# CONTINUOUS LATTICES

# AND

# CONVEXITY THEORY



By

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## ABSTRACT

This thesis deals with continuous lattices and convexity theory in the following original topics.

1) augmented compact spaces.
2) compact star structures.
3) star lattices.
4) convex continuous lattices.

These topics produce plentiful situations where continuous lattices arise from studies of various convexity theories. Conversely, it is shown how the lattices recapture the original theory when they are equipped with the appropriate structure.

The theory of augmented compact spaces provides a framework for the succeeding developments. It encompasses not only constructions from convexity theory but also topology and topological algebra.

The theories are enriched by numerous examples.

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# INTRODUCTION

In the theory of continuous lattices, convexity theory arises from several different topics:

1) Gierz and Keimel [12] show the equivalence of compact posets and distributive continuous lattices in which the set of primes is Lawson-closed. This result is extended in Chapter 1 and in doing so, the situation takes on a convexity theory aspect. The theory of augmented compact spaces is developed and in this theory certain closed subsets of a compact Hausdorff space are distinguished. These subsets then are analogous to closed convex subsets. The equivalence obtained in section 2 of the chapter forms the framework for many of the succeeding developments.

2) Hofmann and Lawson [16] (see also Banaschewski [1]) construct a duality between distributive continuous lattices and locally-compact sober spaces. In this work, we obtain a similar duality in chapter 3. It will turn out that the additional structure created in this duality is a type of convexity structure. The theory of this structure is developed in chapter 2.

3) Lawson [19] showed that the lattice of closed convex subsets of a compact convex subset of a locally-convex topological vector space is a continuous lattice. In

this work, this lattice is shown to have a convexity structure. This creates the theory of convex continuous lattices. The lattice of closed convex subsets is then shown in chapter 4 to create an equivalence. Applications of the Krein-Mil'man theorem to the theory of convex continuous lattices are discussed in section 7 of chapter 4.

Abstract convexity theory has developed along different lines. There is the notion of abstract convex subsets ([9], [10]) and this is similar to our approach in the theory of augmented compact spaces. There is the notion of abstract line segments ([5], [6], [7], [8]) but this idea is not pursued in this work. Also, there is the notion of abstract convex combinations ([13], [24]) and this idea is used in first a restricted sense in chapters 2 and 3 and then unrestricted in chapter 4.

Some of the approaches in this work are similar to those of Bennett [2], [3] and Thornton [25], however the lattices considered there are not complete and the answers they obtain are quite different from ours.

This work provides insight into the following:

1) the creation of continuous lattices through the theory of augmented compact spaces.
2) the structure of lattices of lower-semicontinuous functions and their relationship to the lattice of open sets through the theory of star lattices.

3) the structure of lattices of closed convex sets through the theory of convex continuous lattices.

# CHAPTER 0. BACKGROUND

## Section 1. Continuous Lattices

Scott introduced continuous lattices in 1972 [23]. For a while; they were interesting mainly for their lattice-theoretic properties and relationship to topology. Then it was realized that continuous lattices are precisely the underlying lattices of compact semilattices with small sub-semilattices. Thus, the topological algebra aspect of continuous lattices was born. The theory of continuous lattices has since grown into a rich and varied subject (see [11]). Lately, Lawson has shown how continuous lattices have relationships to functional analysis by means of the lattice of closed convex sets of a compact convex subset of a topological vector space. This construction is studied in great detail in this work.

The merit of continuous lattices can thus be argued for with examples of continuous lattices arising from subjects such as lattice theory, topology, topological algebra, and functional analysis. Furthermore, these examples brighten the future of continuous lattices by suggesting that there are yet to be discovered examples of continuous lattices arising from these and other subjects. Continuous lattices are in the enviable position of being restrictive enough to be interesting but large enough to include numerous examples.

It is not possible to include many of the considerations of continuous lattices in this work, but only what is needed here. Therefore, the reader is encouraged to read the Compendium [11] for more details.

The definition of continuous lattice given here is not the original one due to Scott but is the one most useful to us in this work.

Recall that a directed set is a poset (or even preordered set) in which each finite subset has an upper bound.

1.1 <u>Definition</u>. Given elements $x$ and $y$ of a complete lattice $L$, $x$ is <u>way-below</u> $y$, denoted by $x \ll y$, iff whenever $D$ is a directed subset of $L$ and $y \leq \bigvee D$, then $x \leq d$ for some $d \in D$.

It follows that

1) $0 \ll x$ for all $x \in L$ (since directed sets are non-empty)
2) $x \ll y \leq z$ or $x \leq y \ll z$ implies $x \ll z$
3) $x \ll z$ and $y \ll z$ implies $x \vee y \ll z$

For example, in $[0,1]$, $x \ll y$ if $x = 0$ or $x < y$.

1.2 <u>Definition</u>. A <u>continuous</u> <u>lattice</u> is a complete lattice $L$ such that for all $y \in L$, $y = \bigvee \twoheaddownarrow y$ where $\twoheaddownarrow y = \{x \in L \mid x \ll y\}$.

For example, $[0,1]$ is a continuous lattice. Also, any product of continuous lattices is a continuous lattice. Therefore, $[0,1]^2$ and $[0,1]^3$ are continuous lattices.

The following functions are one type of morphism associated with continuous lattices but there are others.

1.3 Definition. An algebraically-continuous function is a function $f: L \to M$ between complete lattices that preserves arbitrary meets and directed joins (that is, $f(\bigwedge A) = \bigwedge f(A)$ for all $A \subseteq L$ and $f(\bigvee D) = \bigvee f(D)$ for all directed $D \subseteq L$).

It can be shown that $[0,1]$ is the "mother" of all continuous lattices in the following sense: a complete lattice $L$ is continuous iff there exists a one-one algebraically-continuous function $f: L \to [0,1]^J$ for some set $J$.

Recall that an element $p$ of a complete lattice $L$ is a prime (resp. meet-irreducible) element of $L$ iff whenever $F$ is a finite subset of $L$ and $\bigwedge F \leq p$ (resp. $\bigwedge F = p$), then $x \leq p$ for some $x \in F$ (resp. $p \in F$).

In this work therefore, prime and meet-irreducible elements are not allowed to be the greatest element in the lattice.

Continuous lattices have the following important property.

1.5 Proposition [15]. Every element $y$ in a continuous lattice $L$ is the meet of meet-irreducible elements.

pf. Let $x \not\leq y$ in $L$. Inductively define a sequence $z_n$ of elements in $L$ such that

1) $z_0 \ll x$

2) $z_{n+1} \ll z_n$ for all $n$

3) $z_n \not\leq y$ for all $n$.

Let $F = \{z \mid z_n \leq z$ for some $n\}$. Then $F$ is a Scott-open filter in $L$ (that is, a filter in $L$ such that whenever $D$ is a directed set in $L$ and $\bigvee D \in F$, then $D \cap F \neq \emptyset$). Moreover, $x \in F$ and $y \notin F$.

By Zorn's lemma, let $p$ be an element of $L$ maximal such that $y \leq p$ and $p \notin F$. It follows that $p$ is a meet-irreducible element of $L$, $y \leq p$, and $x \not\leq p$. This proves the proposition.

1.6 Definition. A complete lattice has enough primes iff every element in $L$ is the meet of prime elements.

1.7 Proposition. Every distributive continuous lattice has enough primes.

The following topology on a continuous lattice turns it into a compact Hausdorff space (moreover, a compact semilattice with small subsemilattices [11]).

1.8 Definition. The Lawson-topology on a continuous lattice $L$ is defined to be the topology generated by $\{\Uparrow x \mid x \in L\} \cup \{L - \uparrow x \mid x \in L\}$ where $\Uparrow x = \{y \mid x \ll y\}$ and $\uparrow x = \{y \mid x \leq y\}$.

It can be shown that the Lawson-topology on [0,1] and any power of [0,1] is the usual topology.

An algebraically-continuous function between continuous lattices will be continuous when the lattices are given the Lawson-topology.

The following property of primes in a continuous lattice is extremely important to this work.

1.9 Theorem (Gierz and Keimel lemma [12]). If $p$ is a prime element in a continuous lattice $L$ and $C$ is a Lawson-closed subset of $L$ such that $\bigwedge C \leq p$, then $x \leq p$ for some $x \in C$.

pf. Suppose $x \not\leq p$ for all $x \in C$. For all $x \in C$, let $y_x \ll x$ such that $y_x \not\leq p$. Then $\{\Uparrow y_x \mid x \in C\}$ is a Lawson-open cover of $C$ and so must have a finite subcover $\{\Uparrow y_x \mid x \in F\}$. However, this implies $\bigwedge \{y_x \mid x \in F\} \leq \bigwedge C \leq p$. Since $p$ is prime, $y_x \leq p$ for some $x \in F$; a contradiction.

This property also extends to meet-irreducibles.

1.10 Proposition. If $p$ is a meet-irreducible element of a continuous lattice $L$ and $C$ is a Lawson-closed subset of $L$ such that $\bigwedge C = p$, then $p \in C$.

pf. Let $M = \uparrow p$ with the induced order. Then $M$ is a continuous lattice, $p$ is prime in $M$, and $C$ is a Lawson-closed subset of $M$. The proposition follows from 1.9.

One reason continuous lattices are important is because they include all algebraic lattices.

1.11 Definition. An element $x$ in a complete lattice $L$ is compact iff $x \ll x$.

1.12 Definition. An algebraic lattice is a complete lattice $L$ in which every element is the join of compact elements.

Of course, every algebraic lattice is continuous but, for example, $[0,1]$ is not algebraic.

Section 2. Posets.

2.1 Definition. A subset $A$ of a poset $P$ is an up-set (resp. down-set) iff whenever $x \in A$ and $x \leq y$ (resp. $y \leq x$), then $y \in A$.

2.2 Definition. For a subset $A$ of a poset $P$, define $\uparrow A$ (resp. $\downarrow A$) to be $\{x \mid a \leq x$ (resp. $x \leq a$) for some $a \in A\}$.

$\uparrow A$ (resp. $\downarrow A$) is the smallest up-set (resp. down-set) in $P$ containing $A$.

2.3 Definition. A subset $C$ of a poset $P$ is order-convex iff whenever $a, b \in C$ and $a \leq x \leq b$, then $x \in C$.

2.4 Definition. For a subset $A$ of a poset $P$, define the order-convex hull of $A$, denoted by $\mathrm{ocon}_P(A)$, to be $\uparrow A \cap \downarrow A$.

$\mathrm{ocon}_P(A)$ is the smallest order-convex subset of $P$ containing $A$. If $x \leq y$, then $\mathrm{ocon}_P\{x,y\} = [x,y] = \{z \mid x \leq z \leq y\}$.

2.5 Definition. A compact poset is a poset $P$ that is also a compact topological space such that $\{(x,y) \mid x \leq y\}$

is closed in $P \times P$.

Every compact poset is Hausdorff. If $C$ is a closed set in a compact poset $P$, then $\uparrow C$ and $\downarrow C$ are closed in $P$.

2.6 Theorem (see Nachbin [21]). If $C$ is a closed up-set and $D$ is a closed down-set in a compact poset $P$ such that $C \cap D = \emptyset$, then there exists an open up-set $U$ and an open down-set $V$ in $P$ such that $C \subseteq U$, $D \subseteq V$, and $U \cap V = \emptyset$.

2.7 Proposition. If $C$ is a closed up-set in a compact poset $P$, then $C = \cap\{D \mid D$ is a closed up-set in $P$ and $C \subseteq \operatorname{int} D\}$.

pf. If $x \notin C$, then $C \cap \downarrow x = \emptyset$. Let $U$ be an open up-set and $V$ an open down-set such that $C \subseteq U$, $\downarrow x \subseteq V$, and $U \cap V = \emptyset$. Let $D = P - V$. Then $D$ is a closed up-set in $P$, $C \subseteq \operatorname{int} D$, and $x \notin D$.

## Section 3. Prime Spectrum

In chapter 3 we construct a duality for the category of sober spaces. Therefore, here we will recall the facts concerning another such duality so that the two can be compared.

3.1 Definition. Define the lattice of open sets of a topological space $X$, denoted by $O(X)$, to be the set of all open subsets of $X$ ordered by inclusion.

$O(X)$ is a complete lattice with enough primes. We can also take the closed sets in $X$ ordered by reverse-inclusion, denoted by $\Gamma(X)$, and, of course, this is order-isomorphic to $O(X)$.

3.2 Definition. A framed function is a function $f: L \to M$ between complete lattices that preserves arbitrary joins and finite meets.

If $f: X \to Y$ is a continuous function, then $O(f): O(Y) \to O(X)$ defined by $O(f)(U) = f^{-1}(U)$ is a framed function. Hence, $O$ is a contravariant functor from the category of topological spaces to the category of complete lattices and framed functions. Conversely, we can construct the following.

3.3 Definition. Define the spectrum of a complete lattice $L$, denoted by $\mathrm{Spec}(L)$, to be the set of prime elements in $L$ with $\{\sigma(x) \mid x \in L\}$ as the topology where $\sigma(x) = \{p \in \mathrm{Spec}(L) \mid x \nleq p\}$.

$\mathrm{Spec}(L)$ is a sober space. If $f: L \to M$ is a framed function, then $\mathrm{Spec}(f): \mathrm{Spec}(M) \to \mathrm{Spec}(L)$ defined by $\mathrm{Spec}(f)(p) = g(p)$ where $g$ is the upper adjoint of $f$ (that is, $g: M \to L$ is defined by $g(y) = \bigvee\{x \mid f(x) \leq y\}$) is a continuous function.

Hence, Spec is a contravariant functor from the category of complete lattices and framed functions to the category of topological spaces. In fact, Spec and O are adjoint on the right. By restriction, we have the following.

3.4 Theorem. The category of sober spaces is dual to the category of complete lattices with enough primes and framed functions. The duality is given by Spec and O.

3.5 Theorem (Hofmann and Lawson [16]). The category of distributive continuous lattices is dual to the category of locally-compact sober spaces.

## Section 4. Lower-Semicontinuous Functions

The lattice of lower-semicontinuous functions appears in chapter 3 as a star lattice. Therefore, we now consider its properties as a complete lattice.

4.1 Definition. Define the lattice of lower-semicontinuous functions of a topological space $X$, denoted by $LC(X)$, to be the set of all lower-semicontinuous functions $f: X \to [0,1]$ ordered by; $f \le g$ iff $f(x) \le g(x)$ for all $x \in X$.

$LC(X)$ is a complete lattice. In fact, $LC(X)$ has enough primes since for each $x \in X$ and $s \ne 1$ in $[0,1]$, $p: X \to [0,1]$ defined by $p(y) = s$ if $y \in cl\{x\}$ and $1$ otherwise is a prime in $LC(X)$. Moreover, for $X$ sober, we have the following.

4.2 Proposition. If $X$ is a sober space and $p$ is a prime element of $LC(X)$, there exists $x \in X$ and $s \ne 1$ in $[0,1]$ such that $p(y) = s$ if $y \in cl\{x\}$ and $1$ otherwise.

pf. Suppose $y,z \in X$ such that $p(z) \not\le p(y)$. There exists $t \in [0,1]$ such that $t \ll p(z)$ and $t \not\le p(y)$.

Hence, there exists an open neighborhood $U$ of $z$ in $X$ such that $p(U) \subseteq \uparrow t$. Define $f \in LC(X)$ by $f(w) = s$ for all $w \in X$ and define $g \in LC(X)$ to be $\chi_U$. Then $f \wedge g \leq p$ but $f \nleq p$. Hence, $g \leq p$. Therefore, $p(w) = 1$ for all $w \in U$.

