# ADVANCES IN SIMULATION AND OPTIMIZATION OF ELECTRICAL NETWORKS

by

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# ADVANCES IN SIMULATION AND OPTIMIZATION OF ELECTRICAL NETWORKS

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# ABSTRACT

This thesis addresses itself to two main veins of computer-aided design of electrical networks, namely, simulation and optimization. A critical review of the state of the art in simulation approaches to networks for analysis and sensitivity evaluation, design concepts and optimization algorithms, is presented. A new approach for the simulation and design of lumped networks in the time domain is presented. The approach is based on the transmission-line matrix method of numerical analysis. The exploitation of general simulators which can be used as a tool in the integrated design process of electrical networks is given with specific examples. A new approach for the analysis and design of cascaded networks has been developed. This approach proves to be efficient and very useful for sensitivity and tolerance analysis. The approach has also been generalized to 2p-port cascaded networks.

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## TABLE OF CONTENTS

PAGE

ABSTRACT iii

ACKNOWLEDGEMENTS iv

LIST OF FIGURES x

LIST OF TABLES xiv

CHAPTER 1: INTRODUCTION 1

CHAPTER 2: SIMULATION AND OPTIMIZATION OF ELECTRICAL NETWORKS: A CRITICAL REVIEW 5

2.1 Introduction 5

2.2 Methods of Analysis 6

2.2.1 Linear Networks in the Frequency Domain (the A.C. Case) 6
2.2.2 Linear Networks in the Time Domain 9
2.2.3 Nonlinear Networks: the D.C. Case 10
2.2.4 Nonlinear Networks in the Time Domain 12

2.3 Response Function Derivatives 13

2.4 Large-change Sensitivity 16

2.5 Design Specifications and Error Functions 19

2.6 Optimization Approaches in Circuit Design 29

2.6.1 Nonlinear Programming Approach 29
2.6.2 The GRG Method 31
2.6.3 Least pth Optimization 33
2.6.4 Generalized Least pth Objective 34
2.6.5 Least pth Objective and Nonlinear Programming 39
2.6.6 Minimax Approximation via Linear Programming 42
2.6.7 Minimax Optimization of Constrained Problems 46

TABLE OF CONTENTS (continued)

PAGE

2.6.8 Other Methods 47

2.7 Centering, Tolerancing and Tuning 48

2.7.1 Definitions 49
2.7.2 Worst-case Design 53
2.7.3 Fixed Tolerance Problem 54
2.7.4 Variable Tolerance Problem 55
2.7.5 Tolerancing and Tuning 57
2.7.6 Uncertainties 60
2.7.7 Design with Yield Less than 100 Percent 62
2.7.8 Related Work and Extensions 64

CHAPTER 3: TRANSMISSION-LINE MODELING AND SENSITIVITY EVALUATION FOR LUMPED NETWORK SIMULATION AND DESIGN IN THE TIME DOMAIN 67

3.1 Introduction 67

3.2 Transmission-line Modeling 68

3.2.1 Link Modeling 68
3.2.2 Stub Modeling 75

3.3 Discussion 79

3.4 Example 81

3.5 Compensation of Errors 83

3.6 Sensitivity Evaluation 85

3.6.1 First-order Sensitivities 88
3.6.2 Second-order Sensitivities 90

3.7 Examples 90

3.8 Sensitivities w.r.t. Time and T 92

3.9 Conclusions 101

## TABLE OF CONTENTS (continued)

PAGE

CHAPTER 4: EFFICIENT USE OF SIMULATION PROGRAMS IN THE ANALYSIS OF COMPLICATED NETWORKS 102

4.1 Introduction 102

4.2 The Use of General Simulators 103

4.3 Examples 106

4.3.1 An Active Filter 106
4.3.2 A Current Switch Emitter Follower 113

4.4 Conclusions 121

CHAPTER 5: ANALYSIS AND SENSITIVITY EVALUATION FOR CASCADED STRUCTURES 122

5.1 Introduction 122

5.2 Theoretical Foundation 124

5.2.1 Reference Planes 128

5.3 Network Functions in Terms of Elements Under Consideration 131

5.4 First- and Second-order Sensitivities 137

5.5 The Evaluation of $V_L$ and its Sensitivities w.r.t. Design Parameters at all Vertices of the Tolerance Region 138

5.6 Branched Circuits 140

5.7 Algorithms 144

5.7.1 Two Algorithms for Evaluation of Large Changes 144
5.7.2 First- and Second-Order Sensitivities 151
5.7.3 Response Value and its Derivatives w.r.t. all Variable Parameters, at all Vertices of the Tolerance Region 156

5.8 Numerical Example 158

5.9 Cascaded Networks of 2p-Port Elements 166

## TABLE OF CONTENTS (continued)

PAGE

5.10 Conclusions 173

CHAPTER 6: CONCLUSIONS 177

APPENDIX A CENTRAL DIFFERENCE FORMULA APPROXIMATING FIRST-ORDER DERIVATIVE 180

APPENDIX B DATA SUPPLIED TO SPICE2 FOR THE CSEF CIRCUIT 181

B.1 With the Equivalent Circuit of the Emitter Junction 181

B.2 With Built-in Transistor Models 182

APPENDIX C FORMULAS RELATING THE LOAD VOLTAGES OF THE BRANCHES TO THE VARIABLE ELEMENTS 183

C.1 To Obtain $V_{BL}$ as a Function of $V_S$ and $\underset{\sim}{B}$ 183

C.2 To Obtain $V_{BL}$ as a Function of $V_S$ and $\underset{\sim}{C}$ 185

C.3 To Obtain $V_{CL}$ as a Function of $V_S$ and $\underset{\sim}{C}$ 186

C.4 To Obtain $V_{CL}$ as a Function of $V_S$ and $\underset{\sim}{B}$ 187

REFERENCES 188

AUTHOR INDEX 199

SUBJECT INDEX 204

# LIST OF FIGURES

| FIGURE | | PAGE |
|---|---|---|
| 2.1 | A cascaded network, consisting of two-port subnetworks connected in cascade, with conventional directions of currents and voltages | 8 |
| 2.2 | An amplifier design problem indicating (a) an applied voltage $V_1(j\omega)$ and output voltage $V_2(j\omega)$, where $\omega$ is the frequency and $j = \sqrt{-1}$, (b) a possible gain specification for the amplifier | 20 |
| 2.3 | The response function of a bandpass filter violating upper and lower specifications | 22 |
| 2.4 | An example of multiple objectives in filter design, (a) the insertion loss specification in the frequency domain of a lowpass filter, (b) an impulse response specification in the time domain of the lowpass filter | 24 |
| 2.5 | Multidimensional specifications, (a) a possible specification for a two-dimensional digital filter; (b) upper and lower specifications for an amplifier to be designed to operate over a specified temperature range | 28 |
| 2.6 | A tolerance region $R_\varepsilon$ inscribed in the constraint region $R_c$. If $\varepsilon = 0$ the conventional nominal design problem is implied | 52 |
| 2.7 | An illustration of the constraint, tolerance and tuning regions and a possible outcome $\phi$. If $t = \underline{0}$ we recover the essential features of Fig. 2.6 | 59 |
| 3.1 | Lossless transmission-line models of a series inductor and a shunt capacitor | 69 |
| 3.2 | Lumped network and link transmission-line model | 72 |
| 3.3 | Stub models of an inductor and a capacitor | 77 |
| 3.4 | Lumped network and transmission-line stub model | 78 |
| 3.5 | Example of two simple subnetworks | 80 |

## LIST OF FIGURES (continued)

| FIGURE | | PAGE |
|---|---|---|
| 3.6 | LC lowpass filter and different types of models (a) the filter, (b) link model, (c) stub model, (d) mixed model | 82 |
| 3.7 | An inductor with two capacitors representing modeling error | 85 |
| 3.8 | Chebyshev filter with 7 elements | 87 |
| 3.9 | Optimization using TLM analysis. Starting point a: $L_2 = L_4 = 0.5$, $C_3 = 1.0$. Starting point b: $L_2 = L_4 = 0.8$, $C_3 = 2.2$ | 91 |
| 3.10 | Optimization using TLM analysis. Starting point $L_2 = L_4 = C_3 = 1.0$. Solution $L_2 = L_4 = 0.76646$, $C_3 = 2.3739$ | 93 |
| 3.11 | Representation of response w.r.t. t and T | 96 |
| 4.1 | Arrangement of the base points w.r.t. the centers of interpolation regions in (a) two dimensions and (b) three dimensions | 105 |
| 4.2 | A circuit supplied to SPICE2, which is the original circuit repeated (k+1)(k+2)/2 times with appropriate parameter values | 107 |
| 4.3 | Tunable active filter with $R_g = 50\ \Omega$, $R = 75\ \Omega$ and one-pole roll-off for A(S) | 108 |
| 4.4 | Equivalent circuit for nodal analysis of the circuit of Fig. 4.3 | 110 |
| 4.5 | Equivalent circuit for SPICE2 analysis of the circuit of Fig. 4.3 | 111 |
| 4.6 | Response of the active filter at a point in the interpolation region | 112 |
| 4.7 | Current switch emitter follower (CSEF) circuit | 114 |
| 4.8 | The transistor model | 115 |
| 4.9 | Transistor model described to SPICE2 | 119 |

## LIST OF FIGURES (continued)

| FIGURE | | PAGE |
|---|---|---|
| 4.10 | Input voltage and responses of CSEF with different methods of analysis | 120 |
| 5.1 | Notation for an element in the chain, indicating reference directions and voltage and current variables | 124 |
| 5.2 | Forward and reverse analyses of a cascaded network with source and load impedances assumed constant | 126 |
| 5.3 | Subnetwork i cascaded with subnetworks k (at source end) and j (at load end) | 126 |
| 5.4 | Forward iteration for Fig. 5.3, transferring an equivalent source accounting for design variables from subnetwork k from one reference plane to the other | 130 |
| 5.5 | Reverse iteration for Fig. 5.3, transferring an equivalent source accounting for design variables from subnetwork j from one reference plane to the other | 130 |
| 5.6 | An example of a cascaded circuit with a branch connected in series and a branch connected in parallel. Branches are represented in the cascade by their equivalents. Reference planes where different analyses are initiated are labelled | 141 |
| 5.7 | Illustration for a cascade of 6 two-ports of the principal stages in the calculations involved in the multiple pairwise changes algorithm. Three variable elements are considered, hence three sets of simultaneous analyses are effectively performed | 147 |
| 5.8 | Illustration of the principal stages in the calculation of first- and second-order sensitivities w.r.t. three variable elements $g_i \triangleq \partial Q_{11}/\partial\phi_i$, $i = 1, 2, 3$ $S_{ij} \triangleq \partial^2 Q_{11}/\partial\phi_i\partial\phi_j$, $i,j = 1, 2, 3$ | 153 |
| 5.9 | Illustration of the principal stages of Algorithm 4 | 157 |

# LIST OF FIGURES (continued)

| FIGURE | | PAGE |
|---|---|---|
| 5.10 | Seven-section filter containing unit elements and stubs. All sections are quarter-wave at 2.175 GHz | 159 |
| 5.11 | A 2p-port element: a generalization of Fig. 5.1 | 168 |
| 5.12 | Equivalent Thevenin voltages and impedances for a subnetwork consisting of 2p-port elements | 174 |

# LIST OF TABLES

TABLE PAGE

2.1 Types of electrical networks and numerical methods of solution 11

3.1 Incident and reflected pulses of the circuit in Fig. 3.6(b) 84

3.2 Comparison between different methods of integration and TLM modeling with and without compensation 86

3.3 Component values of the filter shown in Fig. 3.8 before and after compensation 87

3.4 $1/k\ \partial f/\partial T|_k$ obtained from the TLM routine versus $\partial f/\partial t$ by central differences where $T = 0.1$ 97

3.5 A comparison between $\partial f/\partial T$ for constant k obtained by TLM and perturbation 99

3.6 Using $\Delta f/\Delta T$ to predict response for $T = 0$ 99

3.7 Using $\partial f/\partial T$ to predict response for $T = 0$ 100

4.1 The center base point and the size of the interpolation region for the quadratic approximation of the response of the active filter of Fig. 4.3 113

4.2(a) Circuit parameter values 116

4.2(b) Diode model parameters 116

4.2(c) Transistor model parameters 117

4.2(d) Transmission-line parameters 117

5.1 Principal concepts involved in the analyses 125

5.2 Notation and implied initial conditions 134

5.3 Analyses required by certain terms 135

## LIST OF TABLES (continued)

| TABLES | | PAGE |
|---|---|---|
| 5.4 | Functions of input current $I_S$ and output voltage $V_L$ for changes in $\underset{\sim}{A}$ only | 136 |
| 5.5 | Comparison of second-order sensitivities with different approaches | 166 |
| 5.6 | The response $V_L$ and its sensitivities at the vertices of the tolerance region at normalized frequency 0.7 | 167 |

# CHAPTER 1

# INTRODUCTION

Methods of analysis and sensitivity evaluation for electrical circuits in the time domain and cascaded circuits in the frequency domain are the subjects of this thesis. Analysis and sensitivity evaluation form an integral part of any computer-aided circuit design scheme.

The circuit design problem can be classified into two types. The first is the classical type, used during the last decade, from which we obtain one set of circuit parameter values. This set of parameter values let the desired circuit response (or responses) meet optimally the given specifications. Converting the results obtained to the real world can be either very difficult or very expensive especially if mass production is anticipated. This is due to the high price of elements, if they are available, with very precise values. This suggests the second type which considers the problem more seriously from the manufacturing point of view. In this case tolerances on design parameters, post-production tuning of certain components and yield maximization can be considered. This in turn leads to a more sophisticated problem where a nominal set of parameters and their associated manufacturing tolerances (and/or tuning) are the outcomes. Both types of design have these steps in common

a) numerical circuit analysis,

b) first-order sensitivities (needed for the optimization process),

c) large-change sensitivities.

This thesis addresses the time-domain analysis of linear lumped networks and the sensitivity of the response w.r.t. design variables, the use of general simulators to obtain the quadratic approximation of a circuit response which is further used in the design procedure, and the response and sensitivity analysis for cascaded networks in the frequency domain.

Chapter 2 presents a review of existing methods of circuit analysis, sensitivity analysis and optimization. Different problem formulations are also given. A section in the chapter is devoted to the presentation and formulation of the problem with practical considerations. Similar problem formulations and methods, which were not developed by electrical engineers but which deal with the same type of problem, are briefly discussed.

A new approach for time-domain analysis and first-order sensitivities of lumped networks is presented in Chapter 3. The lumped elements are modeled by transmission-line sections or stubs and the modeled network is analyzed by the transmission-line matrix (TLM) method, which provides an exact solution to the model. Compensation of errors arising in modeling the network elements is discussed in this chapter. Sensitivities of the model's response w.r.t. design variables, time and time step are derived. Advantages of this method over existing methods are mentioned.

Chapter 4 deals with the use of general simulators and their exploitation to obtain the response (of any general network) at different points in the parameter space. These response values are subsequently used to obtain a multidimensional polynomial approximating the response function within an interpolation region. The design is then performed using the polynomials instead of the real response. For this approach the sensitivities w.r.t. design variables are obtained from the polynomial approximation directly. Two specific examples are given in this chapter.

An exact and efficient approach to network analysis for cascaded structures is presented in Chapter 5. It is very useful for differential and large-change sensitivity evaluations. It facilitates the exploitation of symmetry to reduce computational effort for the analysis. Algorithms for evaluating first- and second-order sensitivities, the effect of a multiple of simultaneous large changes in the variable parameters, and the evaluation of the response, as well as the sensitivity of the response, at the vertices of a tolerance region are given in this chapter. It is also shown how responses at different loads in branched networks, which may be connected in series or in parallel with the main cascade, can be obtained analytically in terms of the variable elements. The approach has also been generalized to deal with 2p-port cascaded elements.

Appendix A includes a finite difference formula used to approximate the first-order derivative of the time-domain response w.r.t. time needed in Table 3.7.

The data supplied to the general simulator SPICE2 for the analysis of the current switch emitter follower example in Chapter 4 is shown in Appendix B. Appendix C includes the derivations for the formulas in Chapter 5, Section 5.6, for the branched circuits.

Original contributions claimed for this thesis are:

(1) A complete exposition of the design problem of electrical circuits and suitable methods of formulation.

(2) A critical review of optimization methods, used in the design of circuits, developed by electrical engineers.

(3) The development of a new method for analysis and sensitivity evaluation of lumped linear circuits in the time domain.

(4) The illustration of efficient exploitation of general circuit simulators in the design procedure.

(5) A new approach to the analysis and sensitivity evaluation of cascaded networks.

(6) Algorithms which employ this new approach and their use in the design of cascaded networks.

# CHAPTER 2

# SIMULATION AND OPTIMIZATION OF ELECTRICAL NETWORKS: A CRITICAL REVIEW

## 2.1 Introduction

The classical computer-aided circuit design problem can be stated as follows: after choosing the appropriate circuit topology (configuration) with known components, find a single set of designable parameter values which let the circuit response or performance optimally meet some given specifications.

The problem may be reformulated as a nonlinear programming problem (minimizing an objective function subject to constraints) where the objective and constraints embody the design criteria. The objective function itself is usually of the least squares, least pth or minimax form.

The evaluation of a suitable objective function involves the evaluation of the response function $F(\underset{\sim}{\phi}, \underset{\sim}{\psi})$, which is a function of the network parameters $\underset{\sim}{\phi}$ (resistors, capacitors, inductors, emitter area of integrated circuits transistor, etc.) and of other independent variables $\underset{\sim}{\psi}$ (frequency, time, temperature, tunable network elements, etc.). The function $F(\underset{\sim}{\phi}, \underset{\sim}{\psi})$ is usually assumed to be continuous in the ranges of $\underset{\sim}{\phi}$ and $\underset{\sim}{\psi}$ of interest. Performance specifications are usually functions of $\underset{\sim}{\psi}$ only, whereas design constraints are generally functions of $\underset{\sim}{\phi}$.

This chapter reviews the methods and techniques of each step involved in the design procedure, namely the response evaluation (or circuit analysis), derivative evaluation (differential sensitivity), large-change sensitivity, objective formulation and design specification, and optimization approaches used in the design of electronic circuits.

The last section deals with optimal design when certain additional practical engineering problems are considered. The centering problem formulated in a nonlinear programming form is presented. Further practical considerations such as tuning, tolerance assignment under model and environmental uncertainties are discussed.

The difficulties facing the designer wishing to avail himself of efficient nonlinear programming aids are elaborated on. Further development of available algorithms and problem formulations which can improve the state of the art are suggested in Chapter 6.

## 2.2 Methods of Analysis

### 2.2.1 Linear Networks in the Frequency Domain (the A.C. Case)

A linear network is described by a set of linear equations of the form

$$\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b}, \qquad (2.1)$$

where $\underset{\sim}{A}$ is the matrix describing the circuit (with complex coefficients) and can be the nodal admittance matrix $\underline{Y}$, the mesh impedance matrix $\underline{Z}$ or the tableau matrix (Hachtel, Brayton and Gustavson 1971). $\underline{x}$ is the unknown vector consisting of voltages, currents or both, $\underline{b}$ is a known vector consisting essentially of sources exciting the circuit. An important feature of the matrix $\underline{A}$ is that it is sparse for large networks. The sparsity of the matrix increases with the size of the network. Sparse matrix techniques (see Duff 1977) for storing the matrix $\underline{A}$ and for the near-optimum ordering of the equations, are usually used. The reordering of the equations is performed so as to preserve the sparsity and to reduce the number of fill-ins (created nonzero elements which were formerly zeros) during the LU decomposition, which is often used to solve these equations. At each frequency point of interest the matrix $\underline{A}$ is rebuilt and the set of equations resolved. Only the numerical values of the entries of the $\underline{L}$ and $\underline{U}$ matrices, where $\underline{A} = \underline{L}\underline{U}$, are changing but their structures remain fixed.

For certain circuits special methods may be more efficient than general methods of analysis. As an example, cascaded networks, such as the one shown in Fig. 2.1, are analyzed by the transmission or chain matrix, where each element is considered as a two-port subnetwork described by a 2x2 matrix of the form

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} , \qquad (2.2)$$

which relates the input to the output of each two-port subnetwork.

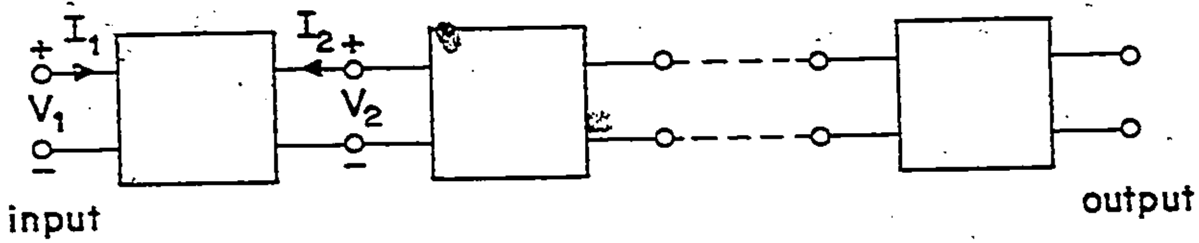


Fig. 2.1 A cascaded network, consisting of two-port subnetworks connected in cascade, with conventional directions of currents and voltages.

The analysis is carried out by assuming a current through the load with a value of one (hence the voltage across the load can be known) and by successive matrix multiplication we can obtain the information at the input (the source) end. Suppose that the computed voltage at the source end is $V_{Sc}$ and the actual source voltage is $V_{Sa}$. Since the network is linear, the actual values for all voltages and currents are found by multiplying the computed values by the factor $V_{Sa}/V_{Sc}$ (Bandler, Rizk and Tromp 1976, Bandler, Popović and Jha 1974, Green 1969, Parker 1969).

A special case of the linear A.C. analysis is the D.C. analysis of resistive networks. The equations, which are real in this case, are set up in the same way as in the A.C. case and then solved once.

### 2.2.2 Linear Networks in the Time Domain

In some problems we are interested in the transients of the circuit and the analysis ~~has~~ to be carried out in the time domain. The network equations describing the linear network, using the state-variable approach (Chua and Lin 1975) which is commonly used, are

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{B}\underset{\sim}{u}, \tag{2.3}$$

$$\underset{\sim}{y} = \underset{\sim}{C}\underset{\sim}{x} + \underset{\sim}{D}\underset{\sim}{u}, \tag{2.4}$$

where $\underset{\sim}{A}$ is a coefficient matrix relating the state vector $\underset{\sim}{x}$ (capacitor voltages and inductor currents, for example) to its time derivative $\dot{\underset{\sim}{x}}$, and $\underset{\sim}{B}$ is a coefficient matrix coupling the effects of the independent source vector $\underset{\sim}{u}$. Equation (2.4) gives the output vector $\underset{\sim}{y}$, where $\underset{\sim}{C}$ and $\underset{\sim}{D}$ are coefficient matrices. Equation (2.3) is a set of first-order differential equations whose solution is given by

$$\underset{\sim}{x}(t) = e^{\underset{\sim}{A}t} \int_{t_o}^{t} e^{-\underset{\sim}{A}\tau}\, \underset{\sim}{B}\, \underset{\sim}{u}(\tau)\, d\tau + e^{\underset{\sim}{A}(t-t_o)}\, \underset{\sim}{x}(t_o), \tag{2.5}$$

and the output vector is

$$\underset{\sim}{y}(t) = \underset{\sim}{C}e^{\underset{\sim}{A}(t-t_o)}\, \underset{\sim}{x}(t_o) + \left\{ \underset{\sim}{C}\, e^{\underset{\sim}{A}t} \int_{t_o}^{t} e^{-\underset{\sim}{A}\tau}\, \underset{\sim}{B}\underset{\sim}{u}(\tau)d\tau + \underset{\sim}{D}\underset{\sim}{u}(t)\right\}. \tag{2.6}$$

Different approaches to evaluating $e^{\underset{\sim}{A}t}$ and the integrals in (2.5) and (2.6) exist (Calahan 1972, Chua and Lin 1975, DeRusso, Roy and Close 1966).

A new method for analyzing lumped, linear networks in the time domain has been developed by Bandler, Johns and Rizk (1977). The lumped elements are modeled by their equivalent distributed transmission-line models. The transmission-line network is then analyzed using the TLM (transmission-line matrix) method. This approach avoids the formulation of the state equations of the original network and the evaluation of $e^{\underset{\sim}{A}t}$ or any integrals.

### 2.2.3 Nonlinear Networks: the D.C. Case

In the nonlinear D.C. case the network equations are expressed in the form

$$\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{0}. \tag{2.7}$$

These equations are usually solved by the Newton-Raphson algorithm (see Table 2.1).

Another method, which is equivalent to the Newton-Raphson method, is to linearize the equations describing the nonlinear elements of the circuit. The linearized formulas are then represented by linear elements, called the discrete or the companion elements (Calahan 1972, Chua and Lin 1975) and the resulting linear circuit is analyzed successively until convergence is reached.

Piecewise-linear analysis is also used in solving nonlinear networks (Chua 1971). Other approaches dealing with circuits with

Table 2.1

TYPES OF ELECTRICAL NETWORKS AND NUMERICAL METHODS OF SOLUTION

| Network Type | | Analysis | | | | Network Equations | | | | | Methods of Solution | | | | | | | | | Features |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Broad electrical network classifications are indicated in this column | | D.C. (direct current) | A.C. (alternating current) | Transient | Steady state | Linear algebraic | Nonlinear algebraic | Linear O.D.E. | Nonlinear O.D.E. | Partial differential equations | Gauss, LU factorization | State equations | Newton-Raphson | Piecewise-linear | Numerical integration | Transmission-line Method | Lumped model | Fast-Fourier transform | Special purpose techniques | Comments are made relating to the physical, mathematical and numerical nature of the analysis problem |
| analog lumped | linear resistive | X | | | | X | | | | | X | | | | | X | | | | no energy storage |
| | linear dynamic | | X | X | | X | | X | | | X | X | | | | X | | | | energy storage, time and complex frequency domains are related by Laplace transform |
| | nonlinear resistive | X | | | | | X | | | | | | X | X | X | | | | | no energy storage, contains nonlinear resistors |
| | nonlinear dynamic | | | X | X | | | | X<br>X | | | X | | | X | X | | | X | energy storage, contains nonlinear capacitors and/or nonlinear inductors |
| analog distributed | linear commensurate and non-commensurate | | X | X | | X | | | | X | X | X<br>X | | | | | X<br>X | X | X | time and complex frequency domains are related by Laplace transform |
| | nonlinear | | X | X | | | X<br>X | | | | | | | | | | X<br>X | X | X<br>X | |

multiple solutions are described in Branin (1972), Chao, Liu and Pan (1975), and Chua and Ushida (1976).

### 2.2.4 Nonlinear Networks in the Time Domain

Nonlinear transient networks may be analyzed by different methods. One method is to formulate the state equations of the network, which are ordinary differential equations in the normal form

$$\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x}, t), \qquad (2.8)$$

where $\underset{\sim}{x}$ is the vector of state variables. Equation (2.8) is then solved by a numerical integration scheme. Stability of the integration and its ability to deal with stiff equations (Gear 1971) are some criteria for choosing the integration scheme for the analysis. The tableau approach (Hachtel, Brayton and Gustavson 1971) is another method for solving nonlinear networks. The method discretizes, at the circuit component (branch) level, the derivative operator d/dt, obtaining nonlinear algebraic difference equations solved by the Newton-Raphson algorithm. The process proceeds in two loops, one for solving the nonlinear algebraic difference equations and the next for the time iteration. In the Newton-Raphson iteration a set of linear equations are repeatedly solved and the sparsity of the coefficient matrix of these equations should be taken into consideration.

The nonlinear network problem in the time domain may be reduced to a sequence of D.C. analyses. This is achieved by discretizing the time derivative operator, then replacing the nonlinear elements by their corresponding companion (linearized) elements and solving a D.C. network. The tableau approach and the companion approach have advantages over the state-space approach in the case of large networks. The reason is that formulating the state equations of a large network requires tremendous effort. The TLM method when used for analyzing nonlinear networks shares this advantage with the former two approaches.

This presentation of different types of networks and methods of solutions is summarized in Table 2.1.

## 2.3 Response Function Derivatives

It is well known that optimization techniques which use derivatives are superior to nongradient techniques if first-order sensitivities are readily available. In order to get the derivatives of the response function $F(\underline{\phi}, \underline{\psi})$, which is a function of certain voltages and/or currents of the circuit, sensitivities of these voltages and/or currents with respect to the variable parameters have to be evaluated. One of the most commonly used approaches to evaluate these sensitivities is the adjoint-network approach (Director and Rohrer 1969a). In this approach an adjoint network is constructed, having the same topology as the original network, and analyzed. The results of both analyses are used to evaluate the required sensitivities.

As an example, in the frequency domain, if the network is represented by its admittance matrix $\underset{\sim}{Y}$ at a frequency point and the equations are $\underset{\sim}{Y}\,\underset{\sim}{V} = \underset{\sim}{I}$, then the equations representing the adjoint network are

$$\underset{\sim}{Y}^T \hat{\underset{\sim}{V}} = \hat{\underset{\sim}{I}}, \tag{2.9}$$

where

$T$ denotes transpose,

$\hat{\underset{\sim}{V}}$ is the vector of node voltages of the adjoint network,

$\hat{\underset{\sim}{I}}$ is the current excitation vector of the adjoint network.

$\underset{\sim}{V}$ and $\hat{\underset{\sim}{V}}$, for example, are substituted into some derived formulas to evaluate the sensitivities (Bandler and Seviora 1970, Director and Rohrer 1969b).

Branin (1973) demonstrated that the sensitivities, in general, can be obtained by matrix manipulation without the need of defining what is termed the adjoint network. Note also that at each frequency two sets of equations are solved. Using the LU decomposition we can achieve some saving by avoiding the decomposition of the matrix transpose (Director 1971). For cascaded networks, an analysis approach newly developed, described in Chapter 5, provides, with little additional computational effort all the information needed to evaluate the required sensitivities. In the linear D.C. case the adjoint network is linear and both original and adjoint networks are analyzed once to calculate the sensitivities. A nonlinear D.C. network will have an associated linear adjoint network which has to be analyzed.

In the time-domain case sensitivities are much more difficult to evaluate because the equations are in the form of ordinary differential equations. Hachtel and Rohrer (1967) used variational techniques to get an adjoint set of equations which, when solved along with the original set, allow sensitivities to be evaluated. In the adjoint-network approach, if the original network is analyzed in the interval $t = [0, t_f]$, the adjoint network is analyzed in the interval $\tau = [0, t_f]$, where $\tau = [t_f - t]$. The integration involving the adjoint network is backward on the time axis. The formulas for the sensitivities are integral formulas, i.e., in evaluating the sensitivities with respect to k variables, k integrations have to be performed after analyzing the original and adjoint networks. Other methods can be used to evaluate the sensitivities (Parker 1971) but they do not appear easier or more efficient than the adjoint-network approach.

The TLM method can, in parallel with the response evaluation, provide the sensitivity of the time response w.r.t. all the design variables (as is shown in Chapter 3).

An approach developed by Bandler and Abdel-Malek (Abdel-Malek 1977, Bandler, Abdel-Malek, Johns and Rizk 1976, Bandler and Abdel-Malek 1978a) avoids the evaluation of the exact response function derivatives. Multidimensional polynomial approximations of the response functions are performed using a minimal number of evaluations of the actual functions within an interpolation region. The approximations are used in the optimization process instead of the actual functions. The derivatives of the approximations are efficiently and rapidly obtained. During optimization the approximation is updated in different regions in the space or in smaller interpolation regions as indicated by the optimization or to obtain higher accuracy, respectively.

In some cases the response derivative is evaluated as a second-order sensitivity. An example of such a case is the group delay which is obtained by finding the sensitivity of the output voltage w.r.t. frequency. An approach which makes use of the adjoint-network concept to find the exact group delay sensitivities is described in Rizk (1975) and Bandler; Rizk and Tromp (1976).

## 2.4 Large-change Sensitivity

Large-change sensitivities are important in the centering and tolerancing problem described in Section 2.7. Here, we are interested in large changes in the variable parameters which often result in a considerable change in the response function.

Fidler (1976) and Schwarz (1977) explored the relationship between large-change and differential sensitivity for bilinear networks (where the network function is a ratio of polynomials).

They showed that two analyses of the network, with the variable element perturbed, in addition to the analysis of the nominal network are required to evaluate any large-change and differential sensitivity of different network functions w.r.t. this element.

Singhal, Vlach and Bryant (1973) expressed the network function (the bilinear function) in terms of the variable parameters explicitly. The approach requires the solution of the network with at most k+1 different excitations (where k is the number of variables), evaluation of some of the principal minors of a matrix of order k+1 and the solution of two triangular systems of equations. Once the coefficients of the bilinear function is obtained any large change can be easily evaluated.

Gadenz, Rezai-Fakhr and Temes (1973) used the adjoint-network concept for evaluating large-change effects. This approach requires k+1 analyses of the adjoint network and the solution of a linear set of equations of order k. For any set of large changes the linear system has to be resolved.

Goddard, Villalaz and Spence (1971) replaced the large change in an element by a current source whose value is identical to the current initially flowing through the change of the element. If the nodal admittance matrix is used for the analysis, only the r.h.s. of the matrix equation is changed. Since the inverse of $\underset{\sim}{Y}$ (or its LU factorization) is obtained previously for the original analysis, only matrix multiplication by the new current vector (or forward and backward substitution) is required to obtain the new network response.

Leung and Spence (1975) used matrix inverse modification methods (Householder relations) to evaluate the change in response due to multiparameter large changes.

The new approach presented in Chapter 5 for cascaded networks provides large-change sensitivities without any additional effort (than the analysis). The reason is that the variable parameter can be related explicitly with the network function and hence any change in this function due to a change in the variable can be easily evaluated.

We have to note that the aforementioned approaches are for linear systems in the frequency domain. Rezai-Fakhr and Temes (1975) partitioned the nonlinear network into two parts. The first is the linear nominal circuit described by its pulse-response matrix and the second consists of all independent sources, all element increments and all nonlinear elements pulled out of the network. Combining the circuit relations a reduced set of nonlinear equations is obtained which has to be solved iteratively, at each time step, for each set of large changes.

In the quadratic approximation approach (Bandler and Abdel-Malek 1978a) once the coefficients of the polynomial are obtained (after (k+1)(k+2)/2 analyses), any large-change sensitivity can be easily obtained by substituting in the multidimensional polynomial the perturbed parameter value. This value has to lie within the limits of the approximation region, where the approximation of the response function is assumed to be valid. This approach is approximate but its advantages are that it is very fast and can be applied in the frequency and time domains (Bandler, Abdel-Malek, Dalsgaard, Elrazaz and Rizk 1978).

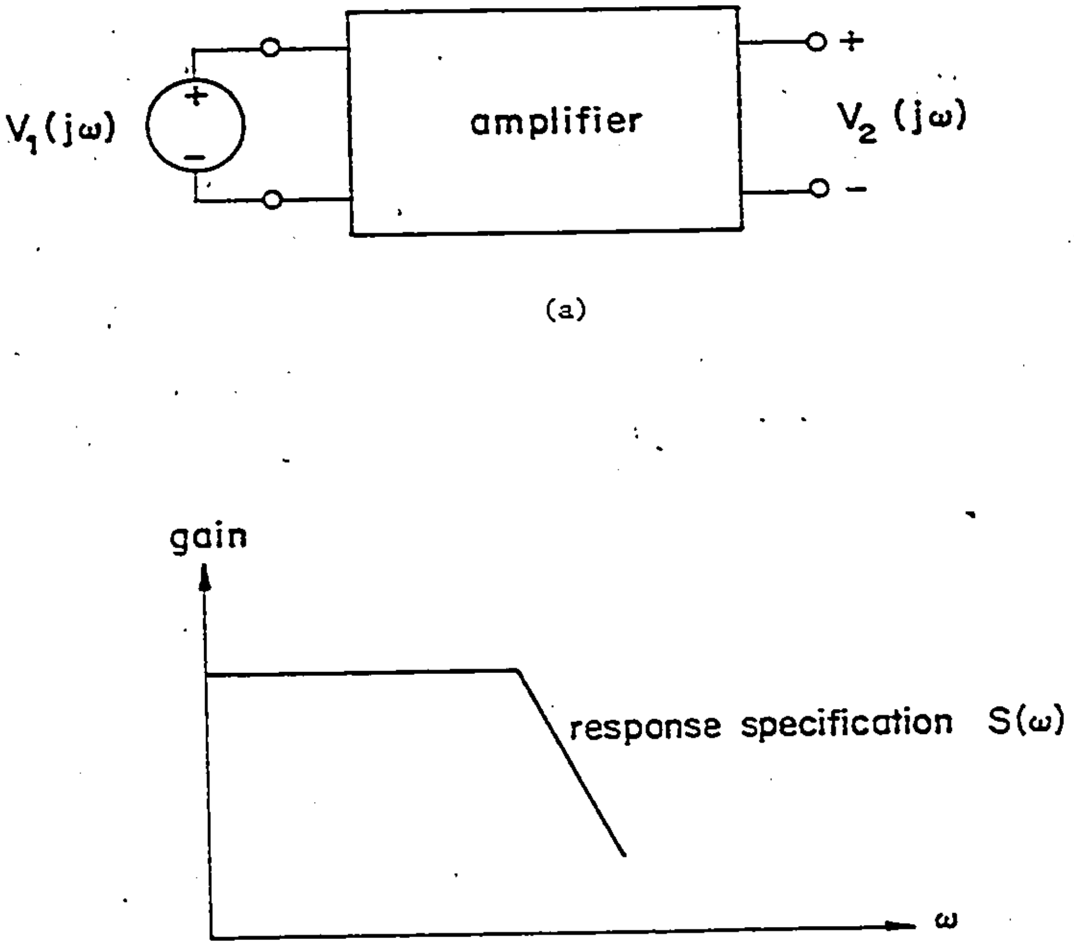
## 2.5 Design Specifications and Error Functions

The problem where the response function has to meet a single specification function $S(\psi)$, assuming we have one independent variable $\psi$, can be demonstrated by an amplifier example. Consider Fig. 2.2(a), in which $V_1(j\omega)$ is the input voltage (voltage of the source) to the amplifier at frequency $\omega$ and $V_2(j\omega)$ is the output voltage at the same frequency. The gain of the amplifier, which is a linear circuit, is usually given by

$$F(\underset{\sim}{\phi}, \psi) = G(\underset{\sim}{\phi}, \omega) \triangleq 20 \log_{10} \left| \frac{V_2(j\omega)}{V_1(j\omega)} \right| . \qquad (2.10)$$

The problem is to obtain $\underset{\sim}{\phi}$ which results in a gain as close as possible, in some sense, to a desired gain, for example, such as the one shown in Fig. 2.2(b).

Another situation which is frequently encountered in practice is the problem defined by upper and lower specifications. In filter design, for example, we are generally interested in two band types (consisting of intervals of frequency $\omega$), namely the


V₁(jω)
+
−
amplifier
+
V₂(jω)
−
gain
response specification S(ω)
ω

(a)

(b)

Fig. 2.2 An amplifier design problem indicating (a) an applied voltage $V_1(j\omega)$ and output voltage $V_2(j\omega)$, where $\omega$ is the frequency and $j = \sqrt{-1}$, (b) a possible gain specification for the amplifier.

stopband and the passband. In the stopband the signal is to be prevented from passing through the filter by making the losses as high as possible. This can be expressed by a lower specification (or bound) of large value. In the passband the situation is reversed and it is expressed by an upper specification (or bound) of a small value. Figure 2.3 shows the upper and lower specifications of a bandpass filter and a response function violating these specifications on the interval $[\psi_\ell, \psi_u]$.

A suitable objective for a problem with upper and lower specifications will reduce the amount by which the actual response fails to meet the specifications, or increase the amount by which the circuit response exceeds the specifications (Bandler 1969).

