PRIME IDEALS IN RINGS SATISFYING POLYNOMIAL IDENTITIES

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ABSTRACT

The systematic study of prime ideals in noncommutative rings satisfying polynomial identities lends itself to diverse methods of approach. The geometric method, which we explore at several points of this thesis, was initiated by C. Procesi in the 1960's. The main results of our chapter on Prime Ideals in Affine Algebras are most naturally developed from this point of view. They lead to certain algebraic questions which are dealt with in this thesis, concerning the structure of prime ideals in affine algebras satisfying polynomial identities. The interplay between the prime ideals in such an algebra, and the center of the algebra is a theme that is exploited frequently, through the use of central polynomials. We also answer a question posed by Procesi concerning the growth of affine, prime algebras satisfying polynomial identities.

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CHAPTER 1

Introduction

The study of rings satisfying polynomial identities has developed rapidly over the last decade into a substantial theory. For an historical survey of the subject, tracing back to Klein's Erlanger program, the interested reader is referred to the survey article of Jacobson [11]. Prime rings satisfying polynomial identities arise frequently in algebra. Classical orders in central simple algebras are perhaps the most immediate examples. The aim of this dissertation is to build on the known body of results concerning prime rings satisfying polynomial identities, in order to elucidate a few aspects of the structure of their prime ideals. In the third chapter, we take the geometric approach in studying the prime ideals of such rings which are affine over a field. We answer several questions concerning prime ideals in this setting. Among other things, we give a sufficient condition for a prime ideal to be finitely generated (as a two-sided ideal of the ring). We remark that it is still an open problem whether or not every prime ideal is finitely generated, in an affine prime p.i. ring. In Chapter 4, we concentrate on the relationship between the prime ideals of a prime p.i. ring and the center of the ring. We investigate conditions under which a prime ideal of maximal p.i. degree is centrally generated, and we also explore the situations where this fails. The main tool in our proofs is the theory of central polynomials. This, and other background is collected together in Chapter 2.

In chapter 5, we investigate the obstructions to localization in a prime p.i. ring. The relevant background from the theory of noncommutative localization is collected together here, so that the chapter is relatively self-contained. We use the results of Mueller [21], and consider the obstructions to localizing a prime ideal as given by the existence of "links" between that prime ideal and other prime ideals of the ring. The main theorem of this chapter, Theorem 5.11, exhibits the behaviour of such links under appropriate ring extensions. This result is useful for computing the structure of links in many examples of prime p.i. rings. For instance, the computations of Mueller [18] are simplified through its use. The methods of chapter 5 are largely module-theoretic, in contrast to the other chapters. In chapter 6, we exploit a result of Bergman and Small in order to obtain a bound on the size of certain link-closed sets of prime ideals of a prime p.i. ring.

Finally, in the seventh chapter, we give an answer to a problem of Procesi concerning the growth of an affine prime p.i. algebra over a field.

CHAPTER 2

Preliminaries

In order to fix some of our notation and terminology, and to clarify some of the known results we shall need, we start with a summary of some of the relevant definitions and basic theorems. All of the results of this chapter can be found, for example, in Jacobson [12] except those for which we give specific references.

For our purposes, rings will be associative, but not necessarily commutative, and will have an identity element, 1, unless specified otherwise. Following the tradition in commutative algebra, the set of all prime ideals of the ring R is called the <u>spectrum</u> of R, denoted Spec(R). The ring R is called a <u>p.i. ring</u> if it satisfies a polynomial identity which is proper on all non-zero factor rings of R; a polynomial identity is proper on a ring if not all of its coefficients annihilate the ring. It follows that any p.i. ring actually satisfies a polynomial identity whose non-zero coefficients are ± 1. Moreover, if $S_n(x_1,\ldots,x_n) = \Sigma(-1)^{\sigma} x_{\sigma(1)}\cdots x_{\sigma(n)}$, where the sum is taken over all permutations σ of $\{1,\ldots,n\}$, then any p.i. ring satisfies some power of S_n. S_n is called the <u>standard identity</u> of degree n. If R is a prime p.i. ring, then in fact R satisfies some standard identity, and the smallest integer m such that R satisfies S_{2m} is called the <u>p.i. degree</u> of R, denoted p.i.deg(R). This integer m is also the smallest integer m such that R satisfies all of the polynomial identities of $m\times m$ matrices over the integers. For any ring R, and $m > 0$, we define

$Spec_m(R) = \{P \in Spec(R) : R/P \text{ is a p.i. ring, and } p.i.deg(R/P) = m\}$.
The following theorem is a sharpening of a theorem originally proved by Posner, in 1960.

Theorem 2.1. Let R be a prime p.i. ring, with center C, and let S = C-{0}. Then the ring of quotients, $S^{-1}R = Q$, is a simple, Artinian ring, with center $S^{-1}C$, the quotient field of C, and Q satisfies exactly the same polynomial identities as R satisfies.

We shall refer to the above simple, Artinian quotient ring as the Posner Quotient ring of R, denoted $\mathcal{Q}(R)$. By the well-known theorem of Kaplansky concerning primitive p.i. rings, $\mathcal{Q}(R)$ is a finite-dimensional algebra over its center. Moreover, the dimension of $\mathcal{Q}(R)$ over its center is m^2, where m = p.i.deg(R).

At the heart of the proof of the above version of Posner's theorem is the theory of central polynomials. A noncommutative polynomial $f(x_1,\ldots,x_k)$ is called a central polynomial for a ring R in case f is not an identity satisfied by R, but $f(x_1,\ldots,x_k)x_{k+1} - x_{k+1}f(x_1,\ldots,x_k)$ is an identity satisfied by R. A nontrivial central polynomial for R is a central polynomial, $f(x_1,\ldots,x_k)$, for R such that $f(0,\ldots,0) = 0$. The fact that nontrivial central polynomials exist for the ring $R = (A)_{m\times m}$ of m×m matrices over a commutative domain A, was discovered in 1972, independently by Formanek [9] and Razmyslov [26].

Definition 2.2. If R is any prime p.i. ring, of p.i. degree m, the Formanek center of R, denoted $F_m(R)$, is defined to be the subring, without 1, of the center of R consisting of all evaluations of nontrivial central polynomials for m×m matrices.

We remark that if R is any prime p.i. ring of p.i. degree m, then $F_m(R)$ is non-zero, and, for any non-zero ideal, I, of R, we have $I \cap F_m(R) \neq \{0\}$. Razmyslov's central polynomial for $m \times m$ matrices will be denoted by $h_m(x_1,\ldots,x_{m^2};y_1,\ldots,y_{m^2+1})$. It enjoys the following properties, which we shall use frequently:

H1. h_m is a nontrivial central polynomial for any prime p.i. ring of p.i. degree m.

H2. h_m is a multilinear in all of its variables.

H3. h_m is alternating in its first m^2 variables $x_1,\ldots,x_{m^2}$.

The following result is a nice illustration of the use of Razmyslov's central polynomials. The proof, which we include, is due to Amitsur, [1].

<u>Theorem 2.3.</u> Let R be a prime p.i. ring, of p.i. degree m, with center C. Let $r_1,\ldots,r_{m^2}$, $s_1,\ldots,s_{m^2+1}$ be any elements of R such that $\delta = h_m(r_1,\ldots,r_{m^2};s_1,\ldots,s_{m^2+1})$ is non-zero. Let $g_i(r) = (-1)^{i+1}h_m(r_1,\ldots,r,\ldots,r_{m^2};s_1,\ldots,s_{m^2+1})$, for any $r \in R$, where r appears as the i^{th} argument in h_m, in place of r_i, $i=1,\ldots,m^2$. Then for all $r \in R$, we have $\delta r = \sum_{i=1}^{m^2} r_i g_i(r)$.

Proof: Consider the polynomial $H(x_0,\ldots,x_{m^2};y_1,\ldots,y_{m^2+1})$ given by

$$H(x_0,\ldots,x_{m^2};y_1,\ldots,y_{m^2+1}) = \sum_{i=1}^{m^2}(-1)^i x_i h_m(x_1,\ldots,x_0,\ldots,x_{m^2};y_1,\ldots,y_{m^2+1}) + x_0 h_m(x_1,\ldots,x_{m^2};y_1,\ldots,y_{m^2+1}),$$

where x_0 appears, in place of x_i, as the i^{th} argument of h_m, in the summation. It is clear, by inspection, that H is an alternating multilinear function of the m^2+1 variables $x_0,\ldots,x_{m^2}$, using H2 and H3. Since the Posner quotient ring of R has dimension only m^2 over its center, H must be identical zero on R. Writing this out, we obtain

$$H(r,r_1,\ldots,r_{m^2};s_1,\ldots,s_{m^2+1}) = 0 = -\sum_{i=1}^{m^2} r_i g_i(r) + \delta r\ ,\quad \text{as required.}$$

Corollary 2.4. If R is any prime p.i. ring, of p.i. degree m, with center C, then, considered as a C-module, R is isomorphic to a submodule of the free C-module C^{m^2}.

Proof: In the notation of the theorem, the r_i's must be linearly independent over the quotient field of C, in $\mathcal{Z}(R)$, since, otherwise, h_m, considered as an alternating multilinear function of its first m^2 variables, defined on the vector-space span of $r_1,\ldots,r_{m^2}$, would necessarily vanish. Therefore, $r_1C + \ldots + r_{m^2}C$ is a direct sum of m^2 copies of C, contained in R. So multiplication by δ gives a C-module homomorphism from R into $r_1C \oplus \ldots \oplus r_{m^2}C$. That this homorphism is one-one follows from the fact that δ is a non-zero-divisor, being a non-zero central element of the prime ring R.

This result was first obtained, using a different approach, by Formanek [10]. We observe immediately:

Corollary 2.5. If C is Noetherian, then R is a finitely-generated C-module, and R is Noetherian.

In many of our considerations, R will be an algebra over some commutative ring A. In this setting, R is called an affine A-algebra if R is finitely-generated, as an A-algebra. When the commutative ring A is clear from the context, we sometimes say, simply, that R is affine. We note that there is some variability in the literature on the use of this terminology, and several writers use the unadorned descriptive "affine" to mean affine over a field in our sense.

We shall occasionally use the following theorem, due to Procesi [24, Theorem 1.2, page 102], which generalizes the Hilbert Nullstellensatz to affine p.i. algebras over a field.

Theorem 2.6. If R is any affine p.i. algebra over a field k, then every prime ideal of R is the intersection of maximal ideals of R, and if M is any maximal ideal of R, then R/M is a finite-dimensional vector space over k.

We conclude our preliminaries with a few observations concerning the Formanek center, $F_m(R)$, of a prime p.i. ring, R, of p.i. degree m.

Lemma 2.7. If R is any prime p.i. ring of p.i. degree m, and if $P \in \mathrm{Spec}_m(R)$, so that R/P is also a prime p.i. ring of p.i. degree m, then the canonical map $R \to R/P$ maps $F_m(R)$ surjectively, onto $F_m(R/P)$.

This result is Theorem 2.1, part (3), in Procesi [24, chapter VIII]. It follows from the observation that, given any evaluation of a central polynomial on R/P, the elements of R/P occurring as the arguments of the central polynomial may be lifted to elements of R, to obtain an evaluation of the same central polynomial, in R, which maps onto the given evaluation in R/P.

Corollary 2.8. Let R be any prime p.i. ring of p.i. degree m, and $P \in \mathrm{Spec}(R)$. Then $P \in \mathrm{Spec}_m(R)$ if and only if $P \not\supseteq F_m(R)$.

Proof: If $P \in \mathrm{Spec}_m(R)$, the lemma tells us that $(F_m(R) + P)/P = F_m(R/P)$, which is non-zero, so $F_m(R) \not\subseteq P$. Conversely, if $F_m(R) \subseteq P$, then

$F_m(R/P) = (F_m(R)+P)/P = 0$, so R/P could not be of p.i. degree m.

<u>Corollary 2.9</u>. Let R be any prime p.i. ring, of p.i. degree m, with center C. Then:

(i) $F_m(R)$ contains a non-zero ideal of C.

(ii) If $P \in \mathrm{Spec}_m(R)$, and if D is the center of R/P, then D lies between $(C+P)/P$ and its quotient field.

Proof: The set, $H_m(R)$, of all evaluations of h_m on R, is contained in $F_m(R)$, and is non-zero by H1. Moreover, by H2, this set of evaluations is an ideal of C, since, for example, $h_m(cr_1,\ldots,r_{m^2};s_1,\ldots,s_{m^2+1}) = c\cdot h_m(r_1,\ldots,r_{m^2};s_1,\ldots,s_{m^2+1})$, for any $c \in C$, and r_i's, s_j's $\in R$. This proves (i). To prove (ii), just notice that $(C+P)/P$ contains $(F_m(R)+P)/P = F_m(R/P)$ which, by (i), contains a non-zero ideal of D. So D has the same quotient field as $(C+P)/P$, since it is an integral domain containing $(C+P)/P$ and $(C+P)/P$ contains a non-zero ideal of D.

<u>Definition 2.10</u>. Let R be a prime p.i. ring of p.i. degree m. $P \in \mathrm{Spec}(R)$ is said to be a prime ideal of <u>maximal p.i. degree</u> if $P \in \mathrm{Spec}_m(R)$. Otherwise, P is said to be of <u>deficient p.i. degree</u>.

With this terminology, Corollary 2.8 just says that the prime ideals, P, of deficient p.i. degree, are exactly those containing $F_m(R)$. We remark that we have called maximal p.i. degree is also called "identity-faithful" in the literature. (Cf. [29]).