From this it follows that

1) there exists $s \neq 1$ in $[0,1]$ such that $p(y) = s$ or $1$ for all $y \in X$.
2) $\{y \in X \mid p(y) = s\}$ is closed.

It is easy to see that $\{y \in X \mid p(y) = s\}$ is irreducible and so, equals $cl\{x\}$ for some $x \in X$.

By defining $LC(f)\colon LC(Y) \to LC(X)$ by $LC(f)(g) = gf$, $LC$ becomes a contravariant functor. In chapter 3, it will be shown how this functor creates a duality in much the same way that $O$ does.

The following proposition says that constant functions and characteristic functions generate $LC(X)$ as a frame.

4.3 Proposition. $LC(X)$ is the only subframe of itself (that is, closed under arbitrary joins and finite meets) that contains all constant functions and characteristic functions of open sets.

pf. Let $f \in LC(X)$. For each $x \in X$ and $s \ll f(x)$ there exists an open neighborhood $U$ of $x$ such that

$f(U) \leq \dagger s$. Then $g \in LC(X)$ defined by $g(y) = 0$ if $y \notin U$ and $s$ if $y \in U$, is the meet of a constant function and a characteristic function. $g \leq f$ and also, by varying $x$ and $s$, the join of all functions obtained in this way equals $f$.

In chapter 3, we will restrict one duality for sober spaces to obtain another one for locally-compact sober spaces. The following fact will be useful.

4.4 <u>Proposition</u>. If $X$ is a locally-compact space, then $LC(X)$ is a continuous lattice.

pf. Let $f \in LC(X)$. For each open set $U$ in $X$ contained in a compact set $K$ in $X$ and for each $s \in [0,1]$ such that $s \ll f(x)$ for all $x \in K$, define $g \in LC(X)$ by $g(x) = s$ if $x \in U$ and $0$ otherwise.

Then $g \ll f$ in $LC(X)$ and the join of all such functions obtained in this way is $f$.

## CHAPTER 1. AUGMENTED COMPACT SPACES

### Section 1: Augmented Compact Spaces

1.1 Definition. An augmented compact space is a pair $(X, \underline{A})$ where $X$ is a compact Hausdorff space and $\underline{A}$ is a set of closed subsets of $X$ such that

1) whenever $\underline{B} \subseteq \underline{A}$, then $\cap \underline{B} \in \underline{A}$.
2) for all $C \in \underline{A}$, $C = \cap\{D \in \underline{A} \mid C \subseteq \mathrm{int}_X D\}$.

$X$ is said to be augmented by $\underline{A}$ and the elements of $\underline{A}$ are called twice-closed sets.

For example, every compact Hausdorff space is augmented by the set of all its closed subsets, every compact convex subset of a locally-convex topological vector space is augmented by the set of all its closed convex subsets, and every compact poset is augmented by the set of all its closed up-sets.

Most often, an augmented compact space $(X, \underline{A})$ will be denoted simply by $X$.

1.2 Definition. If $X$ is an augmented compact space and $A \subseteq X$, then define the twice-closure of $A$ in $X$, denoted by $cl_X^+(A)$, to be $\cap\{C \mid C$ is twice-closed in $X$ and $A \subseteq C\}$.

$cl_X^+(A)$ is the smallest twice-closed subset of $X$, containing $A$. Clearly, $cl_X(A) \subseteq cl_X^+(A)$.

1.3 Definition. A twice-continuous function is a continuous function $f: X \to Y$ between augmented compact spaces such that for all twice-closed subsets $C$ of $Y$, $f^{-1}(C)$ is twice-closed in $X$.

For example, given the augmentations mentioned above, every continuous function between compact Hausdorff spaces is twice-continuous, every continuous affine function between compact convex subsets of locally-convex topological vector spaces is twice-continuous, and every continuous order-preserving function between compact posets is twice-continuous.

Each identity function is twice-continuous and the composition of twice-continuous functions is twice-continuous. Hence, there exists a category whose objects are augmented compact spaces and whose morphisms are twice-continuous functions.

1.4 Definition. A twice-homeomorphism is a homeomorphism $f: X \to Y$ between augmented compact spaces that is twice-continuous and whose inverse is twice-continuous.

The twice-homeomorphisms are the isomorphisms in the category of augmented compact spaces.

Section 2. Lattice of Twice-Closed Sets

2.1 Definition. Define the lattice of twice-closed sets of an augmented compact space $X$, denoted by $\Gamma^+(X)$, to be the set of all twice-closed subsets of $X$ ordered by reverse inclusion (that is, $C \leq D$ in $\Gamma^+(X)$ iff $D \subseteq C$).

2.2 Proposition. $\Gamma^+(X)$ is a continuous lattice.

pf. Clearly, $\Gamma^+(X)$ is a complete lattice where $\bigvee \underline{B} = \cap \underline{B}$ and $\bigwedge \underline{B} = cl_X^+(\cup \underline{B})$ for all $\underline{B} \subseteq \Gamma^+(X)$.

If $C << D$ in $\Gamma^+(X)$, then since $D$ is the directed join in $\Gamma^+(X)$ of all twice-closed sets $E$ with $D \subseteq int_X(E)$, it follows that $E \subseteq C$ for some twice-closed set $E$ with $D \subseteq int_X(E)$. Hence, $D \subseteq int_X(C)$. Conversely, if $C$ and $D$ are twice-closed sets such that $D \subseteq int_X(C)$, let $\underline{B}$ be a directed subset of $\Gamma^+(X)$ such that $D \leq \bigvee \underline{B}$. Then $\cap \underline{B} \subseteq int_X(C)$ and, by compactness, there exists some $E \in \underline{B}$ such that $E \subseteq C$. Therefore $C << D$ in $\Gamma^+(X)$.

Hence, for all $C, D \in \Gamma^+(X)$, $C << D$ iff $D \subseteq int_X(C)$. Since, for each twice-closed set $D$, $D = \cap\{C \in \Gamma^+(X) \mid D \subseteq int_X(C)\}$, it follows that $D = \bigvee\{C \in \Gamma^+(X) \mid C << D\}$ and $\Gamma^+(X)$ is a continuous lattice.

Moreover, it follows that the compact elements of

$\Gamma^+(X)$ are the open twice-closed sets in $X$ and that $\Gamma^+(X)$ is an algebraic lattice iff every twice-closed set in $X$ is the intersection of open twice-closed sets.

2.3 Definition. If $f: X \to Y$ is a twice-continuous function, then define $\Gamma^+(f): \Gamma^+(X) \to \Gamma^+(Y)$ by $\Gamma^+(f)(C) = cl_Y^+(f(C))$ for all $C \in \Gamma^+(X)$.

2.4 Proposition. $\Gamma^+(f): \Gamma^+(X) \to \Gamma^+(Y)$ is an algebraically-continuous function.

pf. Define $\phi: \Gamma^+(Y) \to \Gamma^+(X)$ by $\phi(C) = f^{-1}(C)$ for all $C \in \Gamma^+(Y)$. Then $\phi$ is a lower-adjoint to $\Gamma^+(f)$ that preserves the way-below relation. Hence $\Gamma^+(f)$ is algebraically-continuous.

$\Gamma^+(1_X) = 1_{\Gamma^+(X)}$ and for twice-continuous functions $f: X \to Y$ and $g: Y \to Z$, $\Gamma^+(gf) = \Gamma^+(g)\Gamma^+(f)$. Hence, $\Gamma^+$ is a covariant functor from the category of augmented compact spaces to the category of continuous lattices and algebraically-continuous functions.

The following will be used to recapture the augmented compact spaces from their lattices of twice-closed sets.

2.5 Definition. If $L$ is a complete lattice and $x \in L$, define $\mu(x)$ to be the set of all meet-irreducibles $p$ in $L$ with $x \le p$.

It follows that

1) $\mu(1) = \emptyset$ and $\mu(0)$ is the set of all meet-irreducibles in $L$.

2) $\mu(\bigvee A) = \bigcap\{\mu(x) \mid x \in A\}$ for all $A \subseteq L$.

3) $x \leq y$ implies $\mu(y) \subseteq \mu(x)$.

Moreover, in a continuous lattice $L$, $x \leq y$ iff $\mu(y) \subseteq \mu(x)$.

2.6 Definition. A sheltered continuous lattice is a continuous lattice $L$ in which the set of meet-irreducibles is Lawson-closed.

For example, the SEM lattice, the lattice of streams, and the diamond lattice with a new greatest element added, are sheltered. It will be shown that for certain augmented compact spaces, $\Gamma^+(x)$ is sheltered.

2.7 Definition. Define the space of meet-irreducibles of a sheltered continuous lattice $L$, denoted by $Mi(L)$, to be the set of all meet-irreducibles in $L$ with the topology induced by the Lawson-topology on $L$ and augmented by $\{\mu(x) \mid x \in L\}$.

2.8 Proposition. $Mi(L)$ is an augmented compact space.

pf. Clearly, $Mi(L)$ is a compact Hausdorff space and for all $x \in L$, $\mu(x)$ is closed in $Mi(L)$. It follows that for all $\underline{A} \subseteq \{\mu(x) \mid x \in L\}$, $\cap \underline{A} \in \{\mu(x) \mid x \in L\}$.

If $x \in L$, $p \in Mi(L)$, and $p \notin \mu(x)$, then let $y \in L$ such that $y \ll x$ and $p \notin \mu(y)$. Then $\mu(x) \subseteq int(\mu(y))$ in $Mi(L)$. Therefore, $\mu(x) = \cap\{\mu(y) \mid \mu(x) \subseteq int(\mu(y))\}$. Thus $Mi(L)$ is an augmented compact space.

Moreover, for all $A \subseteq Mi(L)$, $cl^{+}(A) = \mu(\bigwedge A)$.

2.9 <u>Definition</u>. If $f: L \to M$ is an algebraically-continuous function between sheltered continuous lattices, that preserves meet-irreducibles, then define $Mi(f): Mi(L) \to Mi(M)$ by $Mi(f)(p) = f(p)$ for all $p \in Mi(L)$.

2.10 <u>Proposition</u>. $Mi(f): Mi(L) \to Mi(M)$ is a twice-continuous function.

pf. $Mi(f): Mi(L) \to Mi(M)$ is clearly a continuous function. Let $g: M \to L$ be the lower-adjoint of $f$. Then for all $y \in M$, $Mi(f)^{-1}(\mu(y)) = \mu(g(y))$. Hence $Mi(f)$ is twice-continuous.

$Mi(1_L) = 1_{Mi(L)}$ and whenever $f: L \to M$ and $g: M \to N$ are algebraically-continuous functions between sheltered continuous lattices, that preserve meet-irreducibles, then $Mi(gf) = Mi(g)Mi(f)$. Hence, $Mi$ is a covariant functor from the category of sheltered continuous lattices and algebraically-continuous functions that preserve meet-irreducibles to the category of augmented compact spaces.

We cannot consider $Mi(\Gamma^+(X))$ for all augmented compact spaces $X$ because $\Gamma^+(X)$ may not be sheltered. For example, if $L$ is a continuous lattice, then $L$ with the Lawson-topology is augmented by its Lawson-closed filters and furthermore, $\Gamma^+(L) \cong L$. This motivates the next definition.

2.11 <u>Definition</u>. An augmented compact space $X$ is <u>staid</u> iff

1) whenever $x \neq y$ in $X$, then $cl_X^+\{x\} \neq cl_X^+\{y\}$.
2) $cl_X^+\{x\}$ is meet-irreducible in $\Gamma^+(X)$ for all $x \in X$.

The similarity of this definition to the definition of sober topological spaces is clear. The three examples of augmented compact spaces mentioned after 1.1 are all staid.

2.12 <u>Proposition</u>. If $L$ is a sheltered continuous lattice, then $Mi(L)$ is staid.

pf. For $p \in Mi(L)$, $cl^+\{p\} = \mu(p)$. Hence, if $p \neq q$ in $Mi(L)$, then either $p \nleq \mu(q)$ or $q \nleq \mu(p)$. If $\mu(p) = \mu(x) \wedge \mu(y)$ in $\Gamma^+ Mi(L)$, then since $\mu(x) \wedge \mu(y) = \mu(x \wedge y)$, it follows that $p = x \wedge y$. Suppose $p = x$. Then $\mu(p) = \mu(x)$. Hence $\mu(p)$ is meet-irreducible in $\Gamma^+ Mi(L)$ for all $p \in Mi(L)$.

Conversely, we have the following.

2.13 <u>Proposition</u>. If $X$ is a staid augmented compact space, then $\Gamma^+(X)$ is sheltered.

pf. Define $j: X \to \Gamma^+(X)$ by $j(x) = cl_X^+\{x\}$. Then $j$ is a one-one continuous function when $\Gamma^+(X)$ is given the Lawson-topology. By the Gierz and Keimel Lemma, $j(X)$ is equal to the set of meet-irreducibles in $\Gamma^+(X)$. Hence $\Gamma^+(X)$ is sheltered.

If $f: X \to Y$ is a twice-continuous function, then $\Gamma^+(f)(cl_X^+\{x\}) = cl_Y^+(f(cl_X^+\{x\})) = cl_Y^+\{f(x)\}$ for all $x \in X$. Hence, $\Gamma^+$ restricted is a functor from the category of staid augmented compact spaces to the category of sheltered continuous lattices. Likewise, Mi is a functor going in the opposite direction. It will be shown that this pair of functors forms an equivalence of categories.

2.14 <u>Definition</u>. If $X$ is a staid augmented compact space, then define $\varepsilon_X: X \to Mi(\Gamma^+(x)$ by $\varepsilon_X(x) = cl_X^+\{x\}$ for all $x \in X$.

2.15 <u>Theorem</u>. $\varepsilon_X: X \to Mi(\Gamma^+(X))$ is a twice-homeomorphism.

pf. As it has already been mentioned in the proof of 2.13, $\varepsilon_X$ is a homeomorphism.

Since $\varepsilon_X(C) = \mu(C)$ for all twice-closed sets $C$ in $X$, it follows that $\varepsilon_X$ is a twice-homeomorphism.

If $f: X \to Y$ is a twice-continuous function between staid augmented compact spaces, then $Mi(\Gamma^+(f))\varepsilon_X = \varepsilon_Y f$. Hence $\varepsilon: 1 \to Mi\Gamma^+$ is a natural equivalence.

2.16 Definition. If $L$ is a sheltered continuous lattice, then define $\eta_L: L \to \Gamma^+(Mi(L))$ by $\eta_L(x) = \mu(x)$ for all $x \in L$.

2.17 Theorem. $\eta_L: L \to \Gamma^+(Mi(L)$ is an order-isomorphism.

pf. It has been noted that $x \leq y$ in $L$ iff $\mu(y) \subseteq \mu(x)$. That is, $\mu(x) \leq \mu(y)$ in $\Gamma^+(Mi(L))$. $\eta_L$ is clearly one-one and onto.

If $f: L \to M$ is an algebraically-continuous function that preserves meet-irreducibles between sheltered continuous lattices, then $\Gamma^+(Mi(f))\eta_L = \eta_M f$. Hence, $\eta: 1 \to \Gamma^+ Mi$ is a natural equivalence.

These results are summarized in the following theorem.

2.18 Theorem. The category of staid augmented compact spaces is equivalent to the category of sheltered continuous lattices. The equivalence is given by the functors $\Gamma^+$ and Mi with natural equivalences $\varepsilon$ and $\eta$.

