relation which relates the terms of a proposition, yet it was these, and not some other concept or relation that we intended to take as the logical subject in the proposition formulated about them. Second, the constituents of a relational proposition, e.g., "A", "difference", and "B", do not constitute the whole proposition, e.g., "A differs from B".[375] On my view, what is more fundamental than the problem of unity, which Graham Stevens sees as the theme unifying Russell's developments [Stevens 2005], are the difficulties given rise to by the

[374] These arguments are given in PoM, 51. Relations differ from adjectives in that while the former are not concepts with instances, the latter, following Moore's view in "Identity", are thus regarded. For instance, "One" as adjective/meaning corresponds to a class-concept that should be differently instantiated wherever "…is one" is asserted of a logical subject.

[375] Propositions in which an adjective, rather than a relation, is asserted do not give rise to this difficulty. In this case, the subject term can be replaced by a variable and the adjective can be regarded as a predicate constant, e.g., "Socrates is a man" is of the form C(x), where the constant together with the value of the variable does constitute the proposition. Russell's analyses of such propositions are akin to Frege's function-argument analyses, though Russell does not regard predicates as unsaturated functions.

attempt to denote meanings by taking relating relations, which are (complex) concepts, as the logical subjects of propositions. It remains to consider the case of the attempt to denote meanings which *denote*.[376]

We have seen that when a relational proposition is analyzed into its constituent meanings/concepts, e.g, “A”, “B”, and “difference”, the whole proposition is not constituted by these concepts. It seems that what is asserted in a proposition is not the aggregated concepts, but rather the whole complex comprised of these concepts. In PoM, Russell tells us that, to formulate a proposition about that which is asserted in a proposition, turns it into what he calls a propositional concept, e.g., what is asserted in “the table is black” is the propositional concept “the blackness of the table”, what is asserted in “A differs from B” is the propositional concept “The difference of A from B”[377] or, to use Russell’s example, what is asserted in “Caesar died” is the propositional concept “the death of Caesar.” In such cases, it is not the whole proposition that is taken as the logical subject of a proposition about it, e.g., “ ‘Caesar died” is a proposition”, but rather it is the relating relation[378] that is asserted in the proposition that is taken as the logical subject of a proposition formulated about it. The impossibility of giving a relating

[376] In “On the Meaning and Denotation of Phrases”, written in the latter half of 1903 [Papers 4, 283], Russell makes it clear that proper names denote without meaning, relating relations mean without denoting, and the propositional concept (or any denoting phrase, marked by “the”) both means and denotes.
[377] As we have seen in the case of formal implication, propositions themselves can occur as logical subjects of other propositions, e.g., if [P1] A differs from B, then [P2] B differs from A, contain the unasserted propositions “A differs from B” and “B differs from A”.
[378] Russell classes what might be regarded as predicating predicates along with relating relations, e.g., “the death of Caesar” and “the blackness of the table” are no less ‘relating relations’ than “the difference of A and B”.

relation an entity occurrence (i.e., making it a logical subject) without either destroying the *relating* relation one wished to formulate a proposition about or formulating a proposition about the fact of relatedness is analogous to the problem of the impossibility of denoting a denoting complex without turning it into an entity other than that which denotes or else invoking its denotation. For instance, we cannot say "'the death of Caesar' is a propositional concept" without either taking something other than the relating relation as logical subject (i.e., the complex meaning formed by "the" "death" and "Caesar") or else stating that some particular event is a propositional concept. This may be articulated in terms of the distinction between meaning and denotation, as Russell does in OMD. [379] While a propositional concept means its constituents, e.g., "the blackness of the table" has the meaning constituted by "the", "blackness" and "table" and the meaning of "the difference between A and B" is constituted by "the", "difference", "A", and "B", it would seem that what is asserted in the proposition is not the meaning of the propositional concept, but its denotation e.g., the fact of the blackness of the table or the actual

[379] If the meaning and the denotation of the propositional concept are distinguished, it would seem that what is asserted in a relational proposition, e.g., "Caesar died", is not the meaning of the propositional concept, comprised of "the" and "death" and Caesar", but its denotation, the actual event denoted by "the death of Caesar". In PoM, Russell points out the importance of the distinction for Frege, remarking that Frege must hold that, in asserted propositions, it is the meaning (of the unasserted proposition) and not the indication that is asserted, since otherwise, all propositions would assert 'the true' [PoM, 504-5].Russell draws the distinction between the meaning and denotation of propositional concepts clearly in OMD, written in the latter half of 1903 [OMD, 314], where he points out explicitly that if what is affirmed in a proposition is the denotation of the propositional concept, e.g., is the difference of A and B denoted by the propositional concept "The difference of A and B", then the difference of A and B will be denoted even if the proposition "A differs from B" is false [OMD, 323].

difference of A and B.[380] However, if we wish to formulate a proposition about "the difference between A and B", e.g., "'the difference of A and B' is a propositional concept", we shall either have taken as logical subject the meaning comprised of "The" "difference" "A" and "B", which is something other than what is asserted in the proposition "A differs from B", or else we shall invoke the denotation in taking the meaning as logical subject and the proposition about the meaning will thereby state that the actual difference between A and B is a propositional concept, though it is obviously not a concept, but a fact. The difficulty of taking the relating relation asserted in a proposition as the logical subject of a proposition formulated about it is analogous, then, to the problem of taking a *denoting* denoting complex as the logical subject of a proposition. The lesson of the Gray's Elegy argument is precisely this: if we wish to formulate a proposition about a denoting complex, e.g., "'every finite number' is a denoting complex" then we shall either have denoted something other than the denoting complex which denotes, e.g., the complex meaning constituted by "every", "finite", and "number", or else, since the denoting complex denotes by virtue of its meaning, we shall invoke the denotation and our proposition will state that every particular finite number is a denoting complex. The problem of how to denote meanings has been the subject of much attention in accounts of how Russell arrived at his theory of descriptions, but it is

[380] It is technically correct to simply say that what is denoted by "the blackness of the table" or "the difference of A and B" is the blackness of the table and the difference of A and B, respectively. I use the terms 'fact' and 'actual' loosely to convey what is denoted. The question of whether there is such a fact or whether such a relation actually holds cannot be disentangled from the question of whether the proposition is true.

important to recognize that the difficulties with denoting meanings are an extension of the difficulties of a conception of analysis on which whatever occurs as adjective or relating relation in a proposition can be made the logical subject of a proposition formulated about it.

Interestingly, in MTCA, written in the first half of 1903 [Papers 4, 431], Russell gives his reasons for denying that the denotation of the propositional concept is what is asserted in propositions from which the propositional concept is extracted and, moreover, for denying that propositional concepts are anything apart from propositions. In PoM, Russell considers whether what is denoted by "the death of Caesar" is what is asserted in "Caesar died" [PoM, 48]. Russell points out that if what is asserted is the fact denoted by the meaning, e.g., the fact denoted by "the death of Caesar", then what is asserted must be "the truth of the death of Caesar", but if this is so, then truth and falsity apply to the propositional concept though they ought to apply to the proposition. Propositional concepts were supposed to be what is asserted to overcome the failure of analysis given rise to by the fact that the constituent concepts in a relational proposition fail to constitute the whole. However, it turns out that it is equally problematic to suppose there is some fact asserted apart from the (true) proposition. The difficulty with propositional concepts is captured in Russell's remark that "...the inadequacy of analysis appears...in the fact that propositions are true or false, while their constituents...are neither" [MCTA, 453].This difficulty cannot be overcome, Russell tells us, by maintaining that the propositional concept has an external relation to truth or falsity, while the proposition has truth or falsity as an immediate property, for even if it can be maintained that "Caesar died", in case it is true, is equivalent to "the truth of the death of Caesar",[381] it cannot be maintained that "Caesar died", in case it is false, is equivalent to "the falsity of the death of Caesar" [PoM, 48].