CHAPTER 3

Prime Ideals in Affine Algebras: A Geometric Point of View

We use the notation $(R)_{m\times m}$ for the ring of $m\times m$ matrices over the ring R. The procedure of taking a ring R, and forming $(R)_{m\times m}$, is well-known to be functorial. Considering $(-)_{m\times m}$ as a functor from commutative A-algebras to A-algebras, where A is any commutative coefficient ring, Amitsur [3] and Procesi [24] have noted that there is a left adjoint, from A-algebras back to commutative A-algebras, which is denoted $V_{m,A}(-)$, or, sometimes, simply $V_m(-)$. The front adjunction map, from R to $(V_m(R))_{m\times m}$, is denoted ε_R. For example, if $R = A\{X_1,\ldots,X_n\}$ is the free A-algebra in n noncommuting variables $X_1, \ldots, X_n$, then it is easy to see that $V_m(R)$ is isomorphic to the A-algebra $A[x_{ijk}: 1 \le i,\ j \le m,\ 1 \le k \le n]$, of polynomials in m^2n variables, with ε_R given, up to isomorphism, by the map taking X_k to the $m\times m$ matrix with x_{ijk} in the (i,j)-position. It is also straightforward to check that $V_m(R/I)$ is isomorphic to $V_m(R)$ modulo the ideal generated by the entries of $\varepsilon_R(I)$, if I is any ideal of R. This observation shows that $V_m(-)$ takes affine A-algebras to affine commutative A-algebras. Procesi [24] has used the functor $V_m(-)$ extensively in developing the theory of representations of p.i. algebras.

Definition 3.1. The image of ε_R, where R is the free A-algebra in n noncommuting variables, $A\{X_1,\ldots,X_n\}$, is called the A-algebra of n generic $m\times m$ matrices, and is denoted $A[x_1,\ldots,x_n]_{m\times m}$.

The generic matrix algebras play a role, in the theory of p.i. algebras, analogous to that played by the ordinary polynomial algebras in commutative ring theory. If A is a commutative integral domain, then $A[x_1,\ldots,x_n]_{m\times m}$ is an integral domain satisfying the identities of m×m matrices, and, provided n > 1, it is a prime p.i. ring of p.i. degree m. One crucial way in which the generic matrix algebras differ from the ordinary polynomial algebras is that, for m, n > 1, $A[x_1,\ldots,x_n]_{m\times m}$ is not Noetherian. For further information on the generic matrix algebras, we refer the interested reader to Jacobson [12, 1.13 and II.4] and Procesi [25].

If A is any infinite integral domain, one way of defining the ordinary polynomial algebra, in n variables, over A is to take the A-subalgebra of the A-algebra of all functions from A^n to A which is generated by the projection mappings $(x_1,\ldots,x_n) \to x_i$, $i = 1,\ldots,n$. In the context of classical algebraic geometry, where A is usually taken to be an algebraically closed field, one studies affine varieties contained in A^n by means of studying the structure of the corresponding ideals of the polynomial ring which consist of the functions which vanish on them. One proves, for example, that an affine variety $V \subseteq A^n$ is irreducible if and only if the corresponding ideal of polynomials vanishing on V is a prime ideal, and, moreover, every prime ideal of $A[x_1,\ldots,x_n]$ arises in this way. On the other hand, if A is not necessarily algebraically closed, it is not at all well understood which prime ideals of $A[x_1,\ldots,x_n]$ arise as the sets of polynomials vanishing on irreducible affine varieties. The recent work of Silhol [34] develops the foundations of classical algebraic geometry in the context of an arbitrary coefficient field, and there is given, there, a criterion for a prime ideal of $A[x_1,\ldots,x_n]$ to be the ideal of functions vanishing on some subset of A^n. Let us call a prime ideal of $A[x_1,\ldots,x_n]$ geometric if it arises in this way. If A is any field, and $\bar{A}$ is an algebraic closure of A, then Silhol's criterion is that $P \in \text{Spec}(A[x_1,\ldots,x_n])$ is geometric if and only if $P \cdot \bar{A}[x_1,\ldots,x_n]$ is a prime ideal. We shall give a characterization of the geometric prime ideals of $A[x_1,\ldots,x_n]$ in terms of the first-order logic of A.

Now, by way of analogy to this, one can realize the generic matrix algebra $A[x_1,\ldots,x_n]_{m\times m}$, over A, as the algebra of functions from $(A)_{m\times m}{}^n$ to $(A)_{m\times m}$ which is generated by the projection mappings. As long as A is an infinite integral domain, this is a faithful realization. A subset $V \subseteq (A)_{m\times m}{}^n$ is called an affine variety in m×m matrices over A if there is some set of generic matrix polynomials $X \subseteq A[x_1,\ldots,x_n]_{m\times m}$ such that $V = \{(a_{ijk}) \in (A)_{m\times m}{}^n : f((a_{ijk})) = 0, \text{ for all } f \in X\}$. The variety V is called prime in case $\{f \in A[x_1,\ldots,x_n]_{m\times m} : f \text{ vanishes on } V\}$ is a prime ideal. (We avoid using the term "irreducible" since it is possible that V is not expressible as the union of two proper sub-varieties, and yet V is not prime.

A simple example of this is to take for V a single point $(a_{ijk}) \in (A)_{m\times m}{}^n$, where the m×m matrices $(a_{ij1}),\ldots,(a_{ijn})$ generate the algebra of upper triangular m×m matrices. It is a nice exercise to check that if V is any variety in m×m matrices which contains at least one point $(a_{ijk}) \in (A)_{m\times m}{}^n$ such that the matrices $(a_{ij1}),\ldots,(a_{ijn})$ generate the whole algebra $(A)_{m\times m}$, and if V is irreducible, then V is a prime.)

As in the commutative case, where $m = 1$, we call a prime ideal of $A[x_1,\ldots,x_n]_{m\times m}$ <u>geometric</u> if it arises as the ideal of all generic matrix polynomials vanishing on some subset of $(A)_{m\times m}{}^n$. We shall give a characterization of the geometric prime ideals of the generic matrix rings, also in terms of the first order logic of $(A)_{m\times m}$. We first handle the commutative case. Recall that two algebras are called "elementarily equivalent" if they satisfy exactly the same first-order sentences. A good general reference for what we shall need from first-order logic is Bell & Slomson [6].

<u>Theorem 3.2</u>. Let A be any commutative Noetherian domain, and let $P \subseteq A[x_1,\ldots,x_n]$ be any prime ideal in the polynomial ring in n variables over A. Let $R = A[x_1,\ldots,x_n]/P$. Then P is geometric if and only if R is a subalgebra of some A-algebra elementarily equivalent to A.

One particular, and well-known, case of the above theorem is when A is the field of real numbers. The sub-fields of fields elementarily equivalent to the real numbers are exactly the formally real fields, so we recover the result that, in this case, P is geometric if and only if the quotient field of R is formally real. (Cf. Swan [37, Theorem 19.4, page 223]).

Proof of the theorem: The direction => is easy: If P is geometric, then R can be identified with a ring of A-valued functions on some algebraic set $V(P) \subseteq A^n$. Now pointwise evaluations over the points of V(P) give well-defined A-algebra homomorphisms from R to A, and these combine to give an embedding $R \to \prod_{p\in V(P)} A$, by sending $r \in R$ to the family $(r(p))_{p\in V(P)}$. Since R is an integral domain, a result of Amitsur [2, Theorem 3, page 475] tells us that there is some ultrafilter, F, on V(P), such that the composite mapping $R \to \prod_{p\in V(P)} A \to \prod_{p\in V(P)} A/F$ is still an embedding. This shows that R is isomorphic to a subalgebra of some ultrapower of A. We now invoke Keisler's Ultrapower Theorem (Bell & Slomson, [6, Theorem 2.6, page 150]) which allows us to conclude that ultrapowers of A are elementarily equivalent to A. Now to prove the converse implication, we need a lemma about lifting maps from ultrapoducts to products:

<u>Lemma 3.3.</u> Let R be an affine commutative A-algebra, and $(S_\alpha)_{\alpha\in I}$ any family of commutative A-algebras, and let F be any ultrafilter on I. Let $f : R \to \prod_{\alpha\in I} S_\alpha/F$ be any homomorphism. Then:

(1) If J is any subset of I belonging to F, then $F|_J$, the restriction of F to J, is an ultrafilter on J, and the projection mapping $\pi : \prod_{\alpha\in I} S_\alpha \to \prod_{\alpha\in J} S_\alpha$ factors to give an isomorphism $\overline{\pi} : \prod_{\alpha\in I} S_\alpha/F \xrightarrow{\sim} \prod_{\alpha\in J} S_\alpha/F|_J$.

(2) There exists a subset $J \subseteq I$, which belongs to F, and an A-algebra homomorphism $\phi : R \to \prod_{\alpha\in J} S_\alpha$ which makes the following diagram commutative:

$$\begin{array}{ccc} R & \xrightarrow{f} & \prod_{\alpha\in I} S_\alpha/F \\ \downarrow\phi & & \uparrow\overline{\pi} \\ \prod_{\alpha\in J} S_\alpha & \to & \prod_{\alpha\in J} S_\alpha/F|_J . \end{array}$$

Proof of the lemma: (1) is exactly Theorem 2.2, page 123, from Bell & Slomson [6]. To prove (2), first express R as a factor ring of a polynomial ring over A: $R \cong A[x_1,\ldots,x_n]/I$. Now, using the fact that A is Noetherian, together with the Hilbert Basis Theorem, choose a finite system of generators for I, say $q_1,\ldots,q_t$. Choose elements $t_1,\ldots,t_n \in \prod_{\alpha\in I} S_\alpha$ so that each t_i represents, modulo F, the element $f(x_i)$ of the ultraproduct $\prod_{\alpha\in I} S_\alpha/F$. Now let $J = \{\alpha \in I : q_i(t_1(\alpha),\ldots,t_n(\alpha)) = 0,\ i = 1,\ldots,t\}$. By Łoś's Theorem (Bell & Slomson, [6], Theorem 2.1, page 90) J belongs to F. Let $s_j \in \prod_{\alpha\in J} S_\alpha$ be the restriction of t_j to J, $j = 1,\ldots,n$. The mapping $x_1 \to s_1,\ldots, x_n \to s_n$, from $A[x_1,\ldots,x_n]$ to $\prod_{\alpha\in J} S_\alpha$, contains each q_i in its kernel, by construction. Hence, it contains I in its kernel, so we obtain the required lifting $R \cong A[x_1,\ldots,x_n]/I \to \prod_{\alpha\in J} S_\alpha$.

We note that the above lemma can be formulated in the context of universal algebra, rather than the special context of commutative A-algebras. It is clear from the proof that the requirement that R be finitely-generated for the above lifting to be possible is not quite sufficient, if R is a universal algebra of some type in some equational class. R must also be finitely-presented. In particular, the lemma does not hold, in general, in the context of noncommutative A-algebras satisfying the identities of mxm matrices, so we cannot use this technique, directly, to prove the analogue of Theorem 3.2 for generic matrix rings. Now we can prove the implication $\Leftarrow$ for Theorem 3.2. Suppose there is given an embedding from R into some A-algebra B which is elementarily equivalent to A. Again, using Keisler's Ultrapower Theorem, we can replace B by some appropriate ultrapower of A, and so reduce to the case that B is actually an ultrapower of A. Let $\psi : R \to \prod_X A/F$ be the given embedding, where X is some index set, and F some ultrafilter on X. The lemma tells us that ψ can be lifted to a map $\psi : R \to \prod_Y A$, for some subset Y of X belonging to F. But now we have R embedded in a power of A. From this, it is easy to see that P is geometric, as follows: Let $\pi_y : R \to A$ be the composite of ψ with the y^{th} projection mapping from $\prod_Y A$ to A, for each $y \in Y$, and let $V = \{(\pi_y(\overline{x}_1),\ldots,\pi_y(\overline{x}_n)) : y \in Y\}$, where $\overline{x}_i$ denotes the coset, modulo P, of x_i. It is clear that P is the ideal of all polynomials vanishing on V. For if $I = \{f \in A[x_1,\ldots,x_n] : f|_V = 0\}$, then $f \in I \iff f|_V = 0 \iff f(\pi_y(\overline{x}_1),\ldots,\pi_y(\overline{x}_n)) = 0$, for all $y \in Y \iff \pi_y(\overline{f(x_1,\ldots,x_n)}) = 0$, for all $y \in Y \iff f \in \cap\{\mathrm{Ker}(\pi_y) : y \in Y\} = \overline{0}$ $\iff f \in P$. This completes the proof of Theorem 3.2.

Equipped with the above Lemma 3.3, together with the functor $V_m(-)$, the analogue of Theorem 3.2 can now be proved for generic matrix rings.

Theorem 3.4. Let $P \subseteq A[x_1,\ldots,x_n]_{m\times m}$ be any prime ideal of the generic matrix ring over the commutative Noetherian domain A, and let $R = A[x_1,\ldots,x_n]_{m\times m}/P$ be the corresponding affine prime p.i. algebra. Then P is geometric if and only if R is a subalgebra of an A-algebra elementarily equivalent to $(A)_{m\times m}$.