In the next section, more results dealing with this equivalence will be given and it will be shown that this equivalence extends previously obtained results.

## SECTION 3. THE UP AUGMENTATION

The purpose of this section is three-fold:

1) to embed the category of compact posets into the category of staid augmented compact spaces.
2) to characterize those staid augmented compact spaces $X$ such that $\Gamma^+(X)$ is distributive.
3) to show how the equivalence of section 2 extends previously obtained results.

3.1 Definition. Define the up augmentation of a compact poset $P$, denoted by $up(P)$, to be the set $P$ with its original topology and augmented by the set of all closed up-sets in $P$.

$up(P)$ is a staid augmented compact space. $f: P \to Q$ is a continuous order-preserving function iff $f: up(P) \to up(Q)$ is a twice-continuous function. Also, if $up(P) = up(Q)$, then $P = Q$. Hence, $up$ is a full embedding of the category of compact posets into the category of staid augmented compact spaces.

$\Gamma^+(up(P))$ is the lattice of closed up-sets of $P$.

The next theorem characterizes the image of up and also those staid augmented compact spaces $X$ for which $\Gamma^+(X)$ is distributive.

3.2 Theorem. If $X$ is a staid augmented compact space, then the following are equivalent.

1) $X = up(P)$ for some compact poset $P$.
2) for all twice-closed sets $C$ and $D$ in $X$, $C \cup D$ is twice-closed in $X$.
3) $\Gamma^+(X)$ is distributive.

pf. It is clear that 1) implies 2) and that 2) implies 3).

Suppose that $\Gamma^+(X)$ is distributive. Define $P$ to be the set $X$ with its original topology and ordered by $x \le y$ in $P$ iff $y \in cl_X^+\{x\}$. Clearly, every twice-closed set in $X$ is a closed up-set in $P$.

Conversely, if $C$ is a closed up-set in $P$, let $D = \bigwedge\{cl_X^+\{x\} \mid x \in C\}$ in $\Gamma^+(X)$. Then $D$ is twice-closed in $X$ and $C \subseteq D$. If $y \in D$, then $D \le cl_X^+\{y\}$ in $\Gamma^+(X)$ and by the Gierz and Keimel lemma, $cl_X^+\{x\} \le cl_X^+\{y\}$ for some $x \in C$. Thus $y \in cl_X^+\{x\}$ and since $C$ is an up-set, $y \in C$. Therefore $C = D$ and $C$ is twice-closed in $X$.

This theorem shows that the equivalence of section 2 restricted gives an equivalence of the category of compact posets to the category of distributive sheltered continuous lattices. Therefore, we have proven the following result of Gierz and Keimel: every distributive continuous lattice in which the set of primes is Lawson-closed is order-isomorphic to the lattice of closed up-sets of some compact poset.

Moreover, this result is seen to be a special case of the equivalence given in section 2.

The construction of the preceeding theorem motivates the next definition.

3.3 Definition. Define the posetification of a staid augmented compact space $X$, denoted by $P_X$, to be the set $X$ with its original topology and ordered by $x \leq y$ in $P_X$ iff $y \in cl_X^+\{x\}$.

3.4 Proposition. $P_X$ is a compact poset and $p_X$: $up(P_X) \to X$ defined by $p_X(x) = x$ for all $x \in P_X$ is a twice-continuous function. If $Q$ is a compact poset and $f: up(Q) \to X$ is a twice-continuous function, then there exists a unique continuous order-preserving function $q$: $Q \to P_X$ such that $pq = f$.

pf. In the proof of 3.2 it was noticed that $P_X$ is a compact poset and that every twice-closed set in $X$ is a closed up-set in $P_X$.

Define $q: Q \to P_X$ by $q(x) = f(x)$ for all $x \in Q$. If $x \leq y$ in $Q$, then $y \in cl^+\{x\}$ in $up(Q)$. Hence, $f(y) \in cl_X^+\{f(x)\}$ and therefore $q(x) \leq q(y)$ in $P_X$. Uniqueness is clear.

In other words, the category of compact posets is (isomorphic to) a coreflective subcategory of the category of staid augmented compact spaces. By applying the equivalence, it follows that the category of distributive sheltered continuous lattices is coreflective in the category of sheltered continuous lattices.

## Section 4. Lattice of Closed Order-Convex Sets

In this section, the techniques developed in the theory of augmented compact spaces will be used to study another type of continuous lattice arising from compact posets.

4.1 *Definition*. Define the *lattice of closed order-convex subsets* of a compact poset $P$, denoted by $\mathrm{Con}(P)$, to be the set of all closed order-convex subsets of $P$ ordered by reverse-inclusion.

4.2 *Proposition*. $\mathrm{Con}(P)$ is a continuous lattice.

pf. It suffices to show that $P$ is augmented by its closed order-convex subsets. Clearly, the intersection of closed order-convex subsets is closed and order-convex. Let $C$ be a closed order-convex subset of $P$ and suppose $x \notin C$. Then either $x \notin {\downarrow}C$ or $x \notin {\uparrow}C$. In the first case, there exists a closed down-set $D$ in $P$ such that $x \notin D$ and ${\downarrow}C \subseteq \mathrm{int}_P D$. In the second case, there exists a closed up-set $E$ in $P$ such that $x \notin E$ and ${\uparrow}C \subseteq \mathrm{int}_P E$. In either case, there exists a closed order-convex subset of $P$ not containing $x$ but containing $C$ in its interior. Hence $P$ is augmented by its closed order-convex subsets and by 2.2, $\mathrm{Con}(P)$ is a continuous lattice.

However, Con(P) is seldom distributive as is seen from the next proposition.

4.3 Proposition. The following are equivalent for a compact poset P.

1) Con(P) is distributive.
2) there do not exist elements $x,y,z \in P$ such that $x < y < z$.
3) every subset of P is order-convex.

pf. That 2) implies 3) and that 3) implies 1) is clear. If there do exist elements $x,y,z \in P$ such that $x < y < z$, then in Con(P), $\{y\} \vee (\{x\} \wedge \{z\}) = \{y\} \cap [x,z] = \{y\}$ but $(\{y\} \vee \{x\}) \wedge (\{y\} \vee \{z\}) = \emptyset$. Hence, 1) implies 2).

Since P augmented by its closed order-convex subsets is staid, it follows that the meet-irreducible elements of Con(P) are the singleton subsets of P. On the other hand, the prime elements of Con(P) are the singleton subsets $\{m\}$ where m is either maximal or minimal in P.

4.4 Definition. If $f: P \to Q$ is a continuous order-preserving function, then define $Con(f): Con(P) \to Con(Q)$ by $Con(f)(C) = ocon_P(f(C))$ for all $C \in Con(P)$.

From the theory of augmented compact spaces, it follows that $\mathrm{Con}(f): \mathrm{Con}(P) \to \mathrm{Con}(Q)$ is an algebraically-continuous function that preserves meet-irreducibles.

$\mathrm{Con}(1_P) = 1_{\mathrm{Con}(P)}$ and whenever $f: P \to Q$ and $g: Q \to R$ are continuous order-preserving functions, then $\mathrm{Con}(gf) = \mathrm{Con}(g)\mathrm{Con}(f)$. Therefore, Con is a covariant functor from the category of compact posets to the category of sheltered continuous lattices. However, the order on $P$ is somewhat lost since $\mathrm{Con}(P) = \mathrm{Con}(P^{op})$.

# CHAPTER 2. COMPACT STAR STRUCTURES

## Section 1. Star Structures

Recall that the circle operation $\circ\ [0,1] \times [0,1] \to [0,1]$ is defined by $s \circ t = s + t - st$. $[0,1]$ with this operation is a monoid. Although $[0,1]$ with the circle operation is isomorphic to $[0,1]$ with usual multiplication, the circle operation is preferred here when we begin lattice theoretical considerations.

1.1 Definition. A star structure is a triple $(X,\circ,x_0)$ where $X$ is a set, $\circ\colon [0,1] \times X \to X$ is a function (where $\circ(s,x)$ is denoted $s \circ x$), and $x_0 \in X$ such that

1) $0 \circ x = x$ for all $x \in X$.
2) $s \circ (t \circ x) = (s \circ t) \circ x$ for all $s,t \in [0,1]$ and $x \in X$.
3) $s \circ x = s \circ y$ and $s \neq 1$ implies $x = y$.
4) $1 \circ x = x_0$ for all $x \in X$.

$x_0$ is the nucleus of $(X,\circ,x_0)$.

For example, if $X$ is a star shaped subset about $x_0$ of a vector space over the real numbers (that is, $(1 - s)x + sx_0 \in X$ for all $s \in [0,1]$ and $x \in X$), then $(X,\circ,x_0)$ is a star structure where $s \circ x = (1 - s)x + sx_0$.

Most often a star structure $(X, \circ, x_0)$ will be denoted simply by $X$ and whenever $Y$ (resp. $Z$) is given to be a star structure, the nucleus will be denoted by $y_0$ (resp. $z_0$).

If $X$ is a star structure, then

1) $s \circ x_0 = x_0$ for all $s \in [0,1]$.
2) $s \circ x = x_0$ implies that $s = 1$ or $x = x_0$.

1.2 Definition. A subset $S$ of a star structure $X$ is star shaped iff

1) $x_0 \in S$
2) whenever $x \in S$ and $s \in [0,1]$, then $s \circ x \in S$.

Of course, $X$ and $\{x_0\}$ are star shaped subsets of $X$. The intersection and union of any nonempty set of star shaped subsets of $X$ is star shaped.

1.3 Definition. If $A$ is a subset of a star structure $X$, then the star hull of $A$, denoted by $sh(A)$, is defined to be $\{s \circ x \mid s \in [0,1] \text{ and } x \in A\} \cup \{x_0\}$.

$sh(A)$ is the smallest star shaped subset of $X$ containing $A$. $sh(\{x\})$ is denoted by $sh(x)$ for all $x \in X$.

1.4 Definition. A star structure homomorphism is a function $f: X \to Y$ between star structures such that $f(s \circ x) = s \circ f(x)$ for all $s \in [0,1]$ and $x \in X$.

A star structure homomorphism will preserve the nucleus. The identity function on a star structure is a star structure homomorphism and the composition of star structure homomorphisms is a star structure homomorphism. Hence, there exists a category whose objects are star structures and whose morphisms are star structure homomorphisms. The isomorphisms in this category are the one-one onto star structure homomorphisms.

A star structure homomorphism preserves and inversely preserves star shaped subsets.

The following elements will play a central role in what follows.

1.5 <u>Definition</u>. An element $c$ of $X$ is <u>coronal</u> iff whenever $c = s \circ x$, then $s = 0$.

The nucleus is never a coronal element.

1.6 <u>Proposition</u>. An element $c$ of $X$ is coronal iff whenever $s \circ x = t \circ c$, then $s \leq t$.

pf. Let $c$ be a coronal element of $X$ and suppose $s \circ x = t \circ c$ where $s \nleq t$. Then $t < s$ and there exists $r \neq 0$ such that $t \circ r = s$. Hence, $t \circ r \circ x = t \circ c$ and since $t * 1$, $r \circ x = c$. This is a contradiction.

The converse is clear.

## Section 2. Bounded Star Structures

In this section it will be shown that having enough coronal elements is equivalent to being free in the category of star structures. These star structures will be important in what follows.

2.1 Definition. Define the cone over a set $A$, denoted by $\Delta(A)$, to be

1) for $A$ nonempty, $A \times [0,1] / A \times \{1\}$ where $s \circ (a,t) = (a, s \circ t)$ for all $s \in [0,1]$ and $(a,t) \in \Delta(A)$.

2) $\{x_0\}$ if $A$ is empty.

$\Delta(A)$ is a star structure and for $A$ nonempty, the coronal elements in $\Delta(A)$ are the elements $(a,0)$ for some $a \in A$.

If $X$ is a star structure and $f: A \to X$ is a function, there exists a unique star structure homomorphism $q: \Delta(A) \to X$ such that $q|A = f$ (identifying $A$ with the set of elements $(a,0)$ in $\Delta A$). Hence, regarded as a functor from the category of sets to the category of star structures, $\Delta$ is left adjoint to the underlying set functor.

2.2 Definition. A star structure $X$ is bounded iff for all $x \in X$, $x \neq x_0$, there exists a coronal element $c$

of X such that $x = s \circ c$ for some $s \in [0,1]$.

$\Delta(A)$ is bounded for all sets A. Conversely, if X is a bounded star structure, then X is star isomorphic to $\Delta(C)$ where C is the set of coronal elements in X. Hence, being free in the category of star structures is equivalent to being bounded.

## Section 3. Extensive Star Structure Homomorphisms

Here we consider an important type of morphism.

3.1 Definition. A star homomorphism $f\colon X \to Y$ is extensive iff whenever $f(x) = s \circ y$ in $Y$, then $x = s \circ z$ in $X$ for some $z \in X$.

A star structure homomorphism $f\colon X \to Y$ is extensive iff $f^{-1}(y_0) = \{x_0\}$ and whenever $f(x) = s \circ y$ in $Y$ for $s \neq 1$, then $x = s \circ z$ for some $z \in f^{-1}(y)$.

If $X$ is a bounded star structure, then a star structure homomorphism $f\colon X \to Y$ is extensive iff $f$ preserves coronal elements.

Every star structure isomorphism is extensive and the composition of extensive star structure homomorphisms is extensive.

A one-one star structure homomorphism $f\colon X \to Y$ is extensive iff whenever $s \circ y \in f(X)$ and $s \neq 1$, then $y \in f(X)$. This motivates the next definition.

3.2 Definition. A star shaped subset $S$ of $X$ is extensive iff whenever $s \circ x \in S$ and $s \neq 1$, then $x \in S$.

$S$ is extensive iff the inclusion $i\colon S \to X$ is extensive. The star hull of any set of coronal elements is extensive and conversely, in a bounded star structure, every extensive star shaped subset is the star hull of the coronal elements it contains. The intersection of extensive star shaped subsets is extensive.

## Section 4. Star Ordering

In this section a useful partial order will be defined on certain star structures.

4.1 Definition. A star structure $X$ is faithful iff whenever $s \circ x = x$ and $x \neq x_0$, then $s = 0$.

Every bounded star structure is faithful. However, let $X = \{x, x_0\}$ where $s \circ x = x$ for all $s \neq 1$, $1 \circ x = x_0$, and $s \circ x_0 = x_0$ for all $s \in [0,1]$. Then $X$ is a star structure but not faithful.

4.2 Definition. Define the star ordering on a faithful star structure $X$ by $x \leq y$ iff $x = s \circ y$ for some $s \in [0,1]$.

4.3 Proposition. If $X$ is faithful, then $X$ with the star ordering is a poset with zero.

pf. Clearly the star ordering is reflexive and transitive. Suppose $x = s \circ y$ and $y = t \circ x$. Then $x = s \circ t \circ x$. Thus $s \circ t = 0$ and $s = t = 0$. Therefore $x = y$. Hence, the star ordering is a partial order on $X$. Clearly, $x_0 \leq x$ for all $x \in X$.

A star homomorphism between faithful star structures $f: X \to Y$ will become an order-preserving function preserving zero when $X$ and $Y$ are given the star ordering.

The coronal elements of $X$ are the maximal nonzero elements and the star shaped subsets of $X$ are the down-sets containing zero when $X$ is given the star ordering.