In electrical circuit design more than one response function might have to meet given specifications. As an example, a circuit can be designed to meet desired specifications in both frequency and time domains. In this case we have more than one independent variable $\psi$, namely $\psi^1$, $\psi^2$, ..., $\psi^n$, where n is the number of these independent variables. Accordingly, we have n response functions $F^1(\underset{\sim}{\phi}, \psi^1)$, $F^2(\underset{\sim}{\phi}, \psi^2)$, ..., $F^n(\underset{\sim}{\phi}, \psi^n)$ and n specifications $S^1(\psi^1)$, $S^2(\psi^2)$, ..., $S^n(\psi^n)$. The corresponding error functions are given by

$$e^j(\underset{\sim}{\phi}, \psi^j) = w^j(\psi^j)\,(F^j(\underset{\sim}{\phi}, \psi^j) - S^j(\psi^j)),\ j = 1, 2, \ldots, n, \qquad (2.11)$$

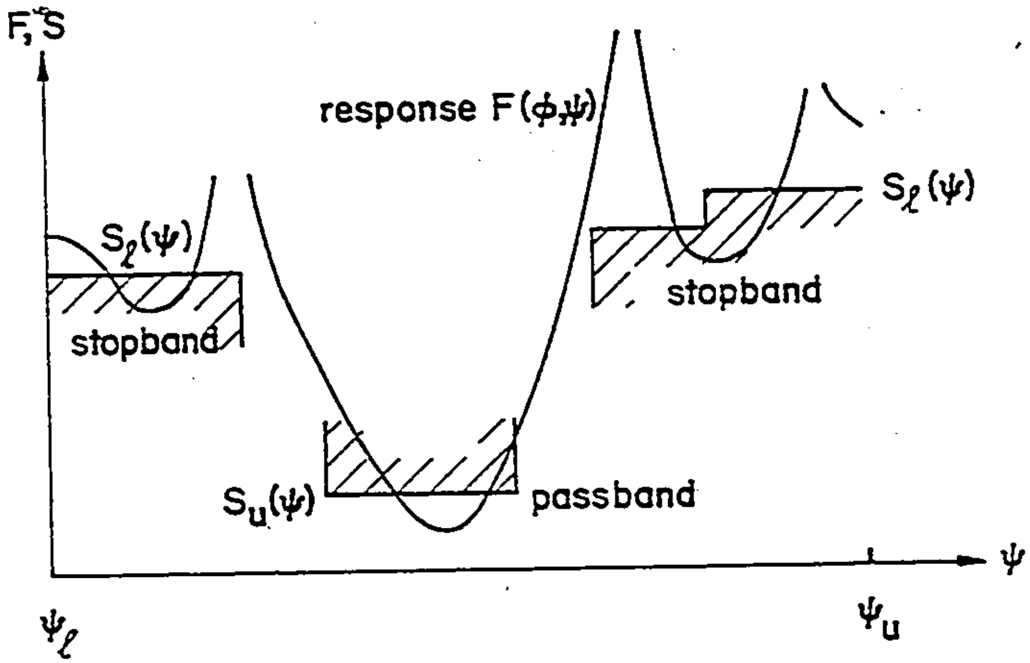

F, S
response $F(\phi,\psi)$
$S_\ell(\psi)$
stopband
$S_\ell(\psi)$
stopband
$S_u(\psi)$
passband
$\psi$
$\psi_\ell$
$\psi_u$

Fig. 2.3 The response function of a bandpass filter violating upper and lower specifications.

for the continuous case with $w^j$ as a positive jth weighting function. It is necessary in practice, on a digital computer, to consider a discrete set of samples of $\underset{\sim}{\psi}$, such that satisfying the specification at these sample points implies satisfying them almost everywhere. Thus, for the discrete case, taking $I^j$ as the index set for the jth functions,

$$e_i^j(\underset{\sim}{\phi}) \triangleq e^j(\underset{\sim}{\phi}, \psi_i^j) = w_i^j(F_i^j(\underset{\sim}{\phi}) - S_i^j), \quad i \in I^j \tag{2.12}$$

is the jth error function evaluated at the ith sample point along the $\psi^j$ axis.

In general, we can have upper and lower specifications for each $\psi^j$. In the design of a lowpass filter, for example, we can have upper and lower specifications in the frequency domain, and a single specification in the time domain. The error functions will be of the form

$$e_u^1(\underset{\sim}{\phi}, \psi^1) = w_u^1(\psi^1) \; (F^1(\underset{\sim}{\phi}, \psi^1) - S_u^1(\psi^1)), \tag{2.13}$$

$$e_\ell^1(\underset{\sim}{\phi}, \psi^1) = w_\ell^1(\psi^1) \; (F^1(\underset{\sim}{\phi}, \psi^1) - S_\ell^1(\psi^1)), \tag{2.14}$$

$$e^2(\underset{\sim}{\phi}, \psi^2) = w^2(\psi^2) \; (F^2(\underset{\sim}{\phi}, \psi^2) - S^2(\psi^2)), \tag{2.15}$$

where the subscripts u and $\ell$ are for upper and lower specifications, respectively, $\psi^1$ is the frequency $\omega$ and $\psi^2$ is the time t. Figures 2.4(a) and 2.4(b) show the specifications in the frequency and time domains, respectively.

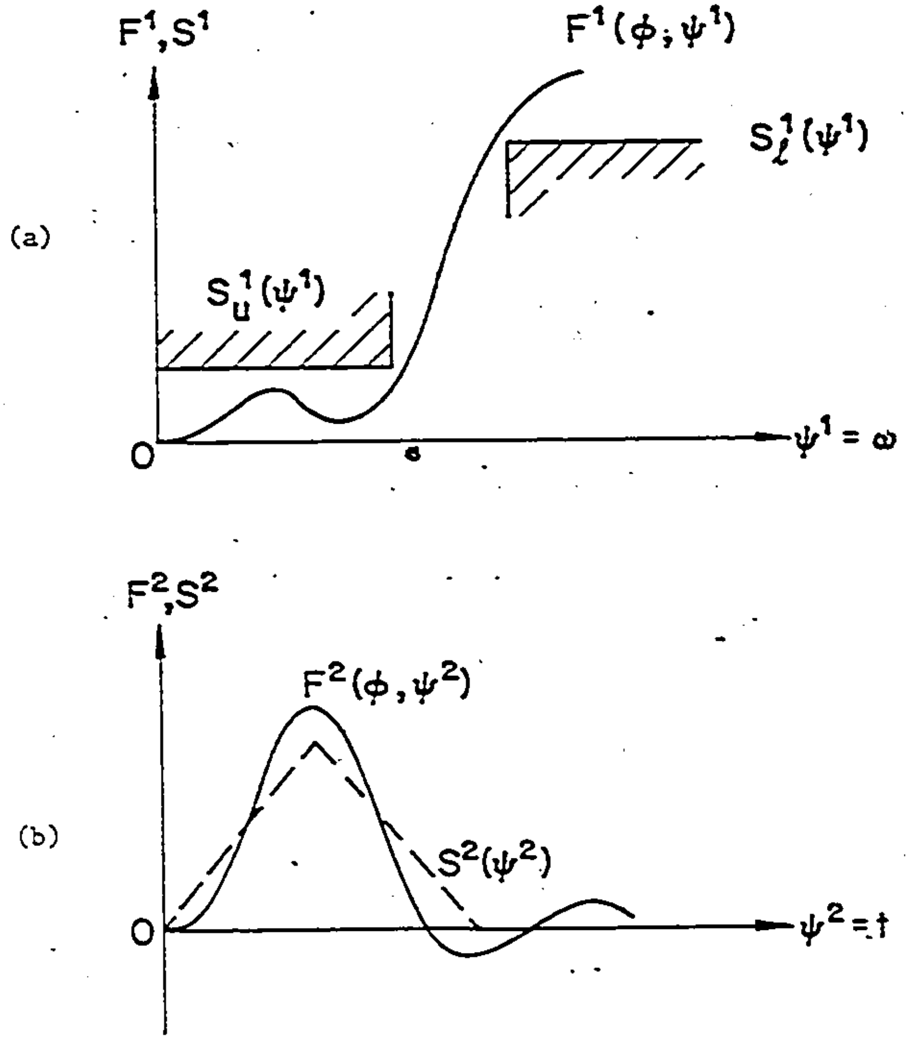

(a)
F¹,S¹
F¹(φ;ψ¹)
S¹ₗ(ψ¹)
S¹ᵤ(ψ¹)
O
ψ¹ = ω
(b)
F²,S²
F²(φ,ψ²)
S²(ψ²)
O
ψ² = t

Fig. 2.4 An example of multiple objectives in filter design, (a) the insertion loss specification in the frequency domain of a lowpass filter, (b) an impulse response specification in the time domain of the lowpass filter.

In the preceding discussion we considered that each response function and each specification is a function of one independent variable $\psi^j$. In some cases we are confronted with response functions and specifications which are functions of the n independent variables. These variables can for instance, be time and temperature; frequency and a tunable circuit parameter; or frequencies in a two-dimensional frequency response of a two-dimensional digital filter. The response function and the specifications will be $F(\underset{\sim}{\phi}, \underset{\sim}{\psi})$ and $S(\underset{\sim}{\psi})$, respectively, where

$$\underset{\sim}{\psi} \triangleq \begin{bmatrix} \psi^1 \\ \psi^2 \\ \cdot \\ \cdot \\ \cdot \\ \psi^n \end{bmatrix} . \tag{2.16}$$

The frequency response function of a two-dimensional lowpass digital filter, for example, of a symmetrically constrained finite impulse response (zero phase) is given by Rabiner, McClellan and Parks (1975), namely,

$$H(e^{j\omega_1}, e^{j\omega_2}) = e^{-j(n_1\omega_1 + n_2\omega_2)} \sum_{k=0}^{n_1} \sum_{\ell=0}^{n_2} a(k,\ell) \cos k\omega_1 \cos \ell\omega_2, \tag{2.17}$$

where the $a(k,\ell)$ are the filter coefficients, and the specifications are

$$S(\omega_1, \omega_2) = \begin{cases} 1 & \omega_1^2 + \omega_2^2 \leq \omega_p^2, \\ 0 & \omega_1^2 + \omega_2^2 \geq \omega_s^2, \end{cases} \tag{2.18}$$

where $\omega_p$ and $\omega_s$ are the edges of passband and stopband, respectively. In the discrete case the response function evaluated at the ith sample point is denoted by

$$F_i(\underset{\sim}{\phi}) \triangleq F(\underset{\sim}{\phi}, \underset{\sim}{\psi}_i), \tag{2.19}$$

for

$$\underset{\sim}{\psi}_i = \begin{bmatrix} \psi_i^1 \\ \psi_i^2 \\ \cdot \\ \cdot \\ \cdot \\ \psi_i^n \end{bmatrix}, \qquad i \in I, \tag{2.20}$$

where $\psi_i^1, \psi_i^2, \ldots, \psi_i^n$ are the values of the independent variables at the ith sample point in the index set I.

In general, where we have upper and lower specifications, the error functions are generalized to

$$e_{ui}(\underset{\sim}{\phi}) \triangleq e_u(\underset{\sim}{\phi}, \underset{\sim}{\psi}_i) = w_{ui}\,(F_i(\underset{\sim}{\phi}) - S_{ui}), \; i \in I_u, \tag{2.21}$$

$$e_{\ell i}(\underset{\sim}{\phi}) \triangleq e_\ell(\underset{\sim}{\phi}, \underset{\sim}{\psi}_i) = w_{\ell i}\,(F_i(\underset{\sim}{\phi}) - S_{\ell i}), \; i \in I_\ell, \tag{2.22}$$

where $I_u$ and $I_\ell$ are index sets, not necessarily disjoint. These can be used in a suitable objective for the approximation problem. Figures 2.5(a) and 2.5(b) show two possible cases in two dimensions.

Suppose

$$f_i = \begin{cases} e_{uj}, & j \in I_u, \\ & \qquad\qquad i \in I, \\ -e_{\ell k}, & k \in I_\ell, \end{cases} \tag{2.23}$$

where

$$I_u = \{1, 2, \ldots, n_u\}, \tag{2.24}$$

$$I_\ell = \{1, 2, \ldots, n_\ell\}, \tag{2.25}$$

$$I = \{1, 2, \ldots, n_u + n_\ell\}, \tag{2.26}$$

and according to a numbering scheme where the error functions for upper specifications are considered first:

$$\begin{aligned} j &= i & &\text{if } i \leq n_u, \\ k &= i - n_u & &\text{if } i > n_u. \end{aligned} \tag{2.27}$$

Let

$$M_f(\underset{\sim}{\phi}) \overset{\Delta}{=} \max_{i \in I} f_i(\underset{\sim}{\phi}). \tag{2.28}$$

Then the sign of $M_f(\underset{\sim}{\phi})$ indicates whether the specifications are satisfied or violated. That is, if

$$M_f(\underset{\sim}{\phi}) \begin{cases} > 0 & \text{the specifications are violated,} \\ = 0 & \text{the specifications are just met,} \\ < 0 & \text{the specifications are satisfied.} \end{cases}$$

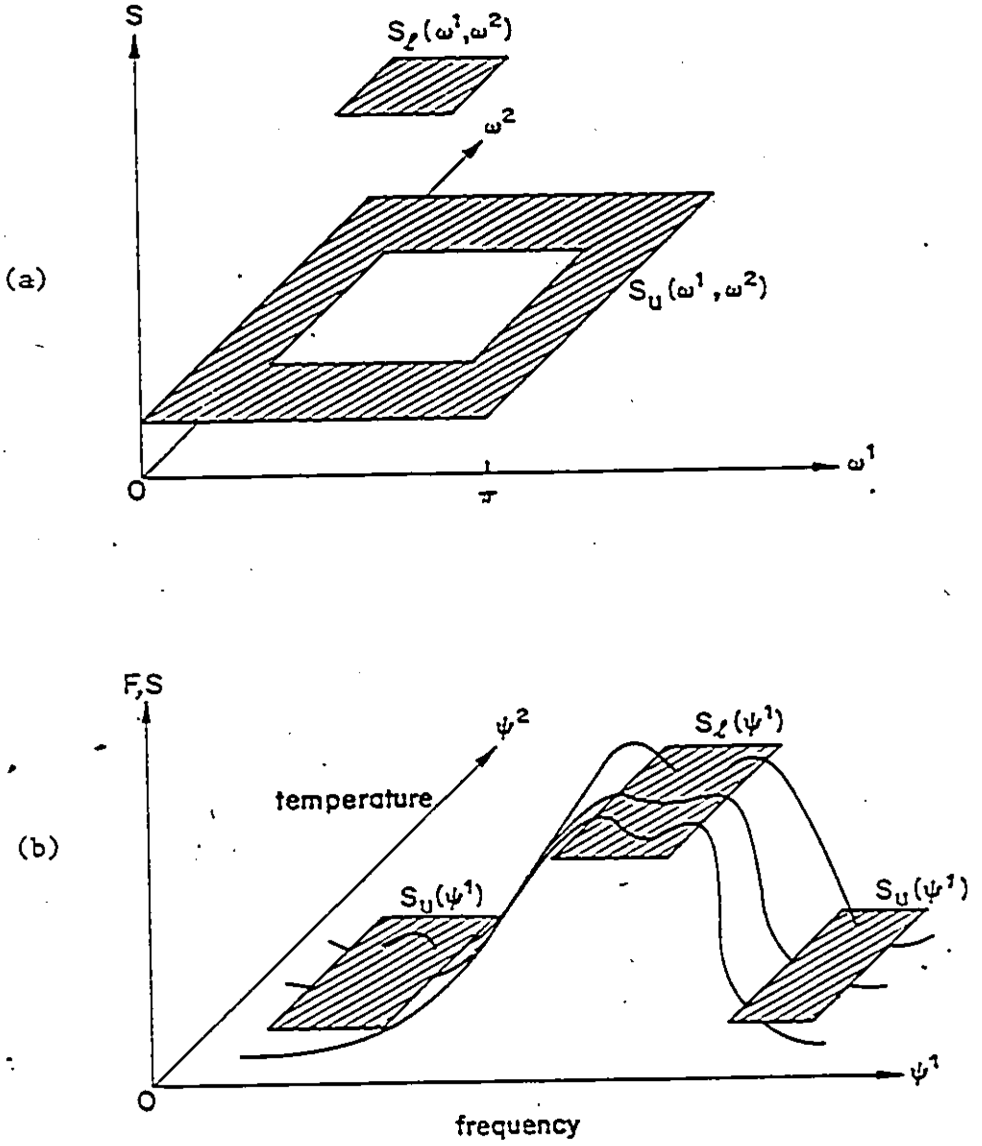

S
$S_\ell(\omega^1,\omega^2)$
$\omega^2$
(a)
$S_u(\omega^1,\omega^2)$
O
$\pi$
$\omega^1$
F,S
$\psi^2$
$S_\ell(\psi^1)$
temperature
(b)
$S_u(\psi^1)$
$S_u(\psi^1)$
O
$\psi^1$
frequency

Fig. 2.5 Multidimensional specifications, (a) a possible specification for a two-dimensional digital filter, (b) upper and lower specifications for an amplifier to be designed to operate over a specified temperature range.

## 2.6 Optimization Approaches in Circuit Design

Optimization approaches which have been used in circuit design are quite numerous. In this section we review the ones which we feel have been the most significant.

### 2.6.1 Nonlinear Programming Approach

Optimal design of filters has been treated as a nonlinear programming problem by Lasdon and Waren (1966). By defining an additional independent variable $\phi_{k+1}$, where k is the number of variables, Waren, Lasdon and Suchman (1967) formulated the problem as the nonlinear program

$$\text{minimize} \quad \phi_{k+1}$$

subject to

$$\phi_{k+1} \geq e_{ui}, \quad i \in I_u, \tag{2.29}$$

$$\phi_{k+1} \geq -e_{\ell i}, \quad i \in I_\ell, \tag{2.30}$$

plus all other constraints. At least one of the constraints has to be active at the optimum, otherwise $\phi_{k+1}$ could be further minimized without violating any of the constraints. If the optimum $\phi_{k+1}$ is negative then the specifications are satisfied, while if it is positive the specifications are violated. Lasdon and Waren applied the interior penalty sequential unconstrained minimization technique by Fiacco and McCormick (1968) along with the Fletcher-Powell variable metric method (Fletcher and Powell 1963) to solve this type of problem. This technique has been applied to the design of cascade crystal-realizable lattice filters, linear arrays (Lasdon, Suchman and Waren 1966), planar arrays (Waren, Lasdon and Suchman 1967), and acoustic sonar transducer arrays (Lasdon, Waren and Suchman 1973).

Other penalty functions can also be used along with the Fiacco-McCormick method like the Zangwill penalty function (Zangwill 1967)

$$P(\underset{\sim}{\phi}, r) = U(\underset{\sim}{\phi}) + (1/r) \sum_{i=1}^{m} [X_i(C_i(\underset{\sim}{\phi}))]^2, \qquad (2.31)$$

where

$$X_i(C_i(\underset{\sim}{\phi})) = [\min(0, C_i(\underset{\sim}{\phi}))], \text{ if } C_i(\underset{\sim}{\phi}) = g_i(\underset{\sim}{\phi}), \qquad (2.32)$$

and

$$X_i(C_i(\underset{\sim}{\phi})) = C_i(\underset{\sim}{\phi}), \qquad \text{if } C_i(\underset{\sim}{\phi}) = h_i(\underset{\sim}{\phi}), \qquad (2.33)$$

$$m = n_h + n_g, \qquad (2.34)$$

which has the advantage of not requiring an initial feasible point and the ability to handle equality constraints. The method is sensitive to the initial choice of r, and ill-conditioning arises when r approaches zero.

Another penalty function is the Powell extension (Powell 1969) to the Zangwill transformation

$$P(\underset{\sim}{\phi}, \underset{\sim}{r}, \underset{\sim}{s}) = U(\underset{\sim}{\phi}) + \sum_{i=1}^{m} \frac{(X_i(C_i(\underset{\sim}{\phi}))+s_i)^2}{r_i} \qquad (2.35)$$

where $s_i$ and $r_i$ are constants during each sequential optimization and $X_i(C_i(\underset{\sim}{\phi}))$ is as defined by (2.32) and (2.33).

The value of $s_i$ is updated by (Powell 1969)

$$s_i^{j+1} = s_i^j + g_i, \qquad (2.36)$$

where j is the present iteration number, and the values of $r_i$ form a decreasing set approaching zero.

(The ill-conditioning problem which arises in penalty function methods when r tends to zero has been studied by Charalambous (1975a), where he extended the work by Powell. The approach is based on the simple idea of perturbing the constraints outwards for the interior penalty function, and inwards for the exterior penalty function by a certain amount so that the r parameter does not have to tend to zero at the optimum. The factor by which the constraints are perturbed and the updating formula are similar to the $s_i$ factor and its updating formula in Powell's transformation.)

### 2.6.2 The GRG Method

Waren et al. (1977) developed a generalized reduced gradient (GRG) algorithm for solving the nonlinear program

$$\text{minimize } U(\underset{\sim}{\phi})$$

subject to

$$h_i(\underset{\sim}{\phi}) = 0, \quad i = 1, 2, \ldots, n_h, \tag{2.37}$$

$$g_i(\underset{\sim}{\phi}) \geq 0, \quad i = 1, 2, \ldots, n_g, \tag{2.38}$$

by converting it to

$$\text{minimize } U(\underset{\sim}{\phi})$$

subject to

$$h_i(\underset{\sim}{\phi}) - \phi_{k+i} = 0, \quad i = 1, 2, \ldots, n_h, \tag{2.39}$$

$$g_i(\underset{\sim}{\phi}) - \phi_{k+n_h+i} = 0, \quad i = 1, 2, \ldots, n_g, \tag{2.40}$$

$$\phi_{\ell i} \leq \phi_i \leq \phi_{ui}, \quad i = 1, \ldots, k + n_h, \tag{2.41}$$

$$\phi_{\ell i} = \phi_{ui} = 0, \quad i = k + 1, \ldots, k + n_h, \tag{2.42}$$

$$\phi_{k+n_h+i} \geq 0, \quad i = 1, 2, \ldots, n_g, \tag{2.43}$$

where

$n_h$ is the number of equality constraints,

$n_g$ is the number of inequality constraints,

k is the number of variables,

$\phi_{k+1}, \ldots, \phi_{k+n_h+n_g}$ are nonnegative slack variables.

At each stage of the optimization process the variables are separated into dependent and independent variables. The number of natural dependent variables is the number of active constraints

$n_a$. The slack variables of the nonactive constraints are the additional dependent variables. All the remaining ones are taken as independent variables. The active constraints are then solved for the natural dependent variables $n_a$ in terms of the natural independent ones $k - n_a$. This reduces the objective function to a function of $k - n_a$ variables only. The generalized reduced gradient algorithm solves the original problem as a sequence of reduced problems. The reduced problems are solved using a variable metric gradient method.

Waren et al. used the GRG method in the design of dielectric interference filters. The problem, defined by inequalities, is reformulated as a nonlinear program (as in Section 2.6.1). The numbers of variables and constraints are considerable. The GRG method apparently handles this large problem efficiently and yields satisfactory results.

### 2.6.3 Least pth Optimization

Temes and Zai (1969) generalized the least squares method of Marquardt (1963) with appropriate damping in the spirit of Levenberg (1944) to a least pth method.

They suggested a simple objective of the form

$$U = \sum_{i \in I} [e_i(\underset{\sim}{\phi})]^p, \qquad (2.44)$$

where the $e_i(\underset{\sim}{\phi})$ are special cases of (2.12) when the number of independent variables is equal to one and $p \geq 2$ is any even number. The method was applied to the optimization of a four-variable RC active equalizer, where p was equal to 10. The maximum deviation from the desired specification for $p = 2$ was found to be 33 percent higher. They also demonstrated the nonuniqueness of the optimum in that particular problem. They obtained different solutions with different starting points.

For large values of p in (2.44) accuracy and convergence problems arise due to very large and very small numbers involved in the calculations. Bandler and Charalambous (1971) alleviated this ill-conditioning by considering the objective

$$U_p = M(\underset{\sim}{\phi}) \left[ \sum_{i \in I} \left| \frac{e_i(\underset{\sim}{\phi})}{M(\underset{\sim}{\phi})} \right|^p \right]^{1/p}, \text{ for } 1 < p < \infty, \qquad (2.45)$$

where

$$M(\underset{\sim}{\phi}) \triangleq \max_{i \in I} |e_i(\underset{\sim}{\phi})|. \qquad (2.46)$$

The error functions, in general, can be real or complex functions. Hebden (1971) employed this type of scaling in some related work.

### 2.6.4 Generalized Least pth Objective

Very recently, Charalambous (1977b) proposed the following generalization of the original generalized least pth objective due to Bandler and Charalambous (1972c) (Charalambous and Bandler

1976)

$$U_p = \begin{cases} M \left[ \sum\limits_{i \in K} u_i \left( \frac{\phi_i}{M} \right)^q \right]^{1/q} & \text{for } M \neq 0, \\ 0 & \text{for } M = 0, \end{cases} \qquad (2.47)$$

where the $\phi_i$ are related to n real, nonlinear functions (assumed differentiable), identified by an index set I, such that

$$\phi_i = f_i - \xi, \qquad (2.48)$$

and where

$$M = \max_{i \in I} \phi_i, \qquad (2.49)$$

$$u_i \geq 0 \; , \; i = 1, 2, \ldots, n, \qquad (2.50)$$

and

if $M > 0$ then $K = J$ and $q = p$,

if $M < 0$ then $K = I$ and $q = -p$,

where

$$J = \{i \mid \phi_i \geq 0\}. \qquad (2.51)$$

When minimizing (2.47) the values of $u_i$, $\xi$ and p are kept fixed. At each optimum point reached a change is made to one or more of $u_i$, $\xi$ and p, such that the sequence of optimum points of $U_p$ tend to a minimax optimum. Depending on which of the parameters we change at each optimum point of $U_p$, different algorithms can be generated, such as the following.

Algorithm 1 (Bandler and Charalambous 1972c) Here, we keep $u_i = 1$, $i = 1, 2, \ldots, n$, and let $\xi = 0$ and strictly increase the value of p at each optimum point of $U_p$ w.r.t. $\underset{\sim}{\phi}$ such that $p \rightarrow \infty$. It should be noted that if $f_i(\underset{\sim}{\phi}) \geq 0$, $i = 1, 2, \ldots, n$, this algorithm turns out to be the well known Polya algorithm (Cheney 1966).

Bandler and Charalambous (1972a, 1973) considered necessary and sufficient conditions for optimality in generalized least pth optimization for $p \rightarrow \infty$ and related them to the conditions for minimax optimality (Bandler 1971, Dem'yanov and Malozemov 1972).

Algorithm 2 Here, we keep $u_i = 1$, $i = 1, 2, \ldots, n$, with p constant and, at each optimum point of $U_p$, change the value of $\xi$ such that it tends to $M_f(\check{\underset{\sim}{\phi}})$, where $\check{\underset{\sim}{\phi}}$ is the solution of the minimax problem. Charalambous and Bandler (1973, 1976) considered the following two variations for changing $\xi$.

Algorithm 2.1

$$\xi^{r+1} \leftarrow M_f(\check{\underline{\phi}}^r) + \epsilon \qquad (2.52)$$

where $\check{\underline{\phi}}^r$ is the solution point of $U_p$ at the rth optimization and $\epsilon$ is a small number.

Algorithm 2.2 This method updates $\xi^r$ as in Algorithm 2.1 if $M_f(\check{\underline{\phi}}^r, \xi^r) < 0$, otherwise

$$\xi^{r+1} \leftarrow (1 - \lambda^r)\ \xi^r + \lambda^r\ M_f(\check{\underline{\phi}}^r), \qquad (2.53)$$

where

$$0 < \lambda^r < 1. \qquad (2.54)$$

In both algorithms, for the first optimization the margin $\xi^1$ is min $[0, M_f(\underline{\phi}^0) + \epsilon]$, where $\underline{\phi}^0$ is the starting point. For r > 1 the first algorithm will let all the $\phi_i$ be negative and be considered in the objective function and the maximum is to be moved away from the margin. In the second algorithm, $\xi$ starts with zero and increases approaching $M_f(\check{\underline{\phi}})$. The small number $\epsilon$ is introduced to avoid M = 0. It is well known that the minimax solution will not change if a constant is added to all the functions $f_i$. If this constant is greater than $|M_f(\check{\underline{\phi}})|$ the second algorithm will be used throughout the whole optimization even if

$M_r(\breve{\underline{\phi}}) \leq 0$.

Algorithm 2.3 Charalambous (1975b) and Bandler et al. (1976a, 1976b) took $\xi^{r+1}$ to be the lower bound on the maximum predicted under convexity assumptions after each optimization. The constant $\xi^r$ is used as a lower bound for the (r+1)th optimization so that all the functions less than this constant are discarded and considered inactive. The associated algorithm is called the $\xi$ algorithm. Any combination of Algorithm 1 and Algorithm 2 can, of course, be used.

Algorithm 3 Recently, Bandler et al. (1976b), Chu (1974) used extrapolation to $p = \infty$, after performing least pth approximation with different values of p, with $u_i = 1$, $i = 1, 2, \ldots, n$, to obtain the minimax solution.

The main drawback of the above three algorithms is that the unconstrained objective function becomes more and more ill-conditioned as we get closer to the minimax solution.

Algorithm 4 (Charalambous 1977b) Very recently Charalambous introduced the parameters $u_i$ into the least pth objective function to overcome the ill-conditioning problem. Any of the foregoing three algorithms can be used in conjunction with the following updating formula for $u_i$ after each optimum point of $U_p$ is reached, namely $\check{\underset{\sim}{\phi}}^r$. Initially, we set $u_i = 1$, $i = 1, 2, \ldots, n$. Subsequently,

$$u_i \leftarrow \frac{1}{\sum\limits_{j=1}^{n} v_j} v_i, \qquad (2.55)$$

where

$$v_i \leftarrow \begin{cases} u_i \left[\dfrac{\phi_i(\check{\underset{\sim}{\phi}}^r, \xi^r)}{M(\check{\underset{\sim}{\phi}}^r, \xi^r)}\right]^{q-1} & i \in K(\check{\underset{\sim}{\phi}}^r, \xi^r) \\ 0 & i \in K(\check{\underset{\sim}{\phi}}^r, \xi^r) \end{cases} \qquad (2.56)$$

From the theoretical and numerical results presented by Charalambous (1977b), it is clear that this algorithm is superior to the other three algorithms.

### 2.6.5 Least pth Objective and Nonlinear Programming

Bandler and Charalambous (Charalambous 1973, Bandler and Charalambous 1974) suggested that the nonlinear programming problem could be solved using minimax techniques by transforming the problem to minimizing w.r.t. $\underset{\sim}{\phi}$ the unconstrained function

$$M(\underset{\sim}{\phi}, \underset{\sim}{\alpha}) = \max_{1 \leq i \leq n_g} [U(\underset{\sim}{\phi}), U(\underset{\sim}{\phi}) - \alpha_i g_i(\underset{\sim}{\phi})], \qquad (2.57)$$

where

$$\underline{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{n_g} \end{bmatrix} \tag{2.58}$$

and

$$\alpha_i > 0 \;, \qquad i = 1, 2, \ldots, n_g. \tag{2.59}$$

(Equality constraints can be transformed to two inequality constraints). They related the Kuhn-Tucker necessary conditions for optimality of the nonlinear programming problem to the necessary conditions for optimality of $M(\underline{\phi}, \underline{\alpha})$. These conditions require that the $\alpha$ parameters be positive and satisfy

$$\sum_{i=1}^{n_g} \frac{\mu_i}{\alpha_i} < 1, \tag{2.60}$$

where the $\mu_i$ are the Kuhn-Tucker multipliers (not known a priori). Sufficiently large values should be assigned to $\underline{\alpha}$ to ensure that the inequality (2.60) is satisfied.

This minimax problem can be solved by least pth optimization (Charalambous 1974a), with large values of p, by letting

$$\phi_1 = U(\underset{\sim}{\phi}), \tag{2.61}$$

$$\phi_{j+1} = U(\underset{\sim}{\phi}) - \alpha_j g_j, \qquad j = 1, 2, \ldots, n_g, \tag{2.62}$$

$$M = M(\underset{\sim}{\phi}, \underset{\sim}{\alpha}). \tag{2.63}$$

Ill-conditioning can arise when the minimax solution is approached because of the tendency of the first partial derivatives to be discontinuous. Charalambous (1977a) attacked the problem by defining a sequence of least pth optimizations where the objective function to be minimized w.r.t. $\underset{\sim}{\phi}$ is

$$U_p = U_p(\underset{\sim}{\phi}, \underset{\sim}{\alpha}^r, \xi^r), \tag{2.64}$$

where

$$u_i = 1\ , \quad i = 1, 2, \ldots, n_g+1, \tag{2.65}$$

$$\phi_1 = \phi_1(\underset{\sim}{\phi}, \xi^r) = U(\underset{\sim}{\phi}) - \xi^r, \tag{2.66}$$

$$\phi_{j+1} = \phi_{j+1}(\underset{\sim}{\phi}, \underset{\sim}{\alpha}^r, \xi^r) = \phi_1 - \alpha_j^r g_j\ , \quad j = 1, 2, \ldots, n_g, \tag{2.67}$$

$$M = M(\underset{\sim}{\phi}, \underset{\sim}{\alpha}^r, \xi^r) = M(\underset{\sim}{\phi}, \underset{\sim}{\alpha}^r) - \xi^r, \tag{2.68}$$

where r is the optimization number. He proved that if

$$\underset{\sim}{\alpha} = \underset{\sim}{\alpha}^r = (n_1 + 1)\ \check{\underset{\sim}{\mu}}, \tag{2.69}$$

the point $\check{\underset{\sim}{\phi}}$ is a stationary point of the function $U_p(\underset{\sim}{\phi}, \underset{\sim}{\alpha}^r, \underset{\sim}{\xi}^r)$ for any p and $\xi$, where $\check{\underset{\sim}{\phi}}$ is the optimum of the nonlinear programming problem, $\check{\underset{\sim}{\mu}}$ are the multipliers at the optimum $\check{\underset{\sim}{\phi}}$, $n_1$ is the number of constraints with multipliers greater than or equal to a certain small number $\epsilon_1$.

An approximation to the multipliers which is an estimate to $\check{\underline{\mu}}$ (since $\check{\underline{\mu}}$ cannot be known beforehand) is used in updating $\underline{\alpha}$.

### 2.6.6 Minimax Approximation via Linear Programming

Ishizaki and Watanabe (1968) had the objective function $M = \max|e_i(\underline{\phi})|$, $i \in I$. They transformed the problem to a nonlinear program of the form of (2.29) and (2.30), with the difference that the upper and lower specifications coincide, and an additional constraint

$$\phi_j/\phi_j^0 \geq 0, \qquad j = 1, 2, \ldots, k. \tag{2.70}$$

The last constraint is to prevent $\phi_j$ from changing sign during the iteration process. By taking the first-order approximation to the constraints at a point $\underline{\phi}^r$, the problem is reduced to a linear program, which is given by

$$\text{minimize } x_{k+1}$$

subject to

$$w_i \sum_{j=1}^{k} \phi_j \frac{\partial F_i(\underline{\phi}^r)}{\partial \phi_j} x_j - x_{k+1} + e_i(\underline{\phi}^r) \leq 0, \; i \in I, \tag{2.71}$$

$$-w_i \sum_{j=1}^{k} \phi_j \frac{\partial F_i(\underline{\phi}^r)}{\partial \phi_j} x_j - x_{k+1} - e_i(\underline{\phi}^r) \leq 0, \; i \in I, \tag{2.72}$$

$$-x_j \leq 1, \quad j = 1, 2, \ldots, k, \tag{2.73}$$

where

$$x_j = \frac{\Delta\phi_j}{\phi_j}\ , \quad x_{k+1} = \phi_{k+1}$$

and $F_i(\underset{\sim}{\phi})$ is the approximating function (or the response function).

The superscript r denotes the iteration number of a sequence of linear programming problems. The linear program is solved by the simplex method. Some examples which include the design of attenuation and group delay equalizers have been presented. A discussion of this method is also presented by Temes and Calahan (1967).

Bandler, Srinivasan and Charalambous (1972) developed the grazor search method for nonlinear minimax optimization. The method is based on a linear programming problem which uses gradient information of one or more near maximum functions to produce a downhill direction followed by a linear search to find a minimum in that direction. They first define a subset $J \subset I$ such that

$$J(\underset{\sim}{\phi}^j, \varepsilon^j) = \{i \mid M_f(\underset{\sim}{\phi}^j) - f_i(\underset{\sim}{\phi}^j) \leq \varepsilon^j,\ i \in I\}, \qquad (2.74)$$

$$\varepsilon^j \geq 0, \qquad (2.75)$$

where $\underset{\sim}{\phi}^j$ denotes a feasible point at the beginning of the jth iteration and $\varepsilon^j$ is the tolerance with respect to the current

$M_f(\underset{\sim}{\phi}^j)$ within which the $f_i$ for $i \in J$ lie. Linearizing $f_i$ at $\underset{\sim}{\phi}^j$

$$\delta f_i(\underset{\sim}{\phi}^j) = \underset{\sim}{\nabla}^T f_i(\underset{\sim}{\phi}^j)\, \Delta\underset{\sim}{\phi}^j\ , \quad i \in J(\underset{\sim}{\phi}^j, \epsilon^j). \tag{2.76}$$

To get $\Delta\underset{\sim}{\phi}^j$ in the descent direction for $M_f(\underset{\sim}{\phi}^j)$

$$\underset{\sim}{\nabla}^T f_i(\underset{\sim}{\phi}^j) \cdot \Delta\underset{\sim}{\phi}^j < 0, \qquad i \in J(\underset{\sim}{\phi}^j, \epsilon^j). \tag{2.77}$$

Considering

$$\Delta\underset{\sim}{\phi}^j = - \sum_{i \in J} \alpha_i^j\, \underset{\sim}{\nabla} f_i(\underset{\sim}{\phi}^j), \tag{2.78}$$

$$\sum_{i \in J} \alpha_i^j = 1, \tag{2.79}$$

$$\alpha_i^j \geq 0, \tag{2.80}$$

(2.77) can be written as

$$- \underset{\sim}{\nabla}^T f_i(\underset{\sim}{\phi}^j) \sum_{i \in J} \alpha_i^j\, \underset{\sim}{\nabla} f_i(\underset{\sim}{\phi}^j) < 0. \tag{2.81}$$

This inequality suggests the linear programming problem

$$\text{maximize } \alpha_{k_r+1}^j (\underset{\sim}{\phi}^j, \epsilon^j) \geq 0$$

subject to

$$- \underset{\sim}{\nabla}^T f_i(\underset{\sim}{\phi}^j) \sum_{i \in J} \alpha_i^j\, \underset{\sim}{\nabla} f_i(\underset{\sim}{\phi}^j) \leq - \alpha_{k_r+1}^j \tag{2.82}$$

and subject to (2.79) and (2.80). $k_r$ denotes the number of elements of $J(\underset{\sim}{\phi}^j, \epsilon^j)$. A golden section search follows each linear program to obtain $\underset{\sim}{\phi}^{j+1}$.

Madsen et al. (1975a, 1975b) developed two minimax algorithms based on successive linearizations of the nonlinear functions and the resulting linear systems are solved in the minimax sense. At the rth stage of the first algorithm a minimax solution $\Delta\underset{\sim}{\phi}^r$ to the linearized system is found subject to the constraints

$$\|\Delta\underset{\sim}{\phi}^r\| = \max_j |\Delta\phi_j^r| \le \lambda^r, \tag{2.83}$$

where $\lambda^r$ is automatically adjusted during the process to satisfy the inequality

$$M_f(\underset{\sim}{\phi}^r + \Delta\underset{\sim}{\phi}^r) < M_f(\underset{\sim}{\phi}^r), \tag{2.84}$$

so that the new point becomes

$$\underset{\sim}{\phi}^{r+1} = \underset{\sim}{\phi}^r + \Delta\underset{\sim}{\phi}^r. \tag{2.85}$$

The choice of $\lambda^r$ gives the flexibility of taking a large step if the linear approximations represent the nonlinear functions well enough. If the decrease in the maximum function (the nonlinear one) does not exceed a small multiple of the decrease predicted by the linear approximations (the maximum of the linearized functions) then $\underset{\sim}{\phi}^{r+1}$ remains $\underset{\sim}{\phi}^r$.

The second algorithm is similar to the first one but does not require derivatives. It uses the Broyden updating formula (Broyden 1965) to approximate the derivatives, where the initial approximation is obtained by perturbation.

Comparison by Madsen et al. of the new algorithms with existing ones has been reported. Design of microwave reflection amplifiers was also carried out. Madsen and Schjaer-Jacobsen (1976) treated common singularities in nonlinear minimax problems by modifying the first algorithm. They developed an automatic procedure to detect ill-conditioning and singularities in a given problem which slow convergence. Intuitively, the reason for slow convergence is that the upper bound on the step taken in each iteration is very small when a narrow valley is reached. However, a common feature of these algorithms is that they have a quadratic final convergence (Madsen and Schjaer-Jacobsen 1978a).

### 2.6.7 Minimax Optimization of Constrained Problems

Bandler and Srinivasan (1974) suggested an unconstrained minimax objective for a constrained minimax problem. The constrained problem is to minimize $M_f$ of (2.28) subject to

$$g_j(\underset{\sim}{\phi}) \geq 0, \qquad j = 1, 2, \ldots, n_g. \tag{2.86}$$

The problem is reduced to

$$\text{minimize } \phi_{k+1}$$

subject to

$$\phi_{k+1} - f_i(\underset{\sim}{\phi}) \geq 0, \quad i \in I \tag{2.87}$$

and (2.86). The problem is then reformulated as an unconstrained minimax problem. We may, for example, minimize M of (2.49) (with $\xi = 0$) w.r.t. $\underset{\sim}{\phi}$ and $\phi_{k+1}$, where

$$\phi_1 = \phi_{k+1} \tag{2.88}$$

$$\phi_{i+1} = \phi_1 - \alpha_1(\phi_1 - f_i(\underset{\sim}{\phi})) , \quad i = 1, 2, \ldots, n, \tag{2.89}$$

$$\phi_{n+i+1} = \phi_1 - \alpha_{i+1}\, g_i(\underset{\sim}{\phi}) , \quad i = 1, 2, \ldots, n_g, \tag{2.90}$$

where

$$\alpha_i > 0 , \quad i = 1, 2, \ldots, n_g + 1 \tag{2.91}$$

and sufficiently large.

Dutta and Vidyasagar (1977) developed two algorithms for solving the nonlinear constrained minimax problem. They are principally a generalization of Morrison's least squares algorithm (Morrison 1968) and are quite similar to Algorithm 2.3 as proposed by Charalambous.

### 2.6.8 Other Methods

Charalambous and Conn (1975, 1978) proposed a minimax optimization algorithm which overcomes the difficulty of discontinuities in the minimax objective's first derivatives. Their approach is direct, unlike the generalized least pth approach.

Einarsson (1975) employed the modified Lagrangians (Rockafellar 1974) (augmented Lagrangians) in solving minimax problems. In his formulation an assumed active function is to be minimized w.r.t. $\underset{\sim}{\phi}$ subject to n-1 nonlinear constraints. If this function is, for example, $f_1(\underset{\sim}{\phi})$ the constraints will be

$$f_i(\underset{\sim}{\phi}) - f_1(\underset{\sim}{\phi}) \leq 0, \quad i = 2, \ldots, n. \tag{2.92}$$

The Hestenes-Powell (Hestenes 1969, Powell 1969) method is used for updating the multipliers. This method requires the constraints to be equalities. The algorithms developed are thus based on knowing the active set of constraints in advance.