Those familiar with OD will have in mind the problem of propositions containing denoting phrases where the (apparent) denotation is absent. In OD, Russell holds that in the absence of the theory of descriptions, such propositions require either the supposition that some non-existent entity is what is denoted or the introduction of some formal denotation, e.g., the null class defined as "'x is an a' is false for all values of x". In MTCA, Russell elaborates the objection from PoM to the notion that the propositional concept is what is asserted. Russell notes that "Meinong appears to hold that when a relation R is affirmed to hold between a and b, as in (say) 'a is the father of b', what is really affirmed is the being or subsistence of the relation" [MTCA, 452]. That is, it would seem that what a relational proposition asserts is the relation rightly relating its terms, i.e., the propositional concept which denotes the relation which actually holds, and not merely the aggregate meaning of the terms and the relation, which fails to constitute the whole. However, the relating relation cannot be what is asserted, for then truth and

[381] On Russell's account, the propositional concept, e.g., "the death of Caesar" is akin to Frege's Gedanke (thought), while "the truth of the death of Caesar" is akin to Frege's and Meinong's Annhame (assumption) [PoM, 503].

falsity apply to the propositional concept, rather than to the proposition. In an important passage, Russell argues that what he calls "particularized relations", which are the denotations of propositional concepts,[382] are not what are asserted in relational propositions. Russell writes:

> **If what is actually meant by a relational proposition is the being of the particularized relation, then, when the proposition in question is not true, it must be meaningless; for it affirms the being of what, *ex hypothesi*, does not have being, and therefore it affirms nothing, and is meaningless. In other words, every constituent of a proposition, whether this proposition be true or false, must have being; consequently, if the particularized relation is a constituent of the proposition in which it is supposed to occur, then, since such a proposition is significant when it is false, the particularized relation has being even when the terms are not related by the relation in question. Hence, the being of the particularized relation is not what is asserted [MTCA, 453].**

The parallel to OD is clear enough. [383] In OD, Russell argues that if what is asserted in a proposition containing a denoting phrase is (the subsistence of) the denotation, then, when the denotation is absent, the proposition would be meaningless where it ought rather to be false. This might be thought insignificant, for in OD, Russell points out that the theory of denoting with which his theory of descriptions will dispense is not a Meinongian theory, but his earlier Fregean theory of denoting. It may then seem that the purpose of the argument given above is merely to deny the subsistence of "false abstractions". The importance of the argument in MTCA consists rather in the fact that Russell seems finally

[382]The term 'particularized relation' is misleading. In PoM, the particularized relations are instances of a common concept [PoM, 55], e.g., the blackness of the table and the blackness of the chair are both instances of the concept 'blackness'. What is meant here is the propositional concept, e.g., to use Russell's own example, "the blackness of my table", which is what is affirmed, even in case Russell's table is brown [MTCA, 470], so that when the concept is given, so too is its (unique) denotation. In PoM, Russell tells us that, although we may begin with the presentation of some object without knowing the concept of which it is *the* instance, definition is not concerned with that object, but with giving a symbolic abbreviation by which the denotation is uniquely determined (as the only instance of some class-concept)[PoM, 63].
[383] This parallel was drawn explicitly by Ronald J Butler [Butler 1954, 356] though it was articulated in terms of Russell's ontological motivations for the theory of descriptions.

to have grasped that what is asserted in a relational proposition is a relation, not a relating relation, from which he appears to conclude that there are no propositional concepts to speak of. In MTCA, Russell tells us that in a relational proposition, "the relation R between a and b" "…is simply the relation R, together with a reminder that a and b are related by it…The point…is that the whole proposition aRb seems essential…Thus there seems no such entity as the blackness of the table: there is blackness, and the table, and the proposition "the table is black" [MTCA, 470-71]. The point seems to be that there are no propositional concepts, only the propositions in which they occur. Russell concludes that "[w]hen the table is black, 'the blackness of the table' is merely another expression for the proposition 'the table is black'" [MTCA, 471]. The idea seems to be that the complex meaning of the proposition, when true, affirms the fact denoted by the complex meaning of the propositional concept, e.g., "the death of Caesar" denotes the same fact which is affirmed by the true proposition "Caesar died". This manner of regarding propositional concepts, however, is insupportable on his current theory of denoting.

Where the propositional concept is a (denoting) complex in which one term may be replaced by a variable, this complex has the structure of an ineliminable (mathematical) function. In PoM, Russell tells us that (single-valued) mathematical functions are derived from propositional concepts where the propositional concept is a complex in which one term may be replaced by a variable, e.g., "the father of x". When the value of x is given, the value of the function f(x) (assuming that f(x) has a single value) is not a proposition, but rather is the term y satisfying the propositional function y=f(x). In such cases, that which is the value of the variable and a constituent of the complex is not a constituent of the value of the function and the complex cannot be regarded as merely another expression for the proposition, e.g., in "y=the father of x", letting the value of x be Solomon, the value of the whole function is David, in which Solomon is not a constituent. In "On Functions, Classes, and Relations", Russell makes it clear that propositional functions contain variables whose values are not constituents in the propositions which are the values of the function. He writes:

> **A function is propositional when its values are complex meanings containing their respective arguments as constituents in the way in which a constituent of a proposition is contained in a proposition. This is not a characteristic of functions in general; for example 'the centre of mass of x' is a function of x, but x is not a constituent of its centre of mass" [Papers 4, 86].**

Propositional concepts or denoting complexes are not like propositional functions, for they contain constituents not contained in their values and they are not like propositions, for propositions do not denote.

In OMD, Russell states what seems obvious, but what seems to him a significant admission: "The terms that a proposition is about are different... from the constituents of the proposition, and the notion of about is different from that of constituent" [OMD, 328].[384] Russell was evidently aware that denoting concepts, insofar as they denote, are

[384] Russell had held that the logical subjects which propositions are about are supposed to themselves be the constituents of propositions, for otherwise we know nothing about them. Famously, Russell remarked to Frege in 1902, "Mont Blanc is more than 4000 meters high…[for] [i]f we do not admit this, then we get the conclusion that we know nothing at all about Mont Blanc" [PMC, 169].

constituents of propositions which are about their denotations, since he defined them in just this way. Why, then, does Russell draw the conclusion anew that the notion of *about* is distinct from that of constituent? Russell seemed to think he could capture something distinctive about denoting complexes containing variables with the notion of *aboutness* by determining the nature of the connection of *aboutness* with the variable. Ordinarily, the logical subjects of propositions may be replaced by variables and the relations or predicates by constants, so that the values of the variable will be constituents in the resulting proposition. The logical subjects of propositions about them occur as entities, then, and can be replaced by entity-variables. In OMD, Russell points out that a proposition is not *about* that which is merely a constituent of the denoting complex contained within it, e.g., "Arthur Balfour is the Prime Minister of England" is not about England, though England is a term which could be replaced by a variable. Russell calls for a new theory of denoting on the grounds that some complexes, e.g., "the Prime Minister of England" have constituents in their meanings that are not constituents of their denotation, e.g., the complex meaning "The present Prime Minister of England" has the denotation of "England" among its constituents, but the denotation of the complex is Arthur Balfour, which does not have England as a constituent [OMD, 320].[385] Whitehead, however, did not regard Russell's as a fruitful line of inquiry. Russell notes the following

[385] Russell concludes that where a proposition containing a denoting phrase does not have its (apparent) denotation as a constituent in its meaning, that constituent ought to be a constituent of the fact described e.g., "The present Prime Minister of England is the nephew of Lord Salisbury" does not have Arthur Balfour as a constituent in its meaning, but ought to have Arthur Balfour as a constituent in the fact described [OMD, 324; 327–8].

criticism: “he [Whitehead] denies that there is any precision in the notion of about; he says ‘the King is the patron of this Society’ is about this Society” [OMD, 356]. The source of concern is denoting complexes/non-propositional functions containing variables, that is, cases in which an assertion f is made of a variable term x, and the variability of the x is a function of the assertion. These functions are the mathematical functions which Russell does not know how to eliminate. It will be the aim of the subsequent section to shed light on this issue.

5.2 MATHEMATICAL FUNCTIONS AND DENOTING COMPLEXES

In this section, I shall explain the reasons for which mathematical functions (derived from propositional concepts) are, on Russell’s conception of them, to be regarded as denoting complexes containing variables. It is precisely these denoting complexes containing variables and having the structure of mathematical functions which could not be assimilated into Russell’s conception of logical analysis. What I hope to show is that Russell’s criticisms of Frege’s approach to analysis, together with his attempt to accommodate his own insights about analysis, preclude his adoption of a Fregean view of (mathematical) functions and motivate him to regard mathematical functions as denoting complexes whose analysis presupposes propositional functions. The idea is that Russell did not arrive at his 1905 theory of descriptions from within a Fregean theory of functions, but from within a theory of denoting complexes whose “meanings in isolation” he hoped to deny. In the preceding section, we saw that Russell wished to regard the relational proposition as fundamental, and the propositional concept which might be supposed to be what is asserted in a relational proposition as only another expression of the proposition when the proposition is true. However, if what I urge is correct, this view was untenable in the light of the fact propositional concepts are of the same form as mathematical functions, that is, they are unambiguously denoting complexes whose values are not propositions. These functions are the problematic sort which his pre-1905 theories of denoting are incapable of reducing. This must be understood in the context of Russell's conception of logical analysis.