## Section 5. Compact Star Structures

5.1. A <u>compact star structure</u> $X$ is a compact Hausdorff space and a star structure such that $\circ: [0,1] \times X \to X$ is a continuous function (where $[0,1]$ has the usual topology).

For example, if $X$ is a compact star shaped subset about $x_0$ of a topological vector space, then $X$ is a compact star structure. The full moon, half moon, and the ball and chain star structures (see examples) are compact star structures.

5.2 <u>Proposition</u>. Every compact star structure $X$ is bounded.

pf. Let $x \in X$ and $x \neq x_0$. Then $\{s \mid s \circ y = x$ for some $y \in X\}$ is a nonempty closed subset of $[0,1]$ and hence has a greatest element $t$. Since $x \neq x_0$, $t \neq 1$. Let $c \in X$ such that $t \circ c = x$. If $s \circ y = c$, then $t \circ s \circ y = x$. Therefore, $t \circ s = t$ and $s = 0$. Hence, $c$ is a coronal element of $X$.

Thus, every compact star structure is faithful and can be ordered with the star ordering.

5.3 Proposition. A compact star structure $X$ with the star ordering is a compact poset.

pf. Suppose $x \nleq y$ in $X$. Then $x \neq s \circ y$ for all $s \in [0,1]$. For each $s \in [0,1]$ let $U_s$ be a neighborhood of $x$ in $X$, $V_s$ a neighborhood of $s$ in $[0,1]$, and $W_s$ a neighborhood of $y$ in $X$ such that $u \neq t \circ w$ for all $u \in U_s$, $t \in V_s$, and $w \in W_s$. By compactness of $[0,1]$, let $F$ be a finite subset of $[0,1]$ such that $\{V_s \mid s \in F\}$ covers $[0,1]$. Then let $U = \cap\{U_s \mid s \in F\}$, and $W = \cap\{W_s \mid s \in F\}$. It follows that $u \neq t \circ w$ for all $u \in U$, $w \in W$, and $t \in [0,1]$.

This proposition is very useful since the following facts are now immediate.

If $X$ is a compact star structure, then

1) whenever $S$ is star shaped in $X$, then $cl_X S$ is star shaped.
2) whenever $C$ is closed in $X$, then $sh(C)$ is closed.
3) $sh(cl_X A) = cl_X(sh(A))$ for all $A \subseteq X$.
4) $X$ is augmented by its closed star shaped subsets.

## Section 6. Star Lattice

6.1 Definition. Define the star lattice of a compact star structure $X$, denoted by $St(X)$, to be the set of all closed star shaped subsets of $X$ ordered by reverse-inclusion.

By the theory of augmented compact spaces we know immediately that $St(X)$ is a distributive continuous lattice. Furthermore, the prime elements of $St(X)$ are the sets $sh(x)$ for some $x \in X$, $x \neq x_0$.

6.2 Definition. If $f: X \to Y$ is a continuous star structure homomorphism, then define $St(f): St(X) \to St(Y)$ by $St(f)(S) = f(S)$ for all $S \in St(X)$.

$St(f): St(X) \to St(Y)$ is an algebraically-continuous function.

$St(1_X) = 1_{St(X)}$ and whenever $f: X \to Y$ and $g: Y \to Z$ are continuous star structure homomorphisms, $St(gf) = St(g)St(f)$. Hence, $St$ is a covariant functor from the category of compact star structures to the category of continuous lattices and algebraically-continuous functions. In later work, this functor will be lifted to a category with more structure.

## Section 7. The Corona and Compact Cone.

In this section, we consider the coronal elements in compact star structures and show that for certain compact star structures, they completely determine the star structure.

7.1 Definition. Define the corona of a compact star structure $X$, denoted by $Cor(X)$, to be the set of coronal elements of $X$ with the topology induced by $X$.

$Cor(X)$ is a Hausdorff space. However, we will restrict $X$ later on so that $Cor(X)$ is locally-compact.

7.2 Definition. If $f: X \to Y$ is a continuous extensive star structure homomorphism, then define $Cor(f): Cor(X) \to Cor(Y)$ by $Cor(f)(c) = f(c)$ for all $c \in Cor(X)$.

$Cor(f): Cor(X) \to Cor(Y)$ is a continuous function. $Cor(1_X) = 1_{Cor(X)}$ and $Cor(gf) = Cor(g)Cor(f)$. Hence, Cor is a covariant functor from the category of compact star structures and continuous extensive star structure homomorphisms to the category of Hausdorff spaces. In order to construct a functor in the opposite direction we must make a restriction.

Recall that a locally-compact Hausdorff space $X$ has a one-point compactification, denoted here by $\alpha(X) = X \cup \{\infty\}$ where the open neighborhoods of $\infty$ are all $(X - K) \cup \{\infty\}$ such that $K$ is a compact subset of $X$. A proper continuous function $f: X \to Y$ between locally-compact Hausdorff spaces has a continuous extension $\alpha(f): \alpha(X) \to \alpha(Y)$.

7.3 <u>Definition</u>. Define the <u>compact</u> <u>cone</u> over a locally-compact Hausdorff space $X$, denoted by $\Delta_0(X)$, to be $\alpha(X) \times [0,1] / \{(x,s) \mid x = \infty \text{ or } s = 1\}$ where $s \circ (x,t) = (x, s \circ t)$ for all $s \in [0,1]$ and $(x,t) \in \Delta_0(X)$.

$\Delta_0(X)$ is a compact star structure. For example, if $X = [0,1)$ with the usual topology, then $\Delta_0(X)$ is the half-moon star structure.

7.4 <u>Definition</u>. If $f: X \to Y$ is a proper continuous function between locally-compact Hausdorff spaces, then define $\Delta_0(f): \Delta_0(X) \to \Delta_0(Y)$ by $\Delta_0(f)(x,s) = (\alpha(f)(x),s)$ for all $(x,s) \in \Delta_0(X)$.

$\Delta_0(f): \Delta_0(X) \to \Delta_0(Y)$ is a continuous extensive star structure homomorphism. $\Delta_0(1_X) = 1_{\Delta_0(X)}$ and $\Delta_0(gf) = \Delta_0(g)\Delta_0(f)$. Hence, $\Delta_0$ is a covariant functor from the category of locally-compact Hausdorff spaces and proper continuous functions to the category of compact star structures and continuous extensive star structure homomorphisms. We will now identify those compact star structures that arise as $\Delta_0(X)$.

7.5 Definition. A compact star structure $X$ is solid iff $\mathrm{Cor}(X) \cup \{x_0\}$ is a closed set in $X$.

The half-moon and full-moon star structures are solid but the ball and chain star structure is not. The compact cone of a locally-compact Hausdorff space is solid and the corona of a solid compact star structure is locally-compact. It will now be shown that Cor and $\Delta_0$ suitably restricted form an equivalence of categories. If $f: X \to Y$ is a continuous extensive star structure homomorphism between solid compact star structures, then $\mathrm{Cor}(f): \mathrm{Cor}(X) \to \mathrm{Cor}(Y)$ is proper.

7.6 Definition. If $X$ is a locally-compact Hausdorff space, define $\gamma_X: \mathrm{Cor}(\Delta_0(X)) \to X$ by $\gamma_X(x,0) = x$ for all $(x,0) \in \mathrm{Cor}(\Delta_0(x))$.

It follows that $\gamma_X: \mathrm{Cor}(\Delta_0(X)) \to X$ is a homeomorphism. If $f: X \to Y$ is a proper continuous function between locally-compact Hausdorff spaces, then $f\gamma_X = \gamma_Y \mathrm{Cor}(\Delta_0(f))$. Hence, $\gamma: \mathrm{Cor}\Delta_0 \to 1$ is a natural equivalence.

7.7 Definition. If $X$ is a solid compact star structure, then define $\delta_X: \Delta_0(\mathrm{Cor}(X)) \to X$ by $\delta_X(c,s) = s \circ c$ provided $c \neq \infty$ and $x_0$ if $c = \infty$.

It follows that $\delta_X: \Delta_0(\mathrm{Cor}(X)) \to X$ is a one-one onto continuous star structure homomorphism. If $f: X \to Y$ is a continuous extensive star structure homomorphism, then $f\delta_X = \delta_Y \Delta_0(\mathrm{Cor}(f))$. Hence, $\delta: \Delta_0\mathrm{Cor} \to 1$ is a natural equivalence.

These results are summarized in the following theorem.

7.8 Theorem. The category of solid compact star structures and continuous extensive star structure homomorphisms is equivalent to the category of locally-compact Hausdorff spaces and proper continuous functions. The equivalence is given by the functors Cor and $\Delta_0$ with natural equivalences $\gamma$ and $\delta$.

Solid compact star structures will appear in later results as well.

# CHAPTER 3. STAR LATTICES

## Section 1. Star Lattices

Here we begin to apply the ideas of star structures to lattices. In this way we are not considering arbitrary convex combinations of elements but only between elements and the greatest element of the lattice.

1.1 Definition. A star lattice is a complete lattice L with enough primes that is also a bounded star structure such that

1) $s \circ \bigwedge A = \bigwedge \{s \circ x \mid x \in A\}$ for all $s \in [0,1]$ and $A \subseteq L$.
2) $\bigwedge A \circ x = \bigwedge \{s \circ x \mid s \in A\}$ for all $A \subseteq [0,1]$ and $x \in L$.
3) the set of primes in L together with one is a star shaped subset of L.
4) the set of coronal elements in L is a down set in L.

Except for conditions 1) and 2), every requirement of a star lattice has to do with either primes or coronal elements and their properties both in the lattice sense and star structure sense.

[0,1], the diamond lattice, half-diamond lattice, and the necktie lattice are star lattices when given the

star structure that makes 1 the nucleus. The pennant lattice is not a star lattice in this way because it does not satisfy part 4) of the definition. Also, the thin diamond lattice is not a star lattice in this way because it does not satisfy either part 3) or part 4) of the definition.

The following describes a very important example of star lattices.

1.2 Proposition. If $X$ is a topological space, then $LC(X)$ is a star lattice where $(s \circ f)(x) = s \circ (f(x))$ for all $s \in [0,1]$, $f \in LC(X)$, and $x \in X$.

pf. $LC(X)$ is a complete lattice with enough primes. $\circ: [0,1] \times LC(X) \to LC(X)$ is a function that makes $LC(X)$ into a star structure. $f$ is a coronal element of $LC(X)$ iff $\bigwedge\{f(x) \mid x \in X\} = 0$. Therefore, $LC(X)$ is bounded.

It is straightforward to check conditions 1), 2), and 4) of the definition.

To check part 3), notice that $X$ can be taken to be a sober space since if $\hat{X}$ denotes the soberfication of $X$ and $p: X \to \hat{X}$ the natural function, then $f: LC(\hat{X}) \to LC(X)$ defined by $f(g) = gp$ is an order-isomorphism and a star structure homomorphism. By the characterization of primes in $LC(\hat{X})$, part 3) follows.

If $L$ is a star lattice, then

1) 1 is the nucleus.

2) $x \leq s \circ x$ for all $x \in L$ and $s \in [0,1]$.

3) whenever $s \circ x \leq s \circ y$ and $s \neq 1$, then $x \leq y$.

4) the set of primes together with one is extensive.

Also, the following are equivalent for $c \in L$.

1) $c$ is coronal.

2) whenever $s \circ 0 \leq c$, then $s = 0$.

3) whenever $s \circ x \leq t \circ c$, then $s \leq t$.

The following property is useful in the study of star lattices.

1.3 <u>Proposition</u> (Divisibility). If $L$ is a star lattice and $s \circ x \leq y$ in $L$, then there exists $z \geq x$ such that $s \circ z = y$.

pf. Suppose that $s \neq 1$ and $y \neq 1$. Let $y = t \circ c$ where $c$ is a coronal element of $L$. Because $s \circ x \leq t \circ c$, $s \leq t$. Hence, $t = s \circ r$. Then $x \leq r \circ c$ and $s \circ (r \circ c) = y$.

1.4 <u>Definition</u>. A <u>star lattice homomorphism</u> is a function $f: L \to M$ between star lattices such that

1) $f(\bigvee A) = \bigvee f(A)$ for all $A \subseteq L$.

2) $f(\bigwedge F) = \bigwedge f(F)$ for all finite $F \subseteq L$.

3) $f(s \circ x) = s \circ f(x)$ for all $s \in [0,1]$ and $x \in L$.

In other words, a star lattice homomorphism is a framed star structure homomorphism.

The identity function on a star lattice is a star lattice homomorphism and the composition of star lattice homomorphisms is a star lattice homomorphism. Hence, there is a category whose objects are star lattices and whose morphisms are star lattice homomorphisms.

If $f: X \to Y$ is a continuous function, then $LC(f): LC(Y) \to LC(X)$ is a star lattice homomorphism. Hence, LC is a contravariant functor from the category of topological spaces to the category of star lattices. It will be shown that this functor has an adjoint on the right and by restriction, provides a duality for the category of sober spaces.

1.5 <u>Proposition</u>. The upper adjoints $g: M \to L$ of star lattice homomorphisms $f: L \to M$ are extensive star structure homomorphisms and preserve prime elements.

pf. Let $s \in [0,1]$ and $y \in M$. Since $s \circ fg(y) \le s \circ y$, then $s \circ g(y) \le g(s \circ y)$. Suppose $s \neq 1$ and by divisibility, let $z \in L$ such that $s \circ z = g(s \circ y)$. Then $s \circ f(z) \le s \circ y$ and thus $f(z) \le y$. Hence, $z \le g(y)$ and $g(s \circ y) = s \circ z \le s \circ g(y)$. Therefore, $g$ is a star structure homomorphism.

If $c$ is coronal in $M$ and $s \circ 0 \le g(c)$, then $s \circ 0 \le c$ in $M$. Hence, $s = 0$ and $g$ is extensive.

Lastly, the upper adjoint of a framed function preserves primes.

This proposition allows us to characterize star lattice homomorphisms $f: L \to [0,1]$.

1.6 <u>Theorem</u>. If $L$ is a star lattice, then the star lattice homomorphisms $f: L \to [0,1]$ are in one-one correspondence with the coronal primes of $L$. More precisely, a function $f: L \to [0,1]$ is a star lattice homomorphism iff there exists a coronal prime $p$ in $L$ such that $f$ is the lower adjoint of $g_p: [0,1] \to L$ defined by $g_p(s) = s \circ p$ for all $s \in [0,1]$.

pf. If $f: L \to [0,1]$ is a star lattice homomorphism, then let $g: [0,1] \to L$ be its upper adjoint and let $p = g(0)$. Then $p$ is a coronal prime in $L$ by 1.5 and $g(s) = g(s \circ 0) = s \circ g(0) = s \circ p$ for all $s \in [0,1]$.

Conversely, given a coronal prime $p$ in $L$ and $g_p$ defined above, then $g_p$ has a lower adjoint since $g_p(\bigwedge A) = \bigwedge A \circ p = \bigwedge\{s \circ p \mid s \in A\} = \bigwedge g_p(A)$ for all $A \subseteq [0,1]$. This lower adjoint $f$ is given by $f(x) = \bigwedge\{s \mid x \leq s \circ p\}$. $f$ preserves arbitrary joins and, because $g_p$ preserves primes, $f$ preserves finite meets. Finally, $f(t \circ x) = \bigwedge\{s \mid t \circ x \leq s \circ p\} = \bigwedge\{t \circ r \mid t \circ x \leq t \circ r \circ p\} = t \circ \bigwedge\{r \mid x \leq r \circ p\} = t \circ f(x)$.