## 2.7 Centering, Tolerancing and Tuning

In the classical design problem we are interested in finding one single point in the feasible region. This kind of solution is impractical from the manufacturing point of view. Many other points (design outcomes) can also meet the required specifications. The designer can take advantage of this fact and assign tolerances on component values (Bandler 1974, Geher 1971, Hersom 1971, Karafin 1971, Seth 1972) so as to minimize production cost. The cost of a component may be assumed, for example, to be inversely proportional to the tolerance associated with it.

The formulation of the design problem considering manufacturing tolerances, post-production tuning and model uncertainties, besides the objective of reducing the cost, renders the design more practical and tends to alleviate realization problems.

In practice, during circuit fabrication components are either specially made, chosen randomly or selectively from stock. These components usually have statistical distributions which have to be considered during the design process for electrical circuit components. The aim of tolerance assignment is, consequently, to obtain a region in which every point represents an outcome optimally taking into consideration the aforementioned concepts. All the outcomes, or at least a large percentage, have to meet the specifications, after tuning if necessary.

### 2.7.1 Definitions

Consider the vector of nominal design parameters

$$\underset{\sim}{\phi}^0 \triangleq \begin{bmatrix} \phi_1^0 \\ \phi_2^0 \\ \vdots \\ \phi_k^0 \end{bmatrix}, \qquad (2.93)$$

defining a nominal point and a vector of associated manufacturing tolerances

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_k \end{bmatrix} , \qquad (2.94)$$

described as the tolerance vector and let

$$I_\phi \triangleq \{1, 2, \ldots, k\}, \qquad (2.95)$$

where k represents the number of network design parameters, assumed independent for simplicity in the ensuing presentation.

A nominal point $\underset{\sim}{\phi}^0$ will have a tolerance region $R_\varepsilon$ associated with it defined, under the assumption of independent variables, as

$$R_\varepsilon \triangleq \{\underset{\sim}{\phi} \mid \underset{\sim}{\phi}^0 - \underset{\sim}{\varepsilon} \leq \underset{\sim}{\phi} \leq \underset{\sim}{\phi}^0 + \underset{\sim}{\varepsilon}\}. \qquad (2.96)$$

This region is a convex regular polytope of k dimensions with sides of length $2\varepsilon_i$, $i \in I_\phi$, and centered at $\underset{\sim}{\phi}^0$. The extreme points of the tolerance region, the vertices, are

$$R_v \triangleq \{\underset{\sim}{\phi} \mid \phi_i = \phi_i^0 + \varepsilon_i \mu_i, \ \mu_i \in \{-1,1\}, \ i \in I_\phi\}, \qquad (2.97)$$

and the index set of the vertices

$$I_v \triangleq \{1, 2, \ldots, 2^k\}. \qquad (2.98)$$

Any point in the tolerance region is a possible outcome given by a point $\underline{\phi}$, which is

$$\underset{\sim}{\phi} = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\,\underset{\sim}{\mu}, \tag{2.99}$$

where

$$\underset{\sim}{E} \triangleq \begin{bmatrix} \varepsilon_1 & & & \\ & \varepsilon_2 & & \\ & & \ddots & \\ & & & \varepsilon_k \end{bmatrix} \tag{2.100}$$

and $\underline{\mu} \in R_\mu$, where

$$R_\mu = \{\underset{\sim}{\mu} \mid -1 \leq \mu_i \leq 1,\ i \in I_\phi\}. \tag{2.101}$$

Figure 2.6 depicts a tolerance region inscribed in the constraint region for a two-dimensional case. In general,

$$R_c \triangleq \{\underset{\sim}{\phi} \mid g_i(\underset{\sim}{\phi}) \geq 0,\ i \in I_c\}, \tag{2.102}$$

where

$$I_c \triangleq \{1, 2, \ldots, m_c\}, \tag{2.103}$$

is the index set for the performance specifications (response constraints) and other parameter constraints, $m_c$ being the total number of constraints.

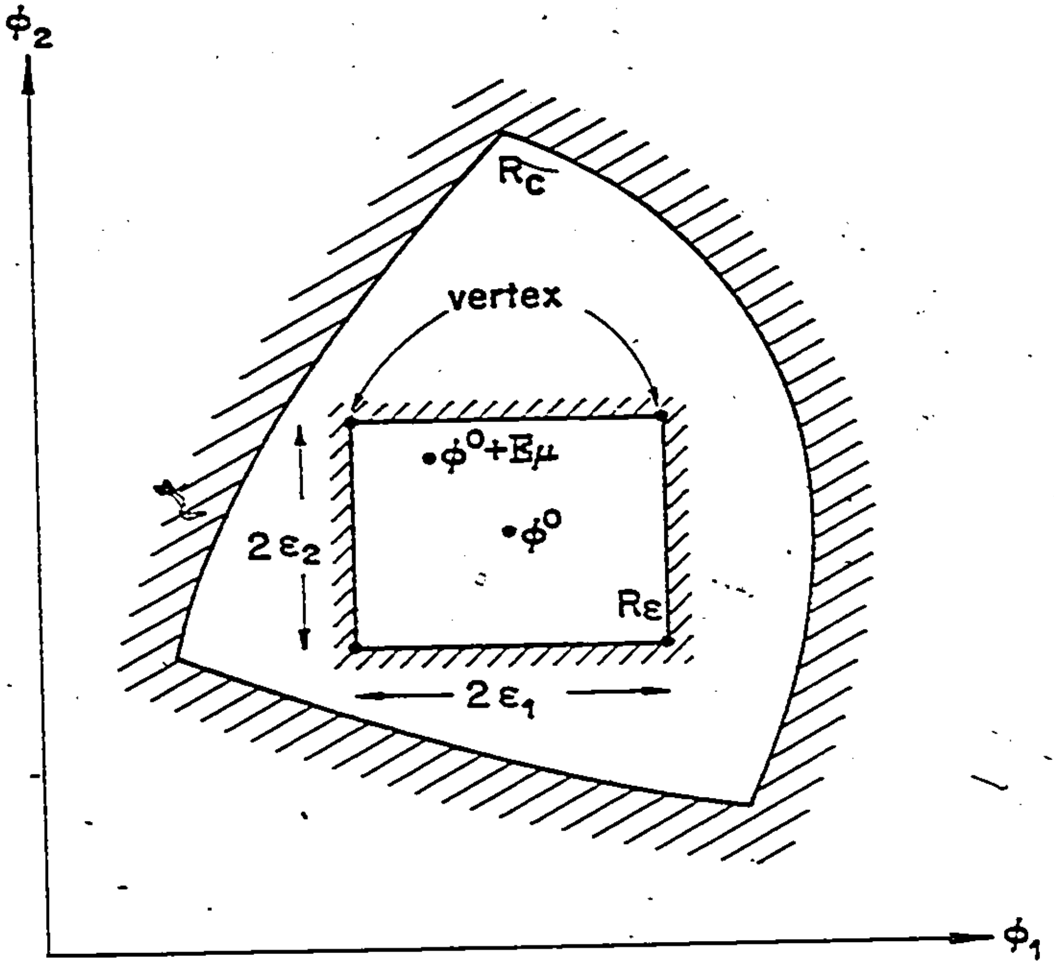

$\phi_2$
$R_c$
vertex
$\phi^0+E\mu$
$\phi^0$
$2\varepsilon_2$
$R_\varepsilon$
$2\varepsilon_1$
$\phi_1$

Fig. 2.6 A tolerance region $R_\varepsilon$ inscribed in the constraint region $R_c$. If $\underline{\varepsilon} = \underline{0}$ the conventional nominal design problem is implied.

### 2.7.2 Worst-case Design

In worst-case design the whole tolerance region has to lie in the constraint region, i.e., it is required that

$$R_\varepsilon \subset R_c. \tag{2.104}$$

This is design with 100% yield, where the yield Y is given by

$$Y \triangleq \frac{\text{number of outcomes which meet specifications}}{\text{total number of outcomes}}$$

The $2^k$ vertices of the tolerance region are usually the points considered as candidates for worst case. There are two main reasons. The first is that it is impractical, or even impossible, to consider explicitly the infinite number of points contained in the tolerance region. The second is that one-dimensional convexity of the constraint region may be assumed. Bandler (1974) proved, in this case, that it is sufficient for worst-case design to require that

$$R_v \subset R_c. \tag{2.105}$$

Bandler and Liu (1975) investigated the validity of these assumptions for networks which possess bilinear dependence on each parameter. In their investigation they studied the behaviour of the modulus squared of the bilinear network function, which is a biquadratic function given by

$$\frac{c\phi^2 + 2d\phi + e}{\phi^2 + 2a\phi + b}, \tag{2.106}$$

and they proved that the worst case assumptions they considered are often valid in the frequency-domain case.

Brayton, Hoffman and Scott (1977) proved, for linear D.C. networks, that if each parameter is at its extreme value the currents and voltages of the network will be at their local or global extrema. The investigation of this kind of problem in nonlinear networks or in the time domain has not yet been reported.

### 2.7.3 Fixed Tolerance Problem

In this problem we want to find $\underset{\sim}{\phi}^0$, the center of the tolerance region, where the manufacturing tolerances on the components are held fixed. The problem is basically a centering problem.

Let us consider a problem with upper and lower performance specifications. The error functions in this case are

$$e_{ui}(\underset{\sim}{\phi}^j) \triangleq w_{ui}(F_i(\underset{\sim}{\phi}^j) - S_{ui}), \; i \in I_u, \; j \in I_v, \tag{2.107}$$

$$e_{\ell i}(\underset{\sim}{\phi}^j) \triangleq w_{\ell i}(F_i(\underset{\sim}{\phi}^j) - S_{\ell i}), \; i \in I_\ell, \; j \in I_v, \tag{2.108}$$

where j denotes the jth vertex contained in $I_v$, and $\underset{\sim}{\phi}^j$ is this vertex. According to a specified vertex numbering scheme, each j will have a corresponding $\underset{\sim}{\mu}$. Any suitable objective function can be formulated to incorporate these error functions and then minimized to obtain the optimal $\underset{\sim}{\phi}^0$. We have to note that a worst-case design, in this case, is not necessarily achievable since we might not be able to inscribe the whole tolerance region, with preselected fixed edges, in the constraint region.

### 2.7.4 Variable Tolerance Problem

In many cases the manufacturing tolerances are considered as variables instead of fixed. The larger they are the cheaper the circuit components will be. The design problem is reformulated as a nonlinear program (Bandler 1974, Bandler 1977, Bandler and Liu 1974, Pinel and Roberts 1972) as follows:

$$\text{minimize } C(\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon})$$

w.r.t. $\underset{\sim}{\phi}^0$ and $\underset{\sim}{\varepsilon}$ subject to

$$\underset{\sim}{\phi} \in R_c \text{ for all } \underset{\sim}{\mu} \in R_\mu, \tag{2.109}$$

where $\underset{\sim}{\phi}$ is as given in (2.99), and

$$\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon} \geq \underset{\sim}{0}. \tag{2.110}$$

The objective function C is directly related to the component cost, and generally possesses the properties

$$C\,(\underset{\sim}{\phi}^0,\ \underset{\sim}{\varepsilon}) \rightarrow \text{constant} \qquad \text{as } \underset{\sim}{\varepsilon} \rightarrow \underset{\sim}{\infty}, \tag{2.111}$$

$$C\,(\underset{\sim}{\phi}^0,\ \underset{\sim}{\varepsilon}) \rightarrow \infty \qquad \text{as } \varepsilon_i \rightarrow 0\ . \tag{2.112}$$

A common form of this objective is

$$\sum_{i=1}^{k} c_i \,\frac{\phi_i^0}{\varepsilon_i}\,, \tag{2.113}$$

where the $c_i$ are constant weights. The number of variables for the optimization is 2k, namely, k independent nominal variables and k associated tolerances.

For large problems, with a large number of variables, the number of vertices of the tolerance region becomes enormous. Selection schemes which include purging (dropping of constraints or vertices) as well as addition of vertices of the tolerance region during the optimization process alleviate the need for considering the $2^k$ vertices (Bandler, Liu and Chen 1975, Bandler, Liu and Tromp 1976b). One of these schemes is based on the iterative solution of necessary conditions for the worst vertex derived from the Kuhn-Tucker conditions. Efficient selection schemes relevant to the tolerance problem are still not well developed.

The tolerance problem described here implicitly solves the centering problem, in which we are interested in finding a "center" of the constraint region. Other centering approaches include the performance contours approach developed by Butler

(1971, 1973), and the simplicial approximation approach (Director and Hachtel 1977).

Madsen and Schjaer-Jacobsen (1978b, 1978c) extended their earlier work on minimax approximation (see Section 2.6.6) to minimax optimization with fixed tolerances and the maximization of a single variable tolerance. (A single degree of freedom in tolerances has also been considered by Bandler et al. (1975).) Centering is implicit in these formulations.

### 2.7.5 Tolerancing and Tuning

Tuning some of the components after production is quite common in electrical circuit fabrication. Considering independent tuning in the design procedure, a tuned design will imply $\underset{\sim}{\phi}$ such that

$$\underset{\sim}{\phi} = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\,\underset{\sim}{\mu} + \underset{\sim}{T}\,\underset{\sim}{\rho}, \qquad (2.114)$$

for some $\underset{\sim}{\rho} \in R_\rho$, with

$$\underset{\sim}{T} \triangleq \begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_k \end{bmatrix}. \qquad (2.115)$$

An example of $R_\rho$ is

$$R_\rho = \{\underset{\sim}{\rho} \mid -1 \leq \rho_i \leq 1,\ i \in I_\phi\}. \qquad (2.116)$$

The corresponding tuning region is defined as

$$R_t(\underset{\sim}{\mu}) = \{\underset{\sim}{\phi} \mid \underset{\sim}{\phi}^0 + \underset{\sim}{E}\,\underset{\sim}{\mu} - \underset{\sim}{t} \le \underset{\sim}{\phi} \le \underset{\sim}{\phi}^0 + \underset{\sim}{E}\,\underset{\sim}{\mu} + \underset{\sim}{t}\}, \qquad (2.117)$$

which is centered at $\underset{\sim}{\phi}^0 + \underset{\sim}{E}\,\underset{\sim}{\mu}$. Figure 2.7 illustrates the constraint, tolerance and the tuning regions.

The design problem in this case is

$$\underline{\text{minimize}}\ C(\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}, \underset{\sim}{t})$$

subject to (2.109), where $\underset{\sim}{\phi}$ is as given in (2.114), and the constraints

$$\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}, \underset{\sim}{t} \ge \underset{\sim}{0} \qquad (2.118)$$

for all $\underset{\sim}{\mu} \in R_\mu$ and some $\underset{\sim}{\rho} \in R_\rho$. C is a function which represents the component cost, for example,

$$\sum_{i=1}^{k} c_i \frac{\phi_i^0}{\varepsilon_i} + \sum_{i=1}^{k} c_i' \frac{t_i}{\phi_i^0}, \qquad (2.119)$$

where the $c_i$ and $c_i'$ are constants. These may be set to zero if the corresponding element is not to be toleranced or tuned, respectively. The worst-case solution of the problem must satisfy

$$R_t(\underset{\sim}{\mu}) \cap R_c \ne \emptyset \qquad (2.120)$$

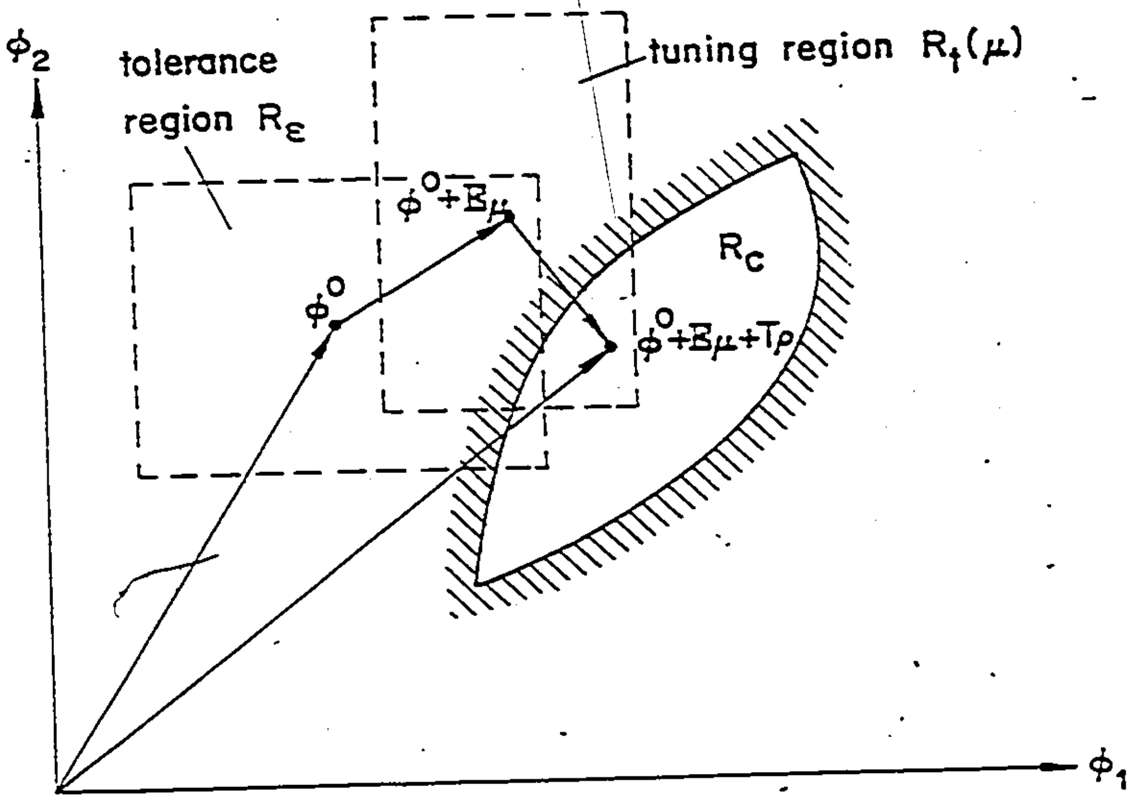

$\phi_2$
tolerance region $R_\varepsilon$
tuning region $R_t(\mu)$
$\phi^0+E\mu$
$R_c$
$\phi^0$
$\phi^0+E\mu+T\rho$
$\phi_1$

Fig. 2.7 An illustration of the constraint, tolerance and tuning regions and a possible outcome $\underset{\sim}{\phi}$. If $\underset{\sim}{t} = \underset{\sim}{0}$ we recover the essential features of Fig. 2.6.

for all $\underline{\mu} \in R_{\mu}$, where $\emptyset$ denotes a null set.

The problem can be reduced by separating the components into effectively tuned and effectively toleranced parameters. Bandler et al. (1976a), Liu (1975) proved that the solution of the reduced problem is the solution of the original one under certain conditions.

### 2.7.6 Uncertainties

The values of $\underline{\phi}$ sufficient to give an acceptable design depend on other uncertainties influencing design performance. In the simulation of actual circuits models or equivalent circuits are used, where uncertainties are associated with the model parameters. In microwave circuit design, for example, parasitic effects exist due to electromagnetic coupling. Models available for common parasitic elements normally include empirical uncertainties on the values of the model parameters. These uncertainties are due to the fact that the model itself is necessarily approximate and that further approximations often have to be made in the implementation of existing model formulas. Non-ideal terminations also alter the performance, i.e., mismatches at the source and the load of the circuit (Bandler, Liu and Tromp 1976c).

In modeling a physical circuit the vector of nominal model parameters $\underline{p}^0$ will have a vector of model uncertainties associated with it, such that the model parameters are described by (Bandler

1977)

$$\underset{\sim}{p} = \underset{\sim}{p}^0(\underset{\sim}{\phi}) + \underset{\sim}{\Delta}(\underset{\sim}{\phi})\ \underset{\sim}{\mu}_\delta, \tag{2.121}$$

where

$$\underset{\sim}{\Delta} \triangleq \begin{bmatrix} \delta_1 & & & \\ & \delta_2 & & \\ & & \ddots & \\ & & & \delta_n \end{bmatrix} \tag{2.122}$$

and for example

$$-\underset{\sim}{1} \leq \underset{\sim}{\mu}_\delta \leq \underset{\sim}{1},$$

where n is the number of model parameters with uncertainties and, in general, n ≠ k. Although the model parameters and the uncertainties are explicit functions of the physical parameters $\underset{\sim}{\phi}$, it is difficult to map the tolerance region from the $\underset{\sim}{\phi}$ space to the $\underset{\sim}{p}$ space in selecting candidates for the worst-case design.

Let $\underset{\sim}{g}(\underset{\sim}{\psi})$ denote a set of nonlinear constraint functions such that

$$\underset{\sim}{g}(\underset{\sim}{\psi}) \geq \underset{\sim}{0} \tag{2.123}$$

represents an acceptable situation for a particular setting of $\underset{\sim}{\psi}$. The nominal performance of the design under ideal environmental effects will be denoted by $\underset{\sim}{g}^0(\underset{\sim}{\psi})$. The measured performance might be described by

$$g_i = g_i^0 (\underset{\sim}{p}, \underset{\sim}{\psi}) + \mu_{g_i} (\underset{\sim}{p}, \underset{\sim}{q}, \underset{\sim}{\psi}), \; i=1,2,\ldots,m(\underset{\sim}{\psi}), \qquad (2.124)$$

where $\mu_{\underset{\sim}{g}}$ is the deviation from the ideal performance and $\underset{\sim}{q}$ is a vector of external parameters, e.g., ones affecting our ability to measure the performance.

### 2.7.7 Design with Yield Less Than 100 Percent

In worst-case design the yield is restricted to 100%. This may render the circuit very expensive due to tight tolerances. The restriction of 100% yield may be relaxed in order to increase the tolerances and reduce the cost of the elements. The overall cost, in this case, although failing circuits are discarded, will have to be lower than the one obtained by worst-case design.

The design problem with a restricted yield can be set up as

$$\underline{\text{minimize}} \; \underset{\sim}{C} (\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon})$$

subject to

$$Y \geq X, \qquad (2.125)$$

where X is the specified percentage. For unrestricted yield the problem might, for example, have the objective function

$$C = \sum_{i=1}^{k} \frac{1}{\varepsilon_i} / Y. \qquad (2.126)$$

In both formulations the yield has to be estimated.

The yield when the parameters are statistically distributed is defined by

$$Y = \int_{R_c} P(\underset{\sim}{\phi})\, d\phi_1\, d\phi_2 \ldots d\phi_k, \qquad (2.127)$$

where $P(\underset{\sim}{\phi})$ is the probability distribution function of the variable parameters. This k-fold integration is not very attractive, especially when the yield estimation is incorporated in an optimization process. Karafin (1974) approximated the yield by computing upper and lower bounds on Y using truncated Taylor series approximations for the constraints. He assumed that each constraint is normally distributed for all choices of component tolerances. The yield estimation problem itself has been treated largely by the Monte Carlo analysis (Elias 1975).

Becker and Jensen (1974) used pattern search for maximizing the yield by finding a set of nominal variables which is optimal for specified tolerances. A feasible solution search precedes the yield optimization.

In the simplicial approximation approach (Director, Hachtel and Vidigal 1978) while finding the center of the constraint region an approximation to this region is also obtained. A crude estimate of the yield can be obtained by performing the Monte Carlo analysis directly in the parameter space. The yield estimation procedure can be improved by testing a sample point (of the Monte Carlo analysis) which lies outside the approximate region to determine whether or not it lies outside the actual region. If it lies inside the actual region it can be used to improve the approximate region.

Bandler and Abdel-Malek (1978a) derived exact formulas for the yield and its sensitivities w.r.t. design parameters. The formulas are based upon multidimensional linear cuts of the tolerance orthotope and uniform distributions of outcomes between tolerance extremes in the orthotope.

This approach has been generalized to estimate the yield when components have arbitrary statistical distributions (Abdel-Malek 1977, Abdel-Malek and Bandler 1978a, 1978b, 1978c).

### 2.7.8 Related Work and Extensions

Tromp (1977, 1978) has generalized the tolerance assignment problem so that physical tolerances, model uncertainties, external disturbing effects and dependently toleranced parameters can be considered in a unified manner. In essence, the approach begins with the definitions of the $k_{0i}$-dimensional vector $\underset{\sim}{\phi}^{0i}$, the $k_i$-dimensional vector $\underset{\sim}{\phi}^{i}$ and the $k_{\mu i}$-dimensional vector $\underset{\sim}{\mu}^{i}$ so that $\underset{\sim}{\phi}^{i}$ is a function of $\underset{\sim}{\phi}^{0i}$ and $\underset{\sim}{\mu}^{i}$ for all $i = 1, 2, \ldots n$, and $\underset{\sim}{\phi}^{0i}$ itself depends on all $\underset{\sim}{\phi}^{i-1}$ for $i = 2, 3, \ldots, n$.

Input parameters, e.g., the physical parameters available to the manufacturer might be identified as $\underset{\sim}{\phi}^{1}$, whereas $\underset{\sim}{\phi}^{n}$ would be the output vector, e.g., the sampled response of a system or the vector of constraints $\underset{\sim}{g}$, which defined $R_c$ of (2.102). The quantities $\underset{\sim}{\phi}^2, \ldots, \underset{\sim}{\phi}^{n-1}$ can be identified, for example, as intermediate or model parameters. The variables $\underset{\sim}{\mu}^i$, i = 1, 2, ..., n, create the unavoidable or undesirable fluctuations and generally embody the unknown or intangible.

The tolerance region in the $\underset{\sim}{\phi}$-space is obviously no longer restricted to be an orthotope in this formulation.

Polak and Sangiovanni-Vincentelli (1978) recently formulated the design centering, tolerancing and tuning problem as a mathematical programming problem in the form

$$\underline{\text{minimize}}\ C(\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}, \underset{\sim}{t})$$

subject to

$$\min_{i \in I_c} \ \min_{\underset{\sim}{\mu} \in R_\mu} \ \max_{\underset{\sim}{\rho} \in R_\rho} \ g_i(\underset{\sim}{\phi}) \geq 0 \tag{2.128}$$

and the constraints (2.118), where $\underset{\sim}{\phi}$ is as given in (2.114). They demonstrated that their formulation is equivalent to the one of Bandler, Liu and Tromp (1976a). They suggested a new algorithm which deals with the nondifferentiable constraints (2.128). The algorithm solves the problem as a sequence of approximating problems with $R_\mu^j \subset R_\mu$ as a discrete set. They showed that, under certain conditions, the accumulation points of the sequence of stationary points of the approximating problems are stationary points of the original problem.

Bandler and Abdel-Malek (1978b) introduced a generalized least pth function of the form of (2.47) to convert a tolerance and tuning problem to an equivalent tolerance problem. An expanded constraint region, namely the tunable constraint region $R_{ct}$, replaces the original region $R_c$. The region is given for $p=\infty$ by

$$R_{ct} \triangleq \{\underset{\sim}{\phi} \mid \max_{\underset{\sim}{\rho}\in R_\rho} \ \min_{i\in I_c} \ g_i(\underset{\sim}{\phi} + \underset{\sim}{T}\underset{\sim}{\rho}) \geq 0\}, \tag{2.129}$$

where $\underset{\sim}{\phi}$ is given by (2.99). They based some definitions of yield upon $R_{ct}$ and described worst-case design and worst-case centering.

Madsen and Schjaer-Jacobsen (1978b, 1978c) proposed the use of interval arithmetic to determine the worst case within the tolerance region. In this case the one-dimensional convexity assumption is not required, and the worst case can lie at an edge of the tolerance region instead of a vertex.

# CHAPTER 3

# TRANSMISSION-LINE MODELING AND SENSITIVITY EVALUATION FOR LUMPED NETWORK SIMULATION AND DESIGN IN THE TIME DOMAIN

## 3.1 Introduction

The transmission-line matrix (TLM) method of numerical analysis provides a new approach to the time-domain analysis of lumped networks. The method has previously been extensively used for solving electromagnetic vector field problems in two and three dimensions (Akhtarzad and Johns 1975). The technique has also been used for solving the diffusion equation (Johns 1975).

In its application to lumped networks (Bandler, Abdel-Malek, Johns and Rizk 1976, Johns 1976), the TLM method has some advantages because it provides an exact solution to the transmission-line networks used to model the actual networks. This chapter demonstrates how the transmission-line models for lumped networks can be obtained and how to compensate for modeling errors in terms of additional network elements.

Unlike the methods mentioned in Section 2.3 the TLM method provides exact sensitivities for the model w.r.t. design variables with some additional effort. No integration schemes are involved. A symmetrical LC lowpass filter has been optimized in the time domain using TLM analysis, the required gradients being obtained from the sensitivities derived.

Sensitivities with respect to the time step are also derived, from which an approximation to the time sensitivities is obtained. Using these formulas and the TLM results, we can extrapolate to the near exact impulse response.

## 3.2 Transmission-line Modeling

The time-domain response of a lumped network can be found using the TLM method, after choosing an appropriate transmission-line model for the network. Inductors and capacitors are represented either by transmission lines or by stubs.

### 3.2.1 Link Modeling

First consider the modeling of a series inductor and a shunt capacitor, each by a transmission line. To simplify the analysis, certain assumptions must be made. We will let all the transmission-line models have the same length, and let the time taken by a pulse to travel along each transmission line be the same, namely, T. The lumped inductor L shown in Fig. 3.1(a) can have the transmission-line model shown in Fig. 3.1(b) with an inductance per unit length $L_d$, where

$$L_d \, \ell = L. \tag{3.1}$$

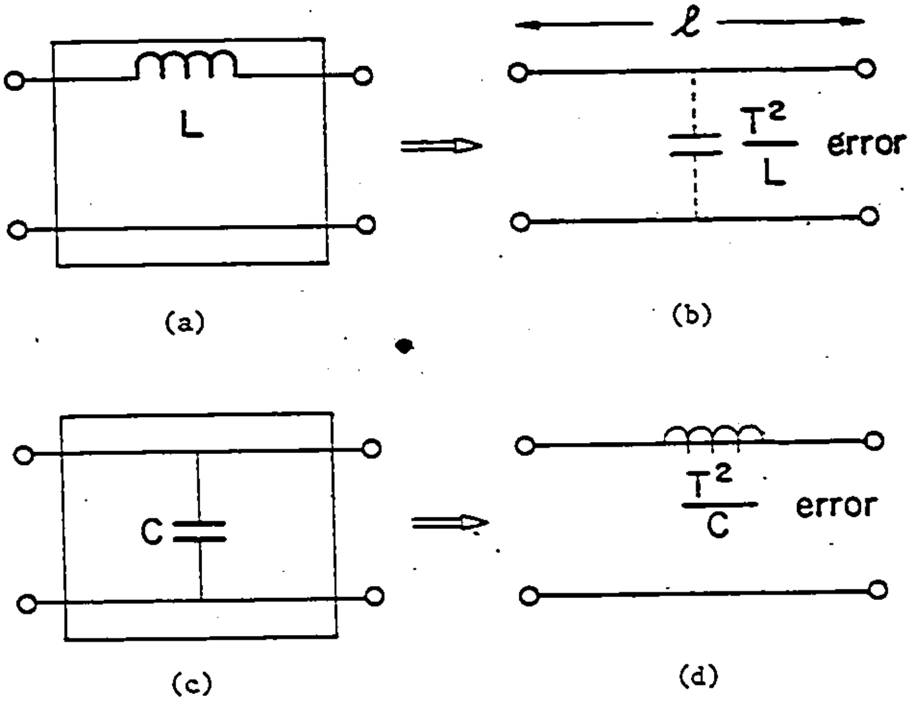

L
$\ell$
$\frac{T^2}{L}$
error
(a)
(b)
C
$\frac{T^2}{C}$
error
(c)
(d)

Fig. 3.1 Lossless transmission-line models of a series inductor and a shunt capacitor.

The velocity of propagation on the transmission line may be expressed as

$$\frac{1}{\sqrt{L_d C_d}} = \frac{\ell}{T} \tag{3.2}$$

and hence the distributed capacitance $C_d$ is given by

$$C_d = \left(\frac{T}{\ell}\right)^2 \frac{1}{L_d} . \tag{3.3}$$

The basic parameter which determines how pulses are scattered throughout a transmission-line network is the characteristic impedance $Z_0$, which for the model of inductor, is obtained from (3.1) and (3.3). Thus,

$$Z_0 = \sqrt{\frac{L_d}{C_d}} = \frac{L}{T} . \tag{3.4}$$

The error associated with the model of the inductor is due to the distributed capacitance given in (3.3). This may be approximated in the lumped circuit by a lumped shunt capacitor $C_e$ representing the error, which is given by

$$C_e = C_d\, \ell = \left(\frac{T}{\ell}\right)^2 \frac{\ell}{L_d} = \frac{T^2}{L} . \tag{3.5}$$

This lumped capacitor is shown dotted in Fig. 3.1(b).

The characteristic impedance for a transmission line modeling a lumped capacitor (Fig. 3.1(c)) may be derived in the same way, the result being

$$Z_0 = \frac{T}{C}, \tag{3.6}$$

and the error this time will be represented by a series lumped inductor $L_e$ (Fig. 3.1(d)) of value

$$L_e = \frac{T^2}{C}. \tag{3.7}$$

It is clear that if T is small then for the model of the inductor $Z_0$ and $L_d$ are large while the unwanted shunt distributed capacitance $C_d$ is small. On the other hand, for the model of the capacitor $Z_0$ and the unwanted $L_d$ will be small if T is small. So, as T becomes smaller, the transmission-line model represents more closely the lumped element.

Consider the lumped network shown in Fig. 3.2(a). It is composed of M simple resistive networks with scattering matrices $\underset{\sim}{S}_1, \underset{\sim}{S}_2, \ldots, \underset{\sim}{S}_M$, connected either by a simple pair of wires or a pair of wires containing a series inductor or a shunt capacitor or both. In the transmission-line model these connections are

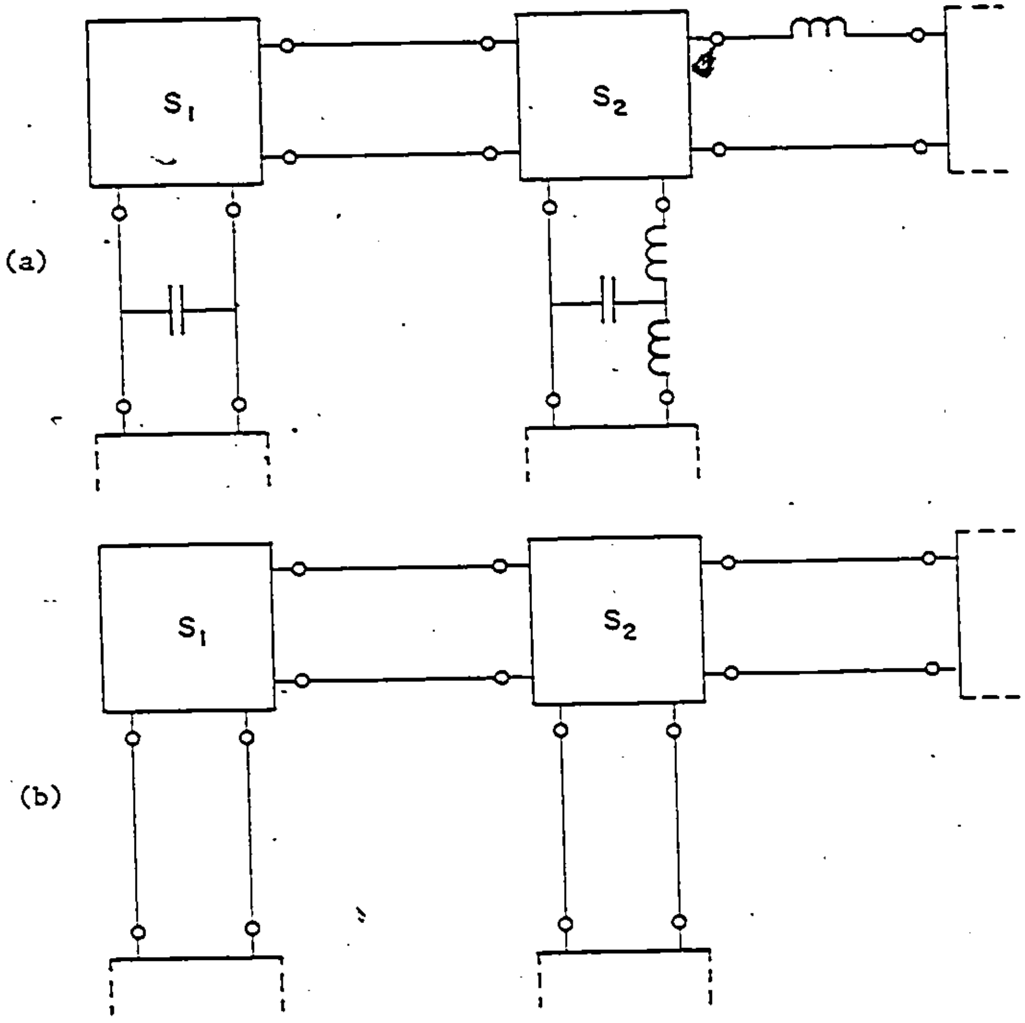

S1
S2
(a)
S1
S2
(b)

Fig. 3.2 Lumped network and link transmission-line model.

replaced by transmission-line sections of propagation time T as shown in Fig. 3.2(b). In this case the model is called a link transmission-line model (Johns 1976).

The numerical method operates by considering a pulse to be injected into the input terminals of the whole network. The pulse scatters on reaching the first subnetwork being partly reflected and partly transmitted. This scattering occurs at every subnetwork, pulses racing to and fro between subnetworks. The output impulse function is the stream of pulses at the output terminals.

If the mth network has N ports with incident and reflected voltages given by (Johns 1976)

$$\underset{\sim}{V}^i_m = \begin{bmatrix} V^i_{m1} \\ V^i_{m2} \\ \vdots \\ V^i_{mN} \end{bmatrix}, \qquad \underset{\sim}{V}^r_m = \begin{bmatrix} V^r_{m1} \\ V^r_{m2} \\ \vdots \\ V^r_{mN} \end{bmatrix} \tag{3.8}$$

then the scattering equation is

$${}_k\underset{\sim}{V}^r_m = \underset{\sim}{S}_m \, {}_k\underset{\sim}{V}^i_m, \tag{3.9}$$

where the subscript k denotes the kth time step. If all the incident and reflected pulses are assembled into the partitioned vectors

$$\underset{\sim}{V}^i = \begin{bmatrix} \underset{\sim}{V}^i_1 \\ \underset{\sim}{V}^i_2 \\ \vdots \\ \underset{\sim}{V}^i_M \end{bmatrix}, \qquad \underset{\sim}{V}^r = \begin{bmatrix} \underset{\sim}{V}^r_1 \\ \underset{\sim}{V}^r_2 \\ \vdots \\ \underset{\sim}{V}^r_M \end{bmatrix} \tag{3.10}$$

then the scattering equation for the entire network is

$$_k\underset{\sim}{V}^r = \underset{\sim}{S}\ _k\underset{\sim}{V}^i, \tag{3.11}$$

where $\underset{\sim}{S}$ in this case is a block diagonal partitioned matrix with $\underset{\sim}{S}_1, \underset{\sim}{S}_2, \ldots, \underset{\sim}{S}_M$ on the diagonal.

The reflected pulses are the incident pulses at the next time step and they are related by

$$_{k+1}\underset{\sim}{V}^i = \underset{\sim}{C}\ _k\underset{\sim}{V}^r, \tag{3.12}$$

where $\underset{\sim}{C}$ is the connection matrix indicating the transmission of reflected pulses from one subnetwork to become incident pulses on a neighbouring subnetwork. The iteration equation is

$$_{k+1}\underset{\sim}{V}^i = \underset{\sim}{C}\ \underset{\sim}{S}\ _k\underset{\sim}{V}^i. \tag{3.13}$$

The method will be unconditionally stable for a passive RLC lumped network and, therefore, it will be useful for stiff networks (Johns 1976).

### 3.2.2 Stub Modeling

A lumped network consisting of resistive, inductive and capacitive elements may also be modeled by stub transmission-lines. In this case, the time taken by a pulse to travel to the end of the stub and back again is T. Following the same procedure used in the link transmission-line models, an inductor is modeled by a short-circuit stub with characteristic impedance

$$Z_0 = \frac{2L}{T} . \tag{3.14}$$

and the modeling error is a capacitor given by

$$C_e = \frac{T^2}{4L} . \tag{3.15}$$

A capacitor is modeled by an open-circuit stub with an impedance

$$Z_0 = \frac{T}{2C} , \tag{3.16}$$

and the modeling error is an inductor given by

$$L_e = \frac{T^2}{4C} . \tag{3.17}$$

The elements with these models are shown in Fig. 3.3.