Recall that, in PoM, Russell thinks that where the independent variability of subject-terms is required, i.e., in propositions of the form xRy, R cannot be the assertion, for it fails to preserve sense, and …Ry cannot be the assertion, since it fails to preserve the independent variability of y [PoM, 505].[386] Rather, the independent variability of x and y requires that the propositional function xRy be regarded as more basic than relations or functions. Though Russell recognizes that it is relations in extension which are important in mathematics, his concern to capture sense in logical analysis leads him to reject Frege's treatment of relations in terms of a double-function determining a double-range, i.e., a class of couples [PoM, 512]. Even in the late additions to PoM, Russell

[386] Recall that, in PoM, Russell tells us that relations in intension must be identified with the class-concepts rather than classes, so that sense, which is expressed in relational propositions, can be preserved in logic, which classes of couples fail to do.As Russell puts it: "…the symbols other than the variable terms (the variable class-concepts and relations) stand for intensions, while the actual objects dealt with are always extensions. Thus in the calculus of relations, it is classes of couples that are relevant, but the symbolism deals with them by means of relations" [PoM, 99].

maintains that, although a propositional function xRy may determine a class of couples, R, of which (x, y) is a member, it is doubtful whether there are any such entities as couples with sense unless these are derived from relational propositions[387] —even the assertion that a is referent and b is relatum with respect to R, he points out, requires a relational proposition [PoM, 99]. In PoM, Russell holds that relational propositions analyzable into y and Rx, e.g., the (unasserted) proposition "David is the father of Solomon" is analyzable into "David" and "being the father of Solomon," are what give rise to those propositional concepts from which are derived the functions f(x) of the sort contained in y=f(x). Russell tells us that "if f(x) is not a propositional function, its value for a given value of x…is the term y satisfying the propositional function y=f(x), i.e., satisfying, for the given value of x, some relational proposition" [PoM, 508]. Propositional concepts, then, have the form of asserting a function f of a variable term x, where this does not yield a proposition for a given value of the variable, e.g., in "the father of x", if x is Solomon, then the value is David, which is not a proposition. Hence, propositional concepts are akin to the functions involved in equations of the form f(x)=y, where the function f(x) is not a propositional function, but a mathematical function. Now the objection might be raised that Russell did not develop his theory of descriptions from

[387] Russell's view is that to know, for instance, that x is the referent and y the relatum with respect to R requires a relational proposition in which the relation is asserted, making relations more basic than classes [PoM, 49]. Russell raises the doubt as to whether there are couples with sense late as his May, 1902 addition of the appendix on Frege [PoM, 512n2]. As we have seen, however, Russell relies increasingly on propositional functions to analyze relations in extension and the view that relations have differing types of occurrence is replaced by the notion that relations occur in propositions of differing types (e.g., the relation to the class as one versus the relation to the class as many).

within the context of this view that propositional concepts and mathematical functions are derived from relational propositions and presuppose propositional functions, but rather from within the context of a "Fregean" theory of denoting. This, we shall see is not the case. Rather, the purpose served by Russell's brief adoption, in 1903, of the Fregean "functional theory" was to convince him of the correctness of his earlier conception of analysis.

Let us briefly consider Russell's motivation for adopting the "functional theory" of 1903 and for regarding the function as a separable entity. Recall that Russell's logicist definition of number trades on the many-one relation between the similar classes and the class with which the number is to be identified. In the light of the Contradiction, Russell required some other means of denoting symbolically what corresponds to "the class as one". In his May 1903 manuscript notes, "Relations",[388] Russell dispenses with Peano's class abstraction notation, x such that φ(x), for the class of x's satisfying φ(x). However, whereas the notation used to indicate that some class of terms satisfying some propositional function, x's such that φ (x), was not null—in Russell's earlier Peanist notation ∃ {x such that φ(x)}— could be straightforwardly replaced by existentially quantified statements of the form (∃x) . φ(x), Russell discovers that the notation for definite descriptions "the φ"—in Russell's Peanist notation

℩x such that φ(x)

[388] The draft of the manuscript was composed earlier, but the revisions dispensing with Peano's class-abstraction notation were made in May, 1903 [Papers 4, 38].

—could not be replaced, and was retained as an ineliminable indefinable. What is interesting in this connection is that the topic of Peano's paper at the Paris congress of 1900 which ushered in the period of Russell's greatest optimism about logicism, was the need for a symbolic expression of "the" in the definition of classes. Quite remarkably, Peano soon articulated his attempted elimination of "the". Peano's proposal for eliminating ⁊ is at least superficially similar to Russell's elimination of "the" in the theory of descriptions. In 1900, Peano pointed out that ⁊ a εb is equivalent to Ǝ (x such that) [a=ιx . x εb] [Peano 1900].[389] In other words, "'the' member of a belongs to b" is equivalent to "the class of x, such that the class a is equal to the unique member x[390] and x belongs to b, is non-empty." Of course, Peano's equivalence statement trades on the notion of classes that the theory of descriptions was to be praised for eliminating.[391] In May, 1903, Russell adopts a Fregean "functional theory" on which class abstracts are to be replaced by function abstracts. In adopting Frege's function notation, Russell initially thought that he could eliminate classes by replacing the class of terms x such that φx by Frege's 'x φx for the value-range of φx. Here, the function φ is a separable entity. Of

[389] For further discussion of this development see Grattan-Guinness 2000, 246 and Rodriguez-Consuegra 2000.

[390] In Peano's notation, this is ιx =y such that (y=x).

[391] It is less clear, however, what prevented its adoption prior to the discovery of the Contradiction. Consider Peano's alternative elimination: ⁊ a εb. =: a=ιx.$\supset_x$. xεb. This states that "'the a' belongs to b" is equivalent to "if, for all x, x is the single member of the unit class a, then x belongs to b".[391] Russell, who believed in 1900 that implications were the essential form of mathematical propositions and whose interest in considering the elimination of "the" would have been related to defining the single-valued functions of mathematics, would perhaps have privileged the elimination by means of implication over that given by existential quantification. Though I am uncertain as to why Russell initially rejected Peano's elimination, there is no question that he would not have accepted it after his discovery of the Contradiction, due to its reliance on class abstraction.

course, Frege had also appreciated the need to symbolize single valued mathematical functions and, in the GG, introduced \ξ in place of 'the', representing a function having for its value the object falling under it in case it is unique (or, in case it is not unique, its extension) [Grattan-Guinness 2000, 191], e.g., the unique object of the square of 2 (or, the extension of the square root of 4). However, by OMD, Russell uses Frege's symbol for single-valued mathematical functions f ξ to symbolize a complex (meaning) of which the variable is a constituent. We may wonder why Russell regarded f ξ as a symbol for a complex meaning having a variable as a constituent. The answer, as we shall see, is that Russell had already decided "in favour of [his] old practice" [OMD, 342] of regarding mathematical functions as denoting complexes containing their variables, which is akin to his manner of construing them in PoM, where he regarded them as derived from propositional concepts.[392]

[392] This is merely to indicate the context in which Russell frames the trouble of denoting complexes and does not suggest any resolution. Recall that it was because truth or falsity belongs to the whole proposition and not to its constituents that Russell denied that the propositional concept is what is asserted in a relational proposition. Where the propositional concept is extracted from a proposition in which it is not a constituent, e.g., "the death of Caesar" from "Caesar died", no difficulty arises, but where the propositional concept occurs as a constituent, e.g., "the death of Caesar was a tragedy", a complex occurs within a complex. The term "Caesar" may be replaced with a variable, but it is within the denoting complex itself, and not the whole proposition, that the variable is replaced, yielding new deaths, not new tragedies, when the value of the variable is given. The constant and the variable, together, are akin to a mathematical function which does not have a proposition as its value when the value of the variable is given. There are cases in which the value of the function when the value of the variable is given fails to be a constituent of the resulting proposition, e.g., "the father of x is wise", when Solomon is given, does not have Solomon as a constituent of the resulting proposition, "David is wise". These problems are those which Russell confronts in the analysis of propositions containing denoting complexes (which contain their variables). The fact, however, that Russell gives prominence to the proposition in regarding propositional concepts as derived from them and as satisfying propositional functions, is important to understanding how his view differed from Frege's conception of mathematical functions and provides an entry-point for understanding the advantages of the solution proposed in the 1905 theory of descriptions.

The significance of Russell's adoption of the functional theory rests in the fact that the problems to which it gave rise led Russell to focus on the problem of denoting in the context of his own conception of analysis. In adopting the functional theory, Russell discovers that if a function is a separable entity, then it can be asserted of itself, in which case, the function "non-assertability of self" can be asserted of itself. Klement locates this discovery in Russell's paper, "No Greatest Cardinal", likely written in May, 1903.

> **Russell considered the function *x*' (~ *x*|*x*),[393] and from it formalized the functions version of the contradiction. This appears explicitly for… the first time in a manuscript entitled "No Greatest Cardinal" (*Papers* 4, 62-3), probably written sometime in the summer of 1903 [Klement 2004, 130].**

As Klement rightly points out, it is precisely by means of his short-lived use of Frege's smooth-breathing abstraction operator, which allowed for (separable) functions, that Russell was able to formulate the functions version of the Contradiction in the first place. Despite the function version of the Contradiction, Russell continued to work on his Fregean functional theory, intending to isolate the properties of non-predicative functions (those functions which do not determine classes) so as to exclude them in the primitive propositions. In a retrospective letter to Jourdain, Russell recalls that it was in the attempt to discern which functions determine classes that he discovered that "…to assume a separable φ in φx is just the same, essentially, as to assume a class defined by φx, and that non-predicative functions [those which do not determine classes] must not be analyzable into φ and x" [R25.03. 1906 in Grattan-Guinness 2000, 79]. While Russell's discovery of

[393] The smooth-breathing mark ' should occur over the x.

the functions version of the Contradiction did not lead him to immediately abandon the functional theory, Russell did find reasons for resisting Frege's view of functions in 1903. These reasons, we shall see, were a product of his own conception of analysis, together with objections to Frege's approach to analysis, and led him to view mathematical functions as denoting complexes containing variables which were derived from propositions and satisfied propositional functions.[394] It remains to show that Russell did not articulate the 1905 theory of descriptions from within a Fregean theory of functions, but from within a conception of analysis which, though relations in intension are subsumed under propositional functions, is consistent with his approach to analysis in PoM.