Therefore we see that coronal elements and prime elements, that were so important in the definition of star lattices, have come together to determine the "characters" of a star lattice.

This characterization of star lattice homomorphisms gives the following representation of star lattices.

1.7 <u>Proposition</u>. For each star lattice $L$, there exists a one-one star lattice homomorphism $j: L \to [0,1]^J$ for some set $J$.

pf. Let $x \nleq y$ in $L$. There exists a coronal prime $p$ in $L$ and $s \in [0,1]$ such that $x \nleq s \circ p$ and $y \leq s \circ p$. Then the star lattice homomorphism $f: L \to [0,1]$ determined by $p$ as in 1.6 is such that $f(x) \nleq s$ and $f(y) \leq s$. Therefore the star lattice homomorphisms from $L$ to $[0,1]$ distinguish points and the proposition follows.

## Section 2. Space of Coronal Primes

In this section coronal primes will be used to construct an adjoint on the right to the LC functor.

2.1 Definition. Define the space of coronal primes of a star lattice $L$, denoted by $CP(L)$, to be the set of all coronal primes in $L$ with $\{\gamma(x) \mid x \in L\}$ as the topology where $\gamma(x) = \{p \in CP(L) \mid x \nleq p\}$.

It follows that

1) $\gamma(1) = CP(L)$ and $\gamma(0) = \emptyset$.
2) $\gamma(\bigvee A) = \bigcup\{\gamma(x) \mid x \in A\}$ for all $A \subseteq L$.
3) $\gamma(x \wedge y) = \gamma(x) \cap \gamma(y)$ for all $x,y \in L$.
4) $x \leq y$ implies that $\gamma(x) \subseteq \gamma(y)$.
5) for all $x \in L$, $\gamma(x) = \gamma(y)$ where $y = \bigwedge\{p \in CP(L) \mid x \leq p\}$.

Hence, $\{\gamma(x) \mid x \in L\}$ is indeed a topology on $CP(L)$.

2.2 Proposition. $CP(L)$ is a sober topological space.

pf. $CP(L)$ is a topological space where $cl\{p\} = CP(L) - \gamma(p)$ for all $p \in CP(L)$. Hence, $CP(L)$ is $T_0$.

Let $CP(L) - \gamma(x)$ be an irreducible closed in $L$

where, without loss of generality, $x$ is the meet of coronal primes in $L$. Then $x \neq 1$ and if $y \wedge z \leq x$, then $CP(L) - \gamma(x) \subseteq (CP(L) - \gamma(y)) \cup (CP(L) - \gamma(z))$. Hence, $\gamma(y) \subseteq \gamma(x)$ or $\gamma(z) \subseteq \gamma(x)$. Then $y \leq x$ or $z \leq x$ and $x$ is shown to be prime in $L$. There exists $p \in CP(L)$ such that $x \leq p$ and thus $x$ is coronal. Hence, $CP(L) - \gamma(x) = cl\{x\}$.

2.3 Definition. If $f: L \to M$ is a star lattice homomorphism, define $CP(f): CP(M) \to CP(L)$ by $CP(f)(p) = g(p)$ where $g$ is the upper adjoint of $f$.

$CP(f): CP(M) \to CP(L)$ is a continuous function. $CP(1_L) = 1_{CP(L)}$ and $CP(gf) = CP(f)CP(g)$. Hence, $CP$ is a contravariant functor from the category of star lattices to the category of sober topological spaces. It will be shown that $CP$ and $LC$ are adjoint on the right.

2.4 Definition. For a topological space $X$, define $\alpha_X: X \to CP(LC(X))$ by $\alpha_x(x)(y) = 0$ if $y \in cl_X\{x\}$ and $1$ otherwise.

$\alpha_X: X \to CP(LC(X))$ is a continuous function which is one-one iff $X$ is $T_0$. If $f: X \to Y$ is a continuous function, then $CP(LC(f))\alpha_X = \alpha_Y f$. Hence, $\alpha: 1 \to CPLC$ is a natural transformation. Note that $f \leq s \circ \alpha_X(x)$ in $LC(X)$ iff $f(x) \leq s$ in $[0,1]$.

2.5 <u>Definition</u>. For a star lattice $L$, define $\beta_L\colon L \to LC(CP(L))$ by $\beta_L(x)(p) = \bigwedge\{s \mid x \le s \circ p\}$ for all $x \in L$ and $p \in CP(L)$.

It follows that $x \le (\beta_L(x)(p)) \circ p$ for all $x \in L$ and $p \in CP(L)$. Thus, to see that $\beta_L(x)\colon CP(L) \to [0,1]$ is lower-semicontinuous, let $0 \neq s \in [0,1]$ such that $s < \beta_L(x)(p)$. Then $x \nleq s \circ p$. Let $y = \bigwedge\{z \mid x \le s \circ z\}$. Then $x \le s \circ y$, $p \in \gamma(y)$, and $\beta_L(x)(\gamma(y)) \le s$. Therefore, $\beta_L(x)$ is lower semicontinuous.

$\beta_L\colon L \to LC(CP(L))$ is a framed function. Since $\beta_L(s \circ x)(p) = \bigwedge\{t \mid s \circ x \le t \circ p\}$ and $(s \circ \beta_L(x))(p) = s \circ \bigwedge\{r \mid x \le r \circ p\} = \bigwedge\{s \circ r \mid x \le r \circ p\}$, it follows that $\beta_L$ is a star lattice homomorphism.

$\beta_L$ is always one-one and $\beta_L(s \circ 0)(p) = s$ for all $s \in [0,1]$ and $p \in CP(L)$. If $f\colon L \to M$ is a star lattice homomorphism, then $LC(CP(f))\beta_L = \beta_M f$. Hence, $\beta\colon 1 \to LCCP$ is a natural transformation.

Note that the upper adjoint of $\beta_L$ is given by $\phi(f) = \bigwedge\{f(p) \circ p \mid p \in CP(L)\}$ for all $f \in LC(CP(L))$.

2.6 <u>Proposition</u>. 1) If $X$ is a topological space, then $LC(\alpha_X)\beta_{LC(X)} = 1_{LC(X)}$.

2) If $L$ is a star lattice, then $CP(\beta_L)\alpha_{CP(L)} = 1_{CP(L)}$.

pf. 1) Let $f \in LC(X)$ and $x \in X$. Then

$LC(\alpha_X)\beta_{LC(X)}(f)(x) = \beta_{LC(X)}(f)(\alpha_X(x)) = \bigwedge\{s \mid f \le s \circ \alpha_X(x)\}$
$= \bigwedge\{s \mid f(x) \le s\} = f(x).$

2) Let $p \in CP(L)$. Then $CP(\beta_L)\alpha_{CP(L)}(p) =$
$\bigwedge\{(\alpha_{CP(L)}(p)(q)) \circ q \mid q \in CP(L)\} = \bigwedge\{0 \circ q \mid q \in cl\{p\}\} =$
$\bigwedge\{q \mid p \le q\} = p.$

The following theorem summarizes these results.

2.7 Theorem. The functors LC and CP are adjoint on the right with natural transformations $\alpha\colon 1 \to CPLC$ and $\beta\colon 1 \to LCCP$.

A topological space X is sober iff $\alpha_X\colon X \to CP(LC(X))$ is a homeomorphism. In the next section there is a characterization of those star lattices L for which $\beta_L\colon L \to LC(CP(L))$ is a star lattice isomorphism. These will then form a category dual to the category of sober spaces.

## Section 3. Fat Star Lattices

Hofmann and Lawson [16] showed that the category of locally-compact sober spaces is dual to the category of distributive continuous lattices, given by the lattice of open sets functor and the prime spectrum functor. Here, we obtain a duality between sober spaces and fat star lattices. By restriction, this will give us a duality of locally-compact sober spaces and continuous fat star lattices.

The following elements are needed to define fat star lattices.

3.1 <u>Definition</u>. A <u>characteristic element</u> $x$ of a star lattice $L$ is an element such that $\uparrow x$ is extensive in $L$.

That is, $x$ is characteristic iff whenever $x \le s \circ y$ and $s \ne 1$, then $x \le y$.

$f$ is a characteristic element in $LC(X)$ iff $f$ is the characteristic function of an open set in $X$.

In a star lattice $L$,

1) 0 and 1 are characteristic elements.
2) the join and finite meet of characteristic elements is characteristic.
3) every characteristic element is the meet of coronal

primes.

The converse of 3) is the basis for the next definition.

3.2 Definition. A star lattice L is fat iff the meet of coronal primes in L is characteristic.

For example, [0,1] is fat as well as the diamond lattice and the half-diamond lattice. However, the necktie lattice is not fat. Also, if X is a topological space, then LC(X) is fat.

3.3 Theorem. A star lattice L is fat iff $\beta_L$: L $\to$ LC(CP(L)) is a star isomorphism.

pf. Suppose L is fat. All that must be shown is that $\beta_L$ is onto (that is, L must be big enough to fill LC(CP(L)). This motivates the name).

As is proven in the introduction, LC(X) for any topological space, is generated as a frame by the constant functions and the characteristic functions. Therefore, it suffices to show that the image of $\beta_L$ contains the constant functions and the characteristic functions.

$\beta_L(s \circ 0)(p) = s$ and so the image of $\beta_L$ contains the constants.

Given an open set $\gamma(x)$ in CP(L), suppose, without loss of generality, that x is the meet of coronal primes. Hence, $x$ is characteristic and this implies $\beta_L(x) = \chi_Y(x)$.

Therefore, $\beta_L$ is a star isomorphism.

The converse is clear.

As examples of this theorem, the diamond lattice is $LC(Z)$ where $Z$ is the two-point discrete space and the half-diamond lattice is $LC(S)$ where $S$ is the Sierpinski space.

Therefore, from this theorem and by preceeding results, we have the following duality.

3.4 Theorem. The category of sober spaces is dual to the category of fat star lattices with the duality given by the functors $LC$ and $CP$ and with natural equivalences $\alpha$ and $\beta$.

It is known that the category of sober spaces is dual to the category of complete lattices with enough primes and framed functions ([26],[21]). In one direction the duality is given by the lattice of open sets functor $O(X)$. This can be viewed as taking all continuous functions from $X$ to $\{0,1\}$ with the Scott-topology. Here, our duality is given by the functor $LC(X)$ which can be viewed as taking all continuous functions from $X$ to $[0,1]$ with the Scott-topology. These two dualities can thus be seen to be similar.

Our duality will now be restricted to give a duality for continuous fat star lattices.

3.5 Proposition. If L is a continuous fat star lattice, then CP(L) is locally-compact.

pf. Let M be the set of characteristic elements of L with the order induced by L. M is closed under arbitrary joins and meets. Hence, M is a distributive continuous lattice. Since $O(CP(L)) \cong M$, it follows by the duality given by Hofmann and Lawson [16] that CP(L) is homeomorphic to Spec(M) and that Spec(M) is locally-compact.

Therefore we have the following.

3.6 Theorem. The category of continuous fat star lattices is dual to the category of locally-compact sober spaces where the duality is given by restriction of the duality in 3.4.

## Section 4. Strong Star Lattices

In this section we introduce a special type of star lattice. They will strongly rely upon the theory of continuous lattices for their properties.

4.1 **Definition.** A *strong star lattice* is a distributive continuous lattice $L$ that is also a star structure such that

1) $\circ: [0,1] \times L \to L$ is a continuous function when $L$ has the Lawson-topology and $[0,1]$ has the usual topology.
2) $s \circ (x \wedge y) = (s \circ x) \wedge (s \circ y)$ for all $s \in [0,1]$ and $x,y \in L$.
3) whenever $s \leq t$ in $[0,1]$ and $x \in L$, then $s \circ x \leq t \circ x$.
4) the set of primes together with one is a Lawson-closed star shaped subset of $L$.
5) the set of coronal primes in $L$ is a down-set.

Every strong star lattice is a star lattice. $[0,1]$, the diamond lattice, half-diamond lattice, and the necktie lattice are strong star lattices. Also, we have the following.

4.2 **Proposition.** If $X$ is a compact star structure, then $St(X)$ is a strong star lattice where $s \circ C = \{s \circ x \mid x \in C\}$ for all $s \in [0,1]$ and $C \in St(X)$.

pf. $St(X)$ is already known to be a distributive continuous lattice. Also, $\circ: [0,1] \times St(X) \to St(X)$ is a well-defined function that makes $St(X)$ into a star lattice.

Since $St(X)$ is a Lawson-closed meet-subsemilattice of $\Gamma(X)$, it follows that $\{m^*(U) \mid U$ is open in $X\} \cup \{c^*(U) \mid U$ is open in $X\}$ is a subbase for the Lawson-topology on $St(X)$ where $m^*(U) = \{C \in St(X) \mid C \cap U \neq \emptyset\}$ and $c^*(U) = \{C \in St(X) \mid C \subseteq U\}$.

If $s \circ C \in m^*(U)$, then let $x \in C$ such that $s \circ x \in U$. Let $V$ be open in $[0,1]$ and $W$ open in $X$ such that $s \in V$, $x \in W$, and $v \circ w \in U$ for all $v \in V$ and $w \in W$. Then $v \circ D \in m^*(U)$ for all $v \in U$ and $D \in m^*(W)$.

If $s \circ C \in c^*(U)$, then, by compactness, let $V$ be open in $[0,1]$ and $W$ open in $X$ such that $s \in V$, $C \subseteq W$, and $t \circ w \in U$ for all $t \in V$ and $w \in W$. Then $t \circ D \in c^*(U)$ for all $t \in V$ and $D \in c^*(W)$.

Therefore, $\circ: [0,1] \times St(X) \to St(X)$ is continuous.

Requirements 2) and 3) of the definition are clearly true here. Since $j: X \to St(X)$ defined by $j(x) = sh(x)$ is continuous, requirement 4) is true here.

If $C$ is coronal in $St(X)$ and $C \subseteq D \in St(X)$, then suppose that $D = s \bullet E$ for some $s \in [0,1]$ and $E \in St(X)$. Let $F = \{x \in E \mid s \circ x \in C\}$. Then $F \in St(X)$

and $s \circ F = C$. Hence $s = 0$ and $D$ is coronal in $St(X)$. Therefore, requirement 5) is true.

Furthermore, if $X$ is a compact star structure, then

1) $C$ is a coronal prime in $St(X)$ iff $C = sh(c)$ for some coronal element $c$ in $X$.
2) $C$ is a characteristic element in $St(X)$ iff $C$ is an extensive subset of $X$.

As soon as we know the morphisms, $St$ will be a covariant functor.

4.3 Definition. A <u>strong star lattice homomorphism</u> is an algebraically-continuous star structure homomorphism that preserves primes and one (that is, the image of a prime is either a prime or one).

Notice that these morphisms are quite different from star lattice homomorphisms.

The identity function on a strong star lattice is a strong star lattice homomorphism and the composition of strong star lattice homomorphisms is a strong star lattice homomorphism. Therefore, there is a category of strong star lattices. However, this category is not a subcategory of the category of star lattices.

If $f: X \to Y$ is a continuous star structure homomorphism, then St(f): St(X) → St(Y) is a strong star lattice homomorphism. Hence, St is a covariant functor from the category of compact star structures to the category of strong star lattices.

A star lattice isomorphism between strong star lattices is a strong star lattice homomorphism. Star lattice isomorphisms between strong star lattices are the isomorphisms in the category of strong star lattices.

## Section 5. Star Structure of Primes

In this section it will be shown that the functor St has a right adjoint and, by suitable restriction of co-domain, that St is an equivalence of categories.