Consider the lumped network in Fig. 3.4(a), which is represented by a resistive network with N pairs of terminals to which all of the inductors and capacitors are connected as shown. A transmission-line model for the circuit is shown in Fig. 3.4(b) in which all of the inductors are replaced by short-circuit stubs and all the capacitors are replaced by open-circuit stubs. The reflected pulses

$$\underset{\sim}{V}^r = \begin{bmatrix} V_1^r \\ V_2^r \\ \vdots \\ V_N^r \end{bmatrix} , \tag{3.18}$$

will be scattered instantaneously into the N stubs. These pulses will travel to the ends of the stubs and be reflected or reflected and inverted for capacitive or inductive stubs, respectively. The pulses then return to the resistive network and become incident pulses

$$\underset{\sim}{V}^i = \begin{bmatrix} V_1^i \\ V_2^i \\ \vdots \\ V_N^i \end{bmatrix} . \tag{3.19}$$

If the scattering matrix of the resistive network is the N x N matrix $\underset{\sim}{S}$ then, at the kth iteration,

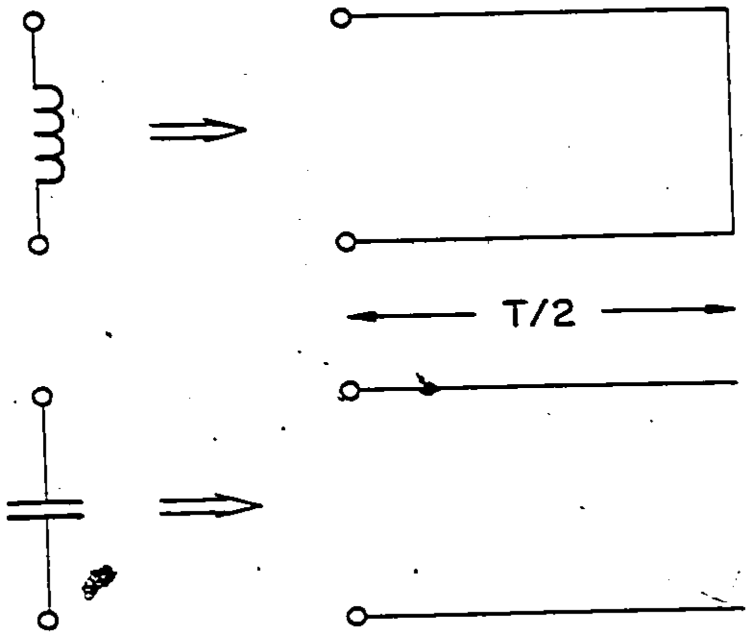
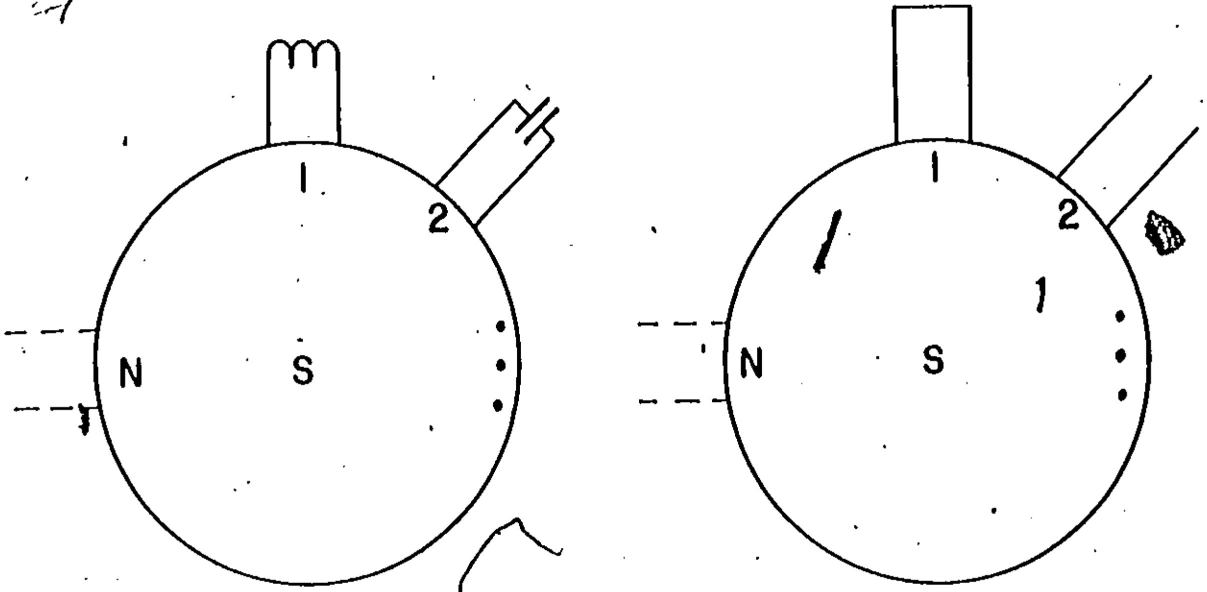


Fig. 3.3 Stub models of an inductor and a capacitor.



(a)

(b)

Fig. 3.4 Lumped network and transmission-line stub model.

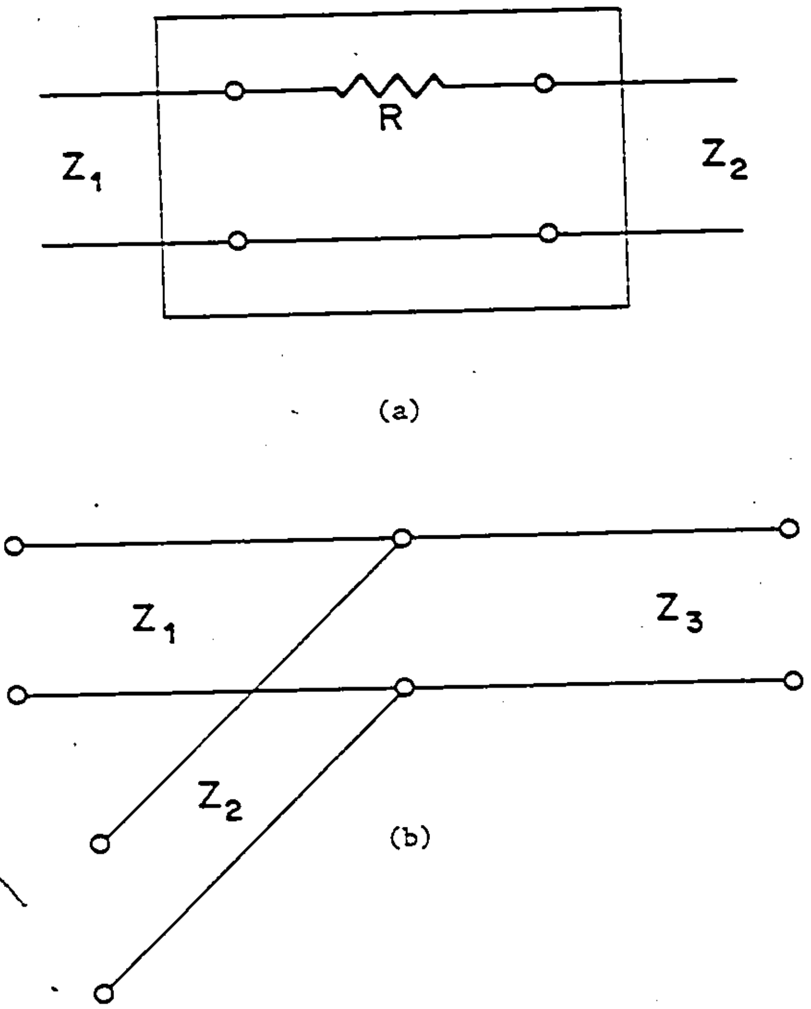
$$_k\underset{\sim}{V}^r = \underset{\sim}{S}\, _k\underset{\sim}{V}^i. \tag{3.20}$$

Reflection of the pulses at the end of the stubs gives the incident pulses at time k+1, obtained using the same formula as (3.12), where $\underset{\sim}{C}$, in this case, is an N x N diagonal matrix with an entry of 1 for a capacitive stub and -1 for an inductive stub. The iteration routine is therefore exactly as (3.13).

To enable the incident pulses $\underset{\sim}{V}^i$ to converge simultaneously it is sufficient that the propagation time T be the same for all the stubs. This propagation time is therefore the same as the iteration time. This method is also unconditionally stable for a lumped network of positive resistors, inductors and capacitors.

## 3.3 Discussion

It should be noted that the stub modeling leads to an implicit routine. The reason is that the scattering matrix $\underset{\sim}{S}$ involves the entire resistive network. Thus, to calculate $\underset{\sim}{S}$ it is necessary to invert a set of simultaneous equations describing the network. If the network is nonlinear, then this inversion is required before every iteration. In link transmission-line modeling, however, the iteration routine is explicit, the complexity of equations being independent of the number of subnetworks or nodes. The scattering matrices of the networks are small enough to be calculated by simple formulas, for example, the scattering matrix of the subnetwork in Fig. 3.5(a) is given by


R
$Z_1$
$Z_2$
$Z_1$
$Z_3$
$Z_2$

(a)

(b)

Fig. 3.5 Example of two simple subnetworks.

$$\frac{1}{R+Z_1+Z_2}\begin{bmatrix} \dfrac{R(R+Z_1+Z_2)+2Z_1(Z_2-Z_1)}{R+2Z_1} & 2Z_1 \\ 2Z_2 & \dfrac{R(R+Z_1+Z_2)+2Z_2(Z_1-Z_2)}{R+2Z_2} \end{bmatrix} \quad (3.21)$$

and the scattering matrix of the subnetwork in Fig. 3.5(b) is

$$\frac{1}{Z_1Z_2+Z_1Z_3+Z_2Z_3}\begin{bmatrix} Z_2Z_3-Z_1Z_2-Z_3Z_1 & 2Z_1Z_3 & 2Z_1Z_2 \\ 2Z_2Z_3 & Z_1Z_3-Z_1Z_2-Z_2Z_3 & 2Z_1Z_2 \\ 2Z_2Z_3 & 2Z_1Z_3 & Z_1Z_2-Z_1Z_3-Z_3Z_2 \end{bmatrix} \quad (3.22)$$

In general, a network may be modeled by either one or both types of model. The LC lowpass filter in Fig. 3.6(a) can have the link model of Fig. 3.6(b) or the stub model of Fig. 3.6(c) or the mixed model of Fig. 3.6(d).

## 3.4 Example

The following example illustrates the TLM routine for link modeling. Consider the circuit of Fig. 3.6(a) (Bandler, Abdel-Malek, Johns and Rizk 1976) and its link transmission-line model in Fig. 3.6(b). Let the time step T be 0.1 second and the component values $L_2 = L_4 = 1$, $C_3 = 2$ and $R_1 = R_5 = 1$. The characteristic impedances $Z_2$, $Z_3$ and $Z_4$ are 10, 0.05 and 10, respectively. An incident pulse of value 0.5 is launched into the

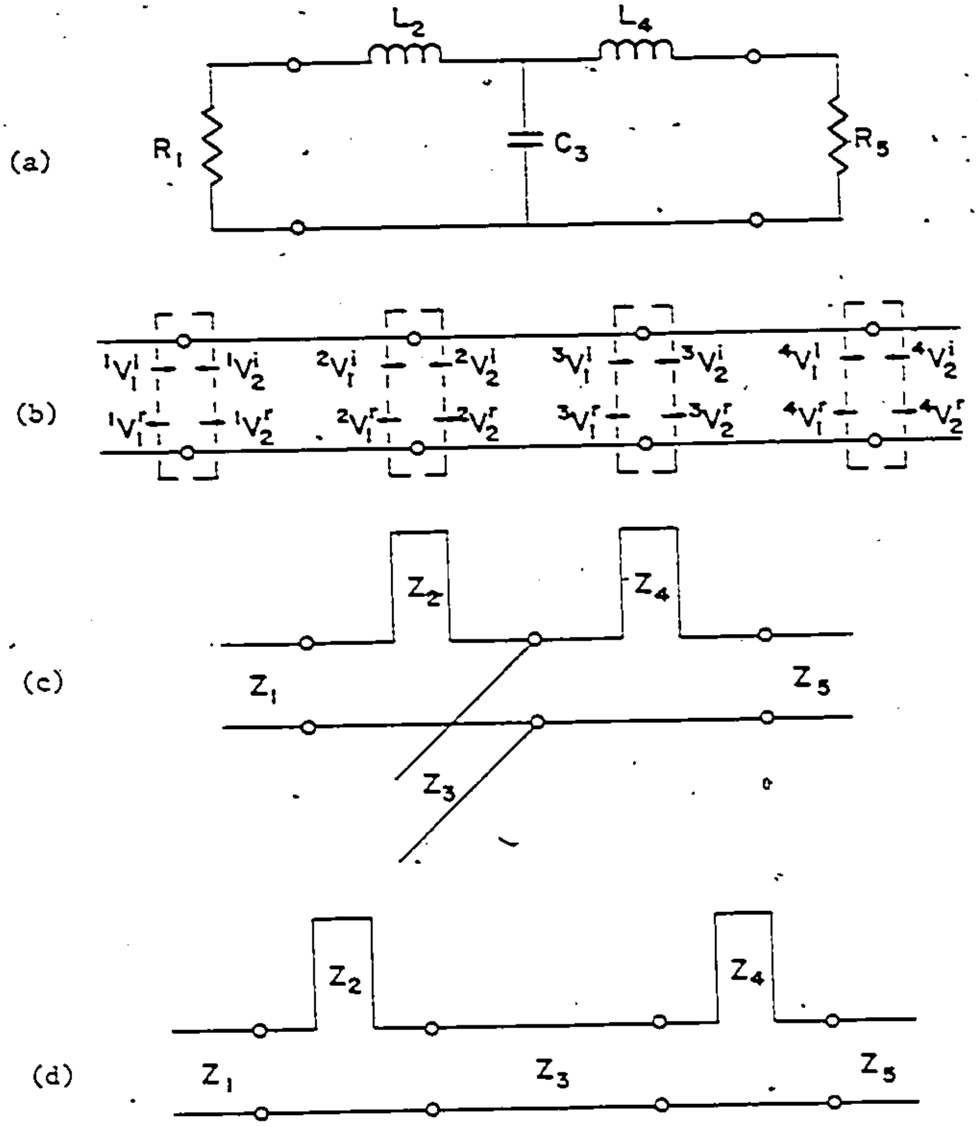

(a)
$L_2$
$L_4$
$R_1$
$C_3$
$R_5$
(b)
$^1V_1^i$
$^1V_2^i$
$^1V_1^r$
$^1V_2^r$
$^2V_1^i$
$^2V_2^i$
$^2V_1^r$
$^2V_2^r$
$^3V_1^i$
$^3V_2^i$
$^3V_1^r$
$^3V_2^r$
$^4V_1^i$
$^4V_2^i$
$^4V_1^r$
$^4V_2^r$
(c)
$Z_2$
$Z_4$
$Z_1$
$Z_5$
$Z_3$
(d)
$Z_2$
$Z_4$
$Z_1$
$Z_3$
$Z_5$

Fig. 3.6 LC lowpass filter and different types of models (a) the filter, (b) link model, (c) stub model, (d) mixed model.

transmission line representing the source resistance and hits the first junction at time t = 0. The pulse scatters producing reflected and transmitted pulses. The transmitted pulses travel towards the output, being scattered at the other junctions. The pulses propagate forward and backward between the junctions. Table 3.1 gives the incident and reflected pulses at the junctions of Fig. 3.6(b) at different times.

### 3.5 Compensation of Errors

Errors in the TLM method arise only from how well the transmission-line model represents the actual circuit. Errors do not arise from the numerical solution of the model. In certain cases the unwanted distributed elements are reduced when the step size T is reduced. A distributed capacitor in modeling an inductor is an example of such a case. We have to note that this capacitor is known before any calculation is started, since T has to be chosen. If the distributed error capacitor is taken to be two lumped capacitors placed at each end of the transmission line, each of these lumped capacitors will have a value of $C_d \ell/2$. The inductor and the two capacitors representing modeling errors are shown in Fig. 3.7. To compensate for modeling error (to some degree) we can subtract the error capacitor from the original neighbouring network components. As T increases the amount to be subtracted increases and it becomes obvious that there may be a limit to such compensation.

### TABLE 3.1

### INCIDENT AND REFLECTED PULSES OF THE CIRCUIT IN FIG. 3.6(b)

| t (s) | $^1V^i_1$ | $^1V^i_2$ | $^1V^r_2$ | $^2V^i_1$ | $^2V^r_1$ | $^2V^i_2$ | $^2V^r_2$ |
|---|---|---|---|---|---|---|---|
| 0.0 | 0.5 | - | 0.90909 | - | - | - | - |
| 0.1 | - | - | - | 0.90909 | -0.90004 | - | 0.00904 |
| 0.2 | - | -0.90004 | 0.73639 | - | - | - | - |
| 0.3 | - | - | - | 0.73639 | -0.71125 | 0.00895 | 0.01619 |
| 0.4 | - | -0.71125 | 0.58193 | - | - | - | - |
| 0.5 | - | - | - | 0.58193 | -0.54452 | 0.01589 | 0.02151 |

| t (s) | $^3V^i_1$ | $^3V^r_1$ | $^3V^i_2$ | $^3V^r_2$ | $^4V^i_1$ | $^4V^r_1$ | $^4V^r_2$ |
|---|---|---|---|---|---|---|---|
| 0.0 | - | - | - | - | - | - | - |
| 0.1 | - | - | - | - | - | - | - |
| 0.2 | 0.00904 | 0.00895 | - | 0.01800 | - | - | - |
| 0.3 | - | - | - | - | 0.01800 | -0.01472 | 0.00327 |
| 0.4 | -0.01619 | 0.01589 | -0.01472 | 0.04680 | - | - | - |
| 0.5 | - | - | - | - | 0.04680 | -0.03829 | 0.00851 |

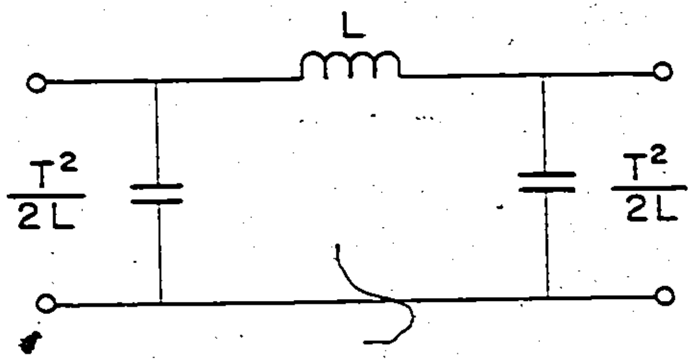


Fig. 3.7 An inductor with two capacitors representing modeling error.

The impulse response of the Chebyshev filter shown in Fig. 3.8 (Matthaei, Young and Jones 1964) was found by Kutta-Simpson, Euler, TLM and TLM with compensation. The results are shown in Table 3.2. The advantage of compensation is clear from the table comparing the percentage error between the Kutta-Simpson integration method and other methods. The actual components and the new components after compensation are given in Table 3.3.

## 3.6 Sensitivity Evaluation

One of the features of the TLM method is that simple calculation of exact sensitivities w.r.t. design variables are possible. Sensitivities are calculated iteratively in the same iteration process for calculating the impulse response.

## TABLE 3.2

## COMPARISON BETWEEN DIFFERENT METHODS OF INTEGRATION AND TLM MODELING WITH AND WITHOUT COMPENSATION

| t (s) | Kutta Simpson | Percentage Error for T = 0.1 | | |
|---|---|---|---|---|
| | | Euler | Link Modeling | Link Modeling With Compensation |
| 1.1 | 0.003981 | - 38.5 | - 8.3 | - 6.9 |
| 2.1 | 0.035665 | - 12.8 | - 2.3 | - 1.4 |
| 3.1 | 0.101499 | 0.0 | - 0.9 | 0.1 |
| 4.1 | 0.160644 | 8.2 | - 0.3 | 0.3 |
| 5.1 | 0.161516 | 13.1 | 0.3 | 0.2 |
| 6.1 | 0.094384 | 11.1 | 1.3 | - 0.3 |
| 7.1 | 0.002772 | 520.3 | 48.9 | -21.8 |
| 8.1 | -0.054462 | 57.9 | - 1.2 | 0.5 |
| 9.1 | -0.051167 | 45.6 | 0.7 | - 0.7 |

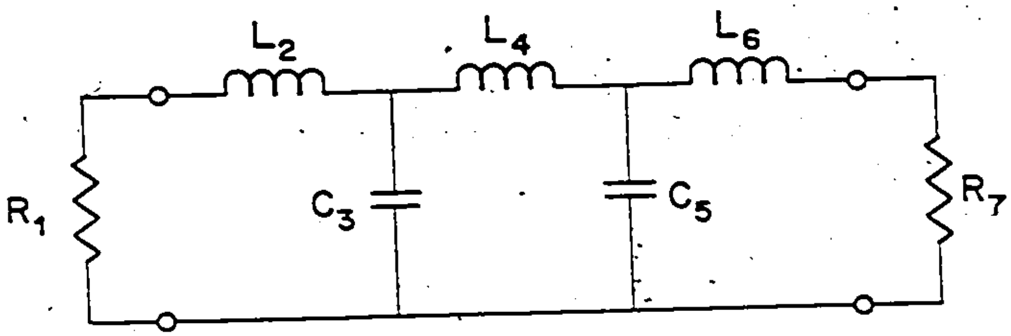


Fig. 3.8 Chebyshev filter with 7 elements.

TABLE 3.3

COMPONENT VALUES OF THE FILTER SHOWN IN

FIG. 3.8 BEFORE AND AFTER COMPENSATION

| Component Values in Ohms, Henries and Farads | | | | | | | |
|---|---|---|---|---|---|---|---|
| State | $R_1$ | $L_2$ | $C_3$ | $L_4$ | $C_5$ | $L_6$ | $R_7$ |
| No Compensation | 1.0 | 1.7058 | 1.2296 | 2.5408 | 1.2296 | 1.7058 | 1.0 |
| With Compensation | 1.0 | 1.7017 | 1.2247 | 2.5327 | 1.2247 | 1.7017 | 1.0 |

### 3.6.1 First-order Sensitivities

Equations (3.11) or (3.20) describe the relationship between appropriate incident and reflected voltages for the whole network and the derivatives w.r.t. the $\kappa$ parameters of the whole network can be written as

$$\begin{bmatrix} {}_{k}\underset{\sim}{V}^{r} \\ \dfrac{\partial\, {}_{k}\underset{\sim}{V}^{r}}{\partial \phi_1} \\ \vdots \\ \dfrac{\partial\, {}_{k}\underset{\sim}{V}^{r}}{\partial \phi_\kappa} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{S} & \underset{\sim}{0} & \cdots & \underset{\sim}{0} \\ \dfrac{\partial \underset{\sim}{S}}{\partial \phi_1} & \underset{\sim}{S} & & \underset{\sim}{0} \\ \vdots & \underset{\sim}{0} & \ddots & \underset{\sim}{0} \\ \dfrac{\partial \underset{\sim}{S}}{\partial \phi_\kappa} & \underset{\sim}{0} & & \underset{\sim}{S} \end{bmatrix} \begin{bmatrix} {}_{k}\underset{\sim}{V}^{i} \\ \dfrac{\partial\, {}_{k}\underset{\sim}{V}^{i}}{\partial \phi_1} \\ \vdots \\ \dfrac{\partial\, {}_{k}\underset{\sim}{V}^{i}}{\partial \phi_\kappa} \end{bmatrix} . \qquad (3.23)$$

The r.h.s. vector is obtained from an equation of the form of (3.12) after differentiating it w.r.t. the jth parameter, viz.,

$$\frac{\partial\, {}_{k}\underset{\sim}{V}^{i}}{\partial \phi_j} = \underset{\sim}{C}\, \frac{\partial}{\partial \phi_j}\, {}_{k-1}\underset{\sim}{V}^{r} , \qquad (3.24)$$

where $\underset{\sim}{C}$ is constant.

It is clear that the matrix in (3.23) is very sparse since, for example, $\partial \underset{\sim}{S}_m / \partial \phi_j$ vanishes if $\underset{\sim}{S}_m$ does not contain the jth parameter. Although this matrix is sparse, the two vectors on the left and right hand sides are full and all the information has to be transferred in each iteration. So in calculating the sensitivities, we have to find the sensitivity of all the incident pulses w.r.t. all the parameters. The sensitivity of the impulse response will be the sensitivity of the stream of pulses at the output port w.r.t. the parameters.

Consider a subnetwork which simply connects two transmission lines having $Z_1$ and $Z_2$ as their characteristic impedances. The scattering matrix $\underset{\sim}{S}_m$ is given by

$$\underset{\sim}{S}_m = \frac{1}{Z_1+Z_2}\begin{bmatrix} Z_2-Z_1 & 2Z_1 \\ 2Z_2 & Z_1-Z_2 \end{bmatrix}. \tag{3.25}$$

Let

$$Z_1 = \phi_j, \quad Z_2 = \phi_{j+1}. \tag{3.26}$$

Then

$$\frac{\partial \underset{\sim}{S}_m}{\partial \phi_j} = \frac{2Z_2}{(Z_1+Z_2)^2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix},$$

$$\frac{\partial \underset{\sim}{S}_m}{\partial \phi_{j+1}} = \frac{2Z_1}{(Z_1+Z_2)^2}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}. \tag{3.27}$$

The expressions (3.25)-(3.27) can be fitted into the scheme of (3.23) for this subnetwork.

### 3.6.2 Second-order Sensitivities

Differentiating (3.11) or (3.20) w.r.t. the jth parameter, we get

$$\frac{\partial\, {}_k\underset{\sim}{V}^r}{\partial \phi_j} = \frac{\partial \underset{\sim}{S}}{\partial \phi_j}\, {}_k\underset{\sim}{V}^i + \underset{\sim}{S}\, \frac{\partial\, {}_k\underset{\sim}{V}^i}{\partial \phi_j}\,, \tag{3.28}$$

where $\partial\, {}_k\underset{\sim}{V}^i / \partial \phi_j$ is found from (3.24).

If we differentiate (3.24) and (3.28) w.r.t. $\phi_\ell$ we get, respectively,

$$\frac{\partial^2\, {}_k\underset{\sim}{V}^i}{\partial \phi_\ell \partial \phi_j} = \underset{\sim}{C}\, \frac{\partial^2\, {}_{k-1}\underset{\sim}{V}^i}{\partial \phi_\ell \partial \phi_j}\,, \tag{3.29}$$

$$\frac{\partial^2_k\, \underset{\sim}{V}^r}{\partial \phi_\ell \partial \phi_j} = \frac{\partial^2 \underset{\sim}{S}}{\partial \phi_\ell \partial \phi_j}\, {}_k\underset{\sim}{V}^i + \frac{\partial \underset{\sim}{S}}{\partial \phi_j}\, \frac{\partial\, {}_k\underset{\sim}{V}^i}{\partial \phi_\ell} + \frac{\partial \underset{\sim}{S}}{\partial \phi_\ell}\, \frac{\partial\, {}_k\underset{\sim}{V}^i}{\partial \phi_j} + \underset{\sim}{S}\, \frac{\partial^2\, {}_k\underset{\sim}{V}^i}{\partial \phi_\ell \partial \phi_j}\,. \tag{3.30}$$

Equation (3.30) holds for subnetworks when subscript m is applied to both sides but some of the derivatives of $\underset{\sim}{S}_m$ are zero.

### 3.7 Examples

The symmetrical LC lowpass filter shown in Fig. 3.6(a) has been optimized in the time domain. Fig. 3.9 shows a specified impulse response for $L_2 = L_4 = 1.0$, $C_3 = 2.0$. Taking 100 sample

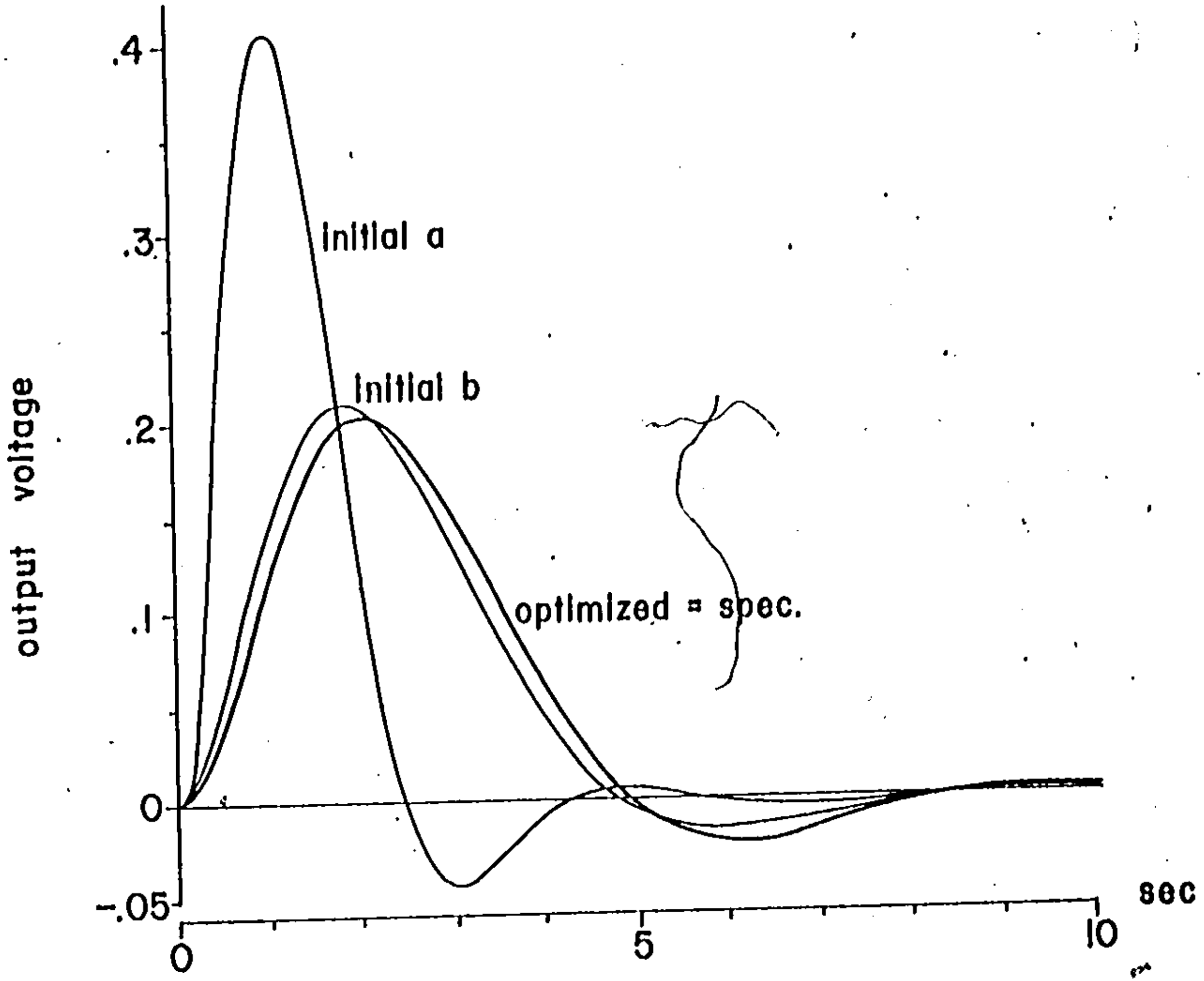


Fig. 3.9 Optimization using TIM analysis. Starting point a: $L_2 = L_4 = 0.5$, $C_3 = 1.0$. Starting point b: $L_2 = L_4 = 0.8$, $C_3 = 2.2$.

points, using TLM analysis, least 4th approximation yielded the solution in 21 s (24 function evaluations) and 17 s (19 function evaluations) from starting points a and b, respectively, with a maximum error of about $3\times10^{-7}$. The specifications of Fig. 3.10 were met with a minimax error of .00219992 after 37 s (46 function evaluations) using 33 sample points for optimization. The starting point was $L_2 = L_4 = C_3 = 1.0$ and the optimum point reached was $L_2 = L_4 = .76645547$ and $C_3 = 2.3739403$. The minimax solution was reached using third-order extrapolation, after a sequence of least pth optimizations where the values of p were 4, 16, 64, 256 and 1024. FLOPT2, a program described in Bandler and Chu (1976), was used in these examples. The computer was a CDC 6400.

## 3.8 Sensitivities w.r.t. Time and T

Differentiating (3.11) or (3.20) w.r.t. T we get

$$\frac{\partial\, {}_k\underset{\sim}{V}^r}{\partial T} = \frac{\partial \underset{\sim}{S}}{\partial T}\, {}_k\underset{\sim}{V}^i + \underset{\sim}{S}\, \frac{\partial\, {}_k\underset{\sim}{V}^i}{\partial T} . \tag{3.31}$$

Usually the scattering matrix $\underline{S}$ includes the parameters $\underline{\phi}$ which are functions of T as obtained from the modeling.

The term $\partial \underline{S}/\partial T$ can be obtained from

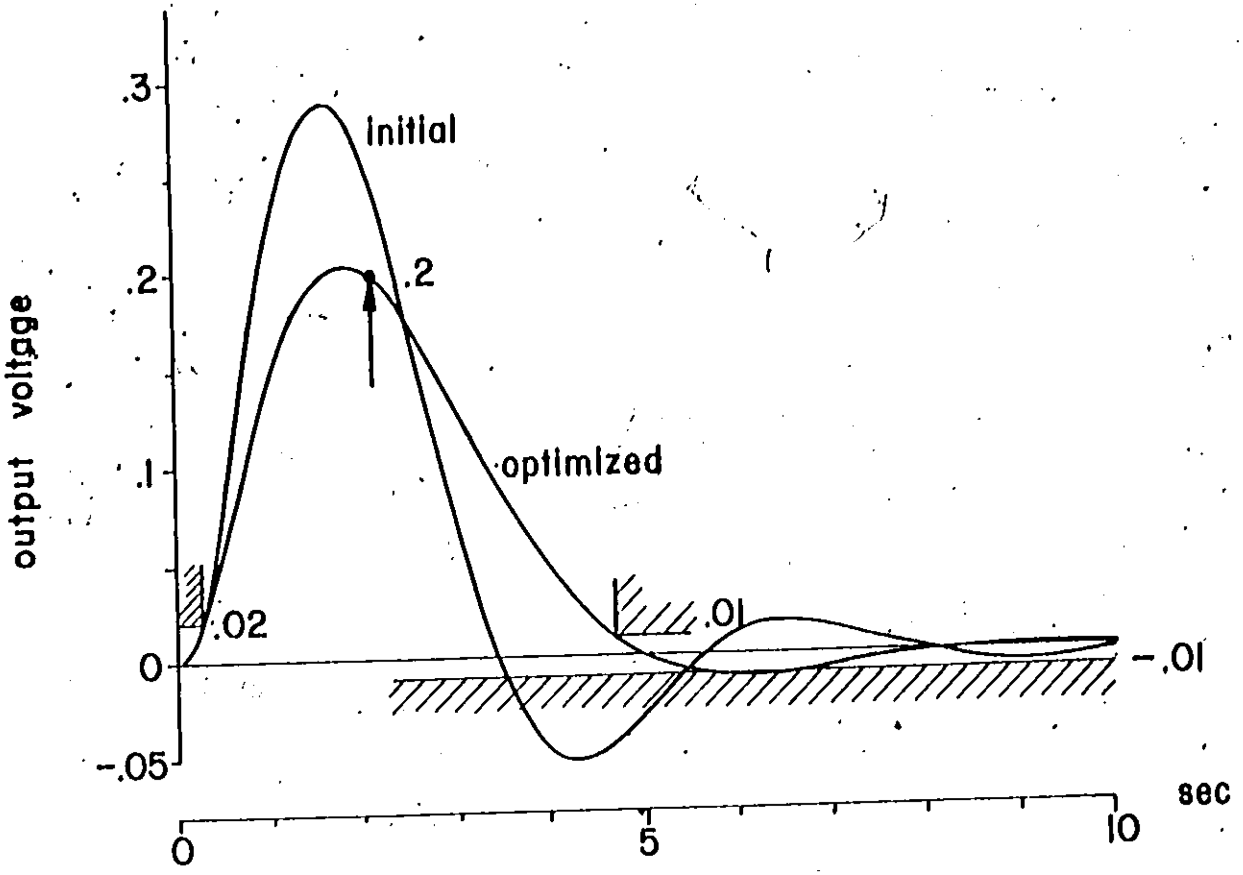


Fig. 3.10 Optimization using TLM analysis. Starting point $L_2 = L_4 = C_3 = 1.0$. Solution $L_2 = L_4 = 0.76646$, $C_3 = 2.3739$.

$$\frac{\partial \underset{\rightarrow}{S}}{\partial T} = \sum_j \frac{\partial \underset{\rightarrow}{S}}{\partial \phi_j} \frac{\partial \phi_j}{\partial T} . \tag{3.32}$$

Suppose $\phi_j$ is the characteristic impedance of a line modeling an inductor. If $\phi_j = L/T$ then $\partial\phi_j/\partial T = -\phi_j/T$. For the capacitive case, $\partial\phi_j/\partial T = \phi_j/T$. The second term on the r.h.s. of (3.31) is obtained from (3.12), where

$$\frac{\partial\, {}_k\underset{\sim}{V}^i}{\partial T} = \underset{\sim}{C} \frac{\partial}{\partial T}\, {}_{k-1}\underset{\sim}{V}^r . \tag{3.33}$$

Note that the differentiation is at discrete time steps and the information is transferred iteratively with the original iteration scheme of the TLM method. Thus the above derivatives can only be obtained at points corresponding to fixed numbers of iterations k, i.e., at $t = kT$, where t is time. Let $f(t,T)$ be an interpolation to the approximation of the impulse response obtained at discrete times $t_1$, $t_2$, $t_3$, ... by the TLM method, where

$$t_j - t_{j-1} = n\,T, \tag{3.34}$$

where n is an integer. The parameter T is chosen arbitrarily, although it is known that the smaller the T the more accurate is the modeling.

Suppose that the analysis is done twice with two different time steps $T_1$ and $T_2$, respectively. In the first analysis we will get $f(t,T_1)$ at points, in general, time $nT_1$ apart, and in the second analysis $f(t,T_2)$ at points $nT_2$ apart. Fig. 3.11 illustrates the situation.

A first-order change in $f(t,T)$ is given by

$$\delta f = \frac{\partial f}{\partial t}\,\Delta t + \frac{\partial f}{\partial T}\,\Delta T, \tag{3.35}$$

where $\Delta t$ and $\Delta T$ are changes in t and T, respectively. Thus,

$$\frac{\delta f}{\Delta T} = \frac{\partial f}{\partial t}\frac{\Delta t}{\Delta T} + \frac{\partial f}{\partial T}\,. \tag{3.36}$$

From the relation $t = kT$ we have

$$\Delta t = k\,\Delta T \tag{3.37}$$

therefore, for a particular k,

$$\left.\frac{\delta f}{\Delta T}\right|_{\Delta T\to 0} = \left.\frac{\partial f}{\partial T}\right|_{k} = k\,\frac{\partial f}{\partial t} + \frac{\partial f}{\partial T}\,. \tag{3.38}$$

The term $\left.\frac{\partial f}{\partial T}\right|_k$ is obtained from (3.31).

Table 3.4 shows, for the circuit of Fig. 3.6(a), where

$$f_j(T) = {}^{4}_{k}V^{r}_{2}/2T,$$

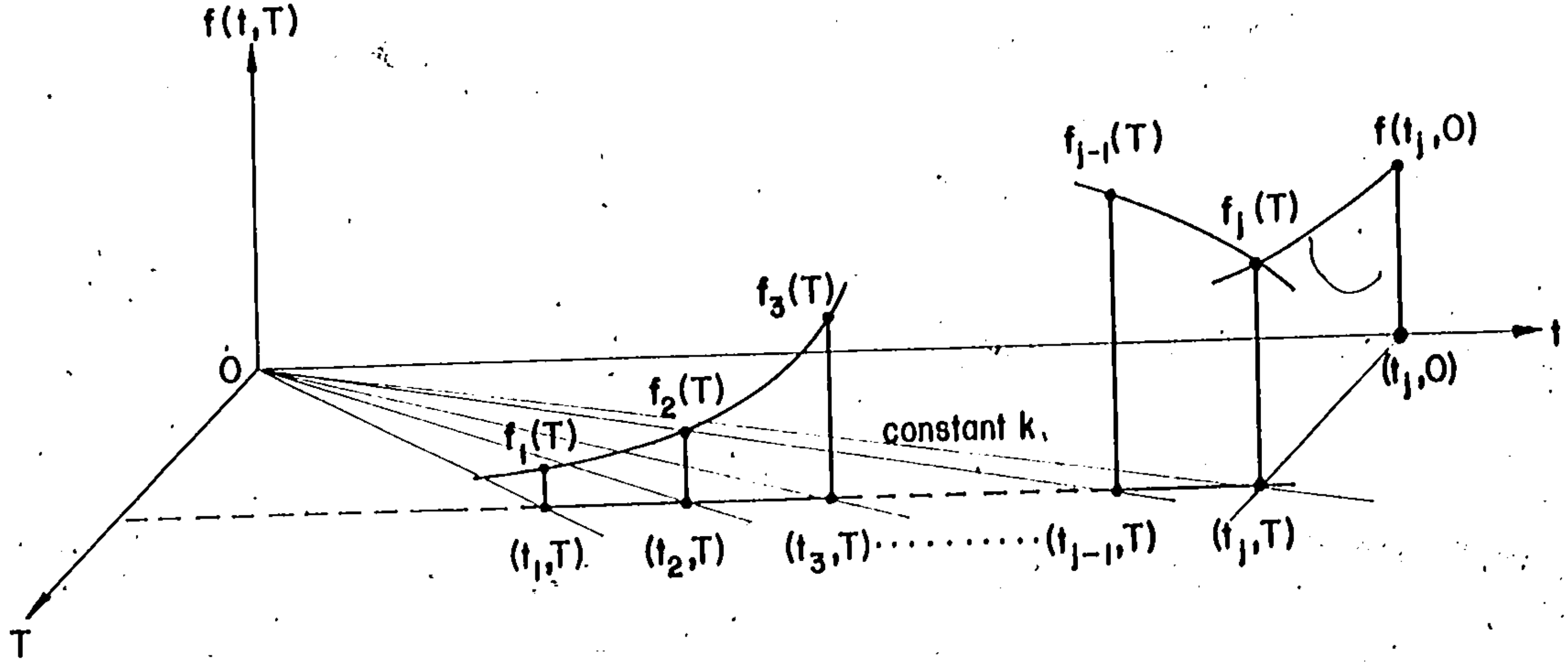

f(t,T)
O
T
t
$f_1(T)$
$f_2(T)$
$f_3(T)$
$f_{j-1}(T)$
$f_j(T)$
$f(t_j,0)$
$(t_j,0)$
constant k.
$(t_1,T)$
$(t_2,T)$
$(t_3,T)$..........$(t_{j-1},T)$
$(t_j,T)$

Fig. 3.11 Representation of response with respect to t and T.

