KINEMATICS IN SPECIAL AND GENERAL RELATIVITY

By

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ABSTRACT

This thesis investigates the problem of motion for extended bodies from the