5.1 Definition. Define the star structure of primes of a strong star lattice $L$, denoted by $Pr(L)$, to be the set of all primes together with one in $L$ where

1) the topology on $Pr(L)$ is the topology induced by the Lawson-topology on $L$.
2) $\circ: [0,1] \times Pr(L) \to Pr(L)$ is defined to be the restriction of $\circ: [0,1] \times L \to L$.

$Pr(L)$ is a compact star structure.

5.2 Definition. If $f: L \to M$ is a strong star lattice homomorphism, then define $Pr(f): Pr(L) \to Pr(M)$ by $Pr(f)(p) = f(p)$ for all $p \in Pr(L)$.

$Pr(f): Pr(L) \to Pr(M)$ is a continuous star structure homomorphism. $Pr(1_L) = 1_{Pr(L)}$ and $Pr(gf) = Pr(g)Pr(f)$. Hence, $Pr$ is a covariant functor from the category of strong star lattices to the category of compact star structures. It will be shown that this functor is a right adjoint to St.

5.3 Definition. For a compact star structure $X$, define $\mu_X: X \to Pr(St(X))$ by $\mu_X(x) = sh(x)$ for all $x \in X$.

$\mu_X: X \to Pr(St(X))$ is a continuous star structure isomorphism. If $f: X \to Y$ is a continuous star structure homomorphism, then $Pr(St(f))\mu_X = \mu_Y f$. Hence, $\mu: 1 \to PrSt$ is a natural equivalence.

5.4 Definition. For a strong star lattice $L$, define $\nu_L: St(Pr(L)) \to L$ by $\nu_L(C) = \bigwedge C$ for all $C \in St(Pr(L))$.

$\nu_L: St(Pr(L)) \to L$ is a strong star lattice homomorphism with lower adjoint $\phi$ defined by $\phi(x) = \{p \in Pr(L) \mid x \le p\}$. If $f: L \to M$ is a strong star lattice homomorphism, then $f\nu_L = \nu_M St(Pr(f))$. Hence, $\nu: StPr \to 1$ is a natural transformation.

5.5 Proposition. 1) If $X$ is a compact star structure, then $\nu_{St(X)} St(\mu_X) = 1_{St(X)}$.

2) If $L$ is a strong star lattice, then $Pr(\nu_L)\mu_{Pr(L)} = 1_{Pr(L)}$.

pf. 1) Let $C \in St(X)$. Then $\nu_{St(X)} St(\mu_X)(C) = \nu_{St(X)}(\{\mu_X(x) \mid x \in C\}) = \bigwedge\{sh(x) \mid x \in C\} = C$.

2) Let $p \in Pr(L)$. Then $Pr(\nu_L)\mu_{Pr(L)}(p) = Pr(\nu_L)(sh(p)) = \nu_L(sh(p)) = \bigwedge sh(p) = p$.

These results are summarized in the following theorem.

5.6 Theorem. The functor Pr is a right adjoint to the functor St with natural transformations $\mu: 1 \to PrSt$ and $\nu: StPr \to 1$.

Moreover, $\mu$ is a natural equivalence. Those strong star lattices $L$ for which $\nu_L$ is a star lattice isomorphism are characterized in the next section.

## Section 6. Thin Star Lattices

Here, we characterize those strong star lattices which are star isomorphic to St(X) for some compact star structure X. In doing so we will determine a category equivalent with the category of compact star structures.

6.1 Definition. A thin star lattice is a strong star lattice L such that whenever p is a prime in L and $p \leq x$, then $x = s \circ p$ for some $s \in [0,1]$.

In other words, $\uparrow p = sh(p)$ for all primes p. Since $sh(p) \subseteq \uparrow p$ always, the definition requires that $\uparrow p$ be "thin"; hence the name.

[0,1] and the diamond lattice are thin star lattices. However, the half-diamond lattice and the necktie lattice are not thin star lattices. If X is a compact star structure, then St(X) is a thin star lattice. Moreover, we have the following theorem.

6.2 Theorem. A strong star lattice L is thin iff $\nu_L$: St(Pr(L)) $\to$ L is a star lattice isomorphism.

pf. Let L be thin and let $C \in$ St(Pr(L)). If $p \in Pr(L)$ and $\bigwedge C \leq p$, then, by the Gierz and Keimel lemma, there exists $q \in C$ such that $q \leq p$. Let $s \in [0,1]$

such that $p = s \circ q$. Hence, because $C$ is star shaped, $p \in C$. This proves that $C = \{p \in Pr(L) \mid \bigwedge C \leq p\}$. Hence, $\nu_L$ is one-one. Therefore, $\nu_L$ is a star lattice isomorphism.

The converse is clear.

In the language of category theory, this theorem says the following.

6.3 <u>Theorem</u>. The category of thin star lattices (as a full subcategory of the category of strong star lattices) is equivalent to the category of compact star structures. The equivalence is given by the functors St and Pr with natural equivalences $\mu$ and $\nu$.

## Section 7. Star Lattice Again

The functor St goes into the category of strong star lattices and we know that the morphisms in this category differ from the morphisms in the category of star lattices. However, by changing the morphisms in the domain of St, it will be shown that St is a contravariant functor into the category of star lattices.

7.1 Definition. If $f: X \to Y$ is a continuous extensive star structure homomorphism, then define $St^0(f): St(Y) \to St(X)$ by $St^0(f)(C) = f^{-1}(C)$ for all $C \in St(Y)$.

Since $f: X \to Y$ is extensive, $St^0(f): St(Y) \to St(X)$ is a star lattice homomorphism. By defining $St^0(X) = St(X)$ for all compact star structures X, it follows that $St^0$ is a contravariant functor from the category of compact star structures and continuous extensive star structure homomorphisms to the category of star lattices. Even though St(X) is always thin, it may occur that St(X) is also fat; an apparent contradiction.

7.2 Theorem. A compact star structure X is solid iff St(X) is fat.

pf. Let X be solid and let A be a set of coronal elements of $X$. Then $\bigwedge\{sh(c) \mid c \in A\} = sh(cl_X A) = sh(cl_X A - \{x_0\})$. Since $cl_X A - \{x_0\}$ is a set of coronal elements in $X$, $sh(cl_X A - \{x_0\})$ is extensive in $X$ and hence characteristic in $St(X)$. Therefore, $St(X)$ is fat.

Conversely, let $St(X)$ be fat and let $A = \{c \mid c$ is a coronal element in $X\} \cup \{x_0\}$. Suppose that $x \in cl_X A$ but $x \notin A$. Let $c$ be a coronal element of $X$ such that $x = s \circ c$ for some $s \in [0,1]$. Then $s \neq 1$ and $x \neq c$. Let $U$ be a neighborhood of $x$ and $V$ a neighborhood of $c$ such that $U \cap V = \emptyset$ and let $B = A - V$. Then $x \in cl_X B$ but $c \notin cl_X B$. Since $sh(cl_X B) = \bigwedge\{sh(b) \mid b \in B\}$, it follows that $sh(cl_X B)$ is extensive in $X$. However, $s \circ c \in sh(cl_X B)$, $s \neq 1$, and $c \notin sh(cl_X B)$; a contradiction. Hence, $X$ is solid.

Since $St^0$ takes us into the category of star lattices, we can follow it with the functor CP. The resulting functor is already known to us.

7.3 <u>Definition</u>. For a compact star structure $X$, define $\tau_X$: $Cor(X) \to CP(St(X))$ by $\tau_X(c) = sh(c)$ for all $c \in Cor(X)$.

$\tau_X$: $Cor(X) \to CP(St(X))$ is a homeomorphism. If $f$: $X \to Y$ is a continuous extensive star structure homomorphism, then $CP(St^0(f))\tau_X = \tau_Y(Cor(f)$. Therefore,

$\tau$: Cor $\to$ CPSt is a natural equivalence. By applying previous results we obtain the following equivalences.

7.4 Theorem. The following pairs of functors are naturally equivalent.

1) Cor and $CPSt^0$.
2) LCCor and $St^0$ for solid compact star structures and continuous extensive star structure homomorphisms.
3) $St_J^0 \Delta_0$ and LC for locally-compact Hausdorff spaces and proper continuous functions.

Therefore, if X is a locally-compact Hausdorff space, LC(X) is a thin star lattice.

## CHAPTER 4. CONVEX CONTINUOUS LATTICES

### Section 1. Convex Sets

Most everything in this section is not new but is included for background purposes. However 1.3 Theorem is new and will be quite important in the theory of convex continuous lattices.

1.1 Definition. A semiconvex set is an ordered pair $(K,\langle\rangle)$ where $K$ is a set and $\langle\rangle: [0,1] \times K \times K \to K$ is a function (sending $(s,x,y)$ to $\langle s,x,y\rangle$) such that

1) $\langle s,x,x\rangle = x$ for all $x \in K$.
2) $\langle 0,x,y\rangle = x$ for all $x,y \in K$.
3) $\langle s,x,y\rangle = \langle 1 - s,y,x\rangle$ for all $s \in [0,1]$ and $x,y \in K$.
4) $\langle s,\langle t,x,y\rangle,z\rangle = \langle s \circ t, x,\langle s/s \circ t, y,z\rangle\rangle$ for all $s,t \in (0,1)$ and $x,y,z \in K$.

A semiconvex set $(K,\langle\rangle)$ will most often be denoted simply by $K$ and $\langle\rangle$ will be referred to as the convexity structure on $K$.

In a semiconvex set $K$,

1) $\langle 1,x,y\rangle = y$ for all $x,y \in K$.
2) $\langle s,x,\langle t,y,z\rangle\rangle = \langle st,\langle s(1 - t)/(1 - st),x,y\rangle,z\rangle$ for all $s,t \in (0,1)$ and $x,y,z \in K$.
3) $\langle s,\langle t,x,y\rangle,\langle t,z,w\rangle\rangle = \langle t,\langle s,x,z\rangle,\langle s,y,w\rangle\rangle$ for all

$s,t \in [0,1]$ and $x,y,z,w \in K$.

We are more interested in convex sets however.

1.2 <u>Definition</u>. A <u>convex</u> <u>set</u> is a semiconvex set $K$ such that $\langle s,x,y\rangle \neq \langle s,x,z\rangle$ whenever $s \neq 0$ and $y \neq z$.

The following theorem is very useful for telling us when a semiconvex set is convex.

1.3 <u>Theorem</u>. A semiconvex set $K$ is a convex set iff for all $x,y,z \in K$, $\{s \mid \langle s,x,y\rangle = \langle s,x,z\rangle\}$ is closed in $[0,1]$.

pf. If $K$ is a convex set, then $\{s \mid \langle s,x,y\rangle = \langle s,x,z\rangle\}$ equals $\{0\}$ if $y \neq z$ and equals $[0,1]$ if $y = z$.

Conversely, suppose that $\langle s,x,y\rangle = \langle s,x,z\rangle$ where $s \neq 0$ and $y \neq z$. Let $t$ be the greatest element of $\{s \mid \langle s,x,y\rangle = \langle s,x,z\rangle\}$. Then $0 < t < 1$. Let $r = t/(t+1)$. Then $0 < r < 1$, $t < t \circ r$, and $r/(t \circ r) = 1/2$.

Hence, $\langle r \circ t, x,y\rangle = \langle r,\langle t,x,y\rangle,y\rangle = \langle r,\langle t,x,z\rangle,y\rangle = \langle r \circ t, x,\langle 1/2,z,y\rangle\rangle = \langle r \circ t, x,\langle 1/2,y,z\rangle\rangle = \langle r,\langle t,x,y\rangle,z\rangle = \langle r,\langle t,x,z\rangle,z\rangle = \langle r \circ t, x,z\rangle$; a contradiction.

We now conclude this section with background material needed in further sections. Much of this information can be found in Gudder [13] and Świrszcz [24].

1.4 <u>Definition</u>. A subset $C$ of a semiconvex set $K$ is <u>convex</u> iff whenever $x,y \in C$, then $\langle s,x,y\rangle \in C$ for all $s \in [0,1]$.

$K$ and $\emptyset$ are convex subsets of $K$ and the intersection of convex subsets of $K$ is a convex subset of $K$.

1.5 <u>Definition</u>. Define the <u>convex hull</u> of a subset $A$ of a semiconvex set $K$, denoted by $\text{con}_K A$, to be $\cap\{C \mid C$ is a convex subset of $K$ and $A \subseteq C\}$.

$\text{con}_K A$ is the smallest convex subset of $K$ containing $A$. If $C$ and $D$ are nonempty convex subsets of $K$, then $\text{con}_K(C \cup D) = \{\langle s,x,y\rangle \mid s \in [0,1],\ x \in C,$ and $y \in D\}$.

1.6 <u>Definition</u>. An <u>affine function</u> is a function $f: K \to L$ between semiconvex sets such that $f(\langle s,x,y\rangle) = \langle s,f(x),f(y)\rangle$ for all $s \in [0,1]$ and $x,y \in K$.

The identity function on a semiconvex set is affine and the composition of affine functions is affine. If $f: K \to L$ is an affine function and $A \subseteq K$, then

$f(\mathrm{con}_K A) = \mathrm{con}_L f(A)$. Furthermore, if $C$ is convex in $K$ (resp. $L$), then $f(C)$ (resp. $f^{-1}(C)$) is convex in $L$ (resp. $K$).

1.7 <u>Definition</u>. An element $e$ of a semiconvex set $K$ is an <u>extreme point</u> iff whenever $e = \langle s,x,y \rangle$, then $e = x$ or $e = y$.

$e$ is an extreme point of $K$ iff $K - \{e\}$ is convex.

1.8 <u>Definition</u>. A subset $F$ of a semiconvex set $K$ is a <u>face</u> iff $F$ is a convex subset and whenever $\langle s,x,y \rangle \in F$ for $0 < s < 1$, then $x,y \in F$.

$K$ and $\emptyset$ are faces of $K$ and the intersection of faces is a face. $e$ is an extreme point iff $\{e\}$ is a face.

Note that extreme points and faces in semiconvex sets are similar to coronal elements and extensive star shaped subsets in star structures.

## Section 2. Compact Convex Sets

Again, the material in this section is background information. However, the definition of compact convex set appears weaker than usual but in fact by 1.3 will be equivalent to the usual definition.

2.1 Definition. A compact convex set is a semiconvex set $K$ that is also a compact Hausdorff space such that $\langle\rangle\colon [0,1] \times K \times K \to K$ is continuous (where $[0,1]$ has the usual topology).

Although a compact convex set is only defined to be a semiconvex set, the next proposition tells us that compact convex sets are true to their name.

2.2 Proposition. Every compact convex set $K$ is a convex set.

pf. It is clear that for all $x,y,z \in K$, $\{s \mid \langle s,x,y\rangle = \langle s,x,z\rangle\}$ is closed in $[0,1]$. Hence, by 1.3, $K$ is a convex set.

This apparent weakening of the definition will be useful in the theory of convex continuous lattices.

Every compact convex subset of a Hausdorff topological vector space is a compact convex set. Conversely, Lawson and Madison [19] have shown that every compact convex set is affinely homeomorphic to a compact convex subset of a Hausdorff topological vector space.

If $K$ is a compact convex set and $C$ is a convex subset of $K$, then $cl_K C$ is convex.

2.3 Definition. Define the closed convex hull of a subset $A$ of a compact convex set $K$, denoted by $\overline{con}_K A$ to be $cl_K(con_K A)$.