## TABLE 3.4

$\frac{1}{k}\frac{\partial f}{\partial T}\Big|_k$ OBTAINED FROM THE TLM ROUTINE VERSUS $\frac{\partial f}{\partial t}$ BY CENTRAL DIFFERENCES WHERE $T' = 0.1$

| t (s) | $\frac{1}{k}\frac{\partial f}{\partial T}\Big\|_k$ | $\frac{\partial f}{\partial t}$ (central differences) | Difference (%) |
|---|---|---|---|
| 0.5 | 0.13999 | 0.14638 | 4.56 |
| 1.1 | 0.13649 | 0.13771 | 0.89 |
| 1.7 | 0.05300 | 0.05315 | 0.28 |
| 2.3 | -0.03086 | -0.03101 | 0.48 |
| 2.9 | -0.08180 | -0.08206 | 1.54 |
| 3.5 | -0.09538 | -0.09569 | 0.32 |
| 4.1 | -0.08170 | -0.08202 | 0.39 |
| 4.7 | -0.05493 | -0.05521 | 0.51 |
| 5.3 | -0.02705 | -0.02727 | 0.81 |
| 5.9 | -0.00535 | -0.00549 | 2.62 |

$\frac{1}{k}\frac{\partial f}{\partial T}\Big|_k$ versus $\partial f/\partial t$ calculated by the central difference formula given in Appendix A. There is a difference between the numbers in the two columns which we can attribute to $\frac{1}{k}\frac{\partial f}{\partial T}$ and the inexactness of calculating $\partial f/\partial t$. However, it is clear that this difference is very small.

Table 3.5 compares the results obtained for $\frac{\partial f}{\partial T}\Big|_k$ obtained from the TLM routine, and the ones obtained by perturbing T to 0.101 and 0.099 from its initial value 0.1 (i.e., repeat the analysis with these new values of T), and using central differences.

Two analyses were performed with two different time steps, namely, 0.1 and 0.07143, and $\partial f/\partial T$ at constant time was estimated by perturbation as $\Delta f/\Delta T$. This $\Delta f/\Delta T$ was used to extrapolate to the exact response. The extrapolation formula

$$f_{extrapolated} = f_{TLM} - \frac{T}{2}\,\frac{\Delta f}{\Delta T}\,. \qquad (3.39)$$

Table 3.6 compares the exact response obtained by the inverse Laplace transform and the extrapolated response. Table 3.7, on the other hand, compares the exact response and the extrapolated one, where $\partial f/\partial T$ was calculated using (3.38) for which $\partial f/\partial t$ is calculated by the central difference formula given in Appendix A.

## TABLE 3.5

## A COMPARISON BETWEEN $\partial f/\partial T$ FOR CONSTANT k OBTAINED BY TLM AND PERTURBATION

| k | f(t,T) |  | $\left.\frac{\partial f}{\partial T}\right\|_k$ (central differences) | $\left.\frac{\partial f}{\partial T}\right\|_k$ | Difference (%) |
|---|---|---|---|---|---|
|  | T=0.101 | T=0.099 |  |  |  |
| 11 | 0.13581327 | 0.13281062 | 1.50132 | 1.50137 | 0.006 |
| 41 | 0.04989793 | 0.05659663 | -3.34935 | -3.34976 | 0.019 |
| 71 | -0.01400295 | -0.01576903 | 0.88304 | 0.88397 | 0.105 |
| 101 | 0.00345630 | 0.00371329 | -0.12850 | -0.129911 | 0.474 |
| 131 | -0.00061299 | -0.00057152 | -0.02073 | -0.02050 | 1.109 |

## TABLE 3.6

## USING $\Delta f/\Delta T$ TO PREDICT RESPONSE FOR T=0

| t (s) | f(t,T) |  | $\frac{\Delta f}{\Delta T}$ | $f_{exact}$ | $f_{extra.}$ | Diff. (%) |
|---|---|---|---|---|---|---|
|  | T=0.1 | T=0.5/7 |  |  |  |  |
| 0.5 | 0.04255324 | 0.04333199 | $-2.72562\times10^{-2}$ | 0.044141 | 0.043916 | 0.510 |
| 1.5 | 0.17926209 | 0.17938925 | $-4.21435\times10^{-3}$ | 0.179524 | 0.179473 | 0.028 |
| 2.5 | 0.19006031 | 0.18991132 | $4.51465\times10^{-3}$ | 0.189777 | 0.189815 | 0.020 |
| 3.5 | 0.10754101 | 0.10727105 | $9.44860\times10^{-3}$ | 0.106988 | 0.107069 | 0.076 |
| 4.5 | 0.02373052 | 0.02339983 | $1.15742\times10^{-2}$ | 0.023053 | 0.023152 | 0.429 |

TABLE 3.7

USING $\partial f/\partial T$ TO PREDICT RESPONSE FOR T=0

| t (s) | $f_{exact}$ | $f_{extrapolated}$ | Difference (%) |
|---|---|---|---|
| 0.5 | 0.044141 | 0.044152 | 0.014 |
| 1.1 | 0.134981 | 0.134992 | 0.008 |
| 1.7 | 0.193099 | 0.193111 | 0.006 |
| 2.3 | 0.198260 | 0.198272 | 0.006 |
| 2.9 | 0.162173 | 0.162183 | 0.006 |
| 3.5 | 0.106988 | 0.106994 | 0.006 |
| 4.1 | 0.052558 | 0.052561 | 0.006 |
| 4.7 | 0.011055 | 0.011054 | 0.009 |
| 5.3 | -0.013432 | -0.013435 | 0.022 |
| 5.9 | -0.022686 | -0.022690 | 0.017 |

## 3.9 Conclusions

The TLM method is a new approach to the analysis of lumped networks. The distinct advantage of the TLM method is that the numerical procedure used solves the transmission-line model exactly. Errors arise only from how well the transmission-line model represents the actual circuit. To a certain limit the compensation of these errors by additional elements can improve the results.

Another advantage is that if the transmission-line network is physically stable, which is true in the case of passive linear networks, then the TLM solution will be stable. This means that stiff networks which give rise to instability in most methods do not cause instability in the TLM method. Different transmission-line models can be obtained for the same network, some of the models can be viewed as implicit methods and some as explicit.

The derived formulas permit sensitivity evaluation of the impulse response with respect to design parameters and makes the TLM method suitable for automated network design. Sensitivities with respect to time and time step can be easily obtained and it has been demonstrated how this information is used to improve accuracy.

Possible developments in the method lie in improving the accuracy by using more complicated transmission-line elements and models and the investigation of limitations on modeling general sets of coupled ordinary differential equations.

# CHAPTER 4

# EFFICIENT USE OF SIMULATION PROGRAMS IN THE ANALYSIS OF COMPLICATED NETWORKS

## 4.1 Introduction

Several general purpose simulators have been developed in the last decade. These simulators are designed to be as general as possible, i.e., to handle any circuit configuration, as many types of electrical elements as possible, to perform D.C., A.C. and time-domain analyses. As a result, these simulators are large, requiring a huge memory and CPU time to perform the analysis of a circuit of a reasonable size. The inclusion of such simulators in an optimization program, where it will be called hundreds of times is an obsolete idea. Another handicap for these simulators is that most of them do not provide sensitivities which are needed for the optimization process.

The trend in circuit design is increasingly towards consideration of production yield, design centering, optimal assignment of component tolerances and post-production tuning in an integrated fashion. The scope and size of the resulting design problems have expanded immensely as a result. The circuit designer confronting the design of a reasonably sized circuit which he has to accomplish in a limited time will be forced to avoid developing his own analysis program.

With the use of the multidimensional approximation approach developed by Bandler and Abdel-Malek (1978a) one can exploit general circuit simulators to perform the design without the explicit requirement of sensitivities. Knowing a nominal solution and the associated tolerances, one run of the simulator at (k+1)(k+2)/2 preselected sets of k parameter values lead to a set of quadratic models of the response w.r.t. the parameters. Those models are subsequently used to carry out the optimization processes.

Problems also arise when the available simulator does not handle (or does not include) one, or more, of the elements in the circuit to be analyzed. This chapter is concerned with the efficient use of these general simulators in the modeling approach and how to overcome the problem of nonexisting elements in the simulator. Two examples are given, one is an active filter and the second is a current switch emitter follower.

## 4.2 The Use of General Simulators

Quadratic models (Bandler and Abdel-Malek 1978a) of the circuit response w.r.t. the parameters at appropriate sample points in the frequency or time domains permit the use of general purpose simulators without explicit requirement of sensitivities. These models are subsequently used to carry out the optimization process. The models may be updated and the process repeated depending on the accuracy required and the conditioning of the

5

problem.

To minimize computational effort, the simulator should provide responses at (k+1)(k+2)/2 base points, where k is the dimension of $\underset{\sim}{\phi}$, suitably arranged within an interpolation region described by Abdel-Malek (1977)

$$\bar{\underset{\sim}{\phi}} - \underset{\sim}{\delta} \leq \underset{\sim}{\phi} \leq \bar{\underset{\sim}{\phi}} + \underset{\sim}{\delta}, \qquad (4.1)$$

where $\bar{\underset{\sim}{\phi}}$ is the center of the interpolation region and $\underset{\sim}{\delta}$ defines the size. Figures 4.1(a) and 4.1(b) depict suitable arrangements of the base points for a two-dimensional and a three-dimensional case, respectively.

The program SPICE2 (Nagel 1975), the available simulator, has been used to obtain circuit responses at the base points needed for the modeling and design of different networks (Bandler, Abdel-Malek, Dalsgaard, Elrazaz and Rizk 1978). The program can be run with different sets of parameter values. In order to reduce the overhead time, and assuming that the circuit is not very large, the program can be used only once by supplying the data in such a way that the circuit is repeated with different sets of nodes (where there is no interconnection between each set of nodes except the ground node) with different sets of parameter values. In the frequency-domain case the overall nodal admittance matrix is, consequently, a block diagonal matrix with each block

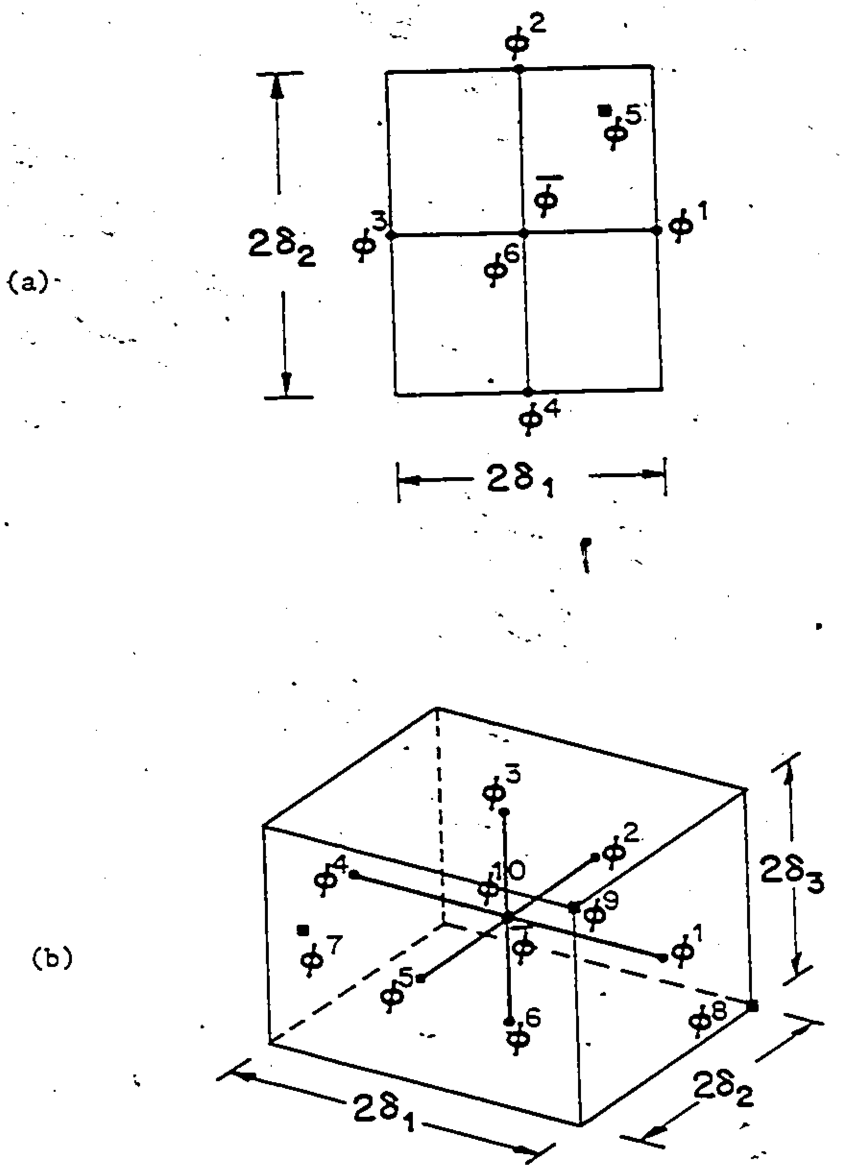

(a)
$\phi^2$
$\phi^5$
$\bar{\phi}$
$\phi^1$
$\phi^3$
$\phi^6$
$\phi^4$
$2\delta_2$
$2\delta_1$
(b)
$\phi^3$
$\phi^2$
$\phi^4$
$\phi^{10}$
$\phi^9$
$\phi^1$
$\phi^7$
$\bar{\phi}$
$\phi^5$
$\phi^6$
$\phi^8$
$2\delta_3$
$2\delta_1$
$2\delta_2$

Fig. 4.1 Arrangement of the base points w.r.t. the centers of interpolation regions in (a) two dimensions and (b) three dimensions.

representing a Y matrix of the circuit. Fig. 4.2 shows such an example. The reordering of the equations to reduce fill-ins, when these equations are solved by LU factorization, would not affect the validity of supplying the data in this way. We have to note here that this can also be done for circuits to be analyzed in the time domain if the companion network or the tableau approach is used for the analysis.

## 4.3 Examples

### 4.3.1 An Active Filter

We consider here the analysis of an active filter (Fig. 4.3) to be designed in the worst-case sense everywhere in the range of a tunable parameter, namely, $R_4$. The active filter is based on an active bandpass realization considered by Budak and Zeller (1972). The operational amplifiers employed are taken as nonideal, in particular, the one-pole roll-off model given by

$$A(s) = \frac{A_0 \, \omega_a}{s + \omega_a}, \tag{4.2}$$

where s is the complex frequency variable, $A_0 = 2 \times 10^5$ is the D.C. gain and $\omega_a = 12\pi$ rad/s the 3 dB radian bandwidth. A nonzero output resistance R is assumed for the operational amplifiers.

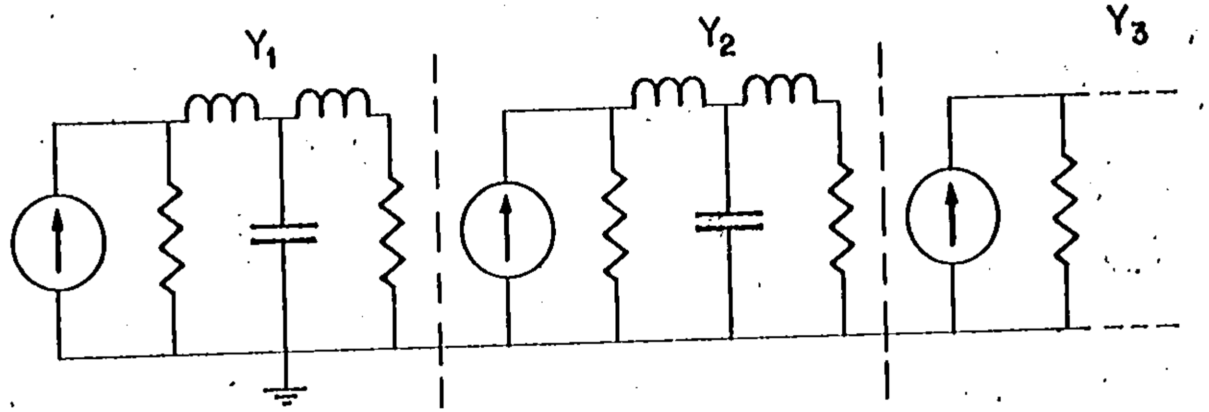


Fig. 4.2 A circuit supplied to SPICE2,which is the original circuit repeated (k+1) (k+2)/2 times with appropriate parameter values.

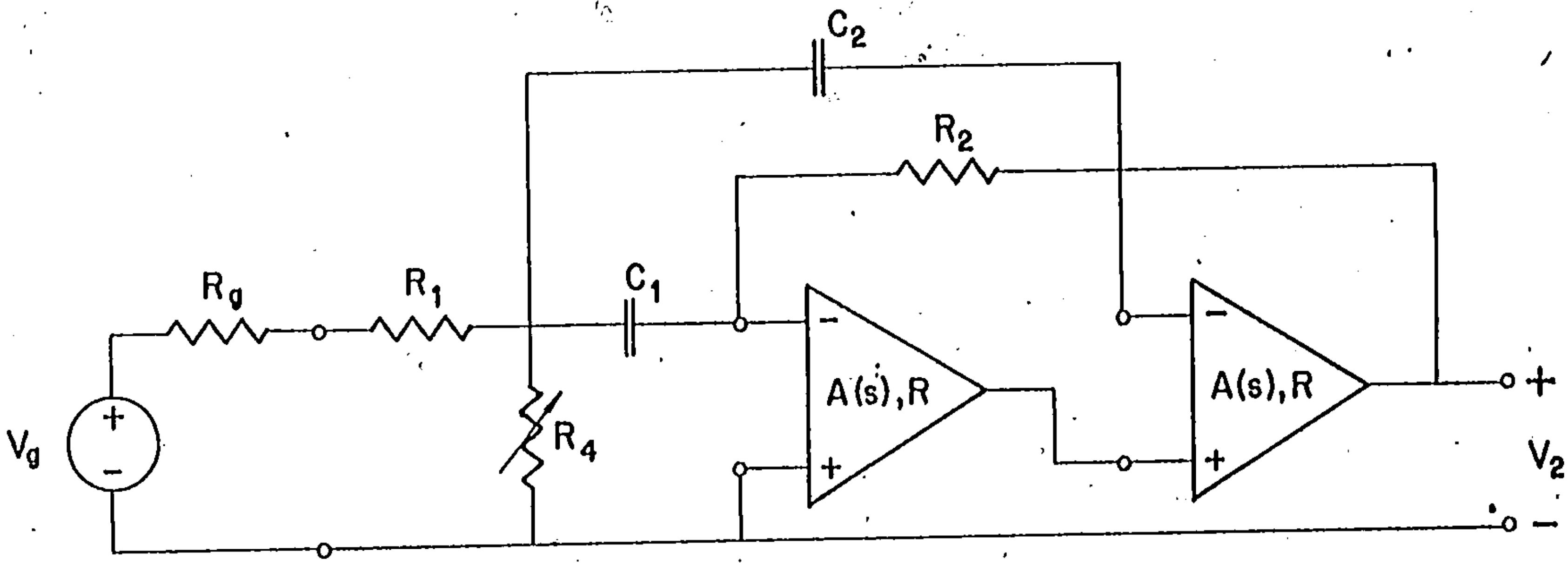

C₂
R₂
Rg
R₁
C₁
A(s),R
A(s),R
Vg
R₄
V₂
+
−

Fig. 4.3 Tunable active filter with $R_g$ = 50 Ω, R = 75 Ω and one-pole roll-off for A(s).

Replacing the operational amplifiers by their equivalent circuits we obtain the circuit of Fig. 4.4. Since the program (SPICE2) does not handle a frequency dependent gain of the form of (4.2), the gain had to be represented by the transfer function of an additional small circuit. The transfer function of a series RL circuit driven by a voltage source of $A_0\ \omega_a$ can represent equation (4.2). This is achieved by choosing a value of 1 H for the inductance, $(\omega_a-1)\Omega$ for the first resistor and a value of 1 Ω for the output resistor. The voltage across the output resistor is the output voltage of the first operational amplifier. The ΔV term at the input to the second amplifier can be modeled by a current leaving a node connecting two voltage controlled current sources which are controlled by the voltages to be subtracted. Figure 4.5 shows the equivalent circuit supplied to SPICE2 to perform the analysis. Fifteen circuits connected in cascade were actually supplied once to the program to obtain the response $|V_2|$ at fifteen base points. The variables are $R_1$, $R_4$, $C_1$ and $C_2$, and $R_2$ is equal to 26.5 kΩ. The center base point and the sizes of the interpolation region are given in Table 4.1. Figure 4.6 shows the response of the filter with $R_1$ = 12.8214 kΩ, $C_2$ = 0.74294 μF, $C_1$ = 0.70106 μF and $R_4$ = 188 Ω (a point in the interpolation region) obtained by SPICE2 and exactly similar to the response obtained by a specially written program.

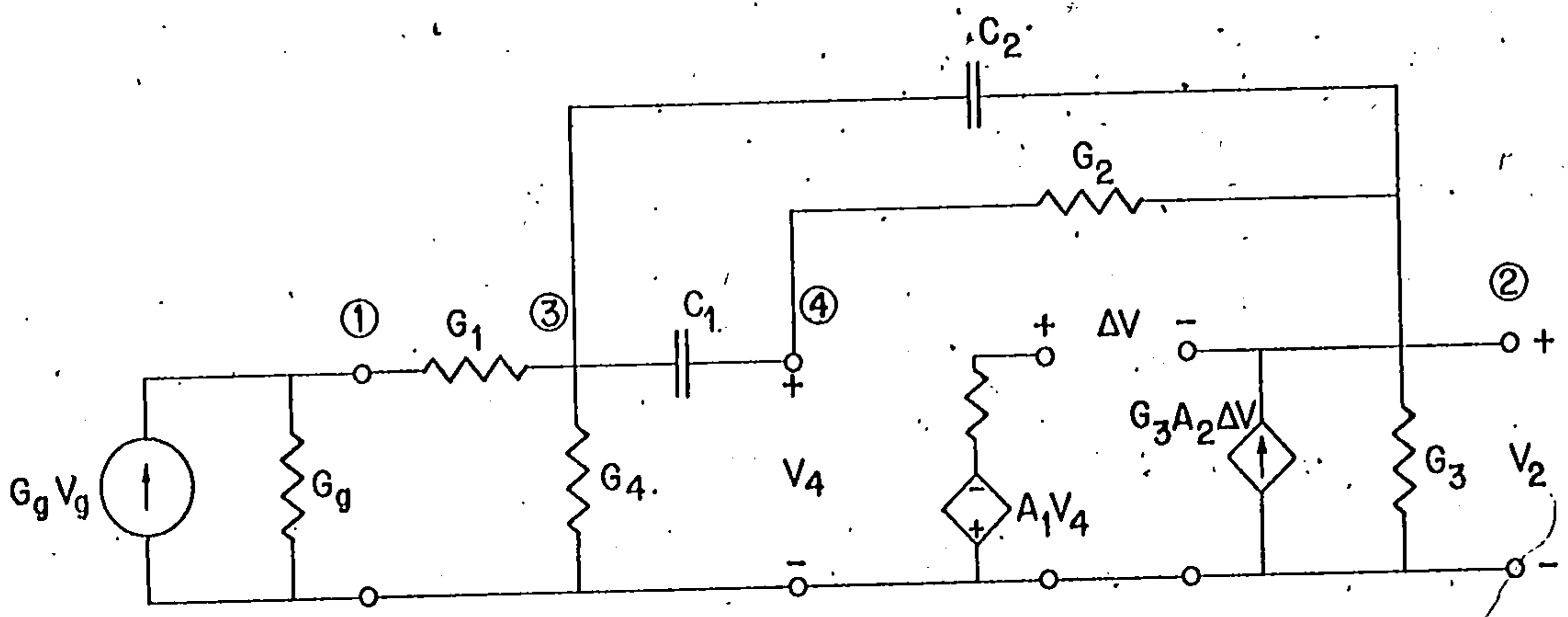

$G_gV_g$
$G_g$
①
$G_1$
③
$G_4$
$C_1$
④
+
$V_4$
−
$C_2$
$G_2$
+
ΔV
−
$A_1V_4$
$G_3A_2\Delta V$
②
+
$G_3$
$V_2$
−

Fig. 4.4 Equivalent circuit for nodal analysis of the circuit of Fig. 4.3.

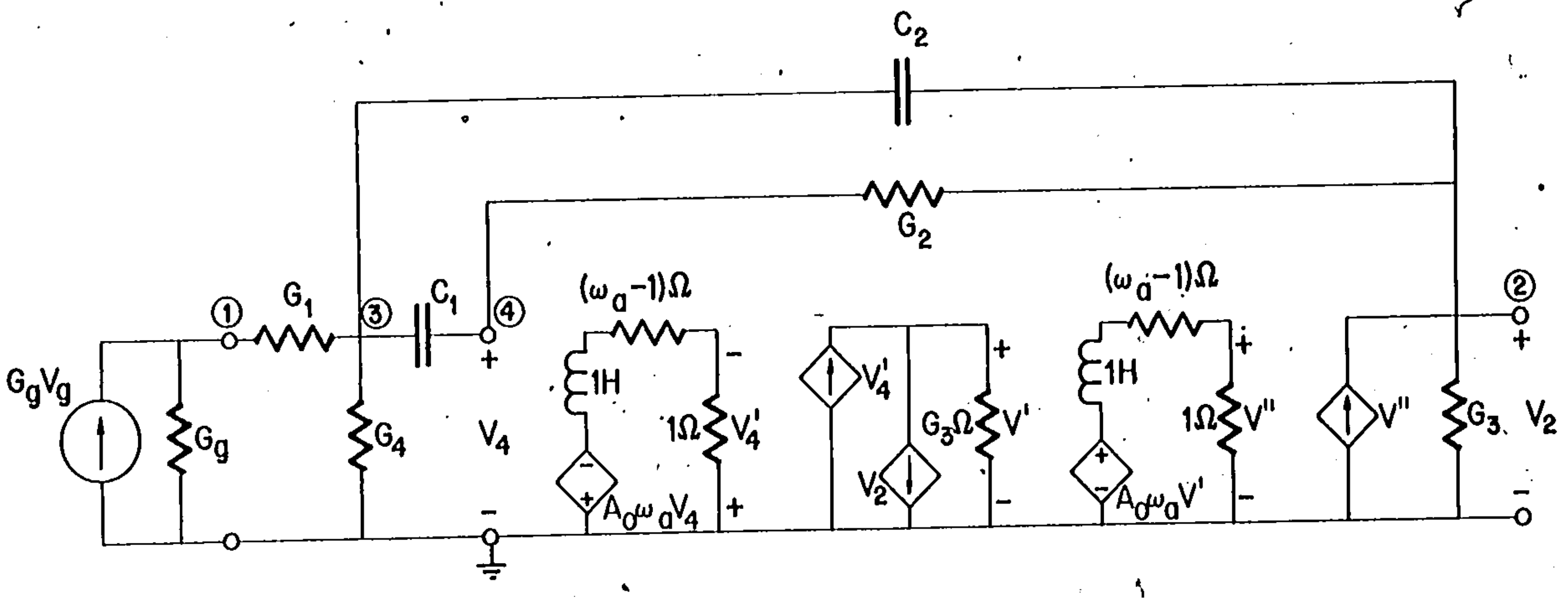
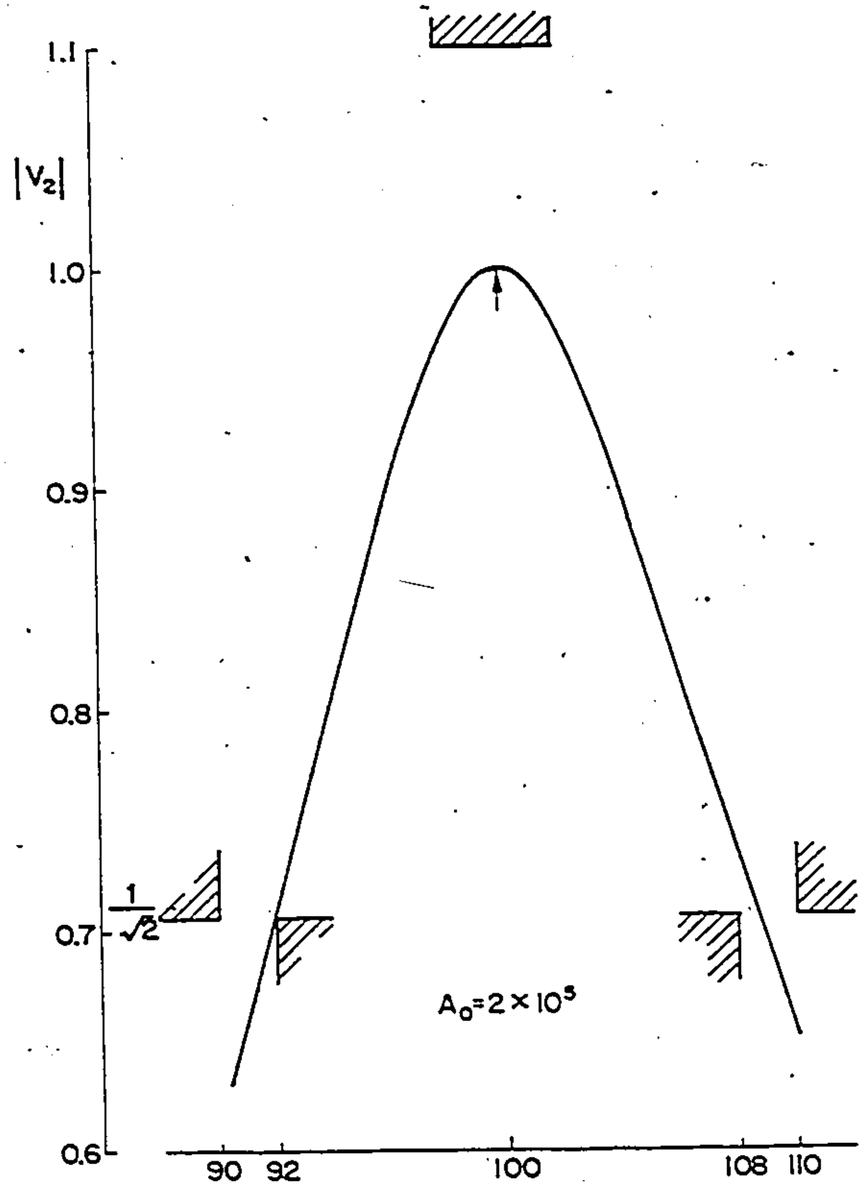

$G_gV_g$
$G_g$
①
$G_1$
③
$G_4$
$C_1$
④
+
$V_4$
−
$C_2$
$G_2$
$(\omega_a-1)\Omega$
1H
$1\Omega$
$V_4'$
$A_0\omega_aV_4$
$V_4'$
$V_2$
$G_3\Omega$
$V'$
$(\omega_a-1)\Omega$
1H
$1\Omega$
$V''$
$A_0\omega_aV'$
$V''$
$G_3$
②
$V_2$

Fig. 4.5 Equivalent circuit for SPICE2 analysis of the circuit of Fig. 4.3.



frequency Hz

Fig. 4.6 Response of the active filter at a point in the interpolation region.

TABLE 4.1

THE CENTER BASE POINT AND THE SIZE OF THE INTERPOLATION REGION FOR THE QUADRATIC APPROXIMATION OF THE RESPONSE OF THE ACTIVE FILTER OF FIG. 4.3

| | $R_1$ (kΩ) | $R_4$ (Ω) | $C_1$ (μF) | $C_2$ (μF) |
|---|---|---|---|---|
| Center Base Point | 10 | 200 | 0.75 | 0.75 |
| Size $\delta$ | 5 | 100 | 0.375 | 0.375 |

### 4.3.2 A Current Switch Emitter Follower

The circuit shown in Fig. 4.7 was employed by Ho (1971) for time-domain sensitivity calculations, and for worst-case design and yield optimization by Abdel-Malek and Bandler (1978c). Here we will consider the analysis of this circuit by SPICE2. Figure 4.8 shows the charge-control model to be used for each transistor. The charge-control diode model corresponds to that of the emitter-base junction. Table 4.2 lists the values of the circuit parameters and model parameters, which were obtained from a worst-case design of this network (Abdel-Malek 1977).

The program SPICE2 could not handle the nonlinear capacitance in the form of the one given in the transistor model of Fig. 4.8. In order to overcome this problem (assuming we want to analyze the network with the given transistor model exactly and

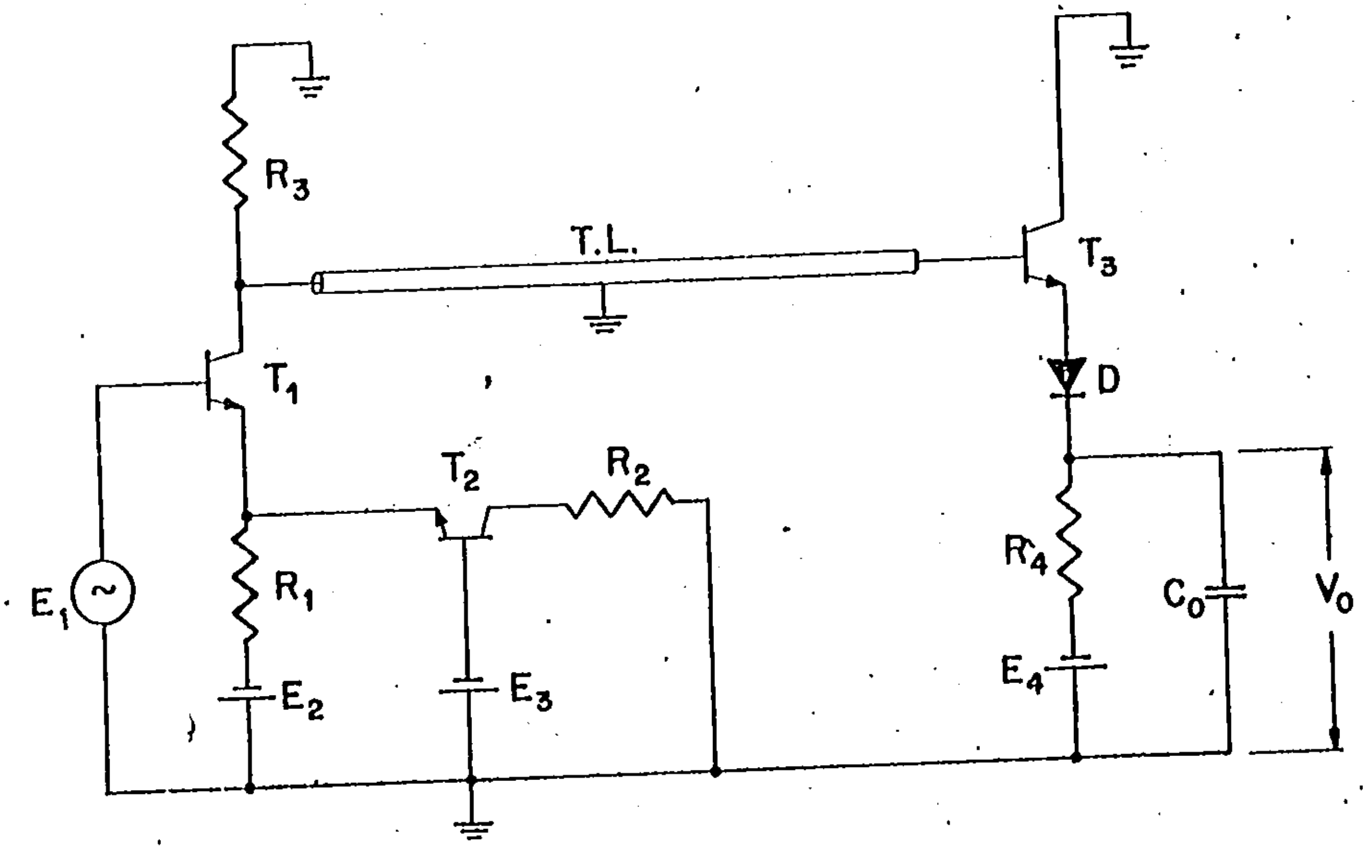

R$_3$
T.L.
T$_3$
T$_1$
D
T$_2$
R$_2$
R$_4$
E$_1$
R$_1$
C$_0$
V$_0$
E$_2$
E$_3$
E$_4$

Fig. 4.7 Current switch emitter follower (CSEF) circuit.

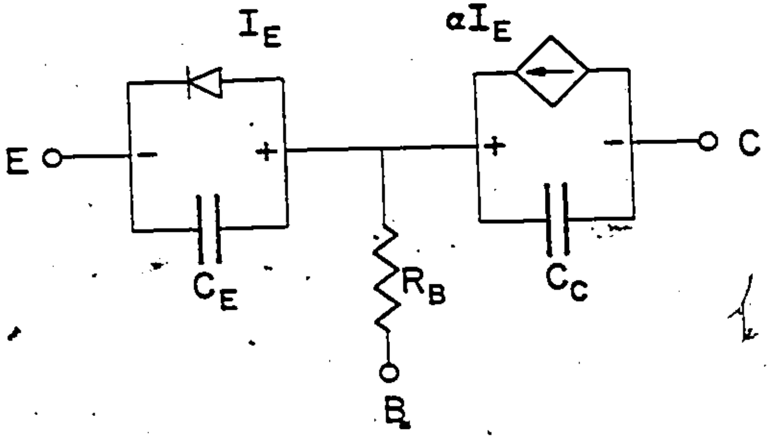

$I_E$
$\alpha I_E$
E
C
B
$C_E$
$R_B$
$C_C$
-
+
+
-

Fig. 4.8 The transistor model.

## TABLE 4.2(a)

## CIRCUIT PARAMETER VALUES

| | |
|---|---|
| $R_1$ | 281.33 Ω |
| $R_2$ | 75.00 Ω |
| $R_3$ | 78.24 Ω |
| $R_4$ | 45.53 Ω |
| $E_2$ | 4.03 V |
| $E_3$ | 1.13 V |
| $E_4$ | 1.66 V |
| $C_0$ | 1.25 pF |

## TABLE 4.2(b)

## DIODE MODEL PARAMETERS

| | | |
|---|---|---|
| $I_{SD}$ | diode saturation current | $0.6 \times 10^{-9}$A |
| $C_{JD}$ | depletion layer capacitance | 0.12 pF |
| $TT_D$ | transit time | 0.01 ns |
| $\theta$ | inverse of thermal potential | 38.688 $V^{-1}$ |

$$I_D = I_{SD}(\exp(\theta V_D)-1)$$

$$C_D = C_{JD} + TT_D \frac{dI_D}{dV_D}$$

TABLE 4.2(c)

TRANSISTOR MODEL PARAMETERS

| | | |
|---|---|---|
| $I_S$ | saturation current | $0.6 \times 10^{-9}$ A |
| $\alpha$ | common base current gain | 0.99 |
| $R_B$ | base resistance | 50.0 Ω |
| $C_C$ | collector junction capacitance | 0.5 pF |
| $C_{JE}$ | emitter junction depletion layer capacitance | 0.12 pF |
| TT | base transit time | 0.01 ns |
| $\theta$ | inverse of thermal potential | 38.668 $V^{-1}$ |

$$I_E = I_S\,(\exp(\theta V_{BE})-1)$$

$$I_C = \alpha\, I_E$$

$$C_E = C_{JE} + TT\,\frac{dI_E}{dV_{BE}}$$

$R_B$ and $C_C$ are assumed zero for transistor $T_3$

TABLE 4.2(d)

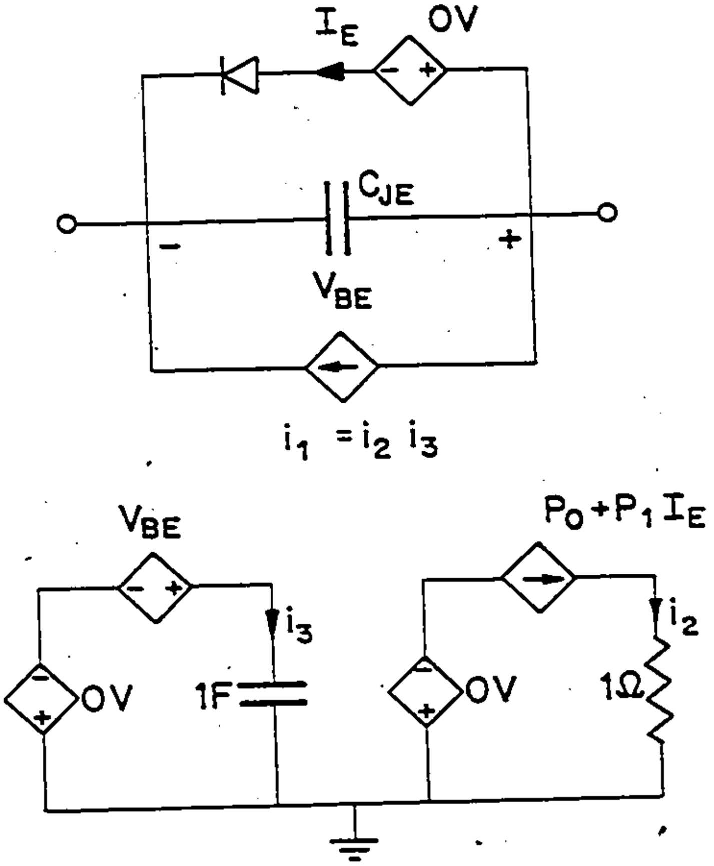
TRANSMISSION-LINE PARAMETERS

| | | |
|---|---|---|
| $Z_0$ | characteristic impedance | 92.004 Ω |
| $\tau$ | delay time | 0.25 ns |

not with any other model) the current passing through the nonlinear part of the capacitance was represented by the current $i_1$ of a two-dimensional current controlled current source. The currents controlling this source are $i_2$ and $i_3$ in two small additional networks as shown in Fig. 4.9. The coefficients of the polynomial representing $i_1$ are all zero except the coefficient of the cross terms which has the value one. In the circuit where $i_2$ is passing $P_0 = P_1 I_S$ so as to let $i_2$ be equal to $P_1 I_S \exp( V_{BE})$. The current $i_3$ will represent $dV_{BE}/dt$. We have to note that the zero valued voltage sources in the additional network have to be introduced since the current controlled sources in SPICE2 can only be controlled by currents passing through independent voltage sources. The results were checked by the companion-network approach (Rizk 1978). .