It is easy to see that the A-algebras elementarily equivalent to $(A)_{m\times m}$ are exactly those of the form $(B)_{m\times m}$, where B is an A-algebra elementarily equivalent to A. So, for instance if A is an algebraically closed field, any $P \in \mathrm{Spec}_m(A[x_1,\ldots,x_n]_{m\times m})$ is geometric. For, by Posner's theorem, the Posner quotient ring of R is a central simple algebra of degree m, and, by choosing a splitting field for this central simple algebra, we can get an embedding of R into the $m\times m$ matrix ring over a field. However, not every prime ideal is geometric, as follows from the following result of Bergman and Small [7, Corollary 7.2]:

Suppose $R \subseteq R'$ are prime p.i. rings, and every non-zerodivisor of R remains a non-zerodivisor in R'. Then p.i.deg(R) divides p.i.deg(R').

Thus, for example, if P is any maximal ideal of $A[x_1,\ldots,x_n]_{m\times m}$ such that the factor ring $R = A[x_1,\ldots,x_n]_{m\times m}/P$ is isomorphic to $(A)_{(m-1)\times(m-1)}$, and if $m > 2$, then R can not be embedded in the ring of $m\times m$ matrices over any field, so P is not geometric.

Proof of Theorem 3.4: The implication $\Rightarrow$ is true for exactly the same reason as before, when m was 1. The fact that P is geometric give rise to an embedding $R \to \prod_{p\in V(P)} (A)_{m\times m}$ and the result of Amitsur still allows us to find an ultrafilter, F, on $V(P)$, such that the composite map $R \to \prod_{p\in V(P)} (A)_{m\times m} \to \prod_{p\in V(P)} (A)_{m\times m}/F$ remains an embedding, and this gives R isomorphic to a subalgebra of an ultrapower of $(A)_{m\times m}$, as required. For the converse, we suppose R is embedded into some A-algebra elementarily equivalent to $(A)_{m\times m}$, and, for the same reason as before, we can assume from the start that R is actually embedded into an ultrapower of $(A)_{m\times m}$. Let $f : R \to \prod_{\alpha\in I} (A)_{m\times m}/F$ be the embedding. Now $\prod_{\alpha\in I} (A)_{m\times m}/F \cong (\prod_{\alpha\in I} A/F)_{m\times m}$, so, up to isomorphism, f is an embedding of R into the $m\times m$ matrix ring over an ultrapower of A. Applying Procesi's functor V_m, we can factor f through a map $\bar{f} : V_m(R) \to \prod_{\alpha\in I} A/F$, so that $(\bar{f})_{m\times m} \circ \varepsilon_R = f$. But now we can use the Lemma 3.3, since f is a map from an affine commutative A-algebra to an ultrapower of A. So we obtain a map $\bar{g} : V_m(R) \to \prod_{\alpha\in J} A$, for some subset J of I, as in the lemma. Now the composite $g = (\bar{g})_{m\times m} \circ \varepsilon_R$ is an embedding of R into a power of $(A)_{m\times m}$, and, as in the commutative case, this gives that P is geometric as required.

Remark 3.5. Notice that, in the proof of this theorem, we could have avoided the use of the functor V_m if we knew that P was a finitely-generated ideal. Now even though the generic matrix rings $A[x_1,\ldots,x_n]_{m\times m}$ are not Noetherian, for m, n > 1, it is an open problem whether or not every prime ideal of $A[x_1,\ldots n_n]_{m\times m}$ is finitely-generated. This question seems to have first appeared in the literature in Small [35]. It is known, for example, that an ideal I of $R = A[x_1,\ldots,x_n]_{m\times m}$ is finitely-generated if the factor ring R/I is right and left Artinian (Cf. Procesi [24, Proposition 5.2, page 118]). The remainder of this chapter will be devoted to an improvement of this result.

Theorem 3.6. Let $R = A\{r_1,\ldots,r_v\}$ be any affine algebra over a commutative Noetherian ring A, and let I be any ideal of R such that R/I is a finitely-generated module over its center. Then I is finitely-generated, as a two sided ideal of R.

Proof: Suppose $y_1,\ldots,y_n$ are elements of R whose cosets, modulo I, generate R/I, considered as a module over its center. Then we have equations of the form: $y_i \cdot r_j \equiv \sum_{k=1}^{n} c_{ijk}y_k$ modulo I, for $1 \le j \le v$, for appropriate elements $c_{ijk} \in R$ which are central modulo I. As a first step, let K be the two-sided ideal of R generated by the elements $(c_{ijk}r_p - r_pc_{ijk})$, for $1 \le i,k \le n$, $1 \le j,p \le v$. Then K is a finitely-generated ideal of R contained in I. Inside R/K, there is the central subring $B = (A/A\cap K))[\bar{c}_{ijk}]$, where $\bar{c}_{ijk}$ denotes the coset, modulo K, of c_{ijk}. By the Hilbert Basis theorem, B is Noetherian. Moreover, the relations $y_ir_j = \sum_{k=1}^{n} c_{ijk}y_k$, modulo I, imply that R/I is a finitely-generated B-module. This situation is handled by the following proposition:

Proposition 3.7. Let B be any commutative Noetherian ring, and $S = B\{s_1,\ldots,s_m\}$ any affine B-algebra. Let J be any right ideal of S such that S/J is a finitely-generated B-module. Then J is finitely-generated, as a right ideal of S.

Applying this proposition to our situation above, we deduce that I/K is a finitely-generated right ideal of R/K. But since K was a finitely-generated ideal, hence I is finitely generated as an ideal of R. This finishes the proof of the Theorem 3.6, provided we prove Proposition 3.7.

Proof of Proposition 3.7: Let $z_r,\ldots,z_r$ be elements of S whose cosets, modulo J, generate S/J as a right B-module. Without loss of generality, we can choose $z_1 = 1$. Then we have $z_i \cdot s_j \equiv \sum_{k=1}^{n} z_k a_{ijk}$ modulo J, for suitable elements $a_{ijk} \in B$. Let J' be the right ideal of S generated by the elements $z_i s_j - \sum_{k=1}^{n} z_k a_{ijk}$, for $1 \le i \le r$ and $1 \le j \le m$. Clearly J' is contained in J, so there is a natural map f : S/J' —>> S/J. Now S/J' is a finitely-generated B-module, since the B-submodule of S/J' generated by the cosets of $z_1,\ldots,z_r$ contains the coset of 1, and is closed under multiplication on the right by $s_1,\ldots,s_m$, so that this B-submodule is also a right S-module. Now the kernel of f is a submodule of the finitely-generated B-module S/J', so it is a finitely-generated right ideal, J/J', of S/J'. Therefore J is a finitely-generated right ideal of S, since it is finitely-generated modulo a finitely-generated right ideal. This completes the proof.

Note: The proof of the above proposition was motivated by Wehrfritz [38, proof of (1), page 22].

In connection with the question of whether or not the prime ideals of $A[x_1,\ldots,x_n]_{m\times m}$ are finitely-generated, Theorem 3.6 gives the following partial affirmative answer:

Corollary 3.8. If $P \subseteq A[x_1,\ldots,x_n]_{m\times m}$ is any prime ideal in the A-algebra of n generic $m\times m$ matrices, where A is any commutative Noetherian domain, then P is finitely-generated if the factor ring, $A[x_1,\ldots,x_n]_{m\times m}/P$, is a finitely-generated module over its center.

This corollary includes, for example, the case where P is a maximal ideal, since, in this case, Kaplansky's theorem on primitive p.i. rings guarantees that the factor ring is a finite-dimensional vector space over its center. Moreover, by Theorem 2.6 if A is a field then the factor ring $A[x_1,\ldots,x_n]_{m\times m}/P$ is a finite-dimensional vector space over A, so Proposition 3.7 gives the additional information that P is finitely-generated as a right ideal. It is also known that if A is a field, and if $R = A\{r_1,\ldots,r_v\}$ is a prime p.i. algebra of dimension one (i.e. with the property that every non-zero prime ideal is maximal), then R is a finitely-generated module over its center. (Cf. Schelter [30], or Malliavin-Brameret [16]. Schelter actually proves it for more general rings A). So we have:

Corollary 3.9. If P is any prime ideal in the generic matrix ring $A[x_1,\ldots,x_n]_{m\times m}$, where A is any field, such that $A[x_1,\ldots,x_n]_{m\times m}/P$ is of dimension one, then P is finitely-generated.

We can also deduce the result of Procesi [24, Proposition 5.2, page 118] that was mentioned before Theorem 3.6, namely:

<u>Corollary 3.10</u>. Let S be an affine algebra over a commutative Noetherian ring A. Let M be an ideal of S such that S/M is an Artinian p.i. ring. Then M is a finitely-generated ideal of S.

Proof: This would be immediate from Theorem 3.6, if we knew that S/M was a finitely-generated module over its center. This does not appear in the literature, but it is not hard to prove:

<u>Proposition 3.11</u>. Let $R = A\{x_1,\ldots,x_n\}$ be any affine p.i. algebra over a commutative ring A, such that R is Artinian. Then R is a finitely-generated module over its center.

Proof of the proposition: First we need a lemma:

<u>Lemma 3.12</u>. Let R be any Artinian ring, with Jacobson radical J, and center C. Let x be any element of C which is a non-zerodivisor modulo $(C \cap J)$. Then x is a non-zerodivisor in R, and hence is invertible in R.

Proof of the lemma: Let $1 = e_1 + \ldots + e_k$ be a decomposition of 1 into a sum of indecomposable, orthogonal idempotents of R. Let them be ordered so that $e_1x, \ldots, e_tx$ are not nilpotent, and $e_{t+1}x, \ldots, e_kx$ are nilpotent. Choose a sufficiently large integer $M > 0$, so that $(e_ix)^M = 0$, for $t+1 \le i \le k$. Let $y = x^M$. Then y is central, and is a non-zerodivisor modulo $(C \cap J)$, and we have that $e_1y, \ldots, e_ty$ are not nilpotent, and $e_{t+1}y = \ldots = e_ky = 0$. Let $e = e_1 + \ldots + e_t$, and $f = e_{t+1} + \ldots + e_k$. Now $y = eye$ is invertible in eRe; for each e_jy is a non-nilpotent, and hence invertible element of the Artinian local ring e_jRe_j, $1 \le j \le t$, so if its inverse is s_j, and we let $s = s_1 + \ldots + s_t$ we see that s is an inverse of y in eRe. Thus, we have $eR \subseteq yR$, and $Re \subseteq Ry$. But this allows us to prove that $eRf = fRe = 0$ (since $eRf \subseteq yRf = Ryf = 0$, and similarly for fRe.) Thus e and f are central. So $yf = 0$, which forces $f \in J$, which forces $f = 0$, and we conclude that $e = 1$, and y is invertible in $eRe = R$. Hence, so is x.

Before continuing the proof of Proposition 3.11, let us note the following interesting little corollary to this lemma:

Corollary 3.13. The center of an Artinian ring is semiprimary.

Proof of this corollary: With the same notation as in the lemma, $C/(C \cap J)$ is a Goldie semiprime ring sitting inside the semisimple Artinian ring R/J, and the above lemma says that every non-zerodivisor in $C/(C \cap J)$ is invertible. Thus, $C/(C \cap J)$ is semisimple Artinian. Moreover, if $m > 0$ is such that $J^m = 0$, then we have $(C \cap J)^m = 0$, so C is semiprimary, as required.

This result is known [14 Folgerung 3.6], but not well-known.

Completion of the proof of Proposition 3.11: R/J, in the situation of the proposition, is a semisimple Artinian p.i. algebra, affine over the semisimple commutative ring $C/(C \cap J)$. By Theorem 2.6, R/J is a finitely-generated $C/(C \cap J)$-module. Also, since each J^i/J^{i+1} is a finitely-generated right R/J-module, it is therefore a finitely-generated $C/(C \cap J)$-module too. By putting together the pieces : $R \supseteq J \supseteq J^2 \supseteq \ldots \supseteq J^m = 0$, we see that R is a finitely-generated C-module.

The following example shows that the affineness assumption in Proposition 3.11 cannot be dropped. It also shows that the center of an Artinian ring need not be Artinian, even if the ring is a p.i. ring:

Example 3.14. Let L be any field, with K_1 and K_2 two subfields of L, of finite index in L, such that $k = K_1 \cap K_2$ has infinite index in L. (For example, take $L = Q(X)$, rational functions in one variable over the field of rational numbers, with $K_1 = Q(X^2)$, and $K_2 = Q(X^2-X)$, and $k = Q$.) We consider the subring, R, of the 2x2 matrix ring over the ring $L[Y]/(Y^2)$ of dual numbers over L, given by:

$$R = \begin{pmatrix} K_1[Y]/(Y^2) & (YL[Y])/(Y^2) \\ 0 & K_2[Y]/(Y^2) \end{pmatrix}.$$

R is an Artinian p.i. ring, and it is easy to see that its center is

$$C = \left\{ \begin{pmatrix} s + Yr_1 & 0 \\ 0 & s + Yr_2 \end{pmatrix} : s \in k,\ r_1 \in K_1,\ r_2 \in K_2 \right\}$$

which is semiprimary, but not Artinian. Moreover, R is not a finitely-generated C-module.

The following theorem gives a criterion that forces an affine prime p.i. algebra over a field to be a finitely-generated module over its center.

Theorem 3.15. Let $R = k\{a_1,\ldots,a_t\}$ be any prime p.i. ring which is an affine algebra over a field k, such that there are only finitely many prime ideals $P \in \mathrm{Spec}(R)$ of deficient p.i. degree. Then R is a finitely-generated module over its center.