In considering Russell's reasons for rejecting Frege's conception of mathematical functions, it will be useful to recall what is distinctive about them. In OMD, Russell points out that any complex containing an independent variable is a dependent variable or function. If the value of the dependent variable is a proposition, i.e., if the function is propositional, no trouble arises, for then the values of the variable are constituents of the resulting proposition or complex, e.g., in "x is mortal", no matter what the value of x, it is

[394] Russell remained concerned with cases where functions, and not their values, were taken as logical subjects. Klement points out that it is only in 1904, when Whitehead introduces the circumflex notation, that Russell has a manner of denoting functions, e.g., "(ˆ*x* is human) is human". Klement points out Russell's remark that: "The circumflex has the same sort of effect as inverted commas have. E.g. we say Any man is a biped; "Any man" is a denoting concept. The difference between $p \supset q \,.\supset.\, q$ and $\hat{p} \supset \hat{q} \,.\supset.\, \hat{q}$ corresponds to the difference between any man and "any man"[Papers 4, 128-9]. Interestingly, Russell makes the distinction, in his July 1904 letter to Couturat, between a function occurring as concept, as in $\boldsymbol{\varphi}$ 'x and a concept occurring as term as in $\boldsymbol{\varphi}$ ' ˆx, pointing out that the function can only be varied when it occurs as term. [CPLP, R05.07.1904]. The circumflexes ˆ should occur over the letters.

a constituent of the value of the function. However, in propositional concepts and mathematical functions, the value of the independent variable does not occur as a constituent in the value of the dependent variable. For instance, in "the Prime Minister of x", if England is the value of x, then the value of the dependent variable is Arthur Balfour, which does not have England as a constituent. Likewise, in "the square of x", letting the value of x be 2, the value of the dependent variable is 4, which does not have 2 as a constituent [OMD, 331], that is, even if these mathematical functions are taken in the context of equations, e.g., $y=x^2$, the value of the variable is still not a constituent of the denotation of the function.[395] To understand how Russell's conception of mathematical functions differs from Frege's, we must now concern ourselves with the question of why the variable appears in the analysis of mathematical functions at all. To address this point, it will be useful to recall his remarks on Frege's function-argument form of analysis in PoM.

On Russell's account in PoM, Fregean analyses into subject and assertion are possible where a proposition either is predicative, e.g., "…is a man" or asserts a fixed

[395] Likewise, for Frege, the value of a function for a given argument may not have either the function or argument as constituent. There is a difficulty, similar to Russell's difficulty that there is "no backwards road" from denotation to meaning, in preserving the connection between a function and its value. If, where one begins with the value, the function cannot be isolated, but if the expression involves the *bedeutung* and not the *sinn* of the names, then what one has is the value in which neither function nor argument appears as constituent. It is for this reason that Russell regards the proposition as fundamental and (propositional) functions as derived from determining what is to be kept constant in the proposition and what is to be varied. For Frege, if the sense of a name, rather than its reference, is the argument, the value is the Gedanke in which the sense is a constituent [Levine 2002, 211-12.]. This allows Frege to account for informative identities e.g., "Arthur Balfour is the present Prime Minister of England" as opposed to "Arthur Balfour is Arthur Balfour" and their intensional aspects, e.g., "Russell was surprised that the number of people at the meeting was greater than 300" from "Russell was surprised that 350 was greater than 300".

relation R to a fixed term b, e.g., "…is greater than a dozen", represented by "…Rb". Frege's function-argument form of analysis is adequate, then, to cases of dependent variables not containing variables, where whatever takes the argument place is a constituent of the value of the function.[396] As we saw above, Russell's analysis into a constant and a variable results in a proposition when the value of the variable is given and this, Russell seems to think, could be carried out just as well by Frege's function-argument approach. It is doubtful whether what Russell has in mind is in fact akin to Frege's function-argument approach, since Russell does not require that concept and object be distinguished or that any type restrictions be placed on the arguments, so that what he construes as a function-argument approach to analysis is much closer to his own conception of analysis into a constant and a variable. Russell accounts for assertion by

[396] Recall that in PoM, Russell still subscribes to the "intensional view of relations", which was developed from a part/whole approach to the analysis of propositions, and was intended to give exact analyses, e.g., of two propositions "hydrogen is lighter than carbon dioxide" and "carbon dioxide is heavier than hydrogen", exhibited two different relations "heavier than" and "lighter than". Even the relations "similar to u" and "similar to v" were thought to differ, so that the class itself and not merely class-concepts had to be invoked to guarantee an object that was the cardinal number. For Frege, the propositions "hydrogen is lighter than carbon dioxide" and "carbon dioxide is heavier than hydrogen" express the same conceptual contents in different ways [Beaney 2009, 7]. Frege's function-argument analysis was developed in the *Begriffsschrift* to permit the substitution of expression in proofs and its merit was supposed to consist precisely in the fact that it permitted conceptual contents to be differently divided. In FA, for instance, the direction of two lines can be differently analyzed as their parallelism: direction of line a=direction of line b iff line a // line b. There is, however, a problem. This allows for the same object (direction of line a) to be identified when it appears "under another guise" (as direction of line b), it correlates parallel lines with the same object (direction) and correlates each new direction with non-parallel lines. It does not, however, tell us what this 'same object' is. There is nothing to prevent England from being the direction of the earth's axis [FA, §66]. The problem, akin to that of the Julius Caesar problem in the definition of number, is that we have no *concept* of direction. It is for this reason that Frege introduces the extension of concepts, e.g., the direction of line *a* as the extension of the concept "parallel to line *a*", and the number of the concept F as the extension of the concept "equinumerous with the concept F" [FA, §68]. With the introduction of the *Sinn*/*Bedeutung* distinction, complexities are introduced into the notion that the same conceptual contents may be differently divided, but I cannot address them here.

considering the constant in a proposition which may be asserted of a variable term: he tells us that "[i]n 'Socrates is a man' we can plainly distinguish Socrates and something that is asserted about him; we should admit unhesitatingly that the *same* thing may be said about Plato or Aristotle" [PoM, 84]. Moreover, he tells us that an assertion is "everything that remains of the proposition when the subject is omitted", i.e., what is obtained "by simply omitting one of the terms occurring in the proposition" [PoM, 85]. In any event, the case against Frege's function-argument analysis is made, in PoM, by considering the case of propositions expressed by formal implications, where structure can only be preserved by propositional functions, e.g., "Socrates is a man implies that Socrates is mortal" can only be captured by the propositional function $\varphi x \supset \psi x$, and not by the Fregean analysis …F⊃…G, which fails to guarantee the reappearance of the same variable [PoM, 509].[397] Functions, prior to Frege, were customarily thought to express the relationship between a dependent and an independent variable, e.g., $x^2 + x$, in equations, e.g., $y = x^2 + x$. Frege held that the correct analysis of statements involved in equations required their analysis into a separable function and an argument place so as not to confuse the function itself with its values given some arbitrary value of x, e.g., the statement $x^2 + x$ would be analyzed into $(\ )^2 + (\ )$. Russell thought this was mistaken for the reason that this fails to guarantee that the same argument appears in each instance and, hence, held that it was necessary to regard the function as containing variables, rather

[397] Likewise, the same variable must reappear in the assertion of a relation of a term to itself.

than regarding the function as a separable entity. It is thus that the variable is included in Russell's functions.[398]

Russell has another, related reason for denying that the function is separable. He points out cases in which the dependent variable does not have a fixed meaning, but varies with the independent variable, e.g., in "if x is a rational number then x^2 is a rational number." Here, x^2 does not mean "the square of anything", but x means "anything" and x^2 means "the square of x", so that the square of x only has a fixed meaning when x is given. To preserve this relationship between the dependent and the independent variable and ensure that the x in the antecedent and the consequent has the same denotation, functions must not be separable and propositional functions must be regarded as fundamental. Russell writes:

> **The point to observe is that an expression containing x must be treated as a whole and must not be regarded as analyzable into bits each of which contains an independent variable, even when every value of the dependent variable is analyzable into bits containing the corresponding value of the independent variable. Now x will always occur in a whole which is propositional; and thus propositional functions are the most fundamental [OMD, 333].**

To see that Russell held that the variable always occurs in the whole proposition, it will be useful to consider the manner in which he returns, in OMD, to the conception of mathematical functions to which he subscribed in PoM. Recall that on Russell's view in

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[398] Beaney writes: "the value of a function does not literally contain its argument(s) as part(s). Russell began to appreciate the power of function-argument analysis after his meeting with Peano in 1900, and as he learnt, developed and applied Peano's logic, he was forced to rethink his adherence to decompositional (whole-part) analysis" [Beaney, 2009, 8]. While I (clearly) share Beaney's view that the importance of part/whole analysis subsided with Russell's discovery of Peano, I do not think the adoption of Peano's approach to analysis, supplemented by propositional functions, represents the embrace of a function-argument approach to analysis, at least not of the sort which resembles Frege's approach.

PoM, mathematical functions which satisfy propositional functions are derived from propositional concepts extracted from relational propositions.