Hence, $\overline{con}_K A$ is the smallest closed convex subset of $K$ containing $A$. If $f\colon K \to L$ is a continuous affine function, then $f(\overline{con}_K A) = \overline{con}_L f(A)$ for all $A \subseteq K$. Note that for closed convex subsets $C$ and $D$ of $K$, $con_K(C \cup D)$ is already closed.

## Section 3. Lattice of Closed Convex Sets

In this section, we consider the lattice of closed subsets of a compact convex set. This construction was used by Lawson to construct a compact semilattice that is not a continuous lattice.

3.1 Definition. Define the lattice of closed convex subsets of a compact convex set $K$, denoted by $\mathrm{Con}(K)$, to be the set of all closed convex subsets of $K$ ordered by reverse-inclusion and with the topology induced by the Lawson-topology on $\Gamma(K)$.

$\mathrm{Con}(K)$ is a compact semilattice where for $C, D \in \mathrm{Con}(K)$, $C \wedge D = \mathrm{con}_K(C \cup D)$ [11].

3.2 Definition. If $f: K \to L$ is a continuous affine function, then define $\mathrm{Con}(f): \mathrm{Con}(K) \to \mathrm{Con}(L)$ by $\mathrm{Con}(f)(C) = f(C)$ for all $C \in \mathrm{Con}(K)$.

$\mathrm{Con}(f): \mathrm{Con}(K) \to \mathrm{Con}(L)$ is a continuous semilattice homomorphism. $\mathrm{Con}(1_K) = 1_{\mathrm{Con}(K)}$ and $\mathrm{Con}(gf) = \mathrm{Con}(g)\mathrm{Con}(f)$. Hence, Con is a covariant functor from the category of compact convex sets to the category of compact semilattices.

## Section 4. Locally-Convex Compact Convex Sets

In this section we consider those compact convex sets for which Con(K) is continuous.

4.1 Definition. A compact convex set K is locally-convex iff every element of K has a neighborhood base of convex sets (not necessarily open).

Every compact convex subset of a locally-convex topological vector space is locally-convex. Conversely, Lawson [19] (see also [18]) has shown that every locally-convex compact convex set is affinely embeddable in a locally-convex topological vector space.

4.2 Theorem. A compact convex set K is locally-convex iff Con(K) is a continuous lattice.

pf. If K is locally-convex, then K may be taken to be a compact convex subset of a locally-convex topological vector space. Therefore, K is augmented by its closed convex subsets and, by the theory of augmented spaces, Con(K) is continuous.

Conversely, suppose that Con(K) is continuous. Let $C \ll D$ in Con(K). Then $\uparrow D \subseteq \hat{\uparrow} C$ in Con(K) and since $j: K \to \mathrm{Con}(K)$ defined by $j(x) = \{x\}$ is continuous when

Con(K) has the Lawson-topology, $D \subseteq j^{-1}(\uparrow D) \subseteq j^{-1}(\hat{\uparrow}C) \subseteq \text{int}_K C$. This proves $C \ll D$ in Con(K) iff $D \subseteq \text{int}_K C$ in K. Hence, K is augmented by its closed convex subsets and therefore locally-convex.

If K is locally-convex, then the meet-irreducibles of Con(K) are the singletons in K and the primes in Con(K) are the singletons of extreme points in K.

## Section 5. Convex Continuous Lattices

Here we introduce the idea of a convex continuous lattice. The motivation for this idea arises from the fact that Con(K), for K locally-convex, is not only a continuous lattice but also inherits a convexity structure from K.

5.1 Definition. A convex continuous lattice is a continuous lattice L such that

1) $L - \{1\}$ is a locally-convex compact convex set with the topology induced by the Lawson-topology on L.
2) $\langle s, x \wedge y, z\rangle = \langle s,x,z\rangle \wedge \langle s,y,z\rangle$ for all $s \in [0,1]$ and $x,y,z \in L - \{1\}$.
3) the set of meet-irreducibles in L is a closed convex subset of $L - \{1\}$.

The SEM lattice, the diaper lattice, the green tetrahedron, and the green diamond lattice are all examples of convex continuous lattices. Also, we have the following important examples.

5.2 Theorem. If K is a locally-convex compact convex set, then Con(K) is a convex continuous lattice where $\langle s,C,D\rangle = \{\langle s,x,y\rangle \mid x \in C \text{ and } y \in D\}$ for all $C,D \in \mathrm{Con}(K) - \{\emptyset\}$ and $s \in [0,1]$.

pf. $Con(K) - \{\emptyset\}$ with the convexity structure described is clearly a semiconvex set. There doesn't seem to be a direct way to prove that it is a convex set. Therefore, we will use the fact that semiconvexity suffices as given in 2.1 and 2.2.

It follows directly that $\langle\rangle$ is continuous. Therefore, $Con(K) - \{\emptyset\}$ is a compact convex set. It also follows directly that $Con(K) - \{\emptyset\}$ is locally-convex.

Let $C,D,E \in Con(K) - \{\emptyset\}$ and let $s \in [0,1]$. Then $\langle s, C \wedge D, E\rangle = \{\langle s,\langle t,x,y\rangle,z\rangle \mid t \in [0,1],\ x \in C,\ y \in D,$ and $z \in E\} = \{\langle t,\langle s,x,z\rangle,\langle s,y,w\rangle\rangle \mid t \in [0,1],\ x \in C,$ $y \in D,$ and $z,w \in E\} = \langle s,C,E\rangle \wedge \langle s,D,E\rangle$.

Because the set of meet-irreducibles in $Con(K)$ is $\{\{x\} \mid x \in K\}$, this set is closed and convex in $Con(K) - \{\emptyset\}$.

Those convex continuous lattices isomorphic to $Con(K)$ for some $K$ will be characterized later on.

If $L$ is a convex continuous lattice, then

1) $x \wedge y \leq \langle s,x,y\rangle$ for all $s \in [0,1]$ and $x,y \in L$.
2) $\langle s,\bigwedge A,y\rangle = \bigwedge\{\langle s,x,y\rangle \mid x \in A\}$ for all $s \in [0,1]$, $\emptyset \neq A \subseteq L - \{1\}$, and $y \in L - \{1\}$.
3) for all filters $F$ in $L$, $F - \{1\}$ is a convex subset of $L - \{1\}$.
4) the set of meet-irreducibles in $L$ is a face of $L - \{1\}$.

5) for all $A \subseteq L - \{1\}$, $\bigwedge A = \bigwedge(\overline{\text{con}A})$.

5.3 Definition. A convex lattice homomorphism is an algebraically-continuous function $f: L \to M$, between convex continuous lattices, that preserves meet-irreducibles and that is affine when restricted to $L - \{1\}$.

The identity function on a convex continuous lattice is a convex lattice homomorphism and the composition of convex lattice homomorphisms is a convex lattice homomorphism. Therefore, there exists a category of convex continuous lattices and convex lattice homomorphisms. If $f: K \to L$ is a continuous affine function between locally-convex compact convex sets, then $\text{Con}(f): \text{Con}(K) \to \text{Con}(L)$ is a convex lattice homomorphism. Therefore, Con is a covariant functor from the category of locally-convex compact convex sets to the category of convex continuous lattices.

## Section 6. Convex Set of Meet-Irreducibles

The convex set of meet-irreducibles allows us to recapture K from Con(K). Conversely, for certain convex continuous lattices L, Con recreates L from its convex set of meet-irreducibles.

6.1 Definition. Define the convex set of meet-irreducibles of a convex continuous lattice L, denoted by Mi(L), to be the set of meet-irreducibles in L with the topology and convexity structure induced by L − {1}.

Mi(L) is a locally-convex compact convex set. For example, [0,1] with its usual topology and convexity structure is Mi(L) where L is the SEM lattice or the diaper lattice.

6.2 Definition. If $f: L \to M$ is a convex lattice homomorphisms, then define $Mi(f): Mi(L) \to Mi(M)$ by $Mi(f)(p) = f(p)$ for all $p \in Mi(L)$.

$Mi(f): Mi(L) \to Mi(M)$ is a continuous affine function. $Mi(1_L) = 1_{Mi(L)}$ and $Mi(gf) = Mi(g)Mi(f)$. Hence, Mi is a covariant functor from the category of convex continuous lattices to the category of locally-convex compact convex sets. It will be shown that this functor is a right adjoint to Con and then by restriction, gives an equivalence of categories.

6.3 <u>Definition</u>. For a locally-convex compact convex set $K$, define $\chi_K: K \to Mi(Con(K))$ by $\chi_K(x) = \{x\}$ for all $x \in K$.

$\chi_K: K \to Mi(Con(K))$ is an affine homeomorphism. If $f: K \to L$ is a continuous affine function, then $Mi(Con(f))\chi_K = \chi_L f$. Hence, $\chi: 1 \to MiCon$ is a natural equivalence.

6.4 <u>Definition</u>. For a convex continuous lattice $L$, define $\kappa_L: Con(Mi(L)) \to L$ by $\kappa_L(C) = \bigwedge C$ for all $C \in Con(Mi(L))$.

$\kappa_L: Con(Mi(L)) \to L$ is a convex lattice homomorphism. If $f: L \to M$ is a convex lattice homomorphism, then $\kappa_M Con(Mi(f)) = f\kappa_L$. Hence, $\kappa: ConMi \to 1$ is natural.

6.5 <u>Proposition</u>. 1) If $K$ is a locally-convex compact convex set, then $\kappa_{Con(K)} Con(\chi_K) = 1_{Con(K)}$.

2) If $L$ is a convex continuous lattice, then $Mi(\kappa_L)\chi_{Mi(L)} = 1_{Mi(L)}$.

pf. 1) Let $C \in Con(K)$. Then $\kappa_{Con(K)} Con(\chi_K)(C) = \kappa_{Con(K)}(\{\{x\} \mid x \in C\}) = \bigwedge\{\{x\} \mid x \in C\} = C$.

2) Let $p \in Mi(L)$. Then $Mi(\kappa_L)\chi_{Mi(L)}(p) = Mi(\kappa_L)(\{p\}) = \bigwedge\{p\} = p$.

These results are summarized in the following theorem.

6.6 Theorem. The functor Mi is a right adjoint to the functor Con. The adjunction is given by the natural transformations $\chi: 1 \to MiCon$ and $\kappa: ConMi \to 1$.

We will now characterize those convex continuous lattices isomorphic to some Con(K) and in doing so, demonstrate an equivalence of categories.

6.7 Definition. A convex continuous lattice L is hard iff whenever C is a closed convex set of meet-irreducibles and p is a meet-irreducible such that $\bigwedge C \leq p$, then $p \in C$.

For example, the diaper lattice is hard. However, neither the SEM lattice, green tetrahedron, or green diamond lattice is hard.

6.8 Proposition. Con(K) is hard for all locally-convex compact convex sets K.

pf. Let C be a closed convex set of meet-irreducibles in Con(K). Then $\bigwedge C = \{x \in K \mid \{x\} \in C\}$. Therefore, if $\bigwedge C \leq \{y\}$, then $\{y\} \in C$.

6.9 Theorem. A convex continuous lattice $L$ is hard iff $\kappa_L$: Con(Mi(L)) $\to$ L is a convex lattice isomorphism.

pf. If $L$ is hard, then $\kappa_L$ is one-one. Hence, $\kappa_L$ is a convex lattice isomorphism.

The converse is clear.

These results are summarized in the following theorem.

6.10 Theorem. The category of locally-convex compact convex sets is equivalent to the category of hard convex continuous lattices. The equivalence is given by the functors Con and Mi.

## Section 7. Primitive Lattices

In this section we consider interpretations and applications of the Kreĭn-Mil'man theorem in the theory of convex continuous lattices.

7.1 Definition. A complete lattice $L$ is primitive iff the meet of all primes in $L$ is $0$.

For example, every complete lattice with enough primes is primitive and so every distributive continuous lattice is primitive. Even though $\mathrm{Con}(K)$ is never distributive (except in the trivial case where $K$ is a singleton set), we have the following.

7.2 Theorem. If $K$ is a locally-convex compact convex set, then $\mathrm{Con}(K)$ is primitive.

pf. By the Kreĭn-Mil'man theorem, $K = \overline{\mathrm{con}}E$ where $E$ is the set of extreme points in $K$. Hence, in $\mathrm{Con}(K)$, $K = \bigwedge\{\{e\} \mid e \in E\}$ and each $\{e\}$ is prime in $\mathrm{Con}(K)$.

In general, in a convex continuous lattice, there is no relationship between the primes and the extreme points which are meet-irreducible. For example, in the green tetrahedron there are meet-irreducible extreme points that are not prime and in the SEM lattice, there are primes that aren't extreme points. Of course, in Con(K), they coincide. In general, we do have the following property of meet-irreducible extreme points.

7.3 <u>Proposition</u>. In a convex continuous lattice $L$, the meet of all meet-irreducible extreme points is $0$.

pf. By the Kreĭn-Mil'man theorem, $Mi(L)$ is the closed convex hull of its extreme points $E$. Hence $\bigwedge E = \bigwedge(\overline{\mathrm{con} E}) = \bigwedge(Mi(L)) = 0$.

Both 7.2 and 7.3 are applications of the Kreĭn-Mil'man theorem in the theory of convex continuous lattices and yet they are faithful interpretations of the Kreĭn-Mil'man theorem in this sense: if either 7.2 or 7.3 could be proven without using the Kreĭn-Mil'man theorem and only with the techniques of lattice theory, then the Kreĭn-Mil'man theorem could be easily proven using them.

This opens the question of a lattice-theoretic proof of the Kreĭn-Mil'man theorem but as yet no solution seems to be known.

The following chart shows that no other implications hold between the properties of distributive, primitive, and hard other than the ones stated.

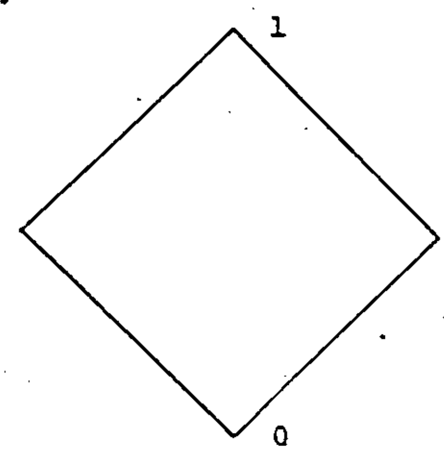
Properties of Convex Continuous Lattices

| | | distributive | primitive | hard |
|---|---|---|---|---|
| two-element lattice | : | yes | yes | yes |
| SEM lattice | : | yes | yes | no |
| diaper lattice | : | no | yes | yes |
| green diamond lattice | : | no | yes | no |
| green tetrahedron lattice | : | no | no | no |

EXAMPLES

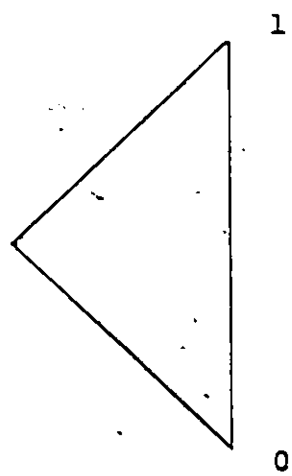
The following lattices are subsets of [0,1], $[0,1]^2$, or $[0,1]^3$ and derive the order on them from these lattices.