Proof: Let $M_1, \ldots, M_s$ be the prime ideals of R of deficient p.i. degree. Since there are only finitely many of them, they must all be maximal. For, by Theorem 2.6, any prime ideal in R is the intersection of maximal ideals; if the prime ideal is not maximal, it takes infinitely many maximal ideals to effect this intersection, and if the prime ideal is also of deficient p.i. degree, all of these maximal ideals must be of deficient p.i. degree. Now these M_i's are the only prime ideals of R containing the set, $H_m(R)$, of evaluations of Razmyslov's central polynomial for R, h_m, where p.i. deg(R) = m. Thus, all prime ideals of $R/(H_m(R)R)$ are maximal. So $R/(H_m(R)R)$ is zero-dimensional. Hence, it is Artinian (by Procesi [24, Theorem 5.4, page 122]). Hence, $H_m(R)R$ is a finitely-generated ideal. So let $c_1, \ldots, c_k$ be evaluations of h_m on R which generate $H_m(R)R$, as an ideal of R. Write $c_i = h_m(x_{1i}, \ldots, x_{m^2 i}; y_{1i}, \ldots, y_{(m^2+1)i})$, for appropriate x_{ji}'s and y_{ji}'s in R. Now let M be the C-submodule of R spanned by all of the x_{ji}'s ($1 \le j \le m^2$, $1 \le i \le k$), where C is the center of R.

Claim: M contains $H_m(R)R$.

Proof of the claim: By Theorem 2.3, we have equations of the form $c_i r = \sum_{j=1}^{m^2} d_{ji} x_{ji}$, for any $r \in R$, and appropriate elements $d_{ji} \in C$. Thus, for all $r \in R$, $c_i r \in M$, so M contains the ideal of R generated by the c_i's, namely $H_m(R)R$.

Conclusion of the proof of the theorem: Theorem 2.6 says that each R/M_i is a finite-dimensional vector space over k. Since $R/(H_m(R)R)$ is Artinian, and has the M_i's for its maximal ideals, it follows that $R/(H_m(R)R)$ is a finite-dimensional vector space over k. Hence so is R/M, since it is a homomorphic image of $R/(H_m(R)R)$. Thus it is also a finitely-generated C-module. So R is a finitely-generated C-module, since it is an extension of the finitely-generated C-module M by the finitely-generated C-module R/M.

Corollary 3.16. Under the above assumptions, R is also Noetherian. In fact, any algebra $R = k\{a_1,\ldots,a_t\}$ which is a finitely-generated module over its center, where k is any commutative Noetherian ring, is Noetherian, and its center is affine over k.

Proof of the corollary: If $y_1,\ldots,y_n$ generate R, considered as a module over its center, C, where we can take $y_1 = 1$ without loss of generality, then $y_i a_j = \sum_{k=1}^{n} c_{ijk} y_k$, for appropriate elements $c_{ijk} \in C$. Letting, $A = k[c_{ijk};\ \text{all } i,j,k]$, we see that the A-submodule of R generated by $y_1,\ldots,y_n$ must be all of R. Thus R is a finitely-generated module over the central, Noetherian ring A, so R is Noetherian, as required. The center of R is an A-submodule of R, so it is finitely-generated A-module. Since A is affine over K, then so is the center of R.

CHAPTER 4

Prime Ideals of Maximal P.I. Degree in Prime P.I. Rings

Beginning this chapter, we study several aspects of the arithmetic and algebraic properties of prime ideals in prime p.i. rings, especially those properties related to localization. To begin with, let R be any prime p.i. ring, of p.i. degree m, and let $P \in \mathrm{Spec}_m(R)$ be any prime ideal of maximal p.i. degree. Small [36] proved the following result.

<u>Theorem 4.1</u>. P is localizable. In other words, the set $\mathcal{C}(P) = \{r \in R :$ is a non-zerodivisor modulo $P\}$ is a multiplicative subset of R which satisfies the Ore conditions in R.

Moreover, the set $\mathcal{C}(P)$ consists of non-zerodivisors in R, so that the localization, R_P, obtained by inverting the elements of $\mathcal{C}(P)$, can be identified with the subring of the Posner quotient ring $\mathcal{Q}(R)$ formed by adjoining to R the elements c^{-1} of $\mathcal{Q}(R)$, for $c \in \mathcal{C}(P)$. Now using central polynomials, this result has been sharpened by Rowen [29].

<u>Theorem 4.2</u>. Let C be the center of R. Then the localization, R_P, can be obtained by adjoining to R only the central elements c^{-1}, $c \in \mathcal{C}(P) \cap C$.

One can think of this result as saying, in the language of torsion theories, that the P-torsion theory is generated by a central multiplicative set. This suggests the following problem, which we shall explore in this chapter:

Problem 4.3. If R is any prime p.i. ring of p.i. degree m, and if $P \in \mathrm{Spec}_m(R)$, how far is P from being centrally generated, as an ideal of R? Under what circumstances must P be centrally generated?

A partial answer to the problem is given by our next result:

Proposition 4.4. As before, let C be the center of R. Then P is centrally generated if P is a maximal ideal, or if C is a Prüfer domain.

To prove this, we shall need a lemma:

Lemma 4.5. If $\pi = P \cap C$, then $P/\pi R$ is a torsion C/π-module.

Proof of the lemma: Recall that, from Theorem 2.3, we have an equation of the form $r\,\delta = \sum_{i=1}^{m^2} r_i g_i(r)$, for all $r \in R$, where δ is any given evaluation of Razmyslov's central polynomial for mxm matrices, the elements r_i are appropriate elements of R, and the functions g_i are C-linear functions from R to C, given by evaluation of Razmyslov's central polynomial with r substituted for one of the arguments. In particular, if $r \in P$, then $g_i(r) \in P \cap C = \pi$. So if we choose δ to be any evaluation of h_m on R which is non-zero in R/P (which is possible, by property H1 of h_m), we see that $P \cdot \delta \in \pi R$. This proves the lemma, since the coset of δ in C/π is non-zero, and it annihilates $P/\pi R$.

Remark 4.6. Notice that the above proof shows that $P/\pi R$ is actually annihilated by any evaluation of h_m. This will be used lated.

Now to prove Proposition 4.4, suppose, first of all, that P is any maximal ideal of R. Then π is a maximal ideal of C. For $F_m(R/P)$, by Corollary 2.9, contains a non-zero ideal of the center of R/P. Since P is maximal, the center of R/P is a field, so $F_m(R/P)$, which equals the image of $F_m(R)$ by Lemma 2.7, is the entire center of R/P. Thus C, which contains $F_m(R)$, also maps onto the entire center of R/P, and so C/π is a field. Now $P/\pi R$, as a torsion module over the field C/π, must be zero. Hence $P = \pi R$, in this case.

This part of the proposition is known, and is proved, using a different approach, in Rowen [29, Theorem 4.16, part (d)].

Next, suppose only that C is a Prüfer domain. Recall that Prüfer domains are characterized by the property that their flat modules are precisely their torsion-free modules. (See, for example, Rotman [27, page 86]). Now R is clearly a torsion-free C-module, since, as we have remarked before, the non-zero elements of C are non-zerodivisors on R. Hence, R is a flat C-module, so that $R \otimes_C C/\pi$ is a flat C/π-module. But C/π is also a Prüfer domain, so that $R \otimes_C C/\pi = R/\pi R$ must be a torsion-free C/π-module. Thus, $P/\pi R$ is both torsion and torsion-free, so it must be zero. So, again in this case, $P = \pi R$.

<u>Example 4.7</u>. A simple example where P is not generated by its intersection with C is the following. Let $C = k[x,y]$ be the ordinary commutative polynomial ring in two variables over a field k, let I be the ideal of C generated by x and y, and let π be the ideal of C generated by x. Consider the ring $R = \begin{pmatrix} C & C \\ I & C \end{pmatrix}$. R is clearly a prime, p.i. ring, of p.i. degree 2, and, in fact, a finitely-generated module over its center, which is just C, embedded diagonally. The ideal $P = \begin{pmatrix} \pi & \pi \\ \pi & \pi \end{pmatrix}$ belongs to $\mathrm{Spec}_2(R)$, but is not generated by its intersection with the center, because $(P \cap C)R = \pi R = \begin{pmatrix} \pi & \pi \\ \pi I & \pi \end{pmatrix} \neq P$.

Notice, however, in this example, that P is the only minimal prime ideal over πR. In other words, $P = \sqrt{\pi R}$. So we ask the question:

Problem 4.3'. When is $P = \sqrt{\pi R}$?

Proposition 4.8. Let R be any prime p.i. ring, whose center C is an integrally closed Noetherian domain. Then, for any $\pi \in \text{Spec}(C)$ and $P \in \text{Spec}(R)$, P is minimal over πR if and only if $P \cap C = \pi$.

This follows straightforwardly from the following theorem of Schelter [31, Theorem 3 (Going Down)]:

Theorem 4.9. If R is a prime p.i. ring, integral over an integrally closed central subring C, then for any primes $\pi_0 \subsetneq \pi_1 \subseteq C$, and $P_1 \subset R$ with $P_1 \cap C = \pi_1$, there exists a prime $P_0 \subsetneq P_1$ of R such that $P_0 \cap C = \pi_0$.

Now in the context of the proposition, R is integral over C since R is, by Corollary 2.5, a finitely-generated C-module. Thus, if P were minimal over πR, and $P \cap C = \pi' \supsetneq \pi$, then Schelter's theorem would give $Q \in \text{Spec}(R)$ with $Q \subsetneq P$ and $Q \cap C = \pi$, contrary to the fact that P was minimal over πR. Conversely, if $P \cap C = \pi$, and P is not minimal over πR, then there would be some prime ideal $Q \subsetneq P$ with $Q \supseteq \pi R$. Then $Q \cap C = \pi$, so, in the prime factor ring R/Q, P/Q would be a non-zero prime ideal having zero intersection with C/π, contrary to the fact that R/Q is integral over C/π.

Corollary 4.10. Let R be any prime p.i. ring, of p.i. degree m, whose center C is an integrally closed Noetherian domain. If $P \in \text{Spec}_m(R)$, then $P = \sqrt{(P \cap C)R}$.

Proof: The proposition tell us that P is a minimal prime over $(P \cap C)R$.

To see that P is the only minimal prime over $(P \cap C)R$, it suffices to show that P is the only prime ideal of R whose intersection with C is $C \cap P$. This follows from the following easy lemma. For a proof, see for example, Rowen [29, Theorem 4.16].

Lemma 4.11. Let R be any prime p.i. ring of p.i. degree m, with center C. Let $V(F_m(R)) = \{\pi \in \mathrm{Spec}(C) : \pi \supseteq F_m(R)\}$. Then the map $P \to P \cap C$, from Spec(R) to Spec(C) restricts to $\mathrm{Spec}_m(R)$ to give a bijective map from $\mathrm{Spec}_m(R)$ onto $\mathrm{Spec}(C) - V(F_m(R))$.

So, in our context, if Q is any other prime ideal of R with $Q \cap C = P \cap C$, then Corollary 2.7 tells us that $Q \in \mathrm{Spec}_m(R)$, and the above lemma then tells us that $Q = P$. This proves the corollary, since now P is the only minimal prime ideal over $(P \cap C)R$.

Corollary 4.12. In the above context, P has the left and right AR properties.

Proof: Since R is Noetherian by Corollary 2.5, some power of P is contained in $(P \cap C)R$. Moreover, $(P \cap C)R$ has the left and right AR properties, since it is centrally generated, by McConnell [17, Corollary 12]. Hence, so does P.

Later, we shall see that the assertion of Corollary 4.10 fails without the assumption that C be integrally closed. We leave as an open question whether the conclusion of Corollary 4.12 holds without the assumption that C be integrally closed. Thus, in response to Problem 4.3, we see that P is far from being centrally generated, in the general case. The following proposition gives a more detailed description of $\sqrt{(P \cap C)R}$:

Proposition 4.13. Let R be a Noetherian, prime p.i. ring, of p.i. degree m, with center C, and let $P \in \mathrm{Spec}_m(R)$, and $\pi = P \cap C$. Then P is a minimal prime ideal over πR, and if the minimal primes over πR are $P, Q_1, \ldots, Q_r$, then all of the other minimal prime ideals Q_i over πR, are of deficient p.i. degree. If R is a finitely-generated C-module, they satisfy $\mathrm{Kd}(R/Q_i) < \mathrm{Kd}(R/P)$, where Kd stands for Krull dimension.

Proof: If Q is any prime ideal of R such that $\pi R \subseteq Q \subseteq P$, then $\pi \subseteq \pi R \cap C \subseteq Q \cap C \subseteq P \cap C = \pi$, so Lemma 4.11 tells us that $Q = P$, and P is indeed minimal over πR. To see the other observations, notice that $\mathrm{ann}_R(P/\sqrt{\pi R}) = Q_1 \cap \ldots \cap Q_r$. The inclusion $\mathrm{ann}_R(P/\sqrt{\pi R}) \supseteq Q_1 \cap \ldots \cap Q_r$ follows from $P\cdot(Q_1 \cap \ldots \cap Q_r) \subseteq P \cap Q_1 \cap \ldots \cap Q_r = \sqrt{\pi R}$. But if $x \in \mathrm{ann}_R(P/\sqrt{\pi R})$ then $Px \subseteq \sqrt{\pi R} \subseteq Q_i$, for all i. So $P\cdot(RxR) \subseteq Q_i$, and $P \not\subseteq Q_i$, which implies that $RxR \subseteq Q_i$ for all i. Thus, $x \in Q_1 \cap \ldots \cap Q_r$. Now remark 4.6 says that the set, $H_m(R)$, of evaluations of Razmyslov's central polynomial h_m, is contained in $\mathrm{ann}_R(P/\pi R)$, so it follows that $Q_1 \cap \ldots \cap Q_r = \mathrm{ann}_R(P/\sqrt{\pi R}) \supseteq \mathrm{ann}_R(P/\pi R) \supseteq H_m(R)$. Thus, every evaluation of h_m is zero modulo each Q_i. By property H1 of h_m, each Q_i must be of deficient p.i. degree. Moreover $Q_i \cap C \supseteq (P \cap C) + F_m(R) \supsetneq P \cap C$, so $\mathrm{Kd}(C/(Q_i \cap C)) < \mathrm{Kd}(C/(P \cap C))$. The additional assumption that R is a finitely-generated C-module gives $\mathrm{Kd}((R/Q_i)_{C/(Q_i \cap C)}) = \mathrm{Kd}(C/(Q_i \cap C))$ and $\mathrm{Kd}((R/P)_{C/\pi}) = \mathrm{Kd}(C/\pi)$. By a result of Segal [33, Lemma 8], we have that the Krull dimension of R/P is the same whether considered as a C/π-module or as a module over itself, and similarly for R/Q_i. This yields the last assertion of the proposition.