The analysis was also performed by SPICE2 using the built-in models. The parameters of these models were fed in the data to match the model as closely as possible to the given model (Fig. 4.8). Responses obtained by the companion network, by SPICE2 and by the state equations (Abdel-Malek 1977) are shown in Fig. 4.10. Note that the two responses obtained by SPICE2 were almost identical.

The running time of SPICE2, where we modeled the nonlinear capacitance, was 92 s, while using the built-in models the running time was only 7 s. This difference is mainly due to the



$I_E$
0V
$C_{JE}$
$V_{BE}$
$i_1 = i_2 i_3$
$V_{BE}$
$i_3$
0V
1F
$P_0 + P_1 I_E$
$i_2$
0V
1Ω

Fig. 4.9 Transistor model described to SPICE2.

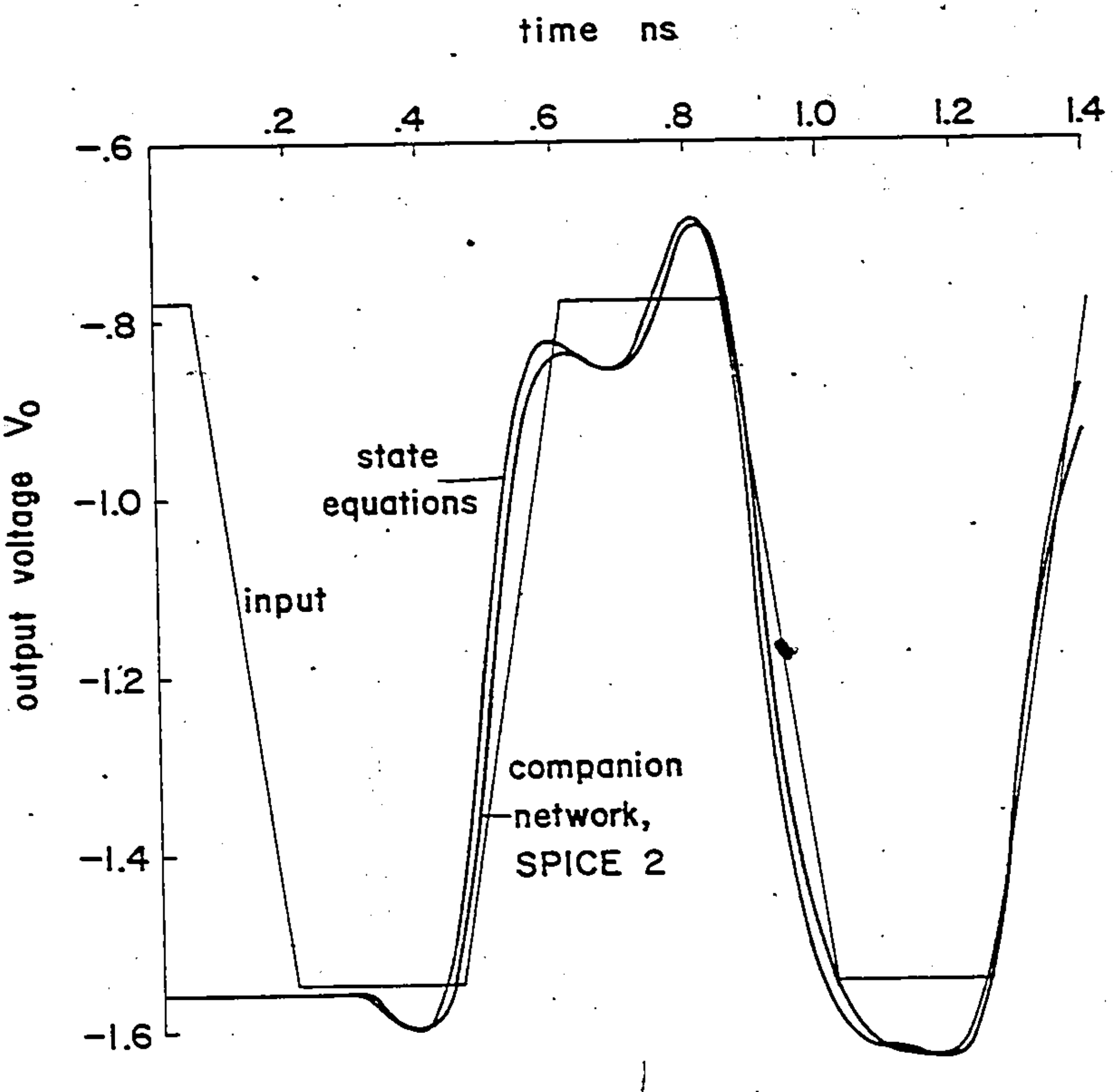


Fig. 4.10 Input voltage and responses of CSEF with different methods of analysis.

additional elements we have introduced in modeling the nonlinear capacitance which resulted in having 12 additional nodes and node voltages. The data supplied to SPICE2 in the two cases is given in Appendix B.

## 4.4 Conclusions

The possible exploitation of general purpose simulators to perform the analysis of circuits (even if they can not handle the circuit directly) and obtain the multidimensional approximation models to carry out sophisticated optimal design problems (design centering, tolerance assignment, post-production tuning, worst-case design and yield optimization) has been described.

# CHAPTER 5

# ANALYSIS AND SENSITIVITY EVALUATION FOR CASCADED STRUCTURES

## 5.1 Introduction

This chapter presents a new and comprehensive treatment of computer-oriented cascaded network analysis. The analysis of cascaded networks plays a very important role in the design and optimization of microwave circuits, so that an attractive approach which facilitates efficient analytical and numerical investigations of response, first- and higher-order sensitivities of response, simultaneous and arbitrary large-change sensitivity evaluation is highly desirable. As is well-known, first-order sensitivities, for example, are useful in network optimization by gradient methods.

In tolerance assignment, the response and its first-order sensitivity at the vertices of the tolerance region are needed. This information is also very useful if a worst-case search algorithm has to identify the worst vertex.

The approach we have developed permits efficient

(a) exact analysis of cascaded networks in any direction,

(b) exact evaluation of first-order response sensitivities at any location,

(c) exact evaluation of the effects of any number of simultaneous large changes in any elements,

(d) the exploitation of network structure:

  1. responses at different loads in branched networks which may be connected in series or in parallel with the main cascade, can be obtained analytically in terms of the variable elements. Sensitivity and large-change effects w.r.t. these variables can be easily evaluated,
  2. symmetry can be taken into consideration to reduce computational effort (Bandler, Biernacki and Rizk 1979),

(e) evaluation of the exact effect due to simultaneously growing elements in appropriate locations.

The conceptual advantages enjoyed by our approach and applicable to 2-port elements are

(a) all calculations are applied directly to the given network: no auxiliary or adjoint network is defined,

(b) all calculations involve at most the premultiplication of two by two matrices by row vectors or postmultiplications by column vectors: no explicit matrix inversion is ever required,

(c) response functions, sensitivities or large-change effects are represented analytically in terms of the parameters to be investigated: all parts of the network to be kept constant are reduced numerically to a few two-element vectors appearing as constants in the formulas,

(d) calculations can be carried out easily by hand, if appropriate, or are readily programmed.

The approach is not confined to 2-port elements. It has been generalized in this chapter to 2p-port elements.

## 5.2 Theoretical Foundation

Consider the two-port element depicted in Fig. 5.1. The basic iteration, also summarized by Table 5.1, is $\underline{\bar{y}} = \underline{A}\ \underline{y}$, where $\underline{A}$ is the transmission or chain matrix, $\underline{y}$ contains the output voltage and current and $\underline{\bar{y}}$ the corresponding input quantities.

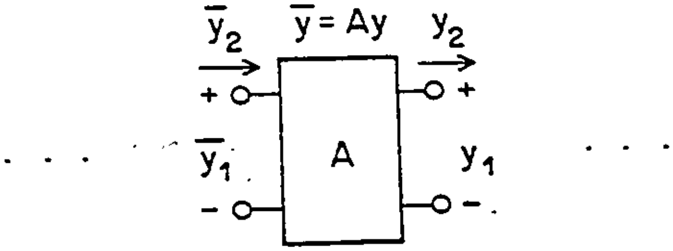


Fig. 5.1 Notation for an element in the chain, indicating reference directions and voltage and current variables.

TABLE 5.1

PRINCIPAL CONCEPTS INVOLVED IN THE ANALYSES

| Concept | Definition | Implication |
|---|---|---|
| Basic iteration | $\bar{\underset{\sim}{y}} = \underset{\sim}{A}\,\underset{\sim}{y}$ | $\underset{\sim}{y} \Longrightarrow \bar{\underset{\sim}{y}}$ |
| Forward operation | $\bar{\underset{\sim}{u}}^T \underset{\sim}{A} = \underset{\sim}{u}^T$ | $\overline{\underset{\sim}{u}^T \underset{\sim}{y}} = \bar{\underset{\sim}{u}}^T \underset{\sim}{A}\underset{\sim}{y} = \underset{\sim}{u}^T \underset{\sim}{y}$ |
| Reverse operation | $\bar{\underset{\sim}{v}} = \underset{\sim}{A}\underset{\sim}{v}$ | $\underset{\sim}{y} = c\underset{\sim}{v} \Longrightarrow \bar{\underset{\sim}{y}} = c\bar{\underset{\sim}{v}}$ |
| Voltage selector | $\underset{\sim}{e}_1 \triangleq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ | $\underset{\sim}{e}_1 \Longrightarrow \underset{\sim}{u}_1$ or $\underset{\sim}{v}_1$ |
| Current selector | $\underset{\sim}{e}_2 \triangleq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ | $\underset{\sim}{e}_2 \Longrightarrow \underset{\sim}{u}_2$ or $\underset{\sim}{v}_2$ |
| Equivalent source | $\underset{\sim}{y} = \begin{bmatrix} V_S - Z_S I_S \\ I_S \end{bmatrix}$ | $\underset{\sim}{e}_1^T \underset{\sim}{y} = V_S - Z_S I_S$, $\underset{\sim}{e}_2^T \underset{\sim}{y} = I_S$ |
| Equivalent load | $\underset{\sim}{y} = \begin{bmatrix} V_L \\ Y_L V_L - I_L \end{bmatrix}$ | $\underset{\sim}{y} = V_L \underset{\sim}{e}_1 + (Y_L V_L - I_L)\underset{\sim}{e}_2$ |

Forward analysis (see Fig. 5.2 and Table 5.1) consists of initializing a $\bar{u}^T$ row vector as either [1 0], [0 1] or a suitable linear combination and successively premultiplying each constant chain matrix by the resulting row vector until an element of interest, a reference plane or a termination is reached.

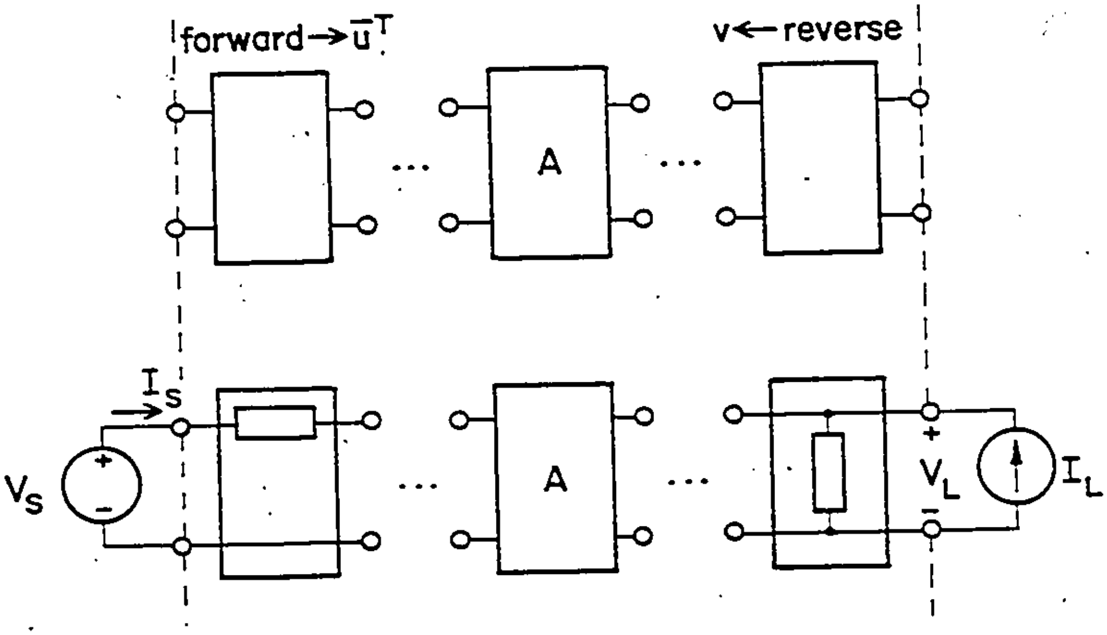


Fig. 5.2 Forward and reverse analyses of a cascaded network with source and load impedances assumed constant.

*Reverse analysis*, which is similar to conventional analysis of cascaded networks, proceeds by initializing a $\underset{\sim}{v}$ column vector as either $[1\ 0]^T$ or $[0\ 1]^T$ or a suitable linear combination and successively postmultiplying each constant matrix by the resulting column vector, again until either an *element of interest*, a reference plane or a termination is reached.

In summary, assuming a cascade of n two-ports we have

$$\underset{\sim}{\bar{y}}^1 = \underset{\sim}{y}^0 = \underset{\sim}{A}^1 \underset{\sim}{A}^2 \ldots \underset{\sim}{A}^i \ldots \underset{\sim}{A}^n \underset{\sim}{y}^n \tag{5.1}$$

and, applying forward and reverse analysis up to $\underset{\sim}{A}^i$, this reduces to an expression of the form

$$d = \underset{\sim}{\bar{u}}^{1^T} \underset{\sim}{\bar{y}}^1 = c\, \underset{\sim}{\bar{u}}^{i^T} \underset{\sim}{A}^i \underset{\sim}{v}^i, \tag{5.2}$$

where

$$\underset{\sim}{y}^n = c\, \underset{\sim}{v}^n \tag{5.3}$$

and c and d relate selected output and input variables of interest explicitly with $\underset{\sim}{A}^i$.

The typical formula will, therefore, contain factors of the form

$$\text{function evaluation:} \qquad \underset{\sim}{\bar{u}}^T \underset{\sim}{A}\, \underset{\sim}{v} \Longrightarrow Q \tag{5.4}$$

$$\text{first-order sensitivity:} \qquad \underset{\sim}{\bar{u}}^T \delta \underset{\sim}{A}\, \underset{\sim}{v} \Longrightarrow \delta Q \tag{5.5}$$

partial derivative: $$\bar{\underset{\sim}{u}}^T \frac{\partial \underset{\sim}{A}}{\partial \phi} \underset{\sim}{v} \implies Q' \qquad (5.6)$$

large-change sensitivity: $$\bar{\underset{\sim}{u}}^T \Delta \underset{\sim}{A}\ \underset{\sim}{v} \implies \Delta Q \ . \qquad (5.7)$$

where the parameter $\phi$ is contained in $\underset{\sim}{A}$. A full reverse analysis taking

$$[\underset{\sim}{v}_1^n \ \ \underset{\sim}{v}_2^n] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

yields

$$[\underset{\sim}{v}_1^i \ \ \underset{\sim}{v}_2^i] = \underset{\sim}{A}^{i+1}\ \underset{\sim}{A}^{i+2} \ldots \underset{\sim}{A}^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and a corresponding full forward analysis taking

$$[\bar{\underset{\sim}{u}}_1^1 \ \ \bar{\underset{\sim}{u}}_2^1]^T = [\underset{\sim}{u}_1^0 \ \ \underset{\sim}{u}_2^0]^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

yields

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underset{\sim}{A}^1\ \underset{\sim}{A}^2 \ldots \underset{\sim}{A}^{i-1} = [\bar{\underset{\sim}{u}}_1^i \ \ \bar{\underset{\sim}{u}}_2^i]^T .$$

### 5.2.1 Reference Planes

In considering more than one element in the cascade we divide the network into subnetworks by reference planes. These in turn are chosen so that no more than one element is to be explicitly considered between any pair of reference planes. In Fig. 5.2 the element $\underset{\sim}{A}$ is the only element whose effect is to be considered. In Fig. 5.3 the elements $\underset{\sim}{A}^k$, $\underset{\sim}{A}^i$ and $\underset{\sim}{A}^j$ are considered in the kth, the ith and the jth subnetworks, respectively. Note that the superscripts of $\underset{\sim}{A}$ here, and from now on, denote the subnetwork and not the element. Forward and reverse analyses are initiated at the reference planes. A forward iteration of the structure of Fig. 5.3 is illustrated in Fig. 5.4, where equivalent (Thevenin) sources are iteratively determined. Reverse iteration is shown in Fig. 5.5, where equivalent (Norton) sources are iteratively determined.

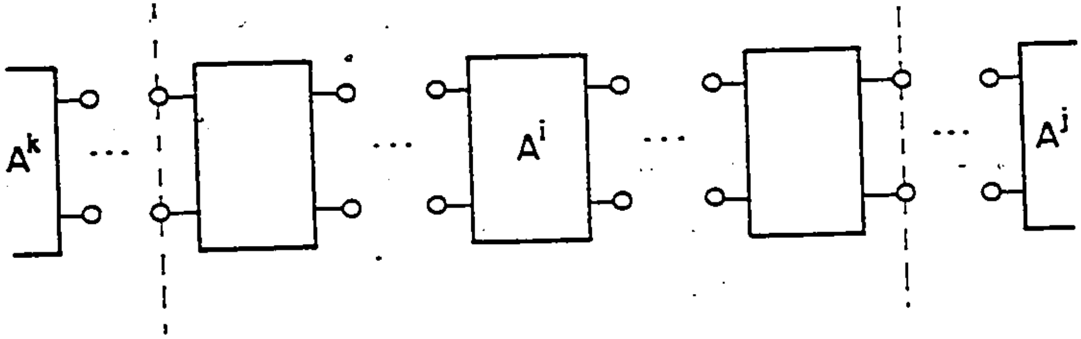


Fig. 5.3 Subnetwork i cascaded with subnetwork k (at source end) and j (at load end).

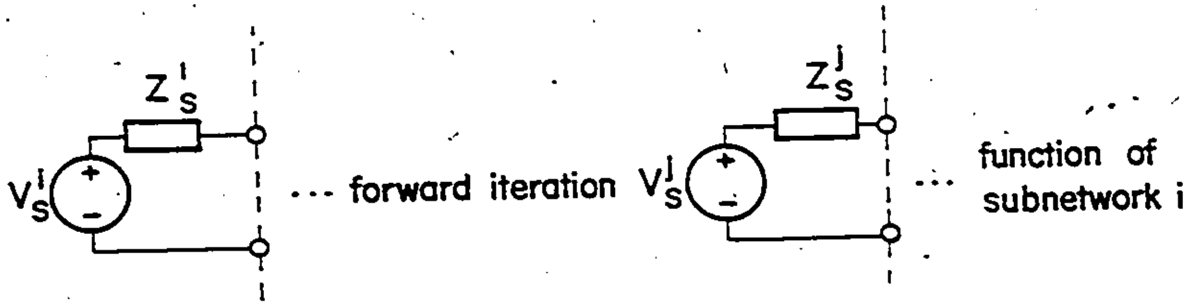

$Z_S^i$
$V_S^i$
... forward iteration
$V_S^j$
$Z_S^j$
... function of subnetwork i

Fig. 5.4 Forward iteration for Fig. 5.3, transferring an equivalent source accounting for design variables from subnetwork k from one reference plane to the other.

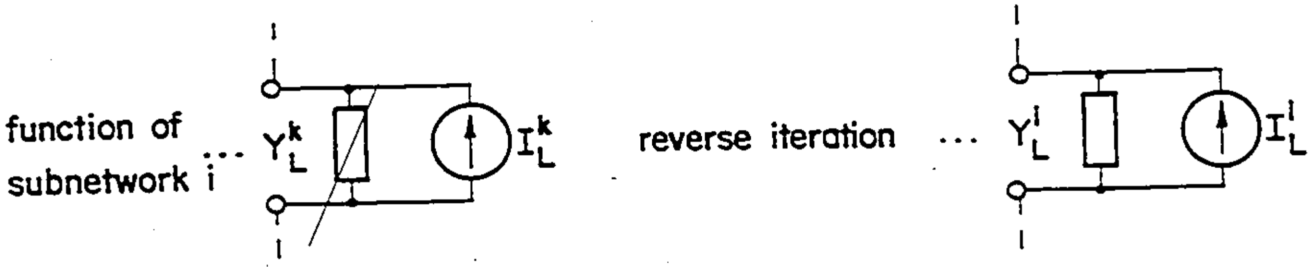

function of subnetwork i ...
$Y_L^k$
$I_L^k$
reverse iteration
... $Y_L^i$
$I_L^i$

Fig. 5.5 Reverse iteration for Fig. 5.3, transferring an equivalent source accounting for design variables from subnetwork j from one reference plane to the other.

## 5.3 Network Functions in Terms of Elements Under Consideration

Performing forward analysis from the source of the ith subnetwork to the input of $\underset{\sim}{A}^i$ and reverse analysis from the load to the output of $\underset{\sim}{A}^i$ we have

$$V_S^i = (\bar{\underset{\sim}{u}}_1 + Z_S^i \, \bar{\underset{\sim}{u}}_2)^T \, \underset{\sim}{A}^i (V_L^i \, \underset{\sim}{v}_1 + (Y_L^i V_L^i - I_L^i) \underset{\sim}{v}_2) = V_L^k + Z_S^i I_S^i \qquad (5.8)$$

and the current through the voltage source of the ith subnetwork

$$I_S^i = \bar{\underset{\sim}{u}}_2^{\,T} \, \underset{\sim}{A}^i (V_L^i \underset{\sim}{v}_1 + (Y_L^i V_L^i - I_L^i) \underset{\sim}{v}_2) = V_L^k Y_L^k - I_L^k . \qquad (5.9)$$

From (5.8), letting $I_L^i = 0$ and $Y_L^i = 0$, we have $I_S^j = 0$ and the Thevenin voltage

$$V_S^j = V_L^i = \frac{V_S^i}{(\bar{\underset{\sim}{u}}_1 + Z_S^i \bar{\underset{\sim}{u}}_2)^T \, \underset{\sim}{A}^i \underset{\sim}{v}_1} = \frac{V_S^i}{Q_{11}^i + Z_S^i Q_{21}^i} , \qquad (5.10)$$

where the Q terms have been defined in (5.4). Letting $V_S^i = 0$ and $Y_L^i = 0$, we have $I_S^j = -I_L^i$ and the output impedance

$$Z_S^j = \frac{V_L^i}{I_L^i} = \frac{(\bar{\underset{\sim}{u}}_1 + Z_S^i \bar{\underset{\sim}{u}}_2)^T \, \underset{\sim}{A}^i \underset{\sim}{v}_2}{(\bar{\underset{\sim}{u}}_1 + Z_S^i \bar{\underset{\sim}{u}}_2)^T \, \underset{\sim}{A}^i \underset{\sim}{v}_1} = \frac{Q_{12}^i + Z_S^i Q_{22}^i}{Q_{11}^i + Z_S^i Q_{21}^i} , \qquad (5.11)$$

where, again, the Q terms of (5.4) are used to obtain a compact expression. These expressions for $V_S^j$ and $Z_S^j$ permit equivalent Thevenin sources to be moved in a forward iteration.

From (5.8) and (5.9), letting $I_L^i = 0$ and $Z_S^i = 0$ we have $I_L^k = 0$ and the input admittance

$$Y_L^k = \frac{I_S^i}{V_S^i} = \frac{\bar{\underset{\sim}{u}}_2^T \underset{\sim}{A}^i(\underset{\sim}{v}_1+Y_L^i\underset{\sim}{v}_2)}{\bar{\underset{\sim}{u}}_1^T \underset{\sim}{A}^i(\underset{\sim}{v}_1+Y_L^i\underset{\sim}{v}_2)} = \frac{Q_{21}^i+Y_L^iQ_{22}^i}{Q_{11}^i+Y_L^iQ_{12}^i} . \qquad (5.12)$$

Letting $V_S^i = 0$ and $Z_S^i = 0$, we have $V_L^k = 0$ and the Norton current

$$I_L^k = -I_S^i = -I_L^i(Y_L^k\bar{\underset{\sim}{u}}_1 - \bar{\underset{\sim}{u}}_2)^T \underset{\sim}{A}^i\underset{\sim}{v}_2 = -I_L^i(Y_L^kQ_{12}^i - Q_{22}^i) . \qquad (5.13)$$

These expressions for $I_L^k$ and $Y_L^k$ permit equivalent Norton sources to be moved (if desired) in a reverse iteration.

The input current $I_S^i$ for $I_L^i = 0$ is obtained via (5.12) as

$$I_S^i = V_S^i\Bigg/\left[Z_S^i + \frac{\bar{\underset{\sim}{u}}_1^T \underset{\sim}{A}^i(\underset{\sim}{v}_1+Y_L^i\underset{\sim}{v}_2)}{\bar{\underset{\sim}{u}}_2^T \underset{\sim}{A}^i(\underset{\sim}{v}_1+Y_L^i\underset{\sim}{v}_2)}\right]$$

$$= \frac{V_S^i\bar{\underset{\sim}{u}}_2^T \underset{\sim}{A}^i(\underset{\sim}{v}_1+Y_L^i\underset{\sim}{v}_2)}{(\bar{\underset{\sim}{u}}_1+Z_S^i\bar{\underset{\sim}{u}}_2)^T \underset{\sim}{A}^i(\underset{\sim}{v}_1+Y_L^i\underset{\sim}{v}_2)} = \frac{V_S^i(Q_{21}^i+Y_L^iQ_{22}^i)}{Q_{11}^i+Y_L^iQ_{12}^i+Z_S^iQ_{21}^i+Z_S^iY_L^iQ_{22}^i} \qquad (5.14)$$

Tables 5.2 and 5.3 summarize the procedures and the effort required in evaluating the different factors in the derived equations.

Useful special cases of these formulas for $I_S$ and $V_L$ in Fig. 5.2 are, from (5.14) and (5.10), respectively,

$$I_S = V_S \frac{\bar{\underset{\sim}{u}}_2^T \underset{\sim}{A} \underset{\sim}{v}_1}{\bar{\underset{\sim}{u}}_1^T \underset{\sim}{A} \underset{\sim}{v}_1} = V_S \frac{Q_{21}}{Q_{11}} \tag{5.15}$$

and

$$V_L = \frac{V_S}{\bar{\underset{\sim}{u}}_1^T \underset{\sim}{A} \underset{\sim}{v}_1} = \frac{V_S}{Q_{11}} . \tag{5.16}$$

Table 5.4 gives some useful formulas which can be obtained for variations in a particular element $\underset{\sim}{A}$. We note, for example, that, since $\underset{\sim}{A}$ is arbitrary and at most only one full analysis yields all $Q_{11}$, $\delta Q_{11}$, $Q_{11}'$ and $\Delta Q_{11}$, the corresponding $V_L$, $\delta V_L$, $\partial V_L/\partial\phi$ and $\Delta V_L$ w.r.t. all possible parameters anywhere in the cascade can be evaluated exactly for one network analysis. This particular special case is equivalent to the results of previous researchers (Bandler and Seviora 1970, Therrien 1974).

## TABLE 5.2

## NOTATION AND IMPLIED INITIAL CONDITIONS

| Factor | Identification | Initial Conditions<br>Forward | <br>Reverse |
|---|---|---|---|
| $\overline{\underset{\sim}{u}}_1^T \, (*) \, \underset{\sim}{v}_1$ | $(\dagger)_{11}$ | voltage | voltage |
| $\overline{\underset{\sim}{u}}_1^T \, (*) \, \underset{\sim}{v}_2$ | $(\dagger)_{12}$ | voltage | current |
| $\overline{\underset{\sim}{u}}_2^T \, (*) \, \underset{\sim}{v}_1$ | $(\dagger)_{21}$ | current | voltage |
| $\overline{\underset{\sim}{u}}_2^T \, (*) \, \underset{\sim}{v}_2$ | $(\dagger)_{22}$ | current | current |

(*) denotes either $\underset{\sim}{A}$, $\delta\underset{\sim}{A}$, $\partial\underset{\sim}{A}/\partial\phi$ or $\Delta\underset{\sim}{A}$

(†) denotes Q, δQ, Q' or ΔQ, as taken from (5.4), (5.5), (5.6) or (5.7), respectively

## TABLE 5.3

## ANALYSES REQUIRED BY CERTAIN TERMS

| Term | Analysis Required |
|---|---|
| $\underset{\sim}{u}^T\underset{\sim}{v}$ | Forward and reverse (*conventional*) cascade analysis to *any* corresponding reference plane, whichever is convenient |
| $\underset{\sim}{u}_1^T\underset{\sim}{v}$, $\underset{\sim}{u}_2^T\underset{\sim}{v}$ | Preferably one *reverse* analysis to source reference plane (avoiding calculation of $\underset{\sim}{u}_1$ and $\underset{\sim}{u}_2$) |
| $\underset{\sim}{u}^T\underset{\sim}{v}_1$, $\underset{\sim}{u}^T\underset{\sim}{v}_2$ | Preferably one *forward* analysis to load reference plane (avoiding calculation of $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$) |
| $\overline{\underset{\sim}{u}}^T \cdot \underset{\sim}{v}$ | One forward analysis to input of $\underset{\sim}{A}$ and one reverse analysis to output of $\underset{\sim}{A}$ |
| $\overline{\underset{\sim}{u}}_1^T \cdot \underset{\sim}{v}$, $\overline{\underset{\sim}{u}}_2^T \cdot \underset{\sim}{v}$ | One full forward analysis to input of $\underset{\sim}{A}$ and one reverse analysis to output of $\underset{\sim}{A}$ |
| $\overline{\underset{\sim}{u}}^T \cdot \underset{\sim}{v}_1$, $\overline{\underset{\sim}{u}}^T \cdot \underset{\sim}{v}_2$ | One full reverse analysis to output of $\underset{\sim}{A}$ and one forward analysis to input of $\underset{\sim}{A}$ |
| $\overline{\underset{\sim}{u}}_1^T \cdot \underset{\sim}{v}_1$, $\overline{\underset{\sim}{u}}_1^T \cdot \underset{\sim}{v}_2$<br>$\overline{\underset{\sim}{u}}_2^T \cdot \underset{\sim}{v}_1$, $\overline{\underset{\sim}{u}}_2^T \cdot \underset{\sim}{v}_2$ | One full forward analysis to input of $\underset{\sim}{A}$ and one full reverse analysis to output of $\underset{\sim}{A}$ |

## TABLE 5.4

## FUNCTIONS OF INPUT CURRENT $I_S$ AND OUTPUT VOLTAGE $V_L$ FOR CHANGES IN $\underset{\sim}{A}$ ONLY

| Variable | Input | Output |
|---|---|---|
| $\underset{\sim}{A}$ | $I_S = V_S \frac{Q_{21}}{Q_{11}}$ | $V_L = \frac{V_S}{Q_{11}}$ |
| $\delta\underset{\sim}{A}$ | $\delta I_S = \frac{V_S \delta Q_{21} - I_S \delta Q_{11}}{Q_{11}}$ | $\delta V_L = - \frac{V_L^2}{V_S} \delta Q_{11}$ |
| $\frac{\partial \underset{\sim}{A}}{\partial \phi}$ | $\frac{\partial I_S}{\partial \phi} = \frac{V_S Q'_{21} - I_S Q'_{11}}{Q_{11}}$ | $\frac{\partial V_L}{\partial \phi} = - \frac{V_L^2}{V_S} Q'_{11}$ |
| $\Delta\underset{\sim}{A}$ | $\Delta I_S = \frac{V_S \Delta Q_{21} - I_S \Delta Q_{11}}{Q_{11} + \Delta Q_{11}}$ | $\Delta V_L = - \frac{V_L^2}{V_L + V_S/\Delta Q_{11}}$ |

## 5.4 First- and Second-order Sensitivities

The first-order sensitivity of $V_L$ w.r.t. a variable parameter $\phi_1$ is given using (5.16) by

$$\frac{\partial V_L}{\partial \phi_1} = \frac{-V_S \dfrac{\partial Q_{11}}{\partial \phi_1}}{Q_{11}^2} . \tag{5.17}$$

Differentiating (5.17) w.r.t. $\phi_2$ we get

$$\frac{\partial^2 V_L}{\partial \phi_2 \partial \phi_1} = - V_S \frac{\partial}{\partial \phi_2} \left[ \frac{\partial Q_{11}}{\partial \phi_1} / \, Q_{11}^2 \right]$$

$$= - V_L \, \frac{Q_{11} \dfrac{\partial^2 Q_{11}}{\partial \phi_2 \partial \phi_1} - 2 \dfrac{\partial Q_{11}}{\partial \phi_1} \dfrac{\partial Q_{11}}{\partial \phi_2}}{Q_{11}^2} . \tag{5.18}$$

The evaluation of $\partial Q_{11}/\partial \phi_1$ and $\partial Q_{11}/\partial \phi_2$ is straightforward (see Table 5.4). For the evaluation of the term $\partial^2 Q_{11}/\partial \phi_2 \partial \phi_1$, we assume that the variables are numbered consecutively from the source end to the load end so that this term is expressed, for example, by

$$\frac{\partial^2 Q_{11}}{\partial \phi_2 \partial \phi_1} = \frac{\partial}{\partial \phi_1} (\underset{\sim}{u}_1^T) \frac{\partial \underset{\sim}{A}}{\partial \phi_2} \; \underset{\sim}{v}_1 . \tag{5.19}$$

Note that $\bar{\underset{\sim}{u}}_1^T$ is a function of a certain chain matrix which contains the variable $\phi_1$, $\underset{\sim}{A}$ is the chain matrix containing $\phi_2$ and $\underset{\sim}{v}_1$ is evaluated at the reference plane following $\underset{\sim}{A}$.

## 5.5 The Evaluation of $V_L$ and its Sensitivities w.r.t. Design Parameters at all Vertices of the Tolerance Region

Algorithms concerned with finding worst vertices of the tolerance region need the value of the response at the vertices (Leung and Spence 1975) as well as the sensitivity of this reponse w.r.t. the design parameters (Bandler, Liu and Chen 1975, Tromp 1978).

Assume that we have partitioned the network by reference planes into subnetworks such that each subnetwork contains one chain matrix containing a variable parameter. Each reference plane is chosen to fall immediately after a variable element.

The Thevenin voltage/impedance of the ith subnetwork is considered as the source voltage/impedance of the (i+1)th subnetwork, given by (5.10) and (5.11), respectively, where j = i+1. We have to note here that the terms $Q_{11}^i$, $Q_{21}^i$, $Q_{12}^i$ and $Q_{22}^i$ are as defined in (5.4) with $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ set to $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$, respectively, since the appropriate reference plane immediately follows the element $\underset{\sim}{A}^i$. The number of pairs of terms $V_S^{i+1}$ and $Z_S^{i+1}$ to be evaluated is $2^i$, since each subnetwork contains one variable element with two extreme values (assuming that each $\underset{\sim}{A}^i$ contains only one variable parameter).

Differentiating (5.10) w.r.t. $\phi_h$, where $\phi_h$ does not belong to $\underset{\sim}{A}^i$, but $V_S^i$ and $Z_S^i$ are functions of $\phi_h$ (i.e., $\phi_h$ is in a subnetwork h before the ith subnetwork) we get

$$\frac{\partial V_S^{i+1}}{\partial \phi_h} = \frac{(Q_{11}^i + Z_S^i\, Q_{21}^i)\, \frac{\partial V_S^i}{\partial \phi_h} - V_S^i\, \frac{\partial Z_S^i}{\partial \phi_h}\, Q_{21}^i}{(Q_{11}^i + Z_S^i\, Q_{21}^i)^2}\,, \tag{5.20}$$

and differentiating (5.11) w.r.t. $\phi_h$, we get

$$\frac{\partial Z_S^{i+1}}{\partial \phi_h} = \frac{(Q_{11}^i + Z_S^i\, Q_{21}^i)\, \frac{\partial Z_S^i}{\partial \phi_h}\, Q_{22}^i - (Q_{21}^i + Z_S^i\, Q_{22}^i)\, \frac{\partial Z_S^i}{\partial \phi_h}\, Q_{21}^i}{(Q_{11}^i + Z_S^i\, Q_{21}^i)^2}$$

$$= \frac{\partial Z_S^i}{\partial \phi_h}\, \frac{(Q_{11}^i Q_{22}^i - Q_{12}^i Q_{21}^i)}{(Q_{11}^i + Z_S^i\, Q_{21}^i)^2}\,. \tag{5.21}$$

On the other hand, the derivatives w.r.t. $\phi_i$ which is contained in $\underset{\sim}{A}^i$ ($Z_S^i$ and $V_S^i$ are not functions of $\phi_i$), are

$$\frac{\partial V_S^{i+1}}{\partial \phi_i} = \frac{-V_S^i\, (\frac{\partial Q_{11}^i}{\partial \phi_i} + Z_S^i\, \frac{\partial Q_{21}^i}{\partial \phi_i})}{(Q_{11}^i + Z_S^i\, Q_{21}^i)^2} \tag{5.22}$$

and

$$\frac{\partial Z_S^{i+1}}{\partial \phi_i} = \frac{(Q_{11}^i + Z_S^i Q_{21}^i)\left(\frac{\partial Q_{21}^i}{\partial \phi_i} + Z_S^i \frac{\partial Q_{22}^i}{\partial \phi_i}\right) - (Q_{12}^i + Z_S^i Q_{22}^i)\left(\frac{\partial Q_{11}^i}{\partial \phi_i} + Z_S^i \frac{\partial Q_{21}^i}{\partial \phi_i}\right)}{(Q_{11}^i + Z_S^i Q_{21}^i)^2}, \tag{5.23}$$

where $\frac{\partial Q_{11}^i}{\partial \phi_i}$, $\frac{\partial Q_{21}^i}{\partial \phi_i}$, $\frac{\partial Q_{12}^i}{\partial \phi_i}$ and $\frac{\partial Q_{22}^i}{\partial \phi_i}$ correspond to (5.6) and Table 5.2. This sensitivity information is carried out through the analysis for each subnetwork. The number of variables for which sensitivities of $V_S^{i+1}$ and $Z_S^{i+1}$ exist at the (i+1)th subnetwork is $i$ so that $2^i \cdot i$ sensitivity calculations are performed. Having $Y_L$ and $I_L$ as zeros, the expression relating $V_L$ and the last sets of $V_S$ and $Z_S$, is given by (5.10), so that $2^k$ values for $V_L$ and its sensitivities can be obtained from appropriate values of $V_S$, $Z_S$ and $\underset{\sim}{A}$.