In OMD, Russell wonders whether propositional functions are more basic than mathematical functions. Recall that in PoM Russell maintained that mathematical functions are derived from relational propositions and satisfy propositional functions y=f(x). In OMD, Russell points out that if denoting is fundamental, then a many-one relation will be expressed by y=(f)x, as an ordinary mathematical function, e.g., y=the father of x. If propositional functions are fundamental, then a many-one relation will be expressed in a relational proposition e.g., "y is the father of x" whose structure is xRy, deriving from this 'the'Ř of x, e.g., "the father of x". The latter was his view from PoM and the difficulty, of course, is that 'the' is ineliminable. Russell writes:

> **[I]f we take *propositional* functions to be fundamental– as I have always done, first consciously and then unconsciously– we must proceed through relations to get to ordinary functions. For then we start with ordinary functions such as "x is a man"; these are originally the only functions of one variable. To get at functions of another sort, we have to pass through *xRy*; but then, with 7, we get all the problems of denoting. And, as we have seen, a a form of denoting more difficult than 7 is involved in the use of variables to start with. Thus denoting seems impossible to escape from [OMD, 340].**

Russell reconsiders his unconsciously held position: "We have been in the habit of defining the relatum by the relation; but this seems to be putting the cart before the horse, if all functions of one variable are equally fundamental" [OMD, 339]. Russell concludes, however, that all functions of one variable are not equally fundamental. The fundamental functions are *propositional* functions of one variable, as in "x is a man" (and propositional functions of two variables as in "x is a son of y") wherein the denoting complexes involved in the propositions, e.g. “a man” (and “a son of y”), denote ambiguously. Where the denoting complexes denote unambiguously, as in “the humanity of x” or “the father of x”, these unambiguously denoting complexes presuppose propositional functions [OMD, 342] whose values are propositions asserting a many-one relation. By the time of his April, 1904 letter to Couturat, Russell has adopted an extensional view of relations[399] and has determined that functions are more basic than relations, but that it is the *propositional function*, not the mathematical function that is “the foundation of the edifice”. Russell comes to hold that relations can be treated in terms of double propositional functions $\boldsymbol{\varphi}!(x, y)$. Recall the letter to Jourdain, wherein Russell writes: “The first thing I discovered in 1904 was that the variable denoting function is to be deduced from the variable propositional function, and is not to be taken as indefinable. I tried to do without ℩ as an indefinable, but failed” [Grattan-Guinness 1977, 79]. This is simply an extension of Russell’s view from OMD. The view is that what are fundamental are single propositional functions $\boldsymbol{\varphi}!(x)$, i.e., assertions containing a single variable as in “x is a man”, and double-propositional functions $\boldsymbol{\varphi}!(x, y)$, i.e., assertions containing two variables as in “x is greater than y” and it is only when there is, for a given value of x, only one value of y satisfying the propositional function $\boldsymbol{\varphi}!(x,y)$

[399] In recalling Russell’s 1904 letter to Couturat, it should be noted that while Russell has accepted an extensional account of relations, he nevertheless remains committed to the view that the fundamental relations of the calculus of relations (i.e., identity and implication) are intensional. Whereas a relation, on the extensional view, merely juxtaposes two terms so that, when the terms are given, the relation holds as a matter of fact, the fundamental intensional relations cannot be so regarded. Implication does not merely assert that there is a couple, propositions p and q, between which the relation of implication, as a matter of fact, holds. Identity statements do not merely assert the two terms or propositions together with a reminder that the identity relation relates them.

that the function is a mathematical function of the sort contained in the equations considered above. That is, when, for a given x, the y satisfying the double propositional function $\boldsymbol{\varphi}$! (x, y) has a unique value, the propositional function is equivalent to y=f (x). The form y=f (x) expresses many-one relations [ORML, 525],[400] but, as Russell points out, it is the general notion of relations with which symbolic logic is concerned [ORML, 524]. This view shows that many-one relations are a special case of relations and that they have the form of mathematical equations involving mathematical functions, but it does not show just how these mathematical or denoting functions are to be reduced to propositional functions. For this, the theory of descriptions is required, which Russell would not arrive at until "On Fundamentals", which he began in June, 1905 [Papers 4, 359]. What is important is that the theory of descriptions does not arise out of a Fregean approach to analysis, but rather is intended to *preserve* Russell's conception of analysis, on which propositional functions are the fundamental sort.

Russell's view that propositional functions are fundamental is the result of a conception of analysis which regards propositions as basic and permits any constituent occurring as a term within it to be replaced by a variable. In PoM, Russell writes: "Accepting as indefinable the notion proposition and the notion constituent of a proposition, we may denote by (a) a proposition in which a is a constituent. We can them transform a into a variable x, and consider $\boldsymbol{\varphi}$(x) where [the value of] $\boldsymbol{\varphi}$(x) is any

[400] In ORML, Russell uses the notation y=f ʻx. Where the correspondence from y to x is one-one if two values of x never produce the same value for y [ORLM, 526].

proposition differing from φ(a), if at all, only by the fact that some other object appears in the place of a" [PoM, 356].[401] Russell adds that "φ(x) is what we called a propositional function". For Frege, linguistic variables in functional expressions do not symbolize non-linguistic variables, so that a concept containing a linguistic variable, e.g., "x is a man" and an unsaturated concept, e.g., "is a man" both express functions [Levine 2002, 213].[402] For Russell, whose logic developed along Peanistic lines, the propositional function containing its variable, e.g., "x is a man" must be the fundamental sort and the variable, we have seen, must occur in the whole proposition if implications are to be intelligible (e.g., x is a man $\supset_x$ x is mortal or x is a rational number $\supset_x$ the square of x is a rational number). Consider again those functions f(x) satisfying propositional functions of the form y=f(x). These are supposed to be derived from relational propositions, e.g., "David is the father of Solomon" involves the propositional concept "the father of Solomon" from which we derive "the father of x". In these cases of mathematical functions or denoting complexes f(x), the variability of the variable is determined by the function so that, though no hypothesis is asserted as in the case of implication considered above, the x in f(x) is the class of terms satisfying the function. For instance, in "The

[401] Landini regards Russell's early approach to the substitution of entities by means of variables and denoting concepts in PoM as the basis for his later substitutional theory. Landini writes: "The substitutional theory emerged from Russell's attempt (in the *Principles)* to use denoting concepts and the notion of the substitution of entities (including denoting concepts themselves) in the explanation of the constituents of propositions named by formulas involving the use of single letters as variables" [Landini 1998, 45].

[402] In a letter to Jourdain, dated January 28, 1914, Frege complains of Russell's notion of the variable as a symbol with an indeterminate meaning in PM, on the grounds that what he really seems to mean is that the letter is a symbol for a symbol (the variable). He complains also that a propositional function, e.g., "x is a man" is a variable whose value determines its meaning, for which reason, its value cannot be said to be ambiguous. "It seems to me", Frege writes, "that the difficulties keep piling up as one penetrates further into Russell's work" [PMC, 81-4].

Prime Minister of x", the variability of x is determined by the function "Prime Minister of", so that the x is restricted to the class of terms satisfying the function, i.e., those which are republics or constitutional monarchies. On what Russell calls "the substitutional view",[403] in OMD, to replace one of the terms of the complex with a variable, we begin with a complex containing constants and replace one of the constants with a variable, e.g., in "the father of Solomon", Solomon is replaced by a variable whose variability is presumed to be determined by the function, which is constant [OMD, 335].[404] However, if a term in a complex containing only constants can be substituted for by another term, replacing the term with a variable requires that we know *what is to be kept constant*, but this seems to be nothing other than a separable function. In a denoting complex of the sort we have been considering, for instance, it seems we must separate off the function, e.g., "The Prime Minister of" from the dependent variable x, before we can substitute France for England [OMD, 339].[405] Hence, the substitution of one entity for another in the complex is permitted by replacing a constant with a variable, but the complex is not the proposition, but a denoting complex. Even if these denoting complexes are derived, in the first instance, from propositions (e.g., "the father of Solomon" from the relational

[403] In his letter to Jourdain, on March 5, 1906, Russell wrote: "About June 1904, I tried hard to construct a substitutional theory more or less like my present theory. But I failed for want of the theory of denoting: also I did not distinguish between *substitution* of a constant for a constant and *determination* of a variable as this or that constant...Then, last autumn, as a consequence of the new theory of denoting, I found at last that substitution would work, and all went swimmingly" [Grattan Guinness 1977, 79-80].

[404] This is not the case in the substitutional view of 1906-1907, but until Russell could eliminate denoting functions, he could not preserve the unrestricted variable, since the variable occurs within the denoting complex and not, as he wished, in the whole proposition.

[405] This is equally true for predicate constants, e.g., to substitute Plato for Socrates in "Socrates is mortal", we need to know that what is to be kept constant in "x is mortal" is "is mortal" (or, as Russell would say "being mortal", to avoid the conflation with assertion).

proposition “David is the father of Solomon” or “the square of 2” from the equation “the square of 2 is 4”), the substitution still takes place within the denoting complex itself which there is no means of eliminating.