We have seen, in Theorem 4.1 that if R is a prime p.i. ring, then the prime ideals of R of maximal p.i. degree are localizable. The well-known theorem of Artin [4] concerning prime p.i. rings states that every prime ideal of R is of maximal p.i. degree precisely when R is an Azumaya algebra over its center. In this case, all prime ideals of R are centrally generated. Suppose, now, that R is any prime p.i. ring of p.i. degree m, and $P \in \mathrm{Spec}_m(R)$. The prime ideals, Q, of the localization R_P, correspond to the prime ideals of R which are contained in P, by the correspondence $Q \to Q \cap R$. Now any such prime ideal $Q \subseteq R_P$, is of maximal p.i. degree because $m = \text{p.i.deg}(R_P) \geq \text{p.i.deg}(R_P/Q) \geq \text{p.i.deg}(R_P/PR_P) = \text{p.i.deg}(\mathcal{Q}(R/P)) = m$. Thus, R_P is an Azumaya algebra over its center. The following additional information is available, in case R is affine over a commutative Noetherian domain.

<u>Proposition 4.14</u>. Let $R = A\{x_1,\ldots,x_n\}$ be any affine, prime p.i. algebra over a commutative Noetherian domain A. Let m be the p.i. degree of R, and let $P \in \mathrm{Spec}_m(R)$ be any prime ideal of maximal p.i. degree. Then the localization of R at P, R_P, is Noetherian.

Proof: It is known that Azumaya algebras are closed under taking coefficients of reduced characteristic polynomials. (See Schelter [31] for a proof of this using only the theory of prime p.i. rings. For general information on Azumaya algebras, see, for example, Knus and Ojanguren [15].) As in the proof of Schelter [32, Proposition 5], let $c_1,\ldots,c_k$ be all of the coefficients of the reduced characteristic polynomials of all monomials of length $\leq m^2$ in $x_1,\ldots,x_n$, and let $B = A[c_1,\ldots,c_k]$ be the subring of R_P generated, over A, by $c_1,\ldots,c_k$. B is Noetherian, by the Hilbert Basis theorem. We now invoke Sirsov's theorem:

Theorem 4.16. Let $S = B\{x_1,\ldots,x_n\}$ be an affine p.i. algebra over the commutative ring B, of p.i. degree $\leq m$. If all of the monomials in $x_1,\ldots,x_n$ of length $\leq m^2$ are integral over B, then S is a finitely-generated B-module.

(For a proof of this, see, for example, Procesi [24, Theorem 3, page 152].)

Applying this to our situation, we take $S = R[c_1,\ldots,c_k]$, and B as before. The monomials in $x_1,\ldots,x_n$ of length $\leq m^2$ satisfy their reduced characteristic polynomials by the Hamilton-Cayley theorem, so they are indeed integral over B. Thus S is a finitely-generated module over the commutative Noetherian ring B, so S is Noetherian. Now R_P is a localization of S, since the multiplicative set of central elements of R not in P is still an Ore set in S. Thus R_P, being a localization of a Noetherian ring, must be Noetherian.

Remark 4.17. Notice that in the above proof, we have shown that if R is any prime p.i. ring which is affine over a commutative Noetherian domain, then there are finitely many elements $c_1,\ldots,c_k$ of the center of the Posner quotient ring of R such that the ring $R[c_1,\ldots,c_k]$ is a finitely-generated module over its center. We shall use this observation later.

Corollary 4.18. In the above setting, $\bigcap_{n=1}^{\infty} P^n = 0$.

Proof: R_P is a fully-bounded Noetherian ring, so, by a result of Jategaonkar [13], $\bigcap_{n=1}^{\infty} J(R_P)^n = 0$, where $J(R_P) = PR_P$ is the Jacobson radical of R_P. Clearly $P^n \subseteq (PR_P)^n$, so we deduce that $\bigcap_{n=1}^{\infty} P^n = 0$.

CHAPTER 5

Obstructions to Localization

We have seen in Theorem 4.1 that any prime ideal of maximal p.i. degree in a prime p.i. ring is localizable. This is not true, in general, for prime ideals of deficient p.i. degree, and in this chapter we shall explore this observation more carefully. Throughout the chapter, we use the machinery of noncommutative localization in fully bounded Noetherian rings, which is developed in Mueller[21], [22]. We start by summarizing the relevant facts we need.

If R is any ring, and M a right R-module, we denote by E(M) the injective envelope of M. If R is right Noetherian, and P is any prime ideal of R, then E(R/P), as an injective module, is a direct sum of finitely may copies of a certain indecomposable injective module, which we denote by E_P. P can be recovered from E_P as the unique prime ideal of R associated to E_P. That is, P is the largest ideal of R which occurs as the annihilator of a non-zero submodule of E_P. Thus, the correspondence $P \rightarrow E_P$ is one-to-one, from Spec(R) into the set of isomorphism classes of injective indecomposable right R-module. R is fully right-bounded exactly when this correspondence is bijective.

If P, Q are prime ideals of the fully bounded Noetherian ring R, a <u>long link</u> exists from P to Q in case $\mathrm{Hom}(E_P, E_Q) \neq 0$ and $\mathrm{Kd}(R/Q) = \mathrm{Kd}(R/Q)$. When this happens, we write $P \rightarrow\rightarrow Q$. Long links provide Spec(R) with the structure of a directed graph, and the set of prime ideals belonging to a given connected component of this directed graph is called a <u>link-component</u>.

The link-component to which a prime ideal $P \in \text{Spec}(R)$ belongs is denoted by comp(P).

The connection between long links and localization which we need is as follows:

L1. (Ref. Mueller [21, Theorem 5]). $P \in \text{Spec}(R)$ is localizable if and only if comp(P) = $\{P\}$. In other words, P is localizable if and only if it is not linked to any other prime ideal of R.

For the next assertion, we need to introduce some terminology: A finite set $\Sigma \subseteq \text{Spec}(R)$, of prime ideals among which there are no inclusion relations, is called a <u>localizable set</u> iff $\cap \Sigma = S$ is a localizable semiprime ideal of R.

L2. (Ref. Mueller [op. cit.]). If comp(P) is finite, then comp(P) is a localizable set, and, moreover, it is the smallest localizable set of prime ideals to which P belongs. In this case, comp(P) is called a <u>clan</u>.

L3. A finite set, Σ, of prime ideals from a given Krull-dimension stratum of Spec(R) is localizable if and only if it is a union of clans.

L4. (Ref. Mueller [22, Theorem 7]). If C is the center of R, and $P, Q \in \text{Spec}(R)$ such that $P \rightarrow\rightarrow Q$, then $P \cap C = Q \cap C$; moreover, if R is a finitely-generated C-module, then comp(P) = $\{Q \in \text{Spec}(R)\,;\, Q \cap C = P \cap C\}$.

In particular, if R is a finitely-generated module over its center, every prime ideal belongs to a localizable set.

Since Noetherian p.i. rings are fully bounded, we have available all of the above results in investigating localization in Noetherian p.i. rings. As an illustration of the use of this machinery, in the setting of Noetherian prime p.i. rings, consider the following problem:

Problem 5.1. Suppose R is a Noetherian prime p.i. ring such that not all prime ideals of R are of maximal p.i. degree. (As remarked in the previous chapter, this means that R is not an Azumaya algebra over its center.) Are there any localizable sets of prime ideals of deficient p.i. degree in Spec(R)?

To see that the answer to this question is affirmative, let S' be the ideal of R which is the intersection of all prime ideals of R of deficient p.i. degree. S' is a proper semiprime ideal of R, since we have assumed that there are some prime ideals of deficient p.i. degree in Spec(R). All of the minimal prime ideals over S' are of deficient p.i. degree. Let them be enumerated $P_1, P_2, \ldots, P_k$ in such a way that $P_1, \ldots, P_t$ are those prime ideals P_i, from among $P_1, \ldots, P_k$ with $Kd(R/P_i)$ as large as possible. Let $S = P_1 \cap \cdots \cap P_t$.

Claim 5.2. S is a localizable semiprime ideal of R. So $\{P_1,\ldots,P_t\}$ is a localizable set of prime ideals of R of deficient p.i. degree.

Proof: Since $Kd(R/P_1) = \ldots = Kd(R/P_t)$, by construction, L3 tells us that we need only to check that $\{P_1,\ldots,P_t\}$ is a union of link-components. Suppose that P_i were linked to some other prime ideal, P, of R, distinct from $P_1 \ldots, P_t$. Then P does not contain S'; for if P contained S', it would contain some prime ideal minimal over S', and, by Krull-dimension considerations, it would have to equal one of $P_1, \ldots, P_t$, contrary to assumption. By definition of S', this forces $P \in \operatorname{Spec}_m(R)$, a prime ideal of maximal p.i. degree. So P is localizable, and, by L1, is not linked to any other prime ideal, contrary to assumption.

Notice that throughout this discussion, we have included the assumption that R must be Noetherian. In fact, the above construction breaks down without this assumption, as the following example shows:

Example 5.4. Let $R_{n,m} = k[x_1, \ldots, x_n]_{m \times m}$ be the k-algebra of a generic $m \times m$ matrices, where k is, for example, any field, and $n > 1$. For $m_1 > m_2$, let $f_{m_1,m_2} : R_{n,m_1} \to R_{n,m_2}$ be the natural homomorphism which takes the generic $m_1 \times m_1$ matrix x_i, in R_{n,m_1} to the generic $m_2 \times m_2$ matrix x_i, in R_{n,m_2}, and let $P_{m_1,m_2} = \ker(f_{m_1,m_2})$. Since f_{m_1,m_2} is surjective, and R_{n,m_2} is a prime p.i. ring of p.i. degree m_2, P_{m_1,m_2} is a prime ideal of R_{n,m_1} of deficient p.i. degree. It is easy to see that $P_{m,m-1}$ is equal to the intersection of all prime ideals of $R_{n,m}$ of deficient p.i. degree. However, if $m > 2$, $P_{m,m-1}$ is not localizable, by the following result due to Bergman and Small [7, Corollary 6.9].

Theorem 5.4. If R is a local, prime p.i. ring, with maximal ideal P, then p.i.deg(R/P) divides p.i. deg(R).

So in our situation, if $P_{m,m-1}$ were localizable, then the localization, R, would be a local, prime p.i. ring of p.i. degree m, with maximal ideal $P = P_{m,m-1} R$, such that R/P is of p.i. degree m - 1. This would contradict the theorem if $m > 2$.

As another illustration, we offer the following result which relates localizability of prime ideals of deficient p.i. degree, and Problem 4.3', of the previous chapter.

Example 5.5. Let R be any Noetherian, prime p.i. ring, which is an affine algebra over a field, and which has a prime ideal P of deficient p.i. degree which does not belong to any localizable set. Let C be the center of R. Then there exists a prime ideal P, of maximal p.i. degree, such that $Q \neq \sqrt{(C \cap Q) R}$.

Proof: By L3, comp(P) must be infinite. By a theorem of Schelter, [31, Lemma 5], there is some prime ideal Q, of maximal p.i. degree, such that $Q \subseteq P$ and $Kd(R/Q) = Kd(R/P) + 1$. Then $Q \cap C \subseteq P \cap C = P' \cap C$ for all $P' \in comp(P)$. So each $P' \in comp(P)$ contains $(Q \cap C) R$, and in order to show that $Q \neq \sqrt{(C \cap Q) R}$, it suffices to show that not every $P' \in comp(P)$ can contain Q. (For in that case, any minimal prime over $(C \cap Q) R$ which is contained in P' will be a minimal prime distinct from Q.) Well if P' contains Q, the P'/Q is a prime ideal of deficient p.i. degree in R/Q. Moreover, it is a height one prime ideal, since $Kd(R/P') = Kd(R/Q) - 1$. However, the prime ideals of deficient p.i. degree in R/Q are exactly those which contaim $F_m(R/Q)$ and there can be only finitely many of these of height one, since R is Noetherian and $F_m(R)$ is non-zero. Thus, only finitely many of the $P' \in comp(P)$ can contain $(C \cap Q)$, so some of these do not.

Thus, we see that the equation $Q = \sqrt{(Q \cap C)R}$ for prime ideals of maximal p.i. degree fails, in fairly general circumstances, and its failure is related to the behaviour of the prime ideals of deficient p.i. degree in R.