## 5.6 Branched Circuits

Consider, as an example, the cascaded circuit shown in Fig. 5.6, which has two branches, one connected in series and one in parallel. In the series and parallel branches we highlight, for example, the elements $\underset{\sim}{B}$ and $\underset{\sim}{C}$, respectively. The series branch can be thought of equivalently as an element consisting of a series impedance connected in cascade with the main circuit as

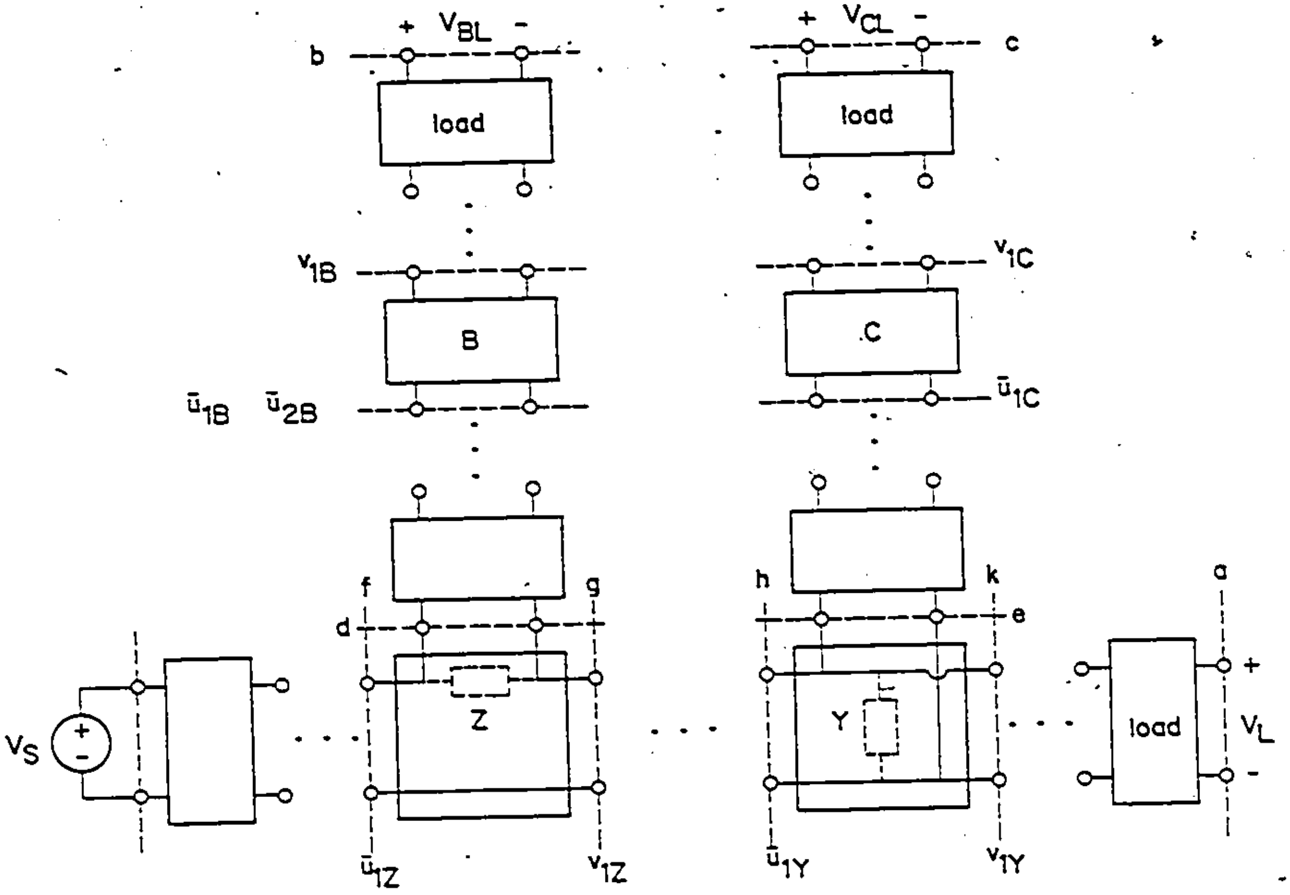

+ $V_{BL}$ -
b
load
$v_{1B}$
B
$\bar{u}_{1B}$ $\bar{u}_{2B}$
+ $V_{CL}$ -
c
load
$v_{1C}$
C
$\bar{u}_{1C}$
f
g
d
h
k
e
a
$V_S$
+
-
Z
Y
load
$V_L$
$\bar{u}_{1Z}$
$v_{1Z}$
$\bar{u}_{1Y}$
$v_{1Y}$

Fig. 5.6 An example of a cascaded circuit with a branch connected in series and a branch connected in parallel. Branches are represented in the cascade by their equivalents. Reference planes where different analyses are initiated are labelled.

shown in Fig. 5.6. This impedance Z may be taken as the inverse of the input admittance derived in (5.12) and is given by

$$Z = \frac{\bar{\underset{\sim}{u}}^T_{1B} \; \underset{\sim}{B} \; \underset{\sim}{v}_{1B}}{\bar{\underset{\sim}{u}}^T_{2B} \; \underset{\sim}{B} \; \underset{\sim}{v}_{1B}} , \qquad (5.24)$$

where the subscript B distinguishes terms associated with the branch from that of the cascaded main circuit. The forward analysis is initiated at reference plane d and the reverse analysis is initiated at reference plane b. (See Fig. 5.6.)

Similarly, the parallel branch can be thought of equivalently as an admittance Y connected in shunt in the cascade. The admittance Y (as in (5.12)) is given by

$$Y = \frac{\bar{\underset{\sim}{u}}^T_{2C} \; \underset{\sim}{C} \; \underset{\sim}{v}_{1C}}{\bar{\underset{\sim}{u}}^T_{1C} \; \underset{\sim}{C} \; \underset{\sim}{v}_{1C}} , \qquad (5.25).$$

where the forward analysis is initiated at reference plane e and the reverse analysis is initiated at reference plane c.

Different formulas relating the load voltages of the branches to the variables can be derived. The load voltage of the series branch can be derived (Appendix C.1) as a function of $\underset{\sim}{B}$ as

$$V_{BL}(\underset{\sim}{B}) = \frac{\underset{\sim}{e}_2^T \, \underset{\sim}{v}_{1Z} \, V_S}{\overline{\underset{\sim}{u}}{}_{2B}^T \, \underset{\sim}{B} \, \underset{\sim}{v}_{1B} \, \overline{\underset{\sim}{u}}{}_{1Z}^T \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} \underset{\sim}{v}_{1Z}}, \tag{5.26}$$

where

$\overline{\underset{\sim}{u}}{}_{1Z}^T$ is the result at reference plane f of a forward analysis initiated at the source,

$\underset{\sim}{v}_{1Z}$ is the result at reference plane g of a reverse analysis initiated at the load reference plane a.

It can also be obtained (Appendix C.2) as a function of $\underset{\sim}{C}$ as

$$V_{BL}(\underset{\sim}{C}) = \frac{[\overline{\underset{\sim}{u}}{}_{1Yf}^T - \overline{\underset{\sim}{u}}{}_{1Yg}^T] \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} \underset{\sim}{v}_{1Y} \, V_S}{\overline{\underset{\sim}{u}}{}_{1B}^T \, \underset{\sim}{B} \, \underset{\sim}{v}_{1B} \, \overline{\underset{\sim}{u}}{}_{1Y}^T \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} \underset{\sim}{v}_{1Y}}, \tag{5.27}$$

where

$\overline{\underset{\sim}{u}}{}_{1Y}^T$ is the result at reference plane h of a forward analysis,

$\underset{\sim}{v}_{1Y}$ is the result at reference plane k of a reverse analysis,

$\overline{\underset{\sim}{u}}{}_{1Yf}^T$ is the result at reference plane h of a forward analysis initiated at reference plane f,

$\overline{\underset{\sim}{u}}{}_{1Yg}^T$ is the result at reference plane h of a forward analysis initiated at reference plane g.

The load voltage of the parallel branch can also be derived

(Appendix C.3) as a function of $\underset{\sim}{C}$ as

$$V_{CL}(\underset{\sim}{C}) = \frac{\underset{\sim}{e}_1^T \; \underset{\sim}{v}_{1Y} \; V_S}{\overline{\underset{\sim}{u}}_{1C}^T \; \underset{\sim}{C} \; \underset{\sim}{v}_{1C} \; \overline{\underset{\sim}{u}}_{1Y}^T \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} \underset{\sim}{v}_{1Y}} , \qquad (5.28)$$

and (Appendix C.4) as a function of $\underset{\sim}{B}$ as

$$V_{CL}(\underset{\sim}{B}) = \frac{\underset{\sim}{e}_1^T \; \underset{\sim}{v}_{1Y} \; V_S}{\overline{\underset{\sim}{u}}_{1C}^T \; \underset{\sim}{C} \; \underset{\sim}{v}_{1C} \; \overline{\underset{\sim}{u}}_{1Z}^T \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} \underset{\sim}{v}_{1Z}} . \qquad (5.29)$$

## 5.7 Algorithms

### 5.7.1 Two Algorithms for Evaluation of Large Changes

The two following algorithms are used to obtain responses at the base points for the multidimensional quadratic interpolation (Bandler and Abdel-Malek 1979). The first is used when one parameter at a time is perturbed and the second is used when pairs of parameters are perturbed simultaneously.

```
Algorithm 1   Multiple One-at-a-time Changes
Step 1        Initialize $\overline{\underset{\sim}{u}}$ and $\underset{\sim}{v}$.
              Set $i \leftarrow 1$, $m \leftarrow 1$, $j \leftarrow n$.
Comment       n is the total number of elements in the cascade and
              m is a counter for the variable elements.
```

```
Step 2     If $i = \ell_m$ go to Step 5.
Comment    $\ell_m$ is an element of L, an index set containing
           superscripts of the k matrices containing the k
           variable parameters and ordered consecutively. It
           is assumed that each matrix contains only one
           variable.
Step 3     $\underline{\bar{u}}^T \leftarrow \underline{\bar{u}}^T \underline{A}^i$.
           $i \leftarrow i + 1$.
Step 4     If $i = \ell_m$ go to Step 5.
           Go to Step 3.
Step 5     Let $\underline{x}^m \leftarrow \underline{\bar{u}}$.
           If $i = \ell_k$ go to Step 7.
Comment    $\underline{x}^1, \underline{x}^2, \ldots, \underline{x}^k$ are working arrays to store the $\underline{\bar{u}}$
           vectors required in the evaluation of the large
           changes taking place.
Step 6     $m \leftarrow m + 1$. Go to Step 3.
Step 7     If $n = \ell_k$ go to Step 10.
Step 8     $\underline{v} = \underline{A}^j \underline{v}$.
           $j \leftarrow j - 1$.
Step 9     If $j = \ell_m$ go to Step 10.
           Go to Step 8.
Step 10    Evaluate Q using the stored $\underline{x}^m$, $\underline{v}$ and the perturbed
           $\underline{A}^j$. If $j = \ell_1$ stop.
Comment    Positive and negative extremes of the variable in $\underline{A}^j$
           are considered simultaneously.
```

Step 11 $m \leftarrow m - 1$. Go to Step 8.

## Algorithm 2 Multiple Pairwise Changes

This algorithm is for evaluating the response at the $k(k-1)/2$ base points where two parameters are perturbed at a time. At the first $k-1$ points following those considered in Algorithm 1 the parameters indicated by the subscripts

$$1,2 \quad 1,3 \quad \ldots \quad 1,k$$

are changed; at the next $k-2$ points the parameters indicated by the subscripts

$$2,3 \quad 2,4 \quad \ldots \quad 2,k$$

are changed, and so on, until the final point at which parameters $k-1$ and $k$ are perturbed. Figure 5.7 serves to illustrate the analyses involved.

Step 1 Initialize $\bar{\underline{u}}_1^0$, $\bar{\underline{u}}_2^0$, $\bar{\underline{u}}_1^1$ and $\bar{\underline{u}}_2^1$.

Set $i \leftarrow 1$, $m \leftarrow 1$, $q \leftarrow 0$, $r \leftarrow 1$ and $s \leftarrow k - 1$.

Comment $\underline{u}_1^1$ and $\underline{u}_2^1$ are vectors to be initialized as $\underline{u}_1^0$ and $\underline{u}_2^0$, respectively. They have the same role as $\underline{u}_1^0$ and $\underline{u}_2^0$ in the forward analysis initiated at a reference plane immediately following the first variable element.

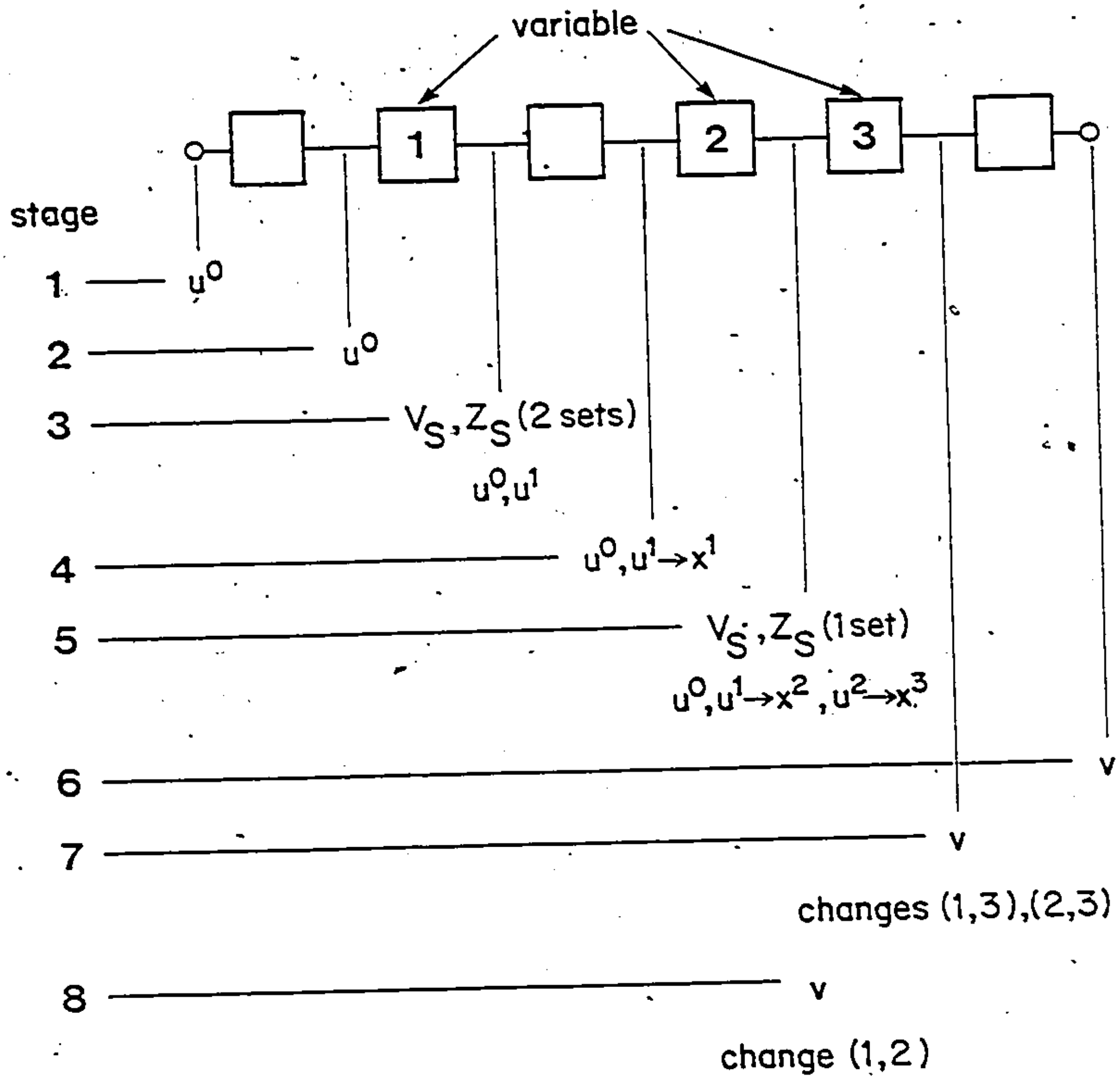

variable
1
2
3
stage
1
$u^0$
2
$u^0$
3
$V_S, Z_S$ (2 sets)
$u^0, u^1$
4
$u^0, u^1 \to x^1$
5
$V_S, Z_S$ (1 set)
$u^0, u^1 \to x^2, u^2 \to x^3$
6
v
7
v
changes (1,3),(2,3)
8
v
change (1,2)

Fig. 5.7 Illustration for a cascade of 6 two-ports of the principal stages in the calculations involved in the multiple pairwise changes algorithm. Three variable elements are considered, hence three sets of simultaneous analyses are effectively performed.

```
Step 2     If $i = \ell_m$ go to Step 4.
Comment    $\ell_m$ is an element of L, an index set containing
           superscripts of the k matrices containing the k
           variable parameters as indicated in Algorithm 1.
Step 3     $\underset{\sim}{u}_1^{0T} \leftarrow \underset{\sim}{u}_1^{0T} \underset{\sim}{A}^i$.
           $\underset{\sim}{u}_2^{0T} \leftarrow \underset{\sim}{u}_2^{0T} \underset{\sim}{A}^i$.
Step 4     If $m = 1$ go to Step 5.
           $\underset{\sim}{u}_1^{1T} \leftarrow \underset{\sim}{u}_1^{1T} \underset{\sim}{A}^i$.
           $\underset{\sim}{u}_2^{1T} \leftarrow \underset{\sim}{u}_2^{1T} \underset{\sim}{A}^i$.
           .
           .
           $\underset{\sim}{u}_1^{qT} \leftarrow \underset{\sim}{u}_1^{qT} \underset{\sim}{A}^i$.
           $\underset{\sim}{u}_2^{qT} \leftarrow \underset{\sim}{u}_2^{qT} \underset{\sim}{A}^i$.
Comment    This step is not performed until we reach a variable
           element, since the analyses involving the $\underset{\sim}{u}^j$ do not
           begin until the jth variable element has been
           considered.
Step 5     Set $i \leftarrow i + 1$.
Step 6     If $i = \ell_m$ go to Step 7.
           Go to Step 3.
Step 7     If $m = k$ go to Step 9.
           Calculate the Thevenin impedances and voltages
           $Z_S(m,1), \ldots, Z_S(m,s)$,
           $V_S(m,1), \ldots, V_S(m,s)$.
           $s \leftarrow s - 1$.
```

```
Comment    For the first variable element k-1 sets of $Z_S$ and $V_S$ have to be evaluated since changes in this element will be coupled one at a time with changes in the next k-1 variable elements.  For the second variable element k-2 sets of $Z_S$ and $V_S$ are calculated and so on.  See Fig. 5.7.
Step 8     If m = 1 go to Step 13.
Step 9     Set $p \leftarrow 1$.
Comment    p is an internal counter.
Step 10    $\underline{x}^r \leftarrow \underline{u}^p$.
           If p = q go to Step 12.
Comment    When the analysis has reached a reference plane immediately preceding an element containing a variable whose change is to be associated with any previously encountered variable a snapshot of the appropriate $\underline{u}$ vectors is taken and stored in the $\underline{x}$ arrays.  See Fig. 5.7.
Step 11    Set $r \leftarrow r + 1$.
           $p \leftarrow p + 1$.
           Go to Step 10.
Step 12    Set $r \leftarrow r + 1$.
Step 13    If m = k go to Step 16.
           $\underline{u}_1^{0T} \leftarrow \underline{u}_1^{0T}\,\underline{A}^i$.
           $\underline{u}_2^{0T} \leftarrow \underline{u}_2^{0T}\,\underline{A}^i$.
```

Step 14 If $m = 1$ go to Step 15.

$\underline{u}_1^{1T} \leftarrow \underline{u}_1^{1T}\, \underline{A}^i$.

$\underline{u}_2^{1T} \leftarrow \underline{u}_2^{1T}\, \underline{A}^i$.

$\vdots$

$\underline{u}_1^{qT} \leftarrow \underline{u}_1^{qT}\, \underline{A}^i$.

$\underline{u}_2^{qT} \leftarrow \underline{u}_2^{qT}\, \underline{A}^i$.

Comment In Step 7 we calculated sets of $Z_S$ and $V_S$ accounting for variations in $\underline{A}^i$. In Steps 13 and 14, however, we carry forward the analyses for which $\underline{A}^i$ is considered fixed.

Step 15 Set $i \leftarrow i + 1$.

$m \leftarrow m + 1$.

$q \leftarrow q + 1$.

Initialize $\underline{u}_1^{qT}$ and $\underline{u}_2^{qT}$ and go to Step 6.

Comment $\underline{u}_1^{qT}$ and $\underline{u}_2^{qT}$ are initialized to start a forward analysis at a reference plane immediately following a variable element $\underline{A}^i$.

Step 16 Set $r \leftarrow r - 1$.

$m \leftarrow m - 1$.

Initialize $\underline{v}_1$ and $\underline{v}_2$.

Comment At this step we start the analysis from the load end.

Step 17 If $n = \ell_k$ go to Step 20.

Set $j \leftarrow n$.

Comment n is the total number of elements in the cascade.

```
Step 18.    $\underline{v}_1 \leftarrow \underline{A}^j \underline{v}_1$.
            $\underline{v}_2 \leftarrow \underline{A}^j \underline{v}_2$.
            $j \leftarrow j - 1$.
Step 19     If $j = \ell_m$ go to Step 20.
            Go to Step 18.
Step 20     $p \leftarrow 1$.
Step 21     Calculate Q using $V_S$, $Z_S$, $\underline{A}^j$ and $\underline{v}$ and the
            appropriate $\underline{x}$.
Comment     When we reach the kth variable element we calculate
            k-1 values of Q, and when the variable element k-1
            is reached we calculate k-2 values of Q and so on as
            illustrated in Fig. 5.7.
Step 22     If $p = q$ go to Step 23.
            Set $r \leftarrow r - 1$.
            $p \leftarrow p + 1$.
            Go to Step 21.
Step 23     If $m = 1$ Stop.
            Set $q \leftarrow q - 1$.
            $m \leftarrow m - 1$.
            Go to Step 18.
```

### 5.7.2 First- and Second-order Sensitivities

The following algorithm, which is similar to Algorithm 2 can be used to obtain the first- and second-order sensitivities of

$V_L$ w.r.t. the design variables. Figure 5.8 illustrates the main stages of the algorithm.

```
Algorithm 3
Step 1      Initialize $\underset{\sim}{u}^0$ and $\underset{\sim}{v}$.
            Set $i \leftarrow 1$, $m \leftarrow 1$, $q \leftarrow 0$, $r \leftarrow 1$, $j \leftarrow n$.
Comment     n is the total number of elements in the cascade.
Step 2      If $i = \ell_m$ go to Step 6.
Comment     $\ell_m$ is an element of L, an index set containing
            superscripts of the k matrices containing the k
            variable parameters and ordered consecutively.
Step 3      $\underset{\sim}{u}^{0T} \leftarrow \underset{\sim}{u}^{0T} \underset{\sim}{A}^i$.
            If m = 1 go to Step 4.
            $\underset{\sim}{u}^{1T} \leftarrow \underset{\sim}{u}^{1T} \underset{\sim}{A}^i$.
            .
            .
            .
            $\underset{\sim}{u}^{qT} \leftarrow \underset{\sim}{u}^{qT} \underset{\sim}{A}^i$.
Comment     $\underset{\sim}{u}^1, \underset{\sim}{u}^2, \ldots, \underset{\sim}{u}^q$ are working arrays used to proceed
            with the evaluation of the gradients of $\underset{\sim}{u}^0$ w.r.t.
            the q variables already passed by the forward
            analysis.
Step 4      Set $i \leftarrow i+1$.
Step 5      If $i = \ell_m$ go to Step 6.
            Go to Step 3.
```

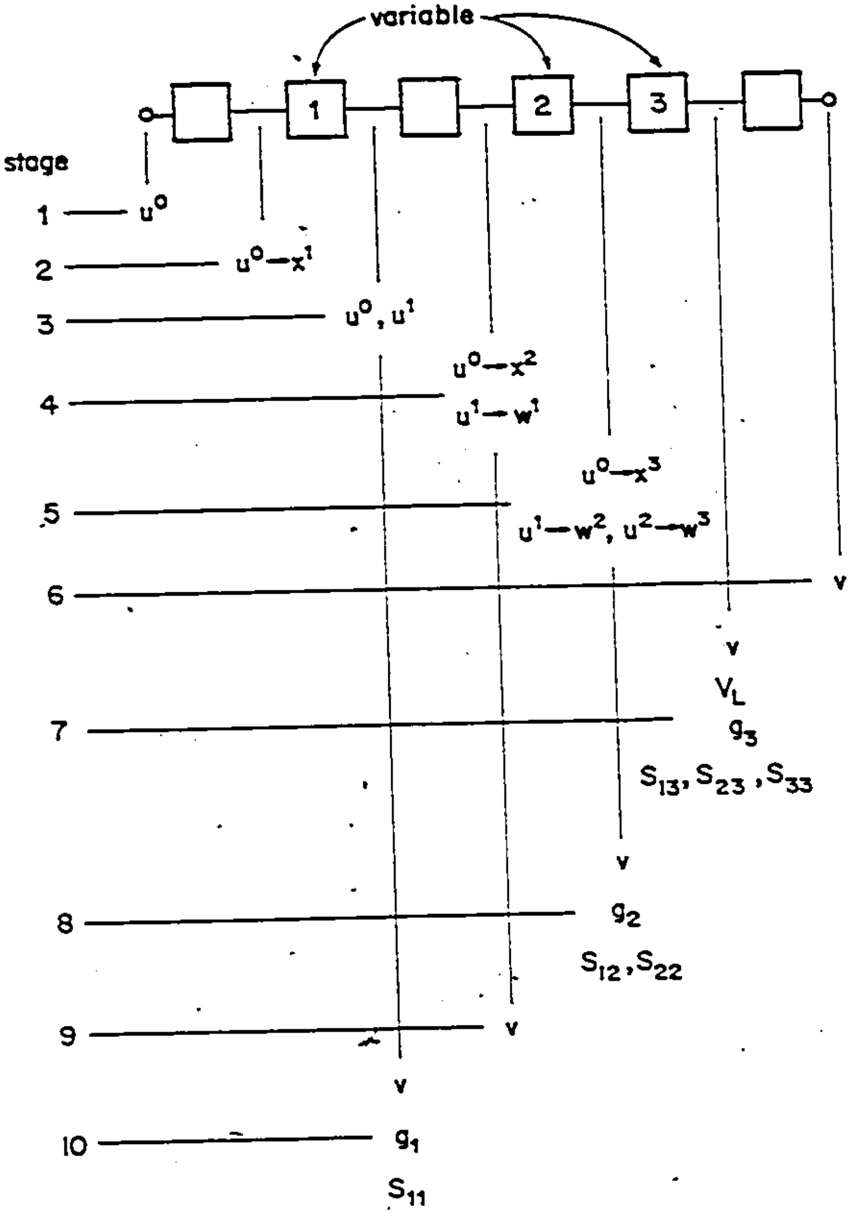

variable
1
2
3
stage
1
u⁰
2
u⁰—x¹
3
u⁰, u¹
4
u⁰—x²
u¹—w¹
5
u⁰—x³
u¹—w², u²—w³
6
v
v
V_L
7
g₃
S₁₃, S₂₃, S₃₃
v
8
g₂
S₁₂, S₂₂
9
v
v
10
g₁
S₁₁

Fig. 5.8 Illustration of the principal stages in the calculation of first- and second-order sensitivities w.r.t. three variable elements.

$g_i \triangleq \partial Q_{11}/\partial \phi_i$ , $i = 1,2,3,$

$S_{ij} \triangleq \partial^2 Q_{11}/\partial \phi_i \partial \phi_j$ , $i,j = 1,2,3.$

```
Step 6      $\underline{x}^m \leftarrow \underline{u}^0$.
            If m = 1 go to Step 10.
Comment     Once a variable element is reached the $\underline{u}^0$ is stored in $\underline{x}^m$ to be used in the calculation of the first-order sensitivity.
Step 7      Set p ← 1.
Step 8      $\underline{w}^r \leftarrow \underline{u}^p$.
            r ← r+1.
            If p = q go to Step 10.
Comment     The $\underline{w}$ arrays are used to store the appropriate gradients of $\underline{u}^0$, namely, $\underline{u}^1$, $\underline{u}^2$, ..., for the calculation of second-order sensitivities.
Step 9      Set p ← p+1.
            Go to Step 8.
Step 10     If m=k go to Step 12.
            $\underline{u}^{mT} \leftarrow \underline{u}^{0T} \dfrac{\partial \underline{A}^i}{\partial \phi_m}$.
            $\underline{u}^{0T} \leftarrow \underline{u}^{0T} \underline{A}^i$.
            $\underline{u}^{1T} \leftarrow \underline{u}^{1T} \underline{A}^i$.
            $\vdots$
            $\underline{u}^{(m-1)T} \leftarrow \underline{u}^{(m-1)T} \underline{A}^i$.
Comment     At this step a new $\underline{u}$ is introduced which is equal to $\underline{u}^0$ multiplied by the derivative of $\underline{A}^i$ w.r.t. $\phi_m$, where $\underline{A}^i$ is a function of $\phi_m$ only.
```

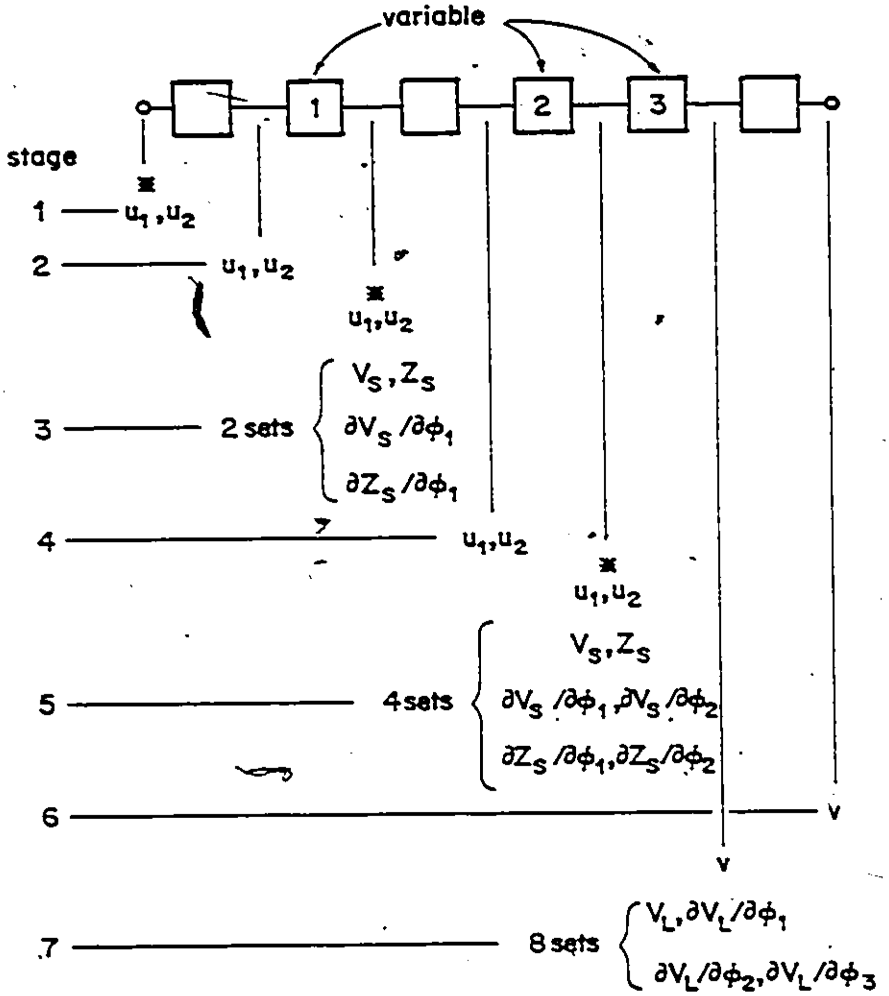
```
Step 11     Set $i \leftarrow i+1$.
            $m \leftarrow m+1$.
            $q \leftarrow q+1$.
            Go to Step 5.
Step 12     Set $r \leftarrow r-1$.
            If $n = \ell_k$ go to Step 15.
Step 13     $\underset{\sim}{v} \leftarrow \underset{\sim}{A}^j \underset{\sim}{v}$.
            $j \leftarrow j-1$.
Comment     This step is concerned with the reverse analysis.
Step 14     If $j = \ell_m$ go to Step 15.
            Go to Step 13.
Step 15     Calculate $\partial Q/\partial\phi_m$ and $\partial^2Q/\partial\phi_m^2$.
Comment     At this point the first-order derivative of Q w.r.t.
            $\phi_m$ can be evaluated, since $\underset{\sim}{u}^0$ and $\underset{\sim}{v}$ at the reference
            planes before and after the element are known.
            $\partial^2Q/\partial\phi_m^2$ is evaluated using $\underset{\sim}{u}^0$, $\underset{\sim}{v}$ and $\partial^2\underset{\sim}{A}^j/\partial\phi_m^2$.
Step 16     If $m = 1$ stop.
            Set $s \leftarrow m - 1$.  $p \leftarrow 1$.
Step 17     Calculate $\partial^2Q/\partial\phi_s\partial\phi_m$.
            If $p=q$ go to Step 19.
Comment     $\partial^2Q/\partial\phi_s\partial\phi_m$ is evaluated using the appropriate $\underset{\sim}{w}^r$,
            $\partial\underset{\sim}{A}^j/\partial\phi_m$ and $\underset{\sim}{v}$, where $s = 1, \ldots, m-1$.
Step 18     Set $p \leftarrow p + 1$.
            $r \leftarrow r - 1$.
            $s \leftarrow s - 1$.
            Go to Step 17.
Step 19     Set $q \leftarrow q - 1$.
            $m \leftarrow m - 1$.
            $r \leftarrow r - 1$.
            Go to Step 14.
```

### 5.7.3 Response Value and its Derivatives w.r.t. all Variable Parameters, at all Vertices of the Tolerance Region

Figure 5.9 shows an example of the stages involved in the following algorithm to obtain the response and its sensitivities at the vertices (3 variables ==> 8 vertices) of the tolerance region.

Algorithm 4

```
Step 1      Initialize $\underset{\sim}{u}_1$, $\underset{\sim}{u}_2$ ~~and $\underset{\sim}{v}$~~.
            Set $i \leftarrow 1$, $m \leftarrow 1$, $j \leftarrow n$.
Step 2      If $i = \ell_m$ go to Step 6.
Step 3      $\overline{\underset{\sim}{u}}_1^T \leftarrow \overline{\underset{\sim}{u}}_1^T \underset{\sim}{A}^i$.
            $\overline{\underset{\sim}{u}}_2^T \leftarrow \overline{\underset{\sim}{u}}_2^T \underset{\sim}{A}^i$.
            Set $i \leftarrow i + 1$.
Step 4      If $i = \ell_m$ go to Step 5.
            Go to Step 3.
Step 5      If m=k go to Step 7.
Step 6      Calculate $V_S$, $Z_S$ ,
            $\frac{\partial V_S}{\partial \phi_1}, \ldots, \frac{\partial V_S}{\partial \phi_m}$ ,
```


variable
1
2
3
stage
1
*
u1,u2
2
u1,u2
*
u1,u2
3
2 sets
Vs,Zs
∂Vs/∂φ1
∂Zs/∂φ1
4
u1,u2
*
u1,u2
5
4 sets
Vs,Zs
∂Vs/∂φ1,∂Vs/∂φ2
∂Zs/∂φ1,∂Zs/∂φ2
6
V
V
7
8 sets
VL,∂VL/∂φ1
∂VL/∂φ2,∂VL/∂φ3

* denotes initialization of $u_1, u_2$

Fig. 5.9 Illustration of the principal stages of Algorithm 4.

```
         $\frac{\partial Z_S}{\partial \phi_1}, \ldots, \frac{\partial Z_S}{\partial \phi_m}$,
         $2^m$ sets all together.
         Set $m \leftarrow m + 1$.
         $i \leftarrow i + 1$.
         Initialize $\underset{\sim}{u}_1$ and $\underset{\sim}{u}_2$ and go to Step 4.
Step 7   If $n = \ell_k$ go to Step 10.
Step 8   $\underset{\sim}{v} = \underset{\sim}{A}^j \underset{\sim}{v}$.
         Set $j \leftarrow j-1$.
Step 9   If $j = \ell_k$ go to Step 10.
         Go to Step 8.
Step 10  Calculate $Q$, $\partial Q/\partial \phi_1, \ldots, \partial Q/\partial \phi_k$ $2^k$ times.
         Stop.
```

## 5.8 Numerical Example

The cascaded seven-section bandpass filter shown in Fig. 5.10 (Horton and Wenzel 1965, Bandler, Charalambous, Chen and Chu 1976) serves as a numerical example. All sections are quarter-wave at 2.175 GHz. The normalized minimax characteristic impedances are (Bandler, Charalambous, Chen and Chu 1976)

$$Z_1^0 = Z_7^0 = 0.606463$$
$$Z_2^0 = Z_6^0 = 0.303051$$
$$Z_3^0 = Z_5^0 = 0.722061$$
$$Z_4^0 = 0.235593$$

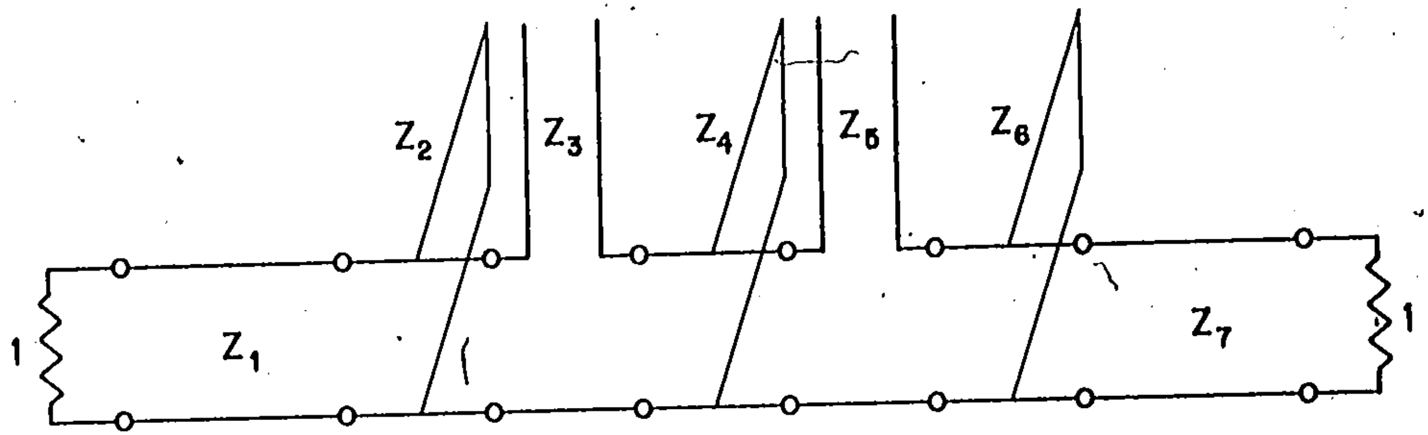

1
Z1
Z2
Z3
Z4
Z5
Z6
Z7
1

Fig. 5.10 Seven-section filter containing unit elements and stubs. All sections are quarter-wave at 2.175 GHz.

The output voltage $V_L$ at a normalized frequency of 0.7 is $0.49740790 - j3.9011594\times10^{-3}$, verified twice using (5.10): once associating $\underline{A}^i$ with $Z_3$ and once with $Z_4$. Furthermore, one analysis yielded

$$V_L(Z_4^0+0.03) = 0.49838950 - j\ 0.034901610$$
$$V_L(Z_4^0-0.03) = 0.49062912 + j\ 0.034959186$$

The open-circuit voltage at the load end was calculated using (5.10) as

$$V_{OC} = 0.98624507 + j\ 0.092266904$$

and the Thevenin impedance using (5.11) is

$$Z_{TH} = 0.98119253 + j\ 0.20103391$$

which further verified $V_L$.

One analysis taking $\epsilon_2 = 0.021$, $\epsilon_5 = 0.024$ yielded

$$V_L(Z_2^0-\epsilon_2, Z_5^0-\epsilon_5) = 0.49719716 + j\ 2.2191360\times10^{-3}$$
$$V_L(Z_2^0+\epsilon_2, Z_5^0-\epsilon_5) = 0.49583538 - j\ 2.3636314\times10^{-2}$$
$$V_L(Z_2^0-\epsilon_2, Z_5^0+\epsilon_5) = 0.49732462 + j\ 1.7909912\times10^{-2}$$
$$V_L(Z_2^0+\epsilon_2, Z_5^0+\epsilon_5) = 0.49751427 - j\ 8.3726470\times10^{-3}$$

A multidimensional quadratic approximation was carried out for $V_L$ following the approach of Bandler and Abdel-Malek (1978a). The variables for the approximation were the characteristic impedances as well as the normalized frequency. The circuit responses at 45 base points (which is equal to (k+1)(k+2)/2, where k is 8) were needed to evaluate the coefficients of the quadratic polynomial approximating the response function (Bandler and Abdel-Malek 1978a). A base point is a point where the approximation and the actual function coincide. The center base point, which is the center of the interpolation region in which the approximation is assumed to be valid, had the characteristic impedances given before and a normalized frequency of 0.7. 16 base points were determined (using Algorithm 1) by varying one parameter at a time by $\pm\delta$ w.r.t. its value at the center of interpolation. For the characteristic impedances $\delta$ was chosen to be 0.03 and for the normalized frequency it was 0.01. At the remaining 28 base points only two parameters were perturbed at a time from their values at the center of interpolation by a percentage of their $\delta$ and Algorithm 2 was used to evaluate the response at these points.

Note that when the normalized frequency was perturbed a whole new analysis had to be performed.

The symmetry of the structure was taken into consideration in choosing these base points. Letting $\underline{\bar{\phi}}$ be the center of the interpolation region, the base points can be expressed by (Abdel-

Malek and Bandler 1978b)

$$[\underset{\sim}{\phi}^1 \quad \underset{\sim}{\phi}^2 \ \cdots \ \underset{\sim}{\phi}^N] = \underset{\sim}{D}\,[\underset{\sim}{1}_k \quad -\underset{\sim}{1}_k \quad \underset{\sim}{B} \quad \underset{\sim}{0}_k] + [\bar{\underset{\sim}{\phi}} \quad \bar{\underset{\sim}{\phi}} \ \cdots \ \bar{\underset{\sim}{\phi}}], \qquad (5.30)$$

where ,

N is equal to 45 in our case,

$\underset{\sim}{1}_k$ is a k-dimensional identity matrix,

$\underset{\sim}{0}_k$ is a zero vector of dimension k,

$$\underset{\sim}{D} = \begin{bmatrix} 0.03 & & & & \\ & 0.03 & & & \\ & & \ddots & & \\ & & & 0.03 & \\ & & & & 0.01 \end{bmatrix},$$

and

$\underset{\sim}{B}$ is a k x [k(k-1)/2] matrix given by

$$\underset{\sim}{B} = \left[\begin{array}{cccccccccccccccccccccccccccc}
.5 & -.2 & .9 & -.2 & .5 & .3 & .7 \\
.7 & & & & & & & .7 & .5 & .7 & .9 & .7 & -.3 \\
 & .8 & & & & & & .8 & & & & & & .7 & -.4 & .8 & .8 & .8 \\
 & & -.5 & & & & & & .9 & & & & & -.9 & & & & & -.9 & .9 & -.5 & .8 \\
 & & & .8 & & & & & & .8 & & & & & .8 & & & & .7 & & & & .8 & .8 & .8 \\
 & & & & .7 & & & & & & -.2 & & & & & .7 & & & & .5 & & & .7 & & & .7 & -.3 \\
 & & & & & -.9 & & & & & & .5 & & & & & -.2 & & & & .9 & & & -.2 & & .5 & & .7 \\
 & & & & & & .6 & & & & & & .4 & & & & & .9 & & & & .4 & & & .9 & & .4 & .6
\end{array}\right]$$

Examining this $\underline{B}$ matrix we note that the entries for perturbing two parameters at a time are the same as for their corresponding symmetrical parameters. The choice of base points given by (5.30) preserves symmetry in the appropriate coefficients of the multidimensional polynomials.