Now, we have seen that the function must not be a separable entity, for the reason that the separable $\boldsymbol{\varphi}$, having unrestricted variability and regarded as entity or logical subject, can be asserted of itself, giving rise to the functions version of the Contradiction. However, in OMD, Russell points out that, on the view that functions are separable which seems required for substitution, “…it seems quite arbitrary to deny that f(f) has meaning” [OMD, 338]. The solution appears to be to avoid the Contradiction by admitting separable functions which can be asserted only of the appropriate values of the variable.[406] To avoid the Contradiction, it is necessary both to preclude functions of the form f($\boldsymbol{\varphi}$), which are the quadratic forms involving functions of variable functions, and also to deny that every function determines a class as entity and are “appropriate to entity variables” [OMD, 338]. In this case, the separability of the function will require that there are, apart from the unrestricted entity-variables, independent function variables. While it is endemic to Frege’s logical system and to his function-notation that functions, which are unsaturated, have values for arguments of the appropriate type, where the order of the

[406] It is clear that Russell’s “Fregean approach” to eschewing the Contradiction differs markedly from Frege’s in that, even where he entertains the possibility of functions which may be asserted only of the appropriate level of argument (objects, functions, second-level functions, etc), he construes these as *denoting complexes* asserted of the *appropriate values of the variable*. “Whatever is asserted of all the values of the variable”, he writes, “must be taken as asserted for all the *appropriate* values of the variable—for other values the complex asserted will be meaningless” [OMD, 338].

function itself gives rise to the restrictions placed on the argument, Russell's approach to analysis, which leads him to view functions as denoting complexes containing variables, offers no solution to the difficulty of determining how restrictions ought to be placed upon the variables. The specification of such restrictions from within Russell's logic, even if it could be carried out, would afford only an *ad hoc* solution.

Let us briefly consider this difficulty in connection with the Contradiction. Russell's remark to Jourdain that, between April 1904 and January 1905, he worked on the question of denoting, which he thought "was probably relevant" to the Contradiction and which "it turned out to be" suggests that Russell saw only a vague connection between the Contradiction and the question of denoting. However, recall the April, 1904 letter to Couturat, in which Russell expresses his dissatisfaction with Fregean functions, writing that Frege has merely expressed the problem that he believes himself to have solved. The problem Frege thinks he has solved is that of non-predicative functions, i.e., those which do not determine classes. On Frege's proposed solution to circumventing the Contradiction, the problematic functions to be excluded are second-level functions taking a function as argument where, when two functions (concepts) with equivalent values are taken as arguments of the second-level function, these determine equivalent values, but the value falls under one function (concept) and not the other. In Russell's reformulation in terms of classes determined by propositional functions, Frege's argument proves that there may be two functions g, $\boldsymbol{\varphi}$, such that x ɜ (gx)= x ɜ ($\boldsymbol{\varphi}$x), but not g { x ɜ ($\boldsymbol{\varphi}$x)}, that is, the functions g and $\boldsymbol{\varphi}$ determine the same class, but x ɜ ($\boldsymbol{\varphi}$x) is a member of one, but not of the other [Papers 4, 608]. The problematic functions, which Russell thinks must be excluded in the primitive propositions, are what, on Russell's account, are called "quadratic forms" which are statements of the form $\boldsymbol{\varphi}$ (f ($\boldsymbol{\varphi}$)) where the argument of the function/assertion varies with the function/assertion.[407] In his notes on Frege's appendix to GG, Russell characterizes such functions by the fact that "[s]uch forms make no fixed assertion concerning the variable term" [Papers 4, 614].[408] The problem which Frege has merely posed is that of denoting. In his own attempted solution to the Contradiction by means of functions in 1903-1904, where the functions admitted in the primitive propositions are to exclude this problematic sort, Russell does not distinguish concept and object or introduce a type-stratification of functions, though, as we have seen, he is well aware that the view that complexes containing variables are analyzable into a separable function and a variable argument leaves open the difficult question of how the variables

[407] Russell notes also that quadratic forms will arise when a relation is asserted to hold between itself and another term.

[408] Landini has provided me with a useful illustration of the manner in which quadratic forms arise and I shall attempt to reconstruct and interpret it here. Consider the characterization of the Union of a class A of classes a_1…to a_n: for all x, x ε UA iff (∃y) (y εA & xεy). Without the existential quantifier, we have yεA & xεy, where x and y are both variables. The trouble arises in attempting this characterization without classes. If the predicate UAẑ is characterized, instead of UA, we have: (∃f) (G(f) & fx), which, without the existential quantifier, leaves G(f) & fx, in which f and x are both variables. It is this sort of function which gives rise to the Contradiction. Consider Russell's: w=cls∩ x ɜ (x ~ εx) . ⊃: w εw ≡ w~εw (this says that if w is the class x such that x is not a member of x, then w is a member of itself if and only if w is not a member of itself. To proceed by predicates or functions instead gives rise to the predicate version, Gx.$\equiv_x$: (∃f) (x=f & ~fx), in which f and x are both variable, and the function version W(f).$\equiv_{\boldsymbol{\varphi}}$. (∃$\boldsymbol{\varphi}$) (f=$\boldsymbol{\varphi}$ &~(f), in which $\boldsymbol{\varphi}$ and f are variable.

of which functions are asserted ought to be restricted to appropriate values. In his July 5, 1904 letter to Couturat, Russell points out that the solution to the Contradiction must be found by placing restrictions on the notion of "a function of x". Interestingly, to make a start on achieving this, Russell employs what is by now a familiar sort of distinction: a function occurring as concept, as in $\boldsymbol{\varphi}$ ‘x, he tells Couturat, must be distinguished from a function occurring as term as in $\boldsymbol{\varphi}$ ‘ ˆx. He points out that the function $\boldsymbol{\varphi}$ can only be varied by turning the proposition into one in which it occurs as term. In this way, Russell believes he can exclude from what might be called "functioning functions" (functions occurring as meanings) those which are the source of the Contradiction, i.e., the quadratic forms in which a variable function is asserted of a variable argument f($\boldsymbol{\varphi}$) [CPLP, R05.07.1904].

This distinction between a denoting complex occurring as meaning and the denoting complex occurring as entity is resumed in "On Fundamentals", where Russell tells us that "what occurs as meaning can't be varied; we must be able to specify what varies, and this can only be done if what varies occurs as entity, not as meaning" [FUND, 362]. Russell's reasoning here comports with his earlier views on the substitution of entities in propositions and the analogy to relations is discernible. To vary a relation, e.g., "differs from" in "x differs from y", it is necessary to take the relation as the propositional concept "difference holds between x and y" so that the relation, now occurring as entity, can be varied [FUND, 380]. However, when that which occurs as meaning/concept is a denoting complex, e.g., “the difference between x and y”, “the father of x”, “the square of x”, the meaning is complex,[409] the concept cannot be replaced by an entity-variable. Russell writes:

> **It is a fallacy to use a single letter to represent an occurrence of a complex as *meaning*, since a single letter will have all entities among its values; moreover, when a complex occurs as meaning, its structure is essential to its significance, and a single letter, since it does not symbolize any structure, destroys the significance [FUND, 374].**

The whole denoting complex, then, cannot be replaced with a variable without destroying the structure of the complex, essential to the significance of the proposition in which it occurs. For instance, if “the author of Waverly” were replaced by a variable in “the author of Waverly is Scott”, and “Scott” is a value for which the resulting proposition is true, then the substitution produces “Scott is Scott”, with a resulting loss of significance.[410]

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[409] Russell writes: “... when complexes occur as *meaning*, their complexity is essential, and their constituents are constituents of any complex containing the said complexes; but when complexes occur as *entities*, their unity is what is essential, and they are not to be split into constituents. Hence generally: When a complex *A* occurs in a complex *B*, if *A* occurs as *meaning*, its constituents are constituents of B, but if it occurs as entity, its constituents are not constituents of B [FUND, 373].

[410] On the 1903 theory of denoting, even if a twofold occurrence of denoting complexes as subject-terms and as complexes which denote is acknowledged, there is nothing to prevent the substitution of the denotation of the denoting complex for the denoting complex as subject term. Insofar as the meaning of the denoting complex contains a denoting concept, an ‘inextricable tangle’ is thus produced in trying to preserve the relation of meaning to denotation, since there is no logical means of exhibiting their difference. On the 1905 theory of OD, if George IV wishes to know whether Scott is the author of Waverly, this is not the same as him wishing to know whether Scott is Scott, and this intensional aspect of identity statements can be captured by giving the exact logical analysis of the proposition. There is at least one x, such that x is author of Waverly, there is only one (if y is author of Waverly, y is x) and that one is Scott--- $\exists x(AW(x)$ & $\forall y(AW(y) \rightarrow x{=}y)$ & $S(x))$. So, the complete analysis does not substitute ‘Scott’ for ‘the author of Waverly’. The trouble with Fregean senses is not that they are ontologically suspect, but that they lead to inexact analyses of propositions containing logical connectives, e.g., of identity statements. Though the adoption of quantification theory in which the variable is not an analyzable entity is crucial to its execution, Russell’s aim is to give the exact logical analysis of (intensional) propositions. Both the 1903 theory of denoting and the 1905 theory of descriptions attempt to capture, in logical terms, the intensional and extensional dimension of the meaning of propositions, only the latter theory gives an exact analysis, where the former

Yet it would seem that if "the author of Waverly" is to have an entity occurrence in the proposition, it ought to be possible to replace "the author of Waverly" with a single entity variable. The problem with denoting complexes is that they undermine Russell's foundational thesis that logic includes only entity variables [Landini 1998, 52] or, to put it in terms of the conception of analysis which has been attributed to Russell throughout this thesis, it destroys Russell's contention that any entity/term is capable of occurrence "as one" in a proposition, that is, that anything can be taken as the logical subject of a proposition without change of significance.