To see that the same phenomenon can arise even when R is a finitely-generated module over its center, we present the following example:

<u>Example 5.6.</u>

$$R = \begin{pmatrix} k[z] + (t^2-1)\,k[z,t] & (t+1)\,k[z,t] \\ (t-1)\,k[z,t] & k[z,t] \end{pmatrix}$$ where k is any field.

R is clearly a prime subring of $(k[z,t])_{2x2}$, with center $C = k[z] + (t^2-1)\,k[z,t]$, embedded diagonally. Notice that C is not integrally closed, so that Corollary 4.10 does not apply, even though R is a finitely-generated C-module. C is the affine coordinate ring of a ruled surface; the plane curve, Γ, defined by $y^2 = x^3 + x^2$ has affine coordinate ring $k + (t^2-1)\,k[t]$, and C is the coordinate ring of the product of Γ with the affine line. The ring R is of the type considered by Mueller [18]. Now let $\pi = C \cap ((z-t)\,k[z,t])$. Clearly π is a prime ideal of C not containing the Formanek center of R (since t^2-1 does not belong to π). Therefore, by Lemma 4.11, there is only one prime ideal of R lying over π, and it is of maximal p.i. degree. If $\pi' = (z-t)\,k[z,t]$, then

$$p = \begin{pmatrix} \pi & (t+1)\pi' \\ (t-1)\pi' & \pi' \end{pmatrix}$$ is the prime ideal of R lying over π. Geometrically, the inclusion $C \subseteq k[z,t]$ corresponds to the mapping from the plane onto the surface Spec(C) given by $(z,t) \to (z, t^2-1, t(t^2-1))$, and the prime

ideal π corresponds to the curve on Spec(C) which is the image of the line $z = t$ under this map. It is clear, from this point of view, that the inverse image of π is comprised of the line $z = t$, together with the two points $(1, -1)$ and $(-1, 1)$. In other words, the minimal prime ideals over $\pi \cdot k[z,t]$ are π, $(z-1, t+1)$, and $(z+1, t-1)$. Note that the latter two prime ideals contain (t^2-1). It follows that the two prime ideals

$$Q_1 = \begin{pmatrix} k[z] + (t^2-1)\,k[z,t] & (t+1)\,k[z,t] \\ (t-1)\,k[z,t] & (t+1,\ z-1) \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} k[z] + (t^2-1)\,k[z,t] & (t+1)\,k[z,t] \\ (t-1)\,k[z,t] & (t-1,\ z+1) \end{pmatrix}$$

are minimal prime ideals over πR. Consequently $P \neq \sqrt{\pi R}$.

To see what goes wrong with the prime ideals of deficient p.i. degree in R, let $\delta = (z-1, t+1) \cap C$, and consider

$$Q_3 = \begin{pmatrix} \delta & (t+1)\,k[z,t] \\ (t-1)\,k[z,t] & k[z,t] \end{pmatrix}.$$ Q_3 is clearly a prime ideal of R of deficient p.i. degree, whose intersection with the center of R is the same as that of Q_1. Thus, we see that Q_1, Q_3 are in the same link-component of Spec(R), P is contained in Q_3, and yet there is no prime ideal in the same link-component as P which is contained in Q_1.

<u>Definition 5.7</u>. We say that a fully bounded Noetherian ring R satisfies <u>Going Down for Link-Components</u> if, whenever T_1, T_2 are prime ideals in the same link-component of Spec(R), and S_1 is any prime ideal of R contained in T_1, then there is some prime ideal of R, S_2, which is contained in T_2

and is in the same link-component as S_1.

Proposition 5.8. If R is any Noetherian ring which is a finitely-generated module over its center, C, then R satisfies Going Down for Link-Components if and only if the inclusion $C \subseteq R$ satisfies the Going Down theorem (ref. Schelter [31, Theorem 3]).

Proof: With the notation of the definition, $T_1 \cap C = T_2 \cap C$ is a prime ideal of Spec(C), containing $S_1 \cap C$, so if $C \subseteq R$ satisfies Going Down, there is some prime ideal, S_2, of R contained in T_2 and lying over $S_1 \cap C$. Conversely, suppose $\pi' \subseteq \pi$ are prime ideals of C, and P is a prime ideal of R lying over π. By the Lying Over theorem (Schelter [32, Theorem 1(3)] there is some prime ideal P' of R lying over π', and by the Going Up theorem (Schelter [32, Theorem 1(1)]) there is a $Q \in \text{Spec}(R)$ containing P' and lying over π. Now P and Q lie in the same link-component of Spec(R), so there is some $Q' \subseteq Q$ in the same link-component of P'. Thus, the Going Down Theorem holds for $C \subseteq R$.

As a consequence, we deduce the following generalization of Corollary 4.10:

Corollary 5.9. If R is any Noetherian prime p.i. ring which is a finitely-generated module over its center, C, and if R satisfies Going Down for Link-Components, then for any prime ideal, P, of maximal p.i. degree in R, we have $P = \sqrt{(C \cap P)R}$.

Proof: If $P \neq \sqrt{(C \cap P)R}$, then Proposition 4.13 tells us that there is some prime ideal Q, minimal over $(C \cap P)R$, and of deficient p.i. degree. Thus $Q \cap C \supsetneq P \cap C$, and the Going Down Theorem says that there must be some prime ideal $Q' \subsetneq Q$ which lies over $P \cap C$, contrary to the fact that Q is minimal over $(C \cap P)R$.

The next main result we are aiming for in this chapter will show how long links behave under the following kind of ring extension:

Definition 5.10. A ring extension $R \to R'$ is called an extension in the sense of Procesi if R' is generated, as a ring over R, by elements which centralize R. (ref. Procesi [24, Definition 6.3 (1), page 52]).

This is clearly equivalent to saying that R' is generated, as an R-module, by elements which centralize R. For short, we call such extensions Procesi extensions. Procesi extensions arise frequently in the setting of p.i. rings. They behave particularly well with respect to prime ideals, since if $R \to R'$ is a Procesi extension, and if P' is a prime ideal of R', then $P' \cap R$ is a prime ideal of R. (ref. Procesi [op. cit., Theorem 6.5]).

Theorem 5.11. Let R be any fully bounded Noetherian ring, and $R \to R'$ any Procesi extension such that R' is a finitely-generated R-module. Then we have the following relationships between the link structure in Spec(R) and that in Spec(R'):

(i) Suppose $P_1, P_2 \in \text{Spec}(R)$, such that $P_1 \twoheadrightarrow P_2$. Then there exist prime ideals $Q_1, Q_2 \in \text{Spec}(R')$ such that $Q_1 \cap R = P_1$, $Q_2 \cap R = P_2$, and $Q_1 \twoheadrightarrow Q_2$.

(ii) Suppose $Q_1, Q_2 \in \text{Spec}(R')$, such that $Q_1 \twoheadrightarrow Q_2$. Then $Q_1 \cap R \twoheadrightarrow Q_2 \cap R$.

The proof of this result rests on the following lemma.

Lemma 5.12. Let P be any prime ideal of R, and Q any prime ideal of R'.

Then Q is an associated prime ideal of $Hom_R(R', E_P)$ if and only if $Q \cap R = P$.

Proof of the lemma: To start with, suppose that Q is any associated prime ideal of $Hom_R(R', E_P)$. Then Q is the annihilator, in R', of some cyclic submodule, say $\psi \cdot R'$, of $Hom_R(R', E_P)$, where $\psi \neq 0$. We need to show that $Q \cap R = P$. The inclusion $Q \cap R \subseteq P$ is true since $q \in Q \cap R \Rightarrow \psi \cdot R' q = 0$, which means that $\psi(x q y) = 0$ for all $x, y \in R'$. In particular, $\psi(R'q) = 0$, which implies that $\psi(R')q = 0$ (since ψ is a right R-homomorphism and $q \in R$). So q belongs to the annihilator, in R, of $\psi(R')$. Since P is maximal among annihilators of non-zero submodules of E_P, we deduce that $q \in P$. Now $Q \cap R$ is a prime ideal of R, since R' is a Procesi extension of R, so in order to show that $Q \cap R = P$, it remains only to show that $Kd(R/(Q \cap R)) = Kd(R/P)$. to do this, we will show that $S^m \subseteq Q \cap R$, where $m > 0$ and S is a semiprime ideal of R whose minimal prime ideals all lie in the same Krull-dimension stratum as P. Since R' is a finitely-generated right R-module, the image of ψ is a finitely-generated submodule of E_P. The results of Jategaonkar [13] on modules over fully bounded Noetherian rings imply that it has a Krull-composition series $\psi(R') = X_m \supsetneq \cdots \supsetneq X_0 = 0$, where the prime ideals $P_i = ann_R(X_i/X_{i-1})$ all lie in the same Krull-dimension stratum as P. Let S be their intersection. Then the image of ψ is annihilated by S^m, for some $m > 0$. But since R' is a Procesi extension of R, we have $S^m R' = R' S^m$, so we have that $\psi(R' S^m R') = \psi(R' S^m) = \psi(R') S^m = 0$, which implies that $\psi \cdot R' \cdot S^m = 0$, so $S^m \subseteq ann_{R'}(\psi \cdot R') = Q$, and we obtain the inclusion $S^m \subseteq Q \cap R$ as required.

We have now shown that any associated prime ideal, Q, to $Hom_R((R', E_P)$ satisfies $Q \cap R = P$. For the converse, suppose Q is any prime ideal of R' such that $Q \cap R = P$. The diagram of inclusions

$$\begin{array}{ccc} R/P & \to & R'/Q \\ \downarrow & & \\ E(R/P) & & \end{array}$$

can be extended to a diagram

$$\begin{array}{ccc} R/P & \to & R'/Q \\ \downarrow & \swarrow \lambda & \\ E(R/P) & & \end{array}$$

where λ is a non-zero right R-homomorphism. Now $E(R/P) \cong {E_P}^k$, for some $k > 0$ and we can compose with one of the projection maps ${E_P}^k \to E_P$ so that the composition $R' \to R'/Q \xrightarrow{\lambda} E(R/P) \cong {E_P}^k \to E_P$ is a non-zero R-homomorphism from R' to E_P containing Q in its kernel. Call this map μ. We have that

$\mu(Q) = 0 \iff \mu(R'QR') = 0 \implies \mu \cdot R' \cdot Q = 0 \iff Q \subseteq \mathrm{Ann}_{R'}(\mu \cdot R')$.

Therefore $Q \subseteq Q^*$ for some prime ideal Q^* which is associated to $\mathrm{Hom}_R(R', E_P)$. But from the part of the lemma we have already proved, we know that $Q^* \cap R = P$. The equality $Q = Q^*$ would follows if we knew that Q and Q^* were in the same Krull-dimension stratum of Spec(R'). This follows from the following observation:

<u>Observation 5.13</u>. For any prime ideal, Q, of R', lying over P, we have $\mathrm{Kd}(R/P_R) = \mathrm{Kd}(R'/Q_{R'})$.

(Proof of the observation: The equality $\mathrm{Kd}(M_R) = \mathrm{Kd}(M_{R'})$ for any finitely-generated right R'-module M is a known result for such Procesi extensions (ref. Segal [33, Lemma 8]). Now $\mathrm{Kd}(R'/Q_R) \geq \mathrm{Kd}(R/P_R)$, since R/P is an R-submodule of R'/Q, and $\mathrm{Kd}(R'/Q_R) \leq \mathrm{Kd}(R/P_R)$, since R'/Q is a finitely-generated R/P module. Thus, we have $\mathrm{Kd}(R'/Q_{R'}) = \mathrm{Kd}(R'/Q_R) = \mathrm{Kd}(R/P_R)$, as required.) This completes the proof of the lemma.

Equipped with this lemma, we can now prove Theorem 5.11.

Proof of Theorem 5.11, part (i): Suppose P_1, P_2, are prime ideal of R such that $P_1 \rightarrow\rightarrow P_2$. Then $Hom_R(E_{P_1}, E_{P_2}) \neq 0$, so let f be any non-zero homomorphism from E_{P_1} to E_{P_2}. Let $E_1' = Hom_R(R', E_{P_1})$ and $E_2' = Hom_R(R', E_{P_2})$. By well-known arguments, these E_i''s are injective, as right R'-modules. Moreover, we have that $Hom_{R'}(E_1', E_2') \neq 0$. For if $x \in E_{P_1}$ is chosen so that $f(x) \neq 0$, the R-homomorphism from R to E_{P_1} sending 1 to x can be extended, by the injectivity of E_{P_1}, to an R-homomorphism, call it g, from R' to E_{P_1}. Then $f \circ g \neq 0$, so that the map $h \rightarrow f \circ h$ is a non-zero element of $Hom_{R'}(E_1', E_2')$. Now we are ready to invoke the lemma. We know that each E_i' is a direct sum of indecomposable injective modules E_Q, corresponding to the associated prime ideals, Q, of E_i', so we can write $E_i' = \sum_{\alpha \in S_i} E_{Q_\alpha}$ (direct), and the lemma tells us that $Q_\alpha \cap R = P_i$ for all $\alpha \in S_i$, i = 1, 2. But a non-zero homomorphism from $\sum_{\alpha \in S_1} E_{Q_\alpha}$ (direct) to $\sum_{\alpha \in S_2} E_{Q_\alpha}$ (direct) gives, by composing with an appropriate injection $E_{Q_\beta} \rightarrow \sum_{\alpha \in S_1} E_{Q_\alpha}$ and an appropriate projection $\sum_{\alpha \in S_2} E_{Q_\alpha} \twoheadrightarrow E_{Q_\delta}$, a non-zero homomorphism from E_{Q_β} to E_{Q_δ}, for some $\beta \in S_1$ and $\delta \in S_2$. Observation 5.13 tells us that Q_β and Q_δ lie in the same Krull-dimension stratum, so we conclude that $Q_\beta \rightarrow\rightarrow Q_\delta$, as desired.