Taking the optimal minimax characteristic impedances (Bandler, Charalambous, Chen and Chu 1976):

$$Z_1 = Z_7 = 0.606595$$
$$Z_2 = Z_6 = 0.303547$$
$$Z_3 = Z_5 = 0.722287$$
$$Z_4 = 0.235183$$

and calculating the group delay using the derivative of $V_L$ w.r.t. $\omega$ obtained from the quadratic approximation yielded

$$T_G = 0.893 \text{ ns},$$

while the exact group delay is (Bandler, Rizk and Tromp 1976)

$$T_{G_{exact}} = 0.895 \text{ ns}.$$

The sensitivity of the output voltage $V_L$ w.r.t. length $\ell_4$ of the fourth section and the sensitivity w.r.t. $Z_4$ are evaluated at a normalized frequency of 0.5 as

$$\frac{\partial V_L}{\partial \ell_4} = -0.2804064 + j0.5161026$$

$$\frac{\partial V_L}{\partial Z_4} = -2.617364 + j4.817395$$

Without any further effort (since the two parameters belong to the same element) we obtain

$$\frac{\partial^2 V_L}{\partial Z_4 \partial \ell_4} = 11.71675 + j5.415667$$

Table 5.5 compares the results obtained by this method and the one obtained by the adjoint-network method (Rizk 1975). Taking two parameters in different elements, for example $Z_4$ and $Z_5$, we obtain the second-order term

$$\frac{\partial^2 V_L}{\partial Z_4 \partial Z_5} = -30.12383 - j7.516802.$$

A tolerance of ±0.03 on $Z_1$, $Z_4$ and $Z_5$ was chosen. Algorithm 4 was used to evaluate $V_L$, $\partial V_L/\partial Z_1$, $\partial V_L/\partial Z_4$ and $\partial V_L/\partial Z_5$ at the eight vertices of the tolerance region ($2^3$ vertices where 3 is the number of toleranced variables). The results are tabulated in Table 5.6. They were checked individually by reanalyzing the circuit at each vertex.

TABLE 5.5

COMPARISON OF SECOND-ORDER SENSITIVITIES WITH DIFFERENT APPROACHES

| Term | Adjoint Network | 1st Order Sensitivity by Adjoint Network 2nd Order Sensitivity by Perturbation | The New Approach |
|---|---|---|---|
| $\frac{\partial^2 V_L}{\partial Z_4 \partial \ell_4}$ | 11.71675+j5.415667 | 11.713232+j5.431066 | 11.71675+j5.415667 |

## 5.9 Cascaded Networks of 2p-port Elements

The approach we have developed can also be utilized in the analysis and design of cascaded networks consisting of 2p-port elements. Consider the 2p-port element shown in Fig. 5.11, possessing p input ports and p output ports. Its transmission matrix is given by

$$\underset{\sim}{A} \triangleq \begin{bmatrix} \underset{\sim}{A}_{11} & \underset{\sim}{A}_{12} \\ \underset{\sim}{A}_{21} & \underset{\sim}{A}_{22} \end{bmatrix},$$

where $\underset{\sim}{A}_{11}$, $\underset{\sim}{A}_{12}$, $\underset{\sim}{A}_{21}$ and $\underset{\sim}{A}_{22}$ are p x p matrices. The input quantities in this case are

TABLE 5.6

THE RESPONSE $V_L$ AND ITS SENSITIVITIES AT THE VERTICES OF THE TOLERANCE REGION AT NORMALIZED FREQUENCY 0.7

| Vertex | $V_L$ | $\partial V_L/\partial Z_1$ | $\partial V_L/\partial Z_4$ | $\partial V_L/\partial Z_5$ | Sign of Tolerance Extreme | | |
|---|---|---|---|---|---|---|---|
| 1 | 0.49135+j0.02351 | -0.02450+j0.05953 | 0.26004-j1.15934 | 0.02549+j0.32944 | - | - | - |
| 2 | 0.48819+j0.02571 | -0.07761+j0.01588 | 0.28346-j1.05326 | 0.00954+j0.34878 | + | - | - |
| 3 | 0.49679-j0.04862 | 0.03751+j0.15916 | -0.06631-j0.94430 | 0.04534+j0.29165 | - | + | - |
| 4 | 0.49677-j0.04046 | -0.03384+j0.11417 | -0.00426-j0.87724 | 0.03578+j0.31848 | + | + | - |
| 5 | 0.49209+j0.04341 | -0.04367+j0.08072 | 0.29407-j1.19530 | -0.00103+j0.33324 | - | - | + |
| 6 | 0.48786+j0.04670 | -0.09378+j0.03123 | 0.32067-j1.07952 | -0.02042+j0.35007 | + | - | + |
| 7 | 0.49889-j0.03101 | 0.02608+j0.18868 | -0.05742-j0.97346 | 0.02462+j0.29494 | - | + | + |
| 8 | 0.49818-j0.02127 | -0.04526+j0.13735 | 0,01132-j0.90191 | 0.01113+j0.32057 | + | + | + |

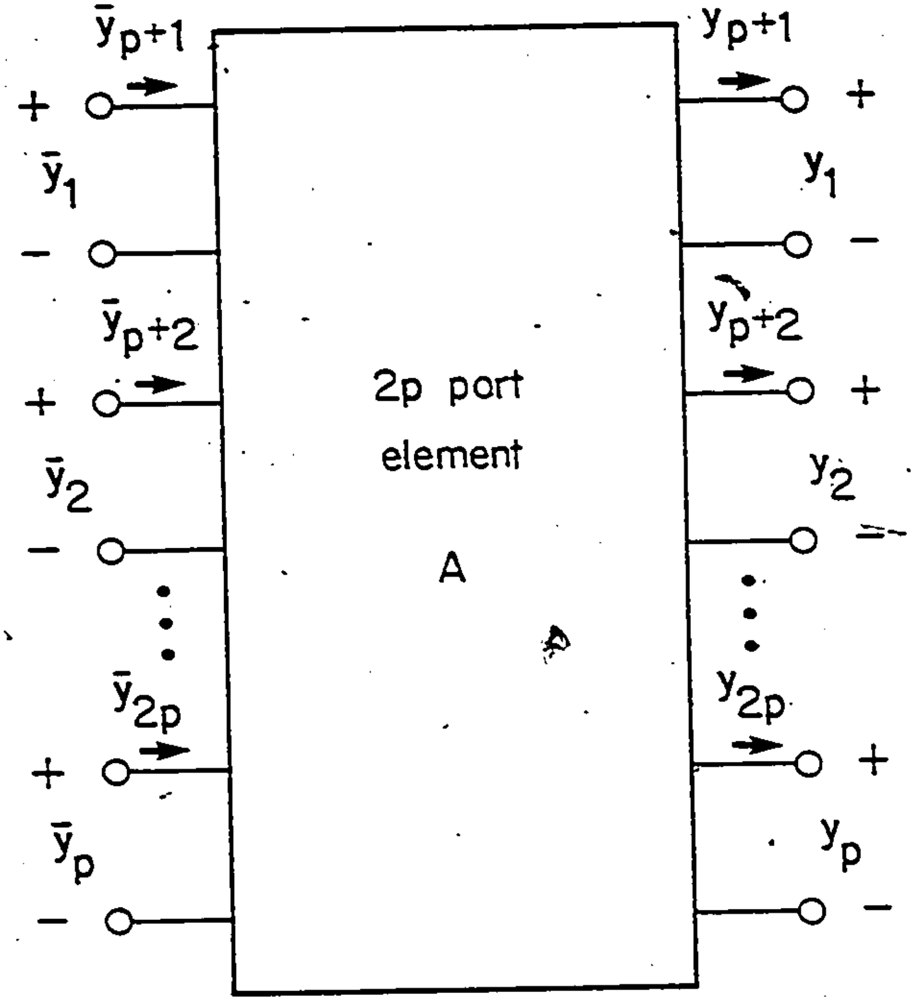

$\bar{y}_{p+1}$
+
$\bar{y}_1$
−
$\bar{y}_{p+2}$
+
$\bar{y}_2$
−
$\bar{y}_{2p}$
+
$\bar{y}_p$
−
2p port
element
A
$y_{p+1}$
+
$y_1$
−
$y_{p+2}$
+
$y_2$
−
$y_{2p}$
+
$y_p$
−

Fig. 5.11 A 2p-port element: a generalization of Fig. 5.1.

$$\bar{\underset{\sim}{y}} = \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \vdots \\ \bar{y}_p \\ \bar{y}_{p+1} \\ \bar{y}_{p+2} \\ \vdots \\ \bar{y}_{2p} \end{bmatrix},$$

and the output quantities are

$$\underset{\sim}{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \\ y_{p+1} \\ y_{p+2} \\ \vdots \\ y_{2p} \end{bmatrix},$$

where the elements with subscripts 1 to p denote voltages and from p+1 to 2p denote currents.

For the forward and reverse analyses the matrices $\bar{\underset{\sim}{U}}_1$, $\bar{\underset{\sim}{U}}_2$, $\underset{\sim}{V}_1$ and $\underset{\sim}{V}_2$ are initialized such that

$$\underset{\sim}{E}_1 \Longrightarrow \underset{\sim}{U}_1 \text{ or } \underset{\sim}{V}_1,$$

$$\underset{\sim}{E}_2 \Longrightarrow \underset{\sim}{U}_2 \text{ or } \underset{\sim}{V}_2,$$

where

$$\underset{\sim}{E}_1 \triangleq \begin{bmatrix} \underset{\sim}{1}_p \\ \underset{\sim}{0}_p \end{bmatrix},$$

and

$$\underset{\sim}{E}_2 \triangleq \begin{bmatrix} \underset{\sim}{0}_p \\ \underset{\sim}{1}_p \end{bmatrix},$$

and where

$\underset{\sim}{1}_p$ is the unit matrix of order p,

$\underset{\sim}{0}_p$ is the null matrix of order p.

We can now derive in an analogous manner to the derivation of (5.8)

$$\underset{\sim}{V}_S = (\overline{\underset{\sim}{U}}{}_1^T + \underset{\sim}{Z}_S\, \overline{\underset{\sim}{U}}{}_2^T)\; \underset{\sim}{A}\; (\underset{\sim}{V}_1\, \underset{\sim}{V}_L + \underset{\sim}{V}_2\, (\underset{\sim}{Y}_L\, \underset{\sim}{V}_L - \underset{\sim}{I}_L)), \qquad (5.31)$$

where

$\overline{\underset{\sim}{U}}_1$, $\overline{\underset{\sim}{U}}_2$, $\underset{\sim}{V}_1$ and $\underset{\sim}{V}_2$ are the matrices obtained from forward and reverse analyses,

$\underset{\sim}{V}_S$ is the vector containing the p source voltages,

$\underset{\sim}{V}_L$ is the vector of load voltages,

$\underset{\sim}{I}_L$ is the vector of current sources at the loads (if any),

$\underset{\sim}{Z}_S$ is a diagonal matrix containing the impedances of the sources,

$\underset{\sim}{Y}_L$ is a diagonal matrix containing the load admittances.

To evaluate the unknowns $\underset{\sim}{V}_L$, having obtained numerical values for (5.31), a system of p linear equations is solved. When $\underset{\sim}{A}$ is perturbed or when derivatives are required, only $6p^3$ additional multiplications and the solution of a p-system of linear equations are needed and not a whole reanalysis of the entire cascaded circuit.

To obtain the Thevenin voltages of the subnetwork on the l.h.s. of the element $\underset{\sim}{A}$, we let $\underset{\sim}{I}_L = \underset{\sim}{0}$ and $\underset{\sim}{Y}_L = \underset{\sim}{0}$ in (5.31), which gives

$$\underset{\sim}{V}_S = (\overline{\underset{\sim}{U}}{}_1^T + \underset{\sim}{Z}_S \overline{\underset{\sim}{U}}{}_2^T)\, \underset{\sim}{A}\, \underset{\sim}{V}_1\, \underset{\sim}{V}_L = (\underset{\sim}{Q}_{11} + \underset{\sim}{Z}_S\, \underset{\sim}{Q}_{21})\, \underset{\sim}{V}_L, \qquad (5.32)$$

where

$$\underset{\sim}{Q}_{11} = \overline{\underset{\sim}{U}}{}_1^T\, \underset{\sim}{A}\, \underset{\sim}{V}_1, \qquad (5.33)$$

$$\underset{\sim}{Q}_{21} = \overline{\underset{\sim}{U}}{}_2^T\, \underset{\sim}{A}\, \underset{\sim}{V}_1, \qquad (5.34)$$

and from (5.32)

$$\underset{\sim}{V}_{TH} = \underset{\sim}{V}_L = (\underset{\sim}{Q}_{11} + \underset{\sim}{Z}_S\, \underset{\sim}{Q}_{21})^{-1}\, \underset{\sim}{V}_S. \qquad (5.35)$$

The output impedance matrix or the Thevenin impedance is obtained one column at a time by letting $\underline{V}_S = \underline{0}$, $\underline{V}_L = \underline{0}$ and $\underline{I}_L = \underline{0}$ except $I_{Li}$ (which is the current source at the load end for the ith port) which leads to

$$\underline{0} = (\underline{Q}_{11} + \underline{Z}_S \underline{Q}_{21}) \underline{V}_L - (\underline{Q}_{12} + \underline{Z}_S \underline{Q}_{22}) \begin{bmatrix} 0 \\ \vdots \\ I_{Li} \\ \vdots \\ 0 \end{bmatrix}, \qquad (5.36)$$

where $\underline{Q}_{11}$ and $\underline{Q}_{21}$ are as defined in (5.33) and (5.34) respectively, and

$$\underline{Q}_{12} = \overline{\underline{U}}_1^T \underline{A} \underline{V}_2. \qquad (5.37)$$

$$\underline{Q}_{22} = \overline{\underline{U}}_2^T \underline{A} \underline{V}_2. \qquad (5.38)$$

Equation (5.36) can be written as

$$(\underline{Q}_{11} + \underline{Z}_S \underline{Q}_{21}) \underline{V}_L = (\underline{Q}_{12} + \underline{Z}_S \underline{Q}_{22}) \begin{bmatrix} 0 \\ \vdots \\ I_{Li} \\ \vdots \\ 0 \end{bmatrix} = \underline{C}_i I_{Li} \qquad (5.39)$$

where $\underline{C}_i$ is the ith column of the matrix $(\underline{Q}_{12} + \underline{Z}_S \underline{Q}_{22})$. From

(5.39) we get

$$\underline{V}_L / I_{Li} = (\underset{\sim}{Q}_{11} + \underset{\sim}{Z}_S \underset{\sim}{Q}_{21})^{-1} \underset{\sim}{C}_i = \begin{bmatrix} Z_{TH_{1i}} \\ Z_{TH_{2i}} \\ \vdots \\ Z_{TH_{pi}} \end{bmatrix} , \qquad (5.40)$$

which is the ith column of the p x p $\underline{Z}_{TH}$ matrix. Figure 5.12 shows the $\underline{Z}_{TH}$ and $\underline{V}_{TH}$ of the subnetwork preceeding the element $\underline{A}$. Similar formulas can be derived (analogous to (5.12) and (5.13)) for the input admittance matrix and the Norton current equivalent matrix.

## 5.10 Conclusions

An important claim we make is that equations (5.8) - (5.14) can be used to generate in a straightforward manner, following differencing or differentiating (as appropriate), any desired exact formulas for multiple network analyses, sensitivity and tolerance analysis with simultaneous large changes. All calculations are carried forward simultaneously and redundant calculations are obviated as demonstrated by the examples and algorithms presented.

The calculation of the first- and second-order sensitivities of a circuit reponse involves one additional

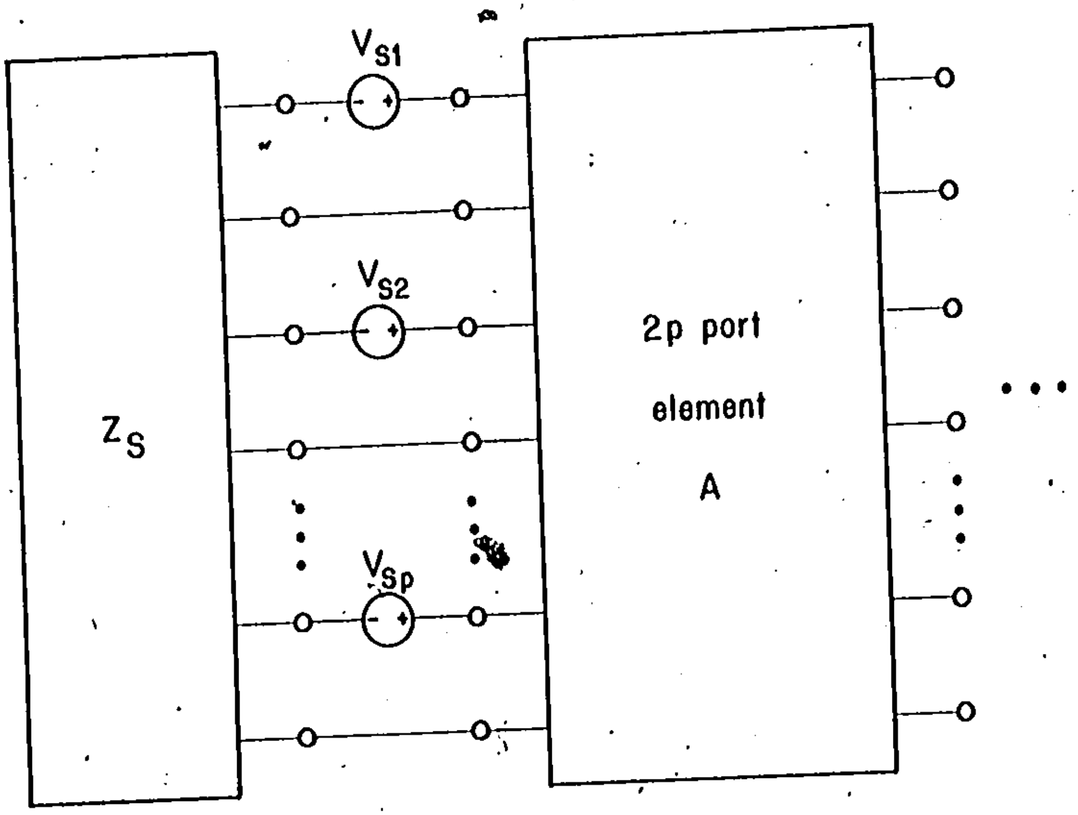

$Z_S$
$V_{S1}$
$V_{S2}$
$V_{Sp}$
2p port
element
A

Fig. 5.12 Equivalent Thevenin voltages and impedances for a subnetwork consisting of 2p-port elements.

analysis of the adjoint network (assuming the analysis of the original network has already been performed) and k(k+1)/2 analyses to find second-order sensitivities calculated by finite differences. A more efficient approach is to calculate these second-order sensitivities using the adjoint-network concept by performing only k analyses. Using the new approach for the analysis of cascaded structures, however, less than k analyses are performed and no additional memory is required.

The algorithm for evaluating the response and its sensitivities at the vertices of the tolerance region proved to be very efficient. The seven-section filter example was run with tolerances on the characteristic impedances of the stubs and transmission lines (all seven). It took 0.269 s CPU time to evaluate only the response at the 128($2^7$) vertices. Using the conventional method of reanalyzing the circuit for different component values would take 0.074 x 128 = 9.472 s CPU, where one analysis is performed in approximately 0.074 s. For the case of evaluating the response and its sensitivities at vertices discussed in Section 5.5 (Algorithm 4), it took 0.118 s CPU time compared with 8 x 0.074 = 0.592 s for 8 analyses. The savings in computational effort is substantial.

Symmetry of the networks analyzed can be exploited leading to saving of computational effort (Bandler, Biernacki and Rizk 1979). Branched circuits can be handled readily. Formulas, similar to (5.26)-(5.29), can be derived for other branched structures using the same concepts so as to render the sensitivity analysis and design of these circuits as simple as possible. The approach should prove to be very suitable for computer-aided design of cascaded microwave circuits and systems consisting of 2-ports. It is also readily extendable to 2p-port networks.

# CHAPTER 6

# CONCLUSIONS

This thesis reflects the current state of the art in computer-aided design of electronic circuits. It clarifies the weaknesses, the disadvantages and advantages of the different methods of analysis, sensitivity evaluation, optimization and problem formulation.

The TLM method proved to be suitable for automated network design in the time domain. This is due to its ability to obtain the response sensitivities w.r.t. the design variables very easily. It avoids the formulation of the state equations which can be rather difficult for large networks. Another advantage of the TLM method is that it is stable for stiff networks which cause instability in most of the numerical methods. The use of more complicated transmission-line elements and models can improve the method's accuracy. The extension of the method to handle nonlinear networks is possible.

The multidimensional polynomial approximation facilitates the design of circuits by exploiting large and general simulation programs to conduct the analyses. Saving in the circuit designer's time and effort is achieved by using this approach.

The new approach for the analysis of cascaded structure, besides the substantial savings in computational effort, gives a deep insight to the design problem. The design variables can be highlighted explicitly in different formulas from which we can extract the exact analysis, differential and large-change sensitivities. The approach permits the exploitation of network structure: symmetry, branches, etc. Different algorithms which employ this approach can be implemented easily on microcomputers since they are simple and compact.

Promising directions for further research and improvements in existing methods and approaches have been revealed by this work. All the algorithms which have been referred to are sequential algorithms. Aside from the human mind, which apparently has been taught to think sequentially, the computers which we now use are substantially sequential. It will not be very long before parallel machines will be widely available. The parallel processors will be much faster than present machines. New optimization algorithms suitable for the new computing machines have to emerge or the existing ones have to be modified. In some of the ill-conditioned problems, for example, it is required to run the problem from different starting points, which might be considered as running different problems in parallel. The tolerance-tuning problem can also be formulated to fit a parallel algorithm because it is inherently a parallel problem.

Automated optimal design of circuits where the topology is not fixed has often been suggested. Presently, the location and type of components to be added to the given network have to be specified a priori. Fully automated design, where the topology can change arbitrarily, and criteria for augmenting or shrinking the circuit are not well established. This will need optimization methods where the number of variables can be automatically increased or decreased in an effective manner during the process.

Other design aids would be

(a) Algorithms which indicate and act upon the existence of symmetry.

(b) Algorithms which stack the constraints in the order of complexity to avoid unnecessary calculations, starting with sets of crude but not necessarily linear approximations.

(c) Algorithms which permit the flexibility of examining the effects of alternative objectives and weights without rerunning the whole problem each time a change is made.

In centering, tolerancing, and tuning problems several concepts need further development. Efficient vertex selection schemes will lead to an enormous reduction of the tolerance problem as well as the possibility of full automation of the whole process. This concept is related to determining active constraints.

The reliability problem is an extension to the tolerance assignment problem. The main difference, we feel, is the redundancy which enhances system reliability and observation that not every component fails simultaneously. Parameter changes to be considered might be much larger than in tolerance assignment.

# APPENDIX A

## CENTRAL DIFFERENCE FORMULA APPROXIMATING FIRST-ORDER DERIVATIVE

The formula used to obtain ∂f/∂t from the response is (Kelly 1967)

$$hf'_j = (\mu\,\delta - \frac{1}{6}\,\mu\,\delta^3) f_j\,, \qquad \text{(A1)}$$

where

$$\mu f(t) = \frac{1}{2}\left[f(t + \frac{h}{2}) + f(t - \frac{h}{2})\right] \qquad \text{(A2)}$$

and

$$\delta f(t) = f(t + \frac{h}{2}) - f(t - \frac{h}{2})\,. \qquad \text{(A3)}$$

Equation (A1) can be rewritten as

$$f'_j = \frac{2}{3h}\left[f(t+h) - f(t-h)\right] - \frac{1}{12h}\left[f(t+2h) - f(t-2h)\right] \qquad \text{(A4)}$$

In Tables 3.4-3.7 (Chapter 3) h was equal to 2T.

# APPENDIX B

## DATA SUPPLIED TO SPICE2 FOR THE CSEF CIRCUIT

### B.1 With the Equivalent Circuit of the Emitter Junction

```
CURRENT SWITCH EMITTER FOLLOWER
===============================
R1     4  5   281.33
R2     7  0   75.
R3     2  0   78.24
R4    10  1   45.533
C1    10  0   1.248E-12
T1     2  0   8  0 Z0=92.004  TD=.25NS
VE2    0  5   DC  4.03
VE3    0  6   DC  1.13
VE4    0  1   DC  1.655
VE1    3  0   PWL(0  -.776 .05NS  -.776  .2NS  -1.55 .45NS -1.55 .6NS
+-.776   .85NS   -.776  1.NS  -1.55  1.25NS  -1.55  1.4NS  -.776)
D1     9  10  DMOD
D2     13  4  DMOD
D3     14  4  DMOD
D4     15  9  DMOD
CE1    11  4  0.12P
CC1    11  2  0.5P
CE2    12  4  0.12P
CC2    12  7  0.5P
CE3    8  9  0.12P
RB1    3  11  50.
RB2    12  6  50.
RT1    17  0  1.
RT2    20  0  1.
RT3    23  0  1.
CT1    25  0  1.E-8
CT2    26  0  1.E-8
CT3    27  0  1.E-8
VT1    11  13  DC  0.
VT2    12  14  DC  0.
VT3    8  15  DC  0.
VT4    16  17  DC  0.
VT5    19  20  DC  0.
VT6    22  23  DC  0.
VT7    18  25  DC 0.
VT8    21  26  DC  0.
VT9    24  27  DC  0.
E1     18  0  13  4  1
E2     21  0  14  4  1
E3     24  0  15  9  1
F1     2  11  VT1  0.99
F2     7  12   VT2   0.99
F3     0  8  VT3  0.99
F4     0  16  VT1  2.3200E-10   3.8668E-2
F5     0  19   VT2  2.32008E-10  3.8668E-2
F6     0  22  VT3  2.32008E-10   3.8668E-2
F7     11  4  POLY(2)  VT4  VT7  0 0 0 0 1
F8     12  4  POLY(2)  VT5  VT8  0 0 0 0 1
F9     8  9  POLY(2)  VT6  VT9  0 0 0 0 1
 .TRAN   0.0125NS  1.4NS
 .PRINT TRAN V(10,0)
 .PLOT  TRAN V(10,0)
 .MODEL  DMOD D IS=.6E-9
 .END
```

## B.2 With Built-in Transistor Models

```
CURRENT SWITCH EMITTER FOLLOWER
==============================
R1     4   5   281.33
R2     7   0   75.
R3     2   0   78.24
R4    10   1   45.533
C1    10   0   1.248E-12
T1     2   0   8  0 Z0=92.004  TD=.25NS
VE2    0   5   DC  4.03
VE3    0   6   DC  1.13
VE4    0   1   DC  1.655
VE1    3   0   PWL(0  -.776 .05NS  -.776 .2NS  -1.55 .45NS -1.55 .6NS
+-.776   .85NS   -.776  1.NS  -1.55  1.25NS  -1.55  1.4NS  -.776)
D1     9   10  DMOD
QT1    2   3   4  QMOD
QT2    7   6   4  QMOD
QT3    0   8   9  QM1
.TRAN   0.0125NS  1.4NS
.PRINT TRAN V(10,0)
.PLOT  TRAN V(10,0)
.MODEL  DMOD D IS=.6E-9
.MODEL    QMOD  NPN  BF=99.  IS=.6E-9   RB=50.  TF=.01NS  CJE=0.12PF
+    CJC=0.5PF  MC=0.  ME=0.
.MODEL    QM1  NPN  BF=99.   IS=.6E-9   TF=.01NS  CJE=0.12PF   ME=0.
+    MC=0.
.END
```

# APPENDIX C

## FORMULAS RELATING THE LOAD VOLTAGES OF THE BRANCHES TO THE VARIABLE ELEMENTS

### C.1 To Obtain $V_{BL}$ as a Function of $V_S$ and $\underset{\sim}{B}$

The voltage across the impedance Z, representing the branched circuit, in terms of $V_{BL}$ is given by

$$V_Z = \underset{\sim}{\bar{u}}{}_{1B}^{T} \, \underset{\sim}{B} \, \underset{\sim}{v}{}_{1B} \, V_{BL}, \tag{C1}$$

and it can be expressed in terms of voltages in the main cascaded circuit as

$$V_Z = \underset{\sim}{e}{}_1^{T} \, [\underset{\sim}{\bar{v}}{}_{1Z} - \underset{\sim}{v}{}_{1Z}] \, V_L, \tag{C2}$$

where $\underset{\sim}{\bar{v}}{}_{1Z}$ is the result of the reverse analysis at reference plane f. So (C2) can be written, substituting for the chain matrix of the element representing the branch, as

$$V_Z = \underset{\sim}{e}{}_1^{T} \left[ \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} \underset{\sim}{v}{}_{1Z} - \underset{\sim}{v}{}_{1Z} \right] V_L \tag{C3}$$

$$= [1 \quad 0] \left( \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \mathbf{v}_{1Z} V_L \tag{C4}$$

$$= \mathbf{e}_2^T \mathbf{v}_{1Z} Z V_L . \tag{C5}$$

The load voltage of the main cascade $V_L$ can be expressed by

$$V_L = \frac{V_S}{\bar{\mathbf{u}}_{1Z}^T \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} \mathbf{v}_{1Z}} \tag{C6}$$

and (C1) can be rewritten as

$$V_{BL} = \frac{V_Z}{\bar{\mathbf{u}}_{1B}^T \mathbf{B} \mathbf{v}_{1B}} . \tag{C7}$$

Substituting for $V_Z$ of (C5) we have

$$V_{BL} = \frac{\mathbf{e}_2^T \mathbf{v}_{1Z} Z V_L}{\bar{\mathbf{u}}_{1B}^T \mathbf{B} \mathbf{v}_{1B}} \tag{C8}$$

and substituting for $V_L$ from (C6) and Z from (5.24), we get

$$V_{BL} = \frac{\mathbf{e}_2^T \mathbf{v}_{1Z} \dfrac{\bar{\mathbf{u}}_{1B}^T \mathbf{B} \mathbf{v}_{1B}}{\bar{\mathbf{u}}_{2B}^T \mathbf{B} \mathbf{v}_{1B}} V_S}{\bar{\mathbf{u}}_{1B}^T \mathbf{B} \mathbf{v}_{1B} \bar{\mathbf{u}}_{1Z}^T \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} \mathbf{v}_{1Z}} , \tag{C9}$$

hence

$$V_{BL}(\underset{\sim}{B}) = \frac{\underset{\sim}{e}_2^T \, \underset{\sim}{v}_{1Z} \, V_S}{\overline{\underset{\sim}{u}}{}_{2B}^T \, \underset{\sim}{B} \, \underset{\sim}{v}_{1B} \, \overline{\underset{\sim}{u}}{}_{1Z}^T \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} \underset{\sim}{v}_{1Z}} \, .$$

## C.2 To Obtain $V_{BL}$ as a Function of $V_S$ and $\underset{\sim}{C}$

From (C7) and (C2) we can write $V_{BL}$ as

$$V_{BL} = \frac{\underset{\sim}{e}_1^T \, [\overline{\underset{\sim}{v}}_{1Z} - \underset{\sim}{v}_{1Z}] \, V_L}{\overline{\underset{\sim}{u}}{}_{1B}^T \, \underset{\sim}{B} \, \underset{\sim}{v}_{1B}} \, . \qquad \text{(C10)}$$

The load voltage $V_L$ can be expressed (compare with (C6)) by

$$V_L = \frac{V_S}{\overline{\underset{\sim}{u}}{}_{1Y}^T \begin{bmatrix} 1 & 0 \\ \underset{\sim}{Y} & 1 \end{bmatrix} \underset{\sim}{v}_{1Y}} \, . \qquad \text{(C11)}$$

We can write, using notation defined for (5.27),

$$\underset{\sim}{e}_1^T \, \overline{\underset{\sim}{v}}_{1Z} = \overline{\underset{\sim}{u}}{}_{1Yf}^T \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} \underset{\sim}{v}_{1Y} \, . \qquad \text{(C12)}$$

Similarly,

$$\underset{\sim}{e}_1^T \, \underset{\sim}{v}_{1Z} = \overline{\underset{\sim}{u}}{}_{1Yg}^T \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} \underset{\sim}{v}_{1Y} \, . \qquad \text{(C13)}$$

Substituting these terms and $V_L$ of (C11) into (C10) we obtain

$$V_{BL}(\underset{\sim}{C}) = \frac{[\bar{\underset{\sim}{u}}^T_{1Yf} - \bar{\underset{\sim}{u}}^T_{1Yg}] \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} \underset{\sim}{v}_{1Y} V_S}{\bar{\underset{\sim}{u}}^T_{1B} \underset{\sim}{B} \underset{\sim}{v}_{1B} \bar{\underset{\sim}{u}}^T_{1Y} \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} \underset{\sim}{v}_{1Y}} . \tag{C14}$$

C.3 To Obtain $V_{CL}$ as a Function of $V_S$ and $\underset{\sim}{C}$

The voltage across Y in terms of $V_{CL}$ is given by

$$V_Y = \bar{\underset{\sim}{u}}^T_{1C} \underset{\sim}{C} \underset{\sim}{v}_{1C} V_{CL} , \tag{C15}$$

and in terms of $V_L$, as

$$V_Y = \underset{\sim}{e}^T_1 \underset{\sim}{v}_{1Y} V_L . \tag{C16}$$

But $V_L$ is also given by

$$V_L = \frac{V_S}{\bar{\underset{\sim}{u}}^T_{1Y} \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} \underset{\sim}{v}_{1Y}} . \tag{C17}$$

So, substituting this $V_L$ into (C16) and the resulting $V_Y$ into (C15) we get

$$V_{CL}(\underset{\sim}{C}) = \frac{\underset{\sim}{e}_1^T \, \underset{\sim}{v}_{1Y} \, V_S}{\overline{\underset{\sim}{u}}_{1C}^T \, \underset{\sim}{C} \, \underset{\sim}{v}_{1C} \, \overline{\underset{\sim}{u}}_{1Y}^T \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} \underset{\sim}{v}_{1Y}} . \qquad \text{(C18)}$$

## C.4 To Obtain $V_{CL}$ as a Function of $V_S$ and $\underset{\sim}{B}$

From (C15), (C16) and (C6) we can write $V_{CL}$ as

$$V_{CL}(\underset{\sim}{B}) = \frac{\underset{\sim}{e}_1^T \, \underset{\sim}{v}_{1Y} \, V_S}{\overline{\underset{\sim}{u}}_{1C}^T \, \underset{\sim}{C} \, \underset{\sim}{v}_{1C} \, \overline{\underset{\sim}{u}}_{1Z}^T \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} \underset{\sim}{v}_{1Z}} . \qquad \text{(C19)}$$

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# AUTHOR INDEX

H.L. Abdel-Malek 15, 18, 19, 64, 65, 67, 81, 103, 104, 113, 118, 144, 161, 162

S. Akhtarzad 67

J.W. Bandler 8, 10, 14, 15, 16, 18, 19, 21, 34, 36, 38, 39, 43, 46, 48, 53, 55, 56, 57, 60, 64, 65, 67, 81, 92, 103, 104, 113, 123, 133, 138, 144, 158, 161, 162, 164

P.W. Becker 63

R.M. Biernacki 123

F.H. Branin, Jr. 12, 14

R.K. Brayton 7, 12, 54

C.G. Broyden 45

P.R. Bryant 17

A. Budak 106

E.M. Butler 56

D.A. Calahan 10, 43

C. Charalambous 31, 34, 36, 38, 39, 40, 41, 43, 47, 158, 164

K.S. Chao 12

J.H.K. Chen 56, 138, 158, 164

E.W. Cheney 36

W.Y. Chu 38, 92, 158, 164

L.O. Chua 9, 10, 12

C.M. Close 10

A.R. Conn 47
P. Dalsgaard 19, 104
V.F. Demy'anov 36
P.M. DeRusso 10
S.W. Director 13, 14, 57, 63
I.S. Duff 7
S.R.K. Dutta 47
O. Einarsson 48
N.J. Elias 63
Z.S. Elrazaz 19, 104
A.V. Fiacco 30
J.K. Fidler 16
R. Fletcher 30
R.N. Gadenz 17
C.W. Gear 12
K. Geher 48
P.J. Goddard 17
P.E. Green 8
F.G. Gustavson 7, 12
G.D. Hachtel 7, 12, 15, 57, 63
M.D. Hebden 34
L.G. Kelly 180
S.E. Hersom 48
M.R. Hestenes 48
C.W. Ho 113

A.J. Hoffman 54

M.C. Horton 158

Y. Ishizaki 42

F. Jensen 63

V.K. Jha 8

P.B. Johns 10, 15, 67, 73, 74, 81

E.M.T. Jones 85

B.J. Karafin 48, 63

L.S. Lasdon 29, 30

K.H. Leung 18, 138

K. Levenberg 33

P.M. Lin 9, 10

D.K. Liu 12

P.C. Liu 53, 55, 56, 60, 65, 138

K. Madsen 45, 46, 57, 66

V.N. Malozemov 36

D.W. Marquardt 33

G.L. Matthaei 85

D.C. McCall 31

J.H. McClellan 25

G.P. McCormick 30

D.D. Morrison 47

L.W. Nagel 104

O. Nielsen 45

C.T. Pan 12

S.R. Parker 15

W.N. Parker 8

T.W. Parks 25

J.F. Pinel 55

E. Polak 65

J.R. Popović 8

M.J.D. Powell 30, 31, 48

L.R. Rabiner 25

M.G. Rezai-Fakhr 17, 18

M.R.M. Rizk 8, 10, 15, 16, 19, 67, 81, 104, 118, 123, 165

K.A. Roberts 55

R.T. Rockafellar 48

R.A. Rohrer 13, 14, 15

R.J. Roy 10

A. Sangiovanni-Vincentelli 65

H. Schjaer-Jacobsen 45, 46, 57, 66

A.F. Schwarz 16

R.T. Scott 54

A.K. Seth 48

R.E. Seviora 14, 133

K. Singhal 17

R. Spence 17, 18, 138

T.V. Srinivasan 43, 46

L.B. Stotts 31

D.F. Suchman 29, 30

G.C. Temes 17, 18, 33, 43
C.W. Therrien 133
L. Thrane 45
H. Tromp 8, 16, 56, 60, 64, 65, 138
A. Ushida 12
L.M. Vidigal 63
M. Vidyasagar 47
P.A. Villalaz 17
J. Vlach 17
J. Voldby 45
A.D. Waren 29, 30, 31
H. Watanabe 42
R.J. Wenzel 158
L. Young 85
D.Y.F. Zai 33
W.I. Zangwill 30
E.R. Zeller 106

## SUBJECT INDEX

Adjoint network, 13, 165

Analysis,

  frequency domain, 6
  time domain, 9, 67
  nonlinear networks (D.C.), 10
  nonlinear networks in the time domain, 12

Base points, 104, 161

Bilinear function, 53

Branched circuits, 140

  admittance, 142
  impedance, 142

Cascaded networks, 124

  analysis, 7
  design, 122
  input impedance, 132
  forward analysis, 126
  of 2p-port elements, 166
  reverse analysis, 127

Centering problem, 56

Conditions,

  optimality, 36
  Kuhn-Tucker, 40, 56

## SUBJECT INDEX (continued)

Constraint,

    equality, 30, 32, 48
    inequality, 32
    region, 51

Convexity, 38

    one-dimensional, 53, 66

Derivatives of response function, 13

Design with yield less than 100%, 62

Equivalent sources,

    Thevenin, 131, 171
    Norton, 131

Error,

    compensation, 83
    functions, 19, 54

Finite difference, 97, 98

General simulator, 103

Generalized,

    least-pth objective, 34
    reduced gradient, 31
    tolerance assignment, 64

Ill-conditioning, 30, 31, 38

SUBJECT INDEX (continued)

Impedance, 142
    output, 131

Interval arithmetic, 66

Least pth optimization, 33

Linear,
    cuts, 64
    programming, 42

Minimax optimization, 40, 42, 46, 47
    of constrained problem, 46

Modeling,
    transmission-line, 68
    link, 68
    stub, 75

Multidimensional approximation, 18, 103, 161

Nonlinear programming, 29, 39

Objective function, 33, 34, 35, 39, 41, 55, 58

Pairwise changes, 146

Penalty function, 29

SUBJECT INDEX (continued)

Quadratic,
    approximation, 18, 161
    models, 103

Reference planes, 128

Scattering matrix, 79

Sensitivity, 13
    at vertices of tolerance region, 138, 156
    differential (1st order), 13, 88, 137, 151
    large change, 16, 144
    second order, 16, 90, 137, 151
    time domain, 85
    w.r.t. time step, 92

Simplicial approximation, 63

Specifications, 9

Tolerance,
    fixed, 54
    problem, 54
    region, 50
    variable, 55

Transmission-line matrix method, 67

Tuning, 57
    region, 58

Uncertainties, 60

## SUBJECT INDEX (continued)

Vertex, 50
    selection scheme, 56

Worst-case design, 53

Yield, 53, 63
    sensitivities, 64