In "On Fundamentals", Russell tries to account for the relationship which the complex meaning of a denoting complex has to its denotation, but arrives nowhere. In "Points About Denoting", written in the latter half of 1903, Russell again attempted to account for the substitution of one term in a complex for another in functions/denoting complexes of the sort which have been the focus of discussion, e.g., England for France in the complex "The present Prime Minister of France". We have seen already that in denoting complexes containing variables (mathematical functions) the value of the variable does not appear as a constituent in the value of the function, e.g. if the value of x is England in "The present Prime Minister of x", England does not appear as a constituent in the value, Arthur Balfour. In PAD, Russell recognized, moreover, that if it is *in the*

yields i. meanings unanalyzable except by virtue of their denotations and ii.. conventional denotations whose connection to the meanings which denote them is logically inscrutable.

denoting complex itself, "p x/y", that one term is to be substituted for another, then the result of the substitution ought to be a denoting complex in which England is a constituent, not "Arthur Balfour" [PAD, 309]. In his July, 1903 notes on "Dependent Variables and Denotation", Russell had considered the case of denoting complexes which denote uniquely and attempted to supply a function to their denotations. Taking dependent variables to be complex meanings which denote, and introducing p x/y for mathematical functions y=f(x), Russell proposed the following function f:

> **In *p x/y* , we want *p* to be a *meaning*. Thence we must go to Dn(*p*), which we must define for all cases. And Dn is an indefinable function. We may put: If *p* is a meaning which unambiguously denotes *q*, then Dn|*p* is to be *q* ; if not, Dn|*p* is to be *p* [Papers 4, 301].**

The denotation of p, which is a meaning, is p, unless this meaning denotes unambiguously, in which case it is its denotation q. The constituents of p are constituents of p, but not of the denotation q. Russell insists, however, that this denoting operator will not do, since, when it is applied to an argument, it just gives rise to a new complex meaning. Russell then introduces "℩ $\boldsymbol{\varphi}$" to symbolize the denotation of the function $\boldsymbol{\varphi}$ whose meaning is comprised by its constituents, but dismisses this on the grounds that "℩" applied to an argument, also gives rise to complex meaning.[411] The difficulty Russell was confronting was precisely that which he described in OD as that of preserving the connection between meaning and denotation without making them one and the same [OD, 421]. The question of how to refer to denoting complexes whose meanings denote

[411] For a discussion of the significance of the introduction of this notation for denoting operators and Russell's reasons for rejecting them, see Klement 2001.

without invoking their denotations would be addressed again, in April, 1904 in connection with the Contradiction and, with greater success, in "On Fundamentals". There, Russell jettisons the distinction between an unambiguously denoting complex C as in "The Author of Waverly is Scott", and the unambiguously denoting complex 'C' as in "'The Author of Waverly' is a denoting concept", which produces two distinct entities whose relation cannot be ascertained[412], and reintroduces "denoting" in a manner akin to the $\iota\varphi$ function. He writes:

> **Let C be an unambiguously denoting complex (we may now drop the inverted commas); then we have:**
>
> **(∃y): C denotes y: C denotes z.$\supset_z$. z=y.**
>
> **Then what is commonly expressed by φ'C will be replaced by**
>
> **(∃y): C denotes y: C denotes z.$\supset_z$. z=y.φ'y**
>
> **Thus, e.g., (the author of Waverly) becomes**
>
> **(∃y): "the author of Waverly" denotes y: "the author of Waverly" denotes z $\supset_z$. z=y: φ'y**
>
> **Thus "Scott is the author of Waverly" becomes**
>
> **(∃y): "the author of Waverly" denotes y: "the author of Waverly" denotes z $\supset_z$. z=y: Scott=y [FUND, 383-4].**

The meanings involved denote unambiguously, which allows Russell to make use of the denoting function, but as soon as he has done so, he can easily see that what is involved is simply the existence condition supplied by the quantifiers and the uniqueness condition

[412] In distinguishing concept and object, Frege holds that a function name may never take the place of a proper name, which Russell denies in the light of the fact that this gives occasion to the familiar contradiction which occurs in taking a meaning/concept as a logical subject, namely, that either it is impossible to formulate a proposition in which it can be denied that "ξ is a proper name" or, in formulating this proposition, it is given an entity occurrence [PMC, 134].

supplied by identity. Russell realizes, then, that the "φ", which marks a class-concept and hence supplies "constituents of the meaning of the denoting complex", together with the quantifiers and identity, suffice. Now it would seem, however, that the complex has dissolved:

φ'ɿ'u = : (∃y) : yεu : zεu. $\supset_z$. z=y: φ'y [FUND, 384].

Of course, this is the rudimentary form of the theory presented in OD, which would allow Russell to dispense with denoting complexes and, hence, mathematical functions. In his November, 1905 paper "On the Relation of Mathematics to Symbolic Logic," Russell reiterates his conclusion from OMD that propositional functions are the fundamental functions with which symbolic logic is concerned and denoting functions, to which the single-valued functions of mathematics belong, are definable by means of them. He writes:

> **The usual functions of mathematics, such as 2x, x^2, sin x, log x, etc., are not propositional functions, but what I call *denoting* functions… [W]e can also define the general concept of a denoting function, as follows. Let φ!(x, y) be a propositional function. It may happen that, for certain values of x, there is one and only one value of y for which φ!(x, y) is true. Hence, for such values, "the y for which φ!(x, y) is true" is a function of x, of the kind which I call a *denoting* function. For all other values of x, that is to say for a value for which φ!(x, y) is not satisfied by any value of y or is satisfied by several, the expression "the y for which φ!(x, y) is true" is meaningless and does not denote anything [ORML, 525].**

However, now denoting functions no longer have "meaning in isolation". Russell tells us, in a note citing his forthcoming article OD, that the denoting function φ ɿ'x is not defined in itself, but the proposition in which it occurs is defined. He writes:

> **Let ψ!y be a propositional function containing y. Then, for each value of x, ψ! φɿ'x means, by definition: "[1] There is one and only one value of y for which φ!(x,y) is true, and [2] this**

> **value satisfies ψ!y…Here, the phrase [1]… is itself defined as: "There is a y such that, for any value of z, *φ*!(x, z) is equivalent to 'x is identical with y'." This by itself has no meaning, but any possible assertion about it has a well defined meaning [ORML, 525 n6].**

Logical analysis, then, by means of propositional functions supplemented by quantifiers, captures the logical form of the whole proposition. In the passages of "On Fundamentals" which precede the elimination of denoting complexes, Russell had thought that what might be required was to supplement meaning and entity occurrence with four additional pairs of kinds of occurrence in complexes which exhibit the conditions for the preservation of truth and identity in substitution [FUND, 374-6]. What turns out to be required, however, is that the constituents of the proposition be ascertained only subsequently to the expression of the true logical form of the proposition.[413]

Recall that, if Allard is correct, then it is for the reason that meanings (and descriptive phrases) are universal and do not denote uniquely, that Bradley held that Reality as a whole must be invoked as the only object that can be uniquely denoted in judgments. For Bradley, as we have seen, adjectives/ideal meanings are universals and do not denote uniquely, but the judgment is irreducibly intensional and is made up of such meanings, so that co-extensive parts cannot be inter-substituted in judgments. On Allard's view, this is precisely the reason for which Bradley holds that the true logical form of the judgment is "Reality is such that S is P", that is, to provide a unique denotation, allowing for the substitution of identicals *salve veritate* [Allard 2005, 80].We saw, in Chapter 1,

[413] On this point, I agree with Beaney, who holds that the theory of descriptions preserves Russell's earlier decompositional conception of analysis [Beaney 2009, 20]. The disagreement, which is substantial, concerns what this earlier conception of analysis consists in.

that Russell's interest in Bradley's logic principally concerns both the issue of whether unique reference is supplied by means of adjectives and the distinction between conceptual and numerical diversity. Where Moore argued, contra Bradley, that the logical idea is not an adjective, Russell pressed this view to its conclusion, insisting on the primitive, non-conceptual diversity of logical subjects. Taking its start from the view that it was contradictory to deny that anything could be the logical subject of a proposition, Russell's 1905 theory of descriptions permitted the true logical form of the proposition to be exhibited, rendering it amenable to decompositional analysis and thereby revealing its proper constituents to be the constituents of reality.