Proof of theorem 5.11 part (ii): Suppose, now, that Q_1 and Q_2 are prime ideals of R' such that $Q_1 \rightarrow\rightarrow Q_2$, and let $P_1 = Q_1 \cap R$, $P_2 = Q_2 \cap R$. The lemma tells us that Q_i is an associated prime ideal to $Hom_R(R', E_{P_i})$,

$i = 1, 2$. By using the assumed non-zero homomorphism from E_{Q_1} to E_{Q_2}, we can easily construct a non-zero homomorphism from $\mathrm{Hom}_R(R', E_{P_1})$ to $\mathrm{Hom}_R(R, E_{P_2})$. (For example, define it to be zero on the complementary direct summand of E_{Q_1}, and equal to the given non-zero homomorphism from E_{Q_1} to E_{Q_2} on E_{Q_1}). Thus, we have $0 \neq \mathrm{Hom}_{R'}(\mathrm{Hom}_R(R', E_{P_1}), \mathrm{Hom}_R(R', E_{P_2}))$ $\cong \mathrm{Hom}_R(\mathrm{Hom}_R(R', E_{P_1}) \otimes R', E_{P_2}) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(R', E_{P_1}), E_{P_2})$. So let $\underline{\overline{\Phi}}$ be a non-zero homomorphism from $\mathrm{Hom}_R(R', E_{P_1})$ to E_{P_2}, and choose any $f \in \mathrm{Hom}_R(R', E_{P_1})$ such that $\underline{\overline{\Phi}}(f) \neq 0$. Since R' is a module-finite Procesi extension of R, there exist elements $x_1, \ldots, x_n \in R'$ which centralize R, and generate R' as an R-module. Let $e_i = f(x_i)$, $i = 1, \ldots, n$. We define a non-zero homomorphism from the submodule of $E_{P_1}{}^n$ generated by $(e_1, \ldots, e_n)$ to E_{P_2}, by sending $(e_1, \ldots, e_n)$ to $\underline{\overline{\Phi}}(f)$. Of course we need to check that this is well-defined. So suppose $x \in R$, such that $(e_1, \ldots, e_n)\, x = 0$. Then $f(x_i)\, x = 0$, for all i. If r' is any element of R', write $r' = \sum_{i=1}^{n} r_i y_i$, for elements $y_i \in R$, and deduce that $f(x \cdot r) = f(\sum_{i=1}^{n} x\, r_i y_i) = f(\sum_{i=1}^{n} r_i x y_i)$ $= \sum_{i=1}^{n} f(r_i)\, x y_i = 0$. Hence $f \cdot x = 0$, and so $\underline{\overline{\Phi}}(f) \cdot x = 0$, as required. Using the injectivity of E_{P_2}, this homomorphism can now be extended to give a non-zero homomorphism from $E_{P_1}{}^n$ to E_{P_2}, and so, by composition with one of the injections $E_{P_1} \to E_{P_1}{}^n$, we obtain a non-zero homomorphism from E_{P_1} to E_{P_2}. Once again, Observation 5.13 implies that P_1 and P_2 lie in the same Krull-dimension stratum of Spec(R), so we conclude $P_1 \rightsquigarrow P_2$ as required.

This theorem furnishes a powerful tool in examining link structure in Noetherian p.i. rings. The examples considered by Mueller, in [19] are more manageable by the use of this theorem. These are rings of the form $R = \begin{pmatrix} A & X \\ Y & B \end{pmatrix}$ where A, B are commutative Noetherian domains contained in a larger commutative Noetherian domain C, so that C is finitely-generated as an A-module and also as a B-module, and X and Y are ideals of C satisfying $XY \subseteq A \cap B$. In this case, by taking $R' = \begin{pmatrix} C & X \\ Y & C \end{pmatrix}$, the extension $R \to R'$ is a module-finite Procesi extension. Since R' is clearly a finitely-generated module over its center, the link-components in Spec(R) are given by L4, and the theorem can be used to compute the link-components in Spec(R).

Schelter [32], investigated the following situation: Let R be any Noetherian prime p.i. ring, and R' the ring operated over R by all central elements of the Posner quotient ring, $\mathcal{Q}(R)$, which are integral over R. The following observations were made:

I1. [op. cit., Proposition 6] If R is affine over a field, then R' is a finitely-generated R-module.

I2. [op. cit., Corollary 2 to Proposition 5] Again assuming R is affine over a field R' is a finitely-generated module over its center.

I3. [op. cit., Proposition 4] Without the affineness assumption, R' need not be finitely-generated module over its center; however, its center is always a Krull domain.

Definition 5.14. R' is called the central integral closure of R. If $R = R'$, R is called centrally integrally closed.

For example, if R is already a finitely-generated module over its center, then R', the central integral closure of R, is just the ring generated by R, together with the integral closure of the center of R, inside $\mathcal{Q}(R)$.

Thus, in many situations there is a module-finite Procesi extension, R', of R such that R' is finitely-generated as a module over its center, and Theorem 5.11 and L4 combine to give information concerning the link structure of R. For instance, we deduce:

<u>Proposition 5.15</u>. Suppose R is any full bounded Noetherian ring such that there is some module-finite Procesi extension, $R \to R'$, such that R' is a finitely-generated module over its center. If P is any prime ideal of R, then there are at most finitely many prime ideals $Q \in \mathrm{Spec}(R)$ such that $P \to\to Q$. In particular, this is true whenever R is any affine Noetherian prime p.i. algebra over a field.

In other words, for such rings R, the directed graph Spec(R) is locally finite. This result is actually known for any Noetherian p.i. ring (ref. Mueller [22]).

Proof: Let $P \in \mathrm{Spec}(R)$. For any long link $P \to\to Q$, Theorem 5.11 part (i) tells us that there exist prime ideals $P', Q' \in \mathrm{Spec}(R')$ lying over P, Q respectively, such that $P' \to\to Q'$. Now there are only finitely many prime ideals P', of R', lying over P, and by L4 there are only finitely many prime ideals Q' in R' linked to each such P'. Hence, by Theorem 5.11 part (ii) there can only be finitely many prime ideals $Q \in \mathrm{Spec}(R)$ such that $P \to\to Q$.

CHAPTER 6

The Size of Clans in Noetherian Prime P.I. Rings

In the last chapter, we saw that for a Noetherian p.i. ring R which is a finitely-generated module over its center, C, every prime ideal belongs to a localizable set of prime ideals, namely $\text{comp}(P) = \{Q \in \text{Spec}(R) : Q \cap C = P \cap C\}$, for example. In this chapter, we derive a bound on the size of comp(P), purely in terms of the p.i. degrees of the prime ideals of R involved, for certain types of prime p.i. rings R. Our starting point is the following result due to Bergman and Small [7, Proposition 6.2].

<u>Proposition 6.1.</u> Let C be a rank one valuation ring, with field of fractions K, and R a finite-dimensional torsion-free prime C-algebra, such that $R \neq S = R \otimes K$; and $m = \text{p.i.deg}(R)$. Let $P_1, \ldots, P_r$ be the prime ideals of R belonging to the maximal ideal, U, of C, and let $m_i = \text{p.i.deg}(R/P_i)$, $i = 1, \ldots, r$. Then there exist non-negative integers c_i such that $m = \sum_{i=1}^{r} c_i m_i$. If C is the center of R, then all c_i can be taken positive simultaneously.

As a first application of this, bringing this together with a result mentioned in the previous chapter, we have:

<u>Corollary 6.2.</u> Let R be any Noetherian prime p.i. ring of p.i. degree m, which is centrally integrally closed, with center C. Let π be any height one prime ideal over C, and let $P_1, \ldots, P_r$ be the prime ideals of R lying over π, and $m_i = \text{p.i.deg}(R/P_i)$. Then there exist integers $c_i > 0$ such that $m = \sum_{i=1}^{r} c_i m_i$. In particular, $r \leq m$.

Proof: The result I3 from the last chapter states that C is a Krull domain, so the localization C_π is a rank one discrete valuation domain. C_π is the center of $R_\pi = R \otimes C_\pi$, and the prime ideals $P'_i = P_i R_\pi$ are the prime ideals over the maximal ideal, πC_π, of C_π. Moreover $m_i = \text{p.i.deg}(R_\pi/P'_i)$, so Proposition 6.1 gives the required equation $m = \sum_{i=1}^{r} c_i m_i$.

This result applies, for example, to the classical situation where C is a Dedekind domain and R is an order in a central simple algebra over the quotient field of C. If R is actually a maximal order, it is known that r = 1 (ref. Auslander and Goldman, [5]). Thus in this case the corollary just gives the known result that for every prime ideal, P, of R, p.i.deg(R/P) divides p.i.deg(R). However, even for the simplest non-maximal orders r can be > 1. For example, the Z-order $\begin{pmatrix} Z & Z \\ 2Z & Z \end{pmatrix}$ possesses the non-trivial clan $P_1 = \begin{pmatrix} 2Z & Z \\ 2Z & Z \end{pmatrix}$, $P_2 = \begin{pmatrix} Z & Z \\ 2Z & 2Z \end{pmatrix}$. We remark also, that, without the assumption that R be centrally integrally closed, the result fails. For example, let $p_n(t) = (t-1) \ldots (t-n) \in k[t]$, let $C = k + p_n(t)k[t]$, and let

$$R = \begin{pmatrix} C & k[t] \\ p_n(t)k[t] & k[t] \end{pmatrix}.$$

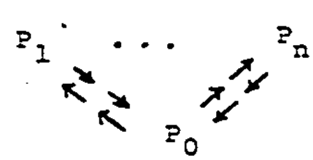
The prime ideals

$$P_0 = \begin{pmatrix} p_n(t)k[t] & k[t] \\ p_n(t)k[t] & k[t] \end{pmatrix},$$

and

$$P_i = \begin{pmatrix} C & k[t] \\ p_n(t)k[t] & (t-i)k[t] \end{pmatrix}$$

i = 1,...,n constitute a clan of (n+1) prime ideals, even though R has p.i. degree 2. The link structure in this example is



as is easily checked, using Theorem 5.11 applied to the extension

$$R \subseteq \begin{pmatrix} k[t] & k[t] \\ p_n(t)k[t] & k[t] \end{pmatrix}.$$

One direction in which Proposition 6.1 can be generalized is the following:

<u>Proposition 6.3</u>. Let R be a prime p.i. ring of p.i. degree m, whose center C is a regular local ring. Let $P_1,\ldots,P_r$ be the prime ideals of R lying over the maximal ideal, U, of C, and let m_i = p.i.deg(R/P_i). Then there exist integers $c_i > 0$ such that

$$m = \sum_{i=1}^{r} c_i m_i .$$

To prove this we need an easy lemma:

<u>Lemma 6.4</u>. Let C be a regular local ring of dimension $>$. 1, with maximal ideal U. Then there are infinitely many pairwise nonassociated elements of $U-U^2$.

Proof of the lemma: Since C has dimension > 1, there exist two elements $x, y \in U$ which are linearly independent module U^2. We claim that for all $n \geq 1$, $x + y^n \in U - U^2$, and no two of these elements are associated. Well, suppose to the contrary that there exist $n, k \geq 1$ such that $(x + y^{n+k}) = c(x + y^n)$, for some invertible element $c \in C$. Then $y^n(y^k - c) = (c - 1)x$. Since y is irreducible, does not divide x, and since C is a unique factorization domain, we deduce that y^n divides $(c - 1)$. Say $(c - 1) = y^n z$, for $z \in C$. Then $x + y^{n+k} = (1 + y^n z)(x + y^n)$, which yields $y^k - zx - y^n z = 1$. This is clearly impossible, since the left side belongs to U.

Proof of the proposition: We go by induction over the dimension of C. If C is one-dimensional, then it is a rank one discrete valuation domain, and the assertion follows from Proposition 6.1 directly, so assume dim(C) > 1. Now $F_m(R)$, as a non-zero subring of C can be contained in only finitely many height one prime ideals of C. But the lemma tells us that there are infinitely many height one prime ideals of C of the form sC, $s \in U - U^2$. So there is some such height one prime ideal, sC, with $sC \not\supseteq F_m(R)$. Now C/sC is a regular local ring of dimension one less than that of C, and since $sC \not\supseteq F_m(R)$, Lemma 4.11 tells us that there is some prime ideal $P \in \mathrm{Spec}_m(R)$ lying over sC. Since R is a finitely-generated C-module, R/P is a finitely-generated C/sC-module, and hence so is the center of R/P. Now Corollary 2.9 (ii) says that the center of R/P lies inside the quotient field of C/sC. Since C/sC is integrally closed, this forces C/sC = center(R/P). Moreover $sC \subseteq U \subseteq \cap P_i \Rightarrow sR \subseteq \cap P_i \Rightarrow P = \sqrt{sR} \subseteq \cap P_i$, by Corollary 4.10. Hence the induction assumption applies, and gives the theorem for R.

Thus, if R is any Noetherian prime p.i. ring which is a finitely-generated module over its center, C, the clans of R are particularly well-behaved at all points $\pi \in \mathrm{Spec}(C)$ such that C_π is a regular local ring. (i.e. the smooth points of Spec(C).)

<u>Problem 6.5</u>. Does the same result, $m = \sum_{i=1}^{r} c_i m_i$, hold over the non-smooth points of Spec(C), assuming, for instance, that C is integrally closed?