Diamond lattice:



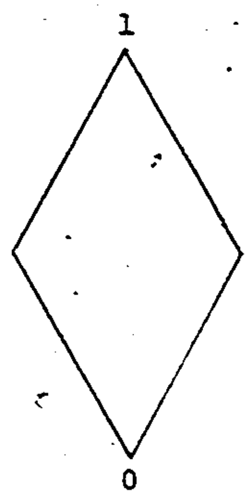

Fat star lattice and thin star lattice.

Half-diamond lattice:



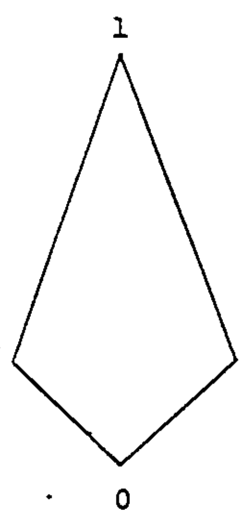

Fat star lattice but not thin.

Thin-diamond lattice:



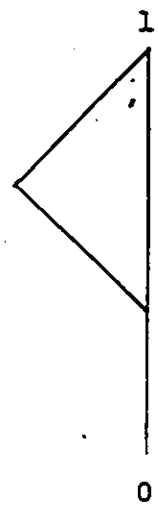

Not a star lattice.

Necktie lattice:



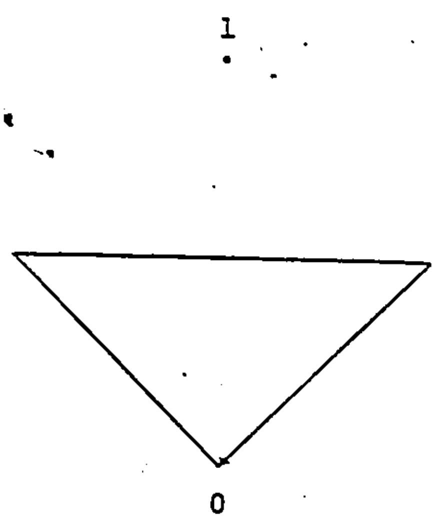

Star lattice but neither fat nor thin.

Pennant lattice:




Not a star lattice.

Diaper lattice:

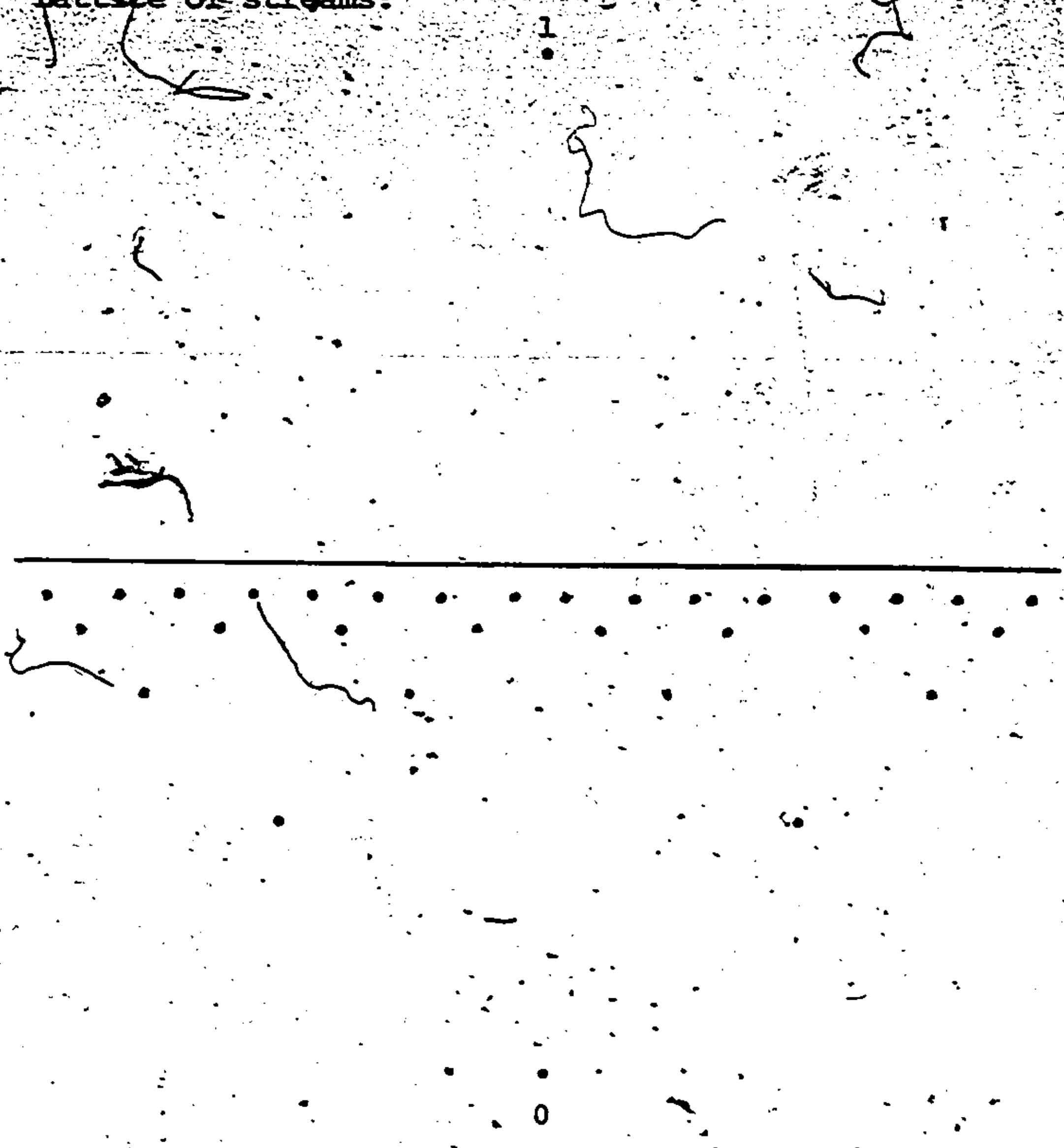



Hard convex continuous lattice.

SEM lattice:


Distributive convex continuous lattice.

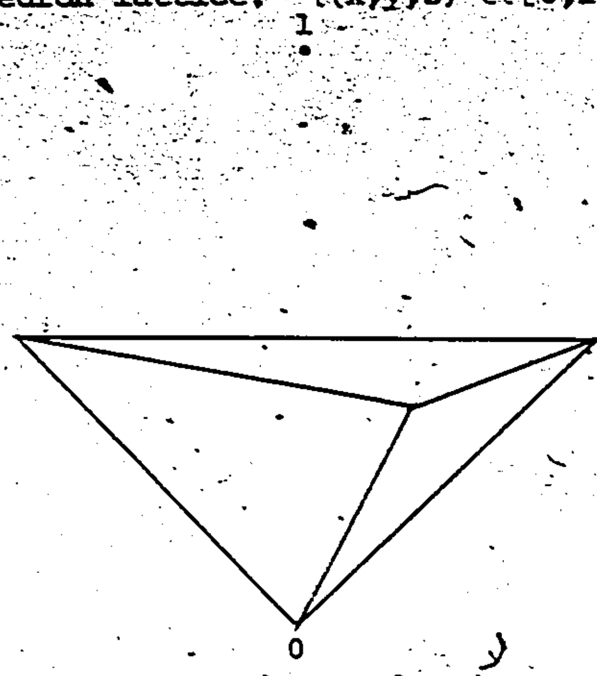
Lattice of streams:




Sheltered algebraic lattice.

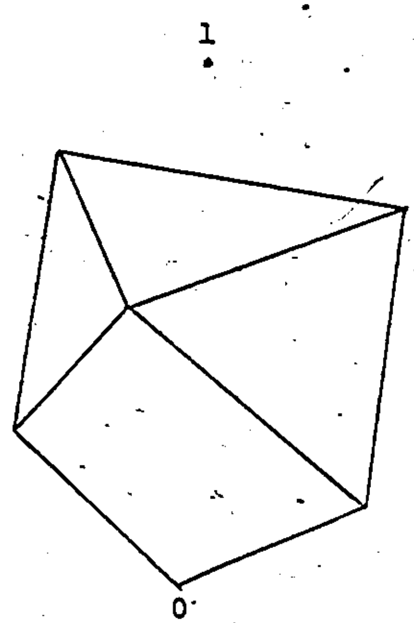
The lattice of streams (due to D. Scott) can be described as all finite and infinite sequences $a_1a_2\ldots$ of 0's and 1's where one sequence $x$ is less than or equal to another sequence $y$ iff $x$ results from $y$ by truncation.

Green tetrahedron lattice: $\{(x,y,z) \in [0,1]^3 \mid x + y + z \leq 1\}$




Convex continuous lattice.

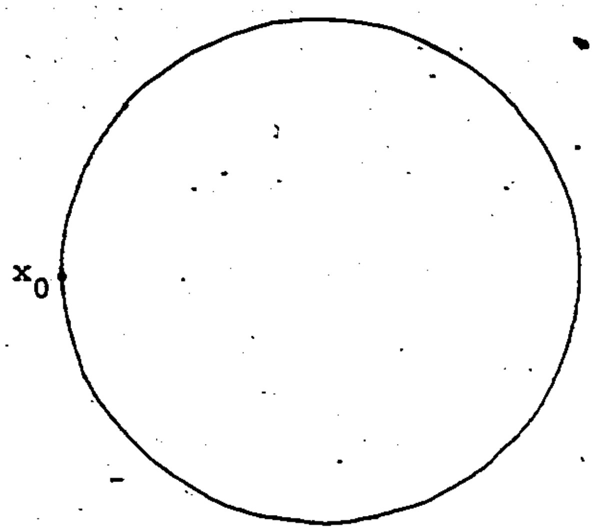
Green diamond lattice: $\{(x,y,z) \in [0,1]^3 \mid x + y + z \leq 2\}$.




Convex continuous lattice.

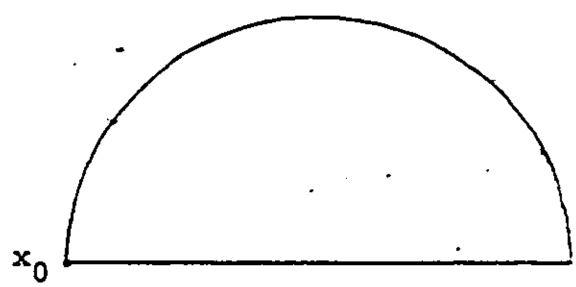
The following star structures are subsets of the plane and obtain their star operation and topology from the plane.

Full moon star structure:



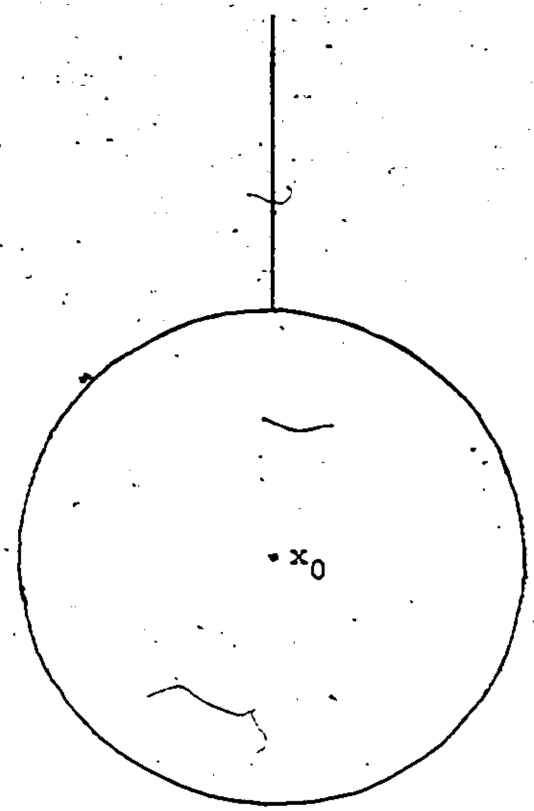

Solid compact star structure.

Half moon star structure:




Solid compact star structure.

Ball and chain star structure:




Compact star structure.

## BIBLIOGRAPHY


1. Banaschewski, B., "The duality of distributive continuous lattices", Canadian Journal of Math., to appear.

2. Bennett, M.K., "Generalized convexity lattices", J. Comb. Theory 10 (1971) 140-144.

3. Bennett, M.K., "Topological convexity lattices", Alg. Univ. 4 (1974) 220-225.

4. Bruns, G., "Distributivität und subdirekte Zerlegbarkeit vollständiger Verbände", Arch. der Math. 12 (1961) 61-66.

5. Bryant, V., "Topological convexity spaces", Proc. Edinburgh Math. Soc. 19 (1974/75) 125-132.

6. Bryant, V.N. and R.J. Webster, "Convexity spaces I. Basic properties", J. Math. Anal. Appl. 37 (1972) 206-213.

7. Bryant, V.N. and R.J. Webster, "Convexity spaces II. Separation", J. Math. Anal. Appl. 43 (1973) 321-327.

8. Bryant, V.N. and R.J. Webster, "Convexity spaces III. Dimension", J. Math. Anal. Appl. 57 (1977) 382-392.

9. de Smet, H.J.P., "On a notion of convexity introduced by B. Fuchssteiner", Simon Stevin 49 (1975/76) 185-198.

10. Fuchssteiner, B., "Verallgemeinerte Konvexitätsbegriffe und der Satz von Krein-Mil'man", Math. Ann. 186 (1970) 149-154.

11. Gierz, G., K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott, A Compendium of Continuous Lattices, to appear.

12. Gierz, G. and K. Keimel, "A lemma on primes appearing in algebra and analysis", Houston J. Math. 3 (1977) 207-224.

13. Gudder, S.P., "Convexity and mixtures", SIAM Rev. 19 (1977) 221-240.

14. Hofmann, K.H., "The space of lower-semicontinuous functions into a CL-object", SCS memo dated 9-20-76.

15. Hofmann, K.H. and J.D. Lawson, "Irreducibility and generation in continuous lattices", Semigroup Forum 13 (1977) 307-353.

16. Hofmann, K.H. and J.D. Lawson, "The spectral theory of distributive continuous lattices", Trans. Amer. Math. Soc., 246 (1978) 285-310.

17. Jamison, R., Ph.D. Thesis, University of Washington,

18. Jamison, R.E., R.D. O'Brien, and P.D. Taylor, "On embedding a compact convex set into a locally-convex topological vector space", Pac. J. Math. 64 (1976) 193-205.

19. Lawson, J.D., "Embeddings of compact convex sets and locally-compact cones", Pac. J. Math. 66 (1976) 443-453.

20. Lawson, J.D. and B. Madison, "On congruences and cones", Math. Z. 120 (1971) 18-24.

21. Nachbin, L., Topology and Order, Van Nostrand, 1965.

22. Papert, S., "Which distributive lattices are lattices of closed sets?", Proc. Comb. Phil. Soc. 55 (1959) 172-176.

23. Scott, D., "Continuous lattices", Toposes, Algebraic Geometry, and Logic, Springer-Verlag, LNM 274 (1972) 97-136.

24. Świrszcz, T., "Monadic functors and categories of convex sets", Institute of Math., Polish Academy of Sciences, Preprint no. 70.

25. Thornton, M.C., "Topological spaces and lattices of lower-semicontinuous functions", Trans. A.M.S. 181 (1973) 495-506.

26. Bruns, G., Darstellungen und Erweiterungen geordneter Mengen. I und II, J.f.d. reine u. angew. Mathematik, 209 (1961) 167-220 and 210 (1962) 1-23.