CONCLUSION

It is well known that the conceptions of analysis which mark the emergence of early analytic philosophy arose out of attempts to analyze the propositions and concepts of mathematics. However, the advantage which Russell's decompositional conception of the analysis of propositions and the attendant theory of terms was supposed to have for the analysis of the propositions and concepts of mathematics is not always easy to ascertain. By comparison to Frege's elegant function-argument approach to analysis, initially invented to provide differing analyses of the same conceptual contents for use in proofs, Russell's conception of analysis as the decomposition of the proposition into its constituent terms and the related notion that the nature of the terms depends upon their manner of occurrence within it, seem almost stultifying to analyses in mathematics. I have attempted to suggest that, to the contrary, the crucial developments in Russell's early logicization of mathematics are endemically linked to his decompositional conception of analysis, characterized by the view that the proposition is the basic element of analysis and the nature of its constituent terms is determined by their manner of occurrence within it.

It is generally recognized that Moore's and Russell's decompositional conception of analysis arose out of their anti-Hegelian commitment to part/whole analysis, on which conceptual differences are not "false abstractions", but real differences which the new logic must preserve. We have seen, however, that in his embrace of the new logic, Russell not only adopted Moore's anti-Bradleian thesis that the logical idea is a concept and not an adjective, but extended the argument to establish, apart from conceptual diversity, the primitive diversity of logical subjects. We have seen that the argument for the primitive diversity of logical subjects which Russell developed in working on Leibniz served, in COR, as the model for the doctrine that relations are external to their terms and irreducible to the properties of relata. Formerly, Russell had subscribed to the view that asymmetrical relations were grounded in conceptual differences in the relata, though no differences were discoverable apart from the adjectives conferred on the relata by the relation. In overturning his own version of the traditional doctrine of relations, however, Russell was able to account for the analysis of mathematical propositions involving asymmetrical, transitive relations of order without appealing to conceptual differences. In committing himself to the primitive diversity of terms, Russell also dispensed with "adjectives of relations", e.g., "A's excess over B", admitting instead the relations in intension whose differences were to be preserved in logical analysis. The intensional view of relations informed Russell's formulation of the logic of relations which supplemented Peano's logic in the early logicist reductions. Though Russell's logicism ceased to hinge upon the intensional view of relations, the insights concerning analysis which this view of relations was intended to accommodate continued to figure centrally in his early logicist program. On the decompositional conception of analysis, as Russell construed it, the proposition was granted primacy as the whole from which all analysis takes its start and it was taken as a central doctrine that any term occurring as concept could be made the logical subject of a proposition. Moreover, intensions were to be captured in the basic apparatus of logic, and extensions, crucial to mathematics, were determined by means of intensions.

In the embrace of Peano's symbolic logic, implication replaced the part/whole relation and part/whole analysis fell by the way, but, with its origins in Boole's propositional calculus, the new logical calculus remained, in the first instance, the logic of propositions. When Russell committed himself to logicism, we have seen, he came to regard pure mathematics as being defined as "the class of all propositions of the form 'a implies b', where a and b are propositions, each containing at least one variable, and containing no constants except logical constants or such as can be defined in terms of logical constants" [Russell 1901c, 185]. In articulating logicism in PoM, Russell criticized Peano's conception of formal implication on the grounds that, in failing to distinguish the class in extension from the intensional class-concept, Peano restricted the variable in the implication to the class of terms defined by the assertion in the antecedent, e.g., the x in "x is a man $\supset_x$ x is mortal" is restricted to the class of men. In this way, formal implication was reduced to the assertion of a relation of inclusion between classes. In keeping with his view that any term is capable of occurrence as entity/logical subject of a proposition, the variable was to be an unrestricted entity-variable. It was the task of symbolic logic to mediate between intensions and extensions, so that "the symbols other than the variable terms…stand for intensions, while the actual objects dealt with are always extensions" [PoM, 99]. While Russell initially held that relations in intension are to be identified with class-concepts [PoM, 514], he came to hold that class-concepts are marked by intensional propositional functions. On the decisive formulation of PoM, Russell's conditionals, we have seen, are universally quantified implications in which the antecedents contain variables ranging over everything and the consequents assert a propositional function of the same variable ("for all x, if x is an *a*, then φx").

The inconsistent axiom systems of non-Euclidean geometry may have led Russell to emphasize the fact that the conditional statements of mathematics assert a relation between the axioms and the theorems of mathematics, without asserting the axioms or the theorems, and without regard for whether such entities as those characterized by the axioms actually exist. As Russell formulated logicism, he held that the non-logical constants in a universally quantified implication can be replaced by variables, so that the axioms in the antecedent formally characterize a class of structures of a certain kind, which must satisfy the propositional function in the consequent, so that the structures have the assigned properties. Since it is the central aim of Russell's logicist project to allow existing mathematics to be true, it would seem that Russell's logicism does not reduce to a formal device: even though his implicit logicist definitions merely supply the formal characterizations of the structures defined, the success of such definitions are judged according to whether they preserve the truths of an existing branch of mathematics. Russell's logicism of PoM is not formalism in any case, since he supplemented his implicit definitions with explicit definitions, which played the marginal but significant role of providing existence theorems for the classes defined. The logicist definitions in the various branches of pure mathematics, then, can be carried out implicitly by means of axioms, and the explicit definitions are needed only for ordinary arithmetic statements and for applications of arithmetic in non-mathematical contexts. Importantly, the explicit definition of number was carried out in accordance with the principle of abstraction, which asserts that there is some entity to which similar classes have a many-one relation. Numbers are thus identified with the *classes* of similar classes which are the logical objects to which similar classes are related. While a characterization of the properties of infinite, well-ordered series suffices for pure arithmetic and mathematical Analysis, ordinary applications of arithmetic require the definition of numbers as logical objects. As Russell later remarked, it is the logicist definition of *number* which renders "the actual world of countable objects intelligible" [PoM, vi]. Of course, the principle of abstraction and the resulting logicist definition of number were undermined by the Contradiction, to which Russell struggled in vain to find a satisfactory solution.

While Russell initially subscribed to a naïve comprehension principle on which every predicate or class-concept determines some class, the contradiction of predicates not predicable of themselves and class-concepts not members of their own extensions led him to reject this principle and, along with it, Frege's analogous principle that every concept (function) indicated by a grammatical predicate has some value-range correlated with its extension. Where Russell invokes classes in his 1901 definition of number by means of the principle of abstraction, it seems reasonable to say that he has adopted the same definition that Frege arrived at by invoking value-ranges. The differences between their respective logicist definitions at first appear to be exhausted by the metaphysical or epistemological issues of whether number is apprehended as a value-range or as a class. In the face of the Contradiction, however, the differences between Frege's and Russell's conception of logic and logical analysis render it doubtful that their logicist definitions of number are the same. We saw that in the face of the Contradiction, Russell initially attempts to treat classes in extension from within an intensional logic of relations, but, recognizing that the intensional logic of relations provides no means of obviating the Contradiction, comes to treat relations in extension from within an intensional logic of propositional functions. It is from the basic notion of a proposition constituted by its constituents and nothing else, and from the analysis of propositions into a constant(s) and a variable term(s), where the value of the variable is a constituent in the proposition that is the value of the function, that Russell extracts the notion of a propositional function. On Russell's approach, in contrast to Frege's analyses into function and argument, the variable is unrestricted and is contained in the propositional function itself. Mathematical functions, which Russell regarded as denoting complexes containing variables, cannot be analyzed in the usual way, since the values of the variables contained in such a function are not constituents in the value of the function. Suspecting that a solution to the

Contradiction depended upon a theory of denoting, Russell concerned himself increasingly with developing such a theory.

Though he briefly countenanced a Fregean theory of functions in 1903, Russell ultimately resisted Frege's approach. Surprisingly, what deterred him was not that treating functions as separable from their arguments led to the functions version of the Contradiction, but rather that the Fregean notion of functions was fundamentally incompatible with his own conception of analysis. On discovering the functions version of the Contradiction, Russell continued to work within the Fregean functional theory and, to avoid the Contradiction, distinguished the function occurring as concept or meaning from the function occurring as entity or logical subject. Since it was only in the latter case that the function could be varied along with the argument, Russell excluded from "functioning functions" those which asserted a variable function of a variable argument, i.e., those which gave rise to quadratic forms. Such solutions, however, were *ad hoc*, and did not comport with Russell's conception of analysis. On Russell's conception of analysis, in contrast to Frege's hierarchy of functions taking arguments of the appropriate type, the variable was to occur in the whole proposition and was to have an unrestricted range of significance. The analysis of denoting complexes into a constant/function and a variable term seemed to require, however, that the function be treated as a separable entity. The Contradiction had shown that not every function determines a class and that functions are thus not appropriate to entity variables. If separable functions were admitted, then there would need to be independent function variables. Denoting complexes were functions, then, whose values were not propositions, and which contained restricted variables whose values were not constituents of the values of the functions. While Russell explicitly held that propositional functions were the fundamental sort and that mathematical functions were derived from propositional functions whose values were propositions asserting many-one relations, he had no logical means of eliminating denoting complexes containing variables. When Russell articulated the 1905 theory of descriptions in its rudimentary form in "On Fundamentals", the logical analysis of propositions containing denoting complexes, by means of propositional functions supplemented by quantifiers, revealed the true logical form of the proposition and showed that the denoting complex formed no part of the proposition so analyzed. The theory of descriptions which allowed mathematical definition to proceed without the introduction of classes-as-entities, both permitted the construction of extensions by a reference to intensions and, at the same time, cleared the way for Russell's decompositional approach to analysis.

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