<u>Example 6.6</u>. Suppose R is a Noetherian prime p.i. ring which is a finitely-generated module over its center, C, which is integrally closed and local. If there is some prime ideal, P, of C such that C/P is regular local, and

$P \supsetneq F_m(R)$, then Problem 6.5 has an affirmative answer for the clan over the maximal ideal of C.

CHAPTER 7

Growth in Affine Prime P.I. Algebras

Let $R = k\{a_1,\ldots,a_t\}$ be any affine algebra over a field k, and let C be any finite-dimensional k-subspace of R which contains 1, and generates R as an algebra. Let C^m denote the k-subspace of R generated by all m-fold products of elements of C, for any positive integer m. Since $1 \in C$, we have $C \subseteq C^2 \subseteq C^3 \subseteq \ldots$, and since C generates R, we have $\bigcup_{m=1}^{\infty} C^m = R$. Define $g_C(m) = \dim_k(C^m)$. for all m. g_C is a function defined on the set N^+ of positive integers, and, in some sense, it keeps track of how quickly the algebra R grows. Notice, however, that it depends on the choice of C. If D is any other finite-dimensional k-subspace of R which contains 1, and generates R as algebra, observe that, for some $m' > 0$, $D^{m'}$ contains C, and hence $D^{m'm}$ contains C^m for all m. In other words, $g_D(m'm) \geq g_C(m)$. This observation motivates the following definitions, due to Borho and Kraft [8].

Definitions 7.1. The set of all monotone non-decreasing functions $f : N^+ \to [0,\infty)$ is endowed with a quasi-ordering by the relation $f_1 \leq f_2$ if, for some $m' \in N^+$, $f_1(m) \leq f_2(m'm)$, for all m. The equivalence class of a function f, determined by this quasi-ordering, namely $\{g : f \leq g$ and $g \leq f\}$, is denoted $\mathcal{W}(f)$, and called the Wachstum (i.e. "growth") of f. The Wachstum of an algebra R, as above, is defined to be $\mathcal{W}(g_C)$, and is denoted $\mathcal{W}(R)$. It is clear, from the above remarks, that $\mathcal{W}(R)$ does not depend on the choice of the generating subspace C.

In this chapter, we shall investigate the Wachstum of affine prime p.i. algebras. First, a few introductory remarks concerning the above notions.

Remarks 7.1. The quasi-ordering, $\leq$, induces a partial ordering, $\leq$, on the above-mentioned equivalence classes, in the usual way; namely $\mathcal{W}(f) \leq \mathcal{W}(g)$ if and only if $f \leq g$.

7.2. If $f(m) = m^n$, where n is any integer ≥ 1, and if $g : N^+ \to [0,\infty)$ is any polynomial function of degree n (with real coefficients) then $\mathcal{W}(f) = \mathcal{W}(g)$. In this case, $\mathcal{W}(f)$ is denoted $\mathcal{P}_n$.

(Proof: Write $g(m) = a_n m^n + \ldots + a_0$. Clearly $a_n > 0$. Choose a positive integer m' to be greater than $(n+1)$ times the maximum of $|a_0|, \ldots, |a_n|$. Then it is easy to see that $g(m) \leq m'm^n \leq (m'm)^n$, which shows that $\mathcal{W}(g) \leq \mathcal{W}(f)$. For the other inequality, $\mathcal{W}(f) \leq \mathcal{W}(g)$, just choose $m' \geq 1$ large enough so that $a_n(m')^n - 1 \geq$ (n times the maximum of $|a_0|$, $|a_1|, \ldots, |a_n|$). It is easy to check that $g(m'm) - m^n \geq 0$.)

This Wachstum, $\mathcal{P}_n$, will be particularly important in our considerations. An affine algebra, R, over a field, such that $\mathcal{W}(R) \leq \mathcal{P}_n$, for some n, is said to be of polynomially bounded growth. If $\mathcal{W}(R) = \mathcal{P}_n$, for some n, R is said to be of polynomial growth, of degree n. The fact that $\mathcal{P}_{n_1} \neq \mathcal{P}_{n_2}$ if $n_1 \neq n_2$ will follow from the next comment, so that there can be at most one value of n such that $\mathcal{W}(R) = \mathcal{P}_n$.

7.3. If $f, g : N^+ \to [0,\infty)$ are non-decreasing functions, and $f \leq g$, then $\limsup_{m\to\infty}(\log(f(m))/\log(m)) \leq \limsup_{m\to\infty}(\log(g(m))/\log(m))$. Moreover, if $f \in \mathcal{P}_n$, then $\lim_{m\to\infty}(\log(f(m))/\log(m))$ exists, and equals n.

(Proof: There is some positive integer m' such that $f(m) \le g(m'm)$, for all m. Taking logarithms, and dividing through by $\log(m)$ gives:

$$\frac{\log(f(m))}{\log(m)} \le \frac{\log(g(m'm))}{\log(m)} = \left(1 + \frac{\log(m')}{\log(m)}\right) \frac{\log(g(m'm))}{\log(m'm)}$$

Letting $m \to \infty$, we conclude that

$\limsup_{m\to\infty} (\log(f(m))/\log(m)) \le \limsup_{m\to\infty}(\log(g(m))/\log(m))$. Now suppose that $f \in \mathcal{P}_n$. We must prove that $\liminf_{m\to\infty}(\log(f(m))/\log(m)) \ge n$. Let M be a positive integer such that $m^n \le f(Mm)$ for all $m \in N^+$. Let $m \in N^+$ be arbitrary, and let p be the greatest integer less than or equal to (m/M). Then $Mp \le m$, so since f is nondecreasing, this implies $p^n \le f(Mp) \le f(m)$. Taking logarithms, and dividing through by $\log(m)$ gives $n \cdot (\log(p)/\log(m)) \le (\log(f(m))/\log(m))$. Observing that $(\log(p)/\log(m))$ goes to 1 as $m \to \infty$, we get the desired result.)

<u>Definition 7.4</u>. If R is any affine algebra over a field K, the <u>Gelfand-Kirillov</u> dimension of R is defined to be $\limsup_{m\to\infty}(\log(f(m))/\log(m))$, where f is any non-decreasing function in $\mathcal{H}(R)$. (For example, we could take $f = g_C$, as before.) We denote this number by G.K.dim(R). The above remark, 7.3., shows that it does not depend on the choice of f. Moreover, it shows that if R has polynomial growth, of degree n, then R has Gelfand-Kirillov dimension n. The converse of this assertion does not hold, in general.

<u>Example 7.5</u>. If $R = K[x_1,\ldots,x_p]$ is the commutative polynomial ring in p unknowns, over K, then $\mathcal{H}(R) = \mathcal{P}_p$. For, if we take C to be the K-span of $\{1, x_1, \ldots, x_p\}$, it is easily computed that $\dim_K(C^m) = \binom{m+p}{p}$, which is a polynomial function of degree p in m. Note that, in this case,

G.K.dim(R) coincides with the Krull dimension of R. It was proved in 1976, by Malliavin-Brameret [16] that if R is an affine prime p.i. algebra over a field, and has Krull dimension p, then G.K.dim(R) = p. We shall prove the sharper result that such R is of polynomial growth, of degree p. This result is a partial answer to the following problem posed by Procesi [24, page 185, problem c]:

"If $R = k\{a_1,\ldots,a_n\}$ is a finitely-generated, graded p.i. algebra, let $f(m) = \dim_k(R_m)$ what can one say about the function f(m)? Is it a polynomial function in some sense?"

Our result will answer the latter question in the affirmative, in the case where R is prime, if "in some sense" is interpreted in the sense of being of polynomial growth. We remark that in the case that R is not prime, the above question has a negative answer, as is shown by the examples of Borho and Kraft [8, Satz 2.10]. The following remark connects Procesi's question to the question of polynomial growth:

<u>Remark 7.6</u>. Suppose $R = \overset{\infty}{\underset{m=0}{\oplus}} R_m$ is a graded, affine algebra over a field k, that R is generated as an algebra by the finite-dimensional subspace R_1, of homogeneous elements of degree 1, and that $R_0 = k$. (We presume that these assumptions were implicit in Procesi's question). If $f(m) = \dim_k(R_m)$, for $m \geq 1$, then $\mathcal{H}(f) = \mathcal{P}_n$ if and only if R is of polynomial growth, of degree (n+1).

Proof: If we take C to be the finite-dimensional k-subspace $R_0 \oplus R_1$, then contains 1, and generates R as an algebra over k. Moreover, $g_C(m) = \dim_k(C^m) = \dim_m(R_0 \oplus \ldots \oplus R_m) = 1 + f(1) + \ldots + f(m)$. So we need to show that $\mathcal{N}(g_C) = \mathcal{P}_{n+1}$ if and only if $\mathcal{N}(f) = \mathcal{P}_n$. Suppose first of all, that $\mathcal{N}(f) = \mathcal{P}_n$. Then there exist integers $M, M' \geq 1$, such that $m^n \leq f(Mm)$ and $f(m) \leq (M'm)^n$, for all $m \geq 1$. We have the inequalities:

$g_C(Mm) \geq 1 + f(M) + \ldots + f(Mm) \geq \sum_{j=1}^{m} j^n$ (which is a polynomial of degree n+1 in m) and $g_C(m) \leq M1^n + \ldots + (M'm)^n \leq (m+1)(M'm)^n$. These inequalities combine to show that $g_C \in \mathcal{P}_{n+1}$. The other implication, $g_C \in \mathcal{P}_{n+1}$ implies $f \in \mathcal{P}_n$, is proved similarly.

Thus, in order to answer Procesi's question for affine, prime p.i. algebras, it will suffice to show that such algebras are of polynomial growth.

<u>Proposition 7.7</u>. Let $R = k\{a_1,\ldots,a_p\}$ be any affine algebra over a field k, and let $R \subseteq R'$ be any Procesi extension of R such that R' is a finitely-generated R-module. If R is of polynomial growth, of degree $n \geq 1$, then so is R'.

Proof: Since R is a subalgebra of R', it is clear that $\mathcal{N}(R) \leq \mathcal{N}(R')$, so it only remains to check that $\mathcal{N}(R') \leq \mathcal{P}_n$. Let $x_1,\ldots,x_t$ be elements of R' which centralize R, and which generate R' as an R-module; choose $x_1 = 1$, without loss of generality. Now write $x_i x_j = \sum_{k=1}^{t} r_{ijk} x_k$, for $1 \leq i, j \leq t$, for appropriate elements $r_{ijk} \in R$. Let C be any finite-dimensional subspace of R which contains 1, as well as all of the r_{ijk}'s, and which generates R as a k-algebra. Let D be the k-subspace of R' spanned by $x_1 C \cup \ldots \cup x_t C$. D is a finite-dimensional subspace of R' which contains 1 and generates R' as an algebra, and by our choice of C, it is clear that D^m is just the k-subspace of R' spanned by $x_1 C^m \cup \ldots \cup x_t C^m$, for all $m > 0$. Therefore, $g_D(m) = \dim_k(D^m) \le t \cdot \dim_k(C^m) = t \cdot g_C(m)$. Now $g_C \in \mathcal{P}_n$ implies $t \cdot g_C \in \mathcal{P}_n$, and this shows that $\mathcal{H}(R') = \mathcal{H}(g_D) \le \mathcal{P}_n$.

Notice that, in general, if f is any nondecreasing function from N^+ to $[0,\infty)$, and t any positive integer, it is not necessarily the case that $t \cdot f \in \mathcal{H}(f)$. However, we do have that $f \in \mathcal{P}_n$ implies $t \cdot f \in \mathcal{P}_n$, for any $n \ge 1$, and this is the key observation to the above proof. We have immediately, from the above proposition:

Corollary 7.8. If R is any affine commutative integral domain, over a field k, then R has polynomial growth, of degree equal to its Krull dimension.

Proof: By the Noether Normalization lemma, there is a subalgebra $S \subseteq R$, which is a commutative polynomial algebra over k, and such that R is a finitely-generated S-module. Thus, the assertion of the corollary follows immediately from the above lemma, together with our previous example (7.5) of commutative polynomial algebras.

We are now equipped to prove the main theorem of this chapter:

Theorem 7.9. Let R be any affine, prime p.i. algebra over a field k. Let n be the transcendency degree of the quotient field of the center of R, over k. Then $\mathcal{H}(R) = \mathcal{P}_n$.

Proof: To see that $\mathcal{H}(R) \ge \mathcal{P}_n$, just choose a transcendency basis, $c_1,\ldots,c_n$, for the quotient field of the center of R, such that each c_i belongs to the center of R. Then since R contains $k[c_1,\ldots,c_n]$, we have $\mathcal{H}(R) \ge \mathcal{H}(k[c_1,\ldots,c_n]) = \mathcal{P}_n$.

For the other inequality, $\mathcal{N}(R) \leq \wp_n$, we make use of Remark 4.17. It tells us that there are finitely many elements $c_1, \ldots, c_k$ from the center of the Posner quotient ring of R such that the extension ring $R' = R[c_1, \ldots, c_k]$ is a finitely-generated module over its center, C. Now C is a commutative domain whose quotient field is the same as that of the center of R. Moreover, by Corollary 3.16, C is affine over k. Thus, by Corollary 7.8, $\mathcal{N}(C) = \wp_n$. Applying Proposition 7.7 to the extension $C \subseteq R'$, we deduce that $\mathcal{N}(R') = \wp_n$. Therefore, $\mathcal{N}(R) \leq \mathcal{N}(R') = \wp_n$, as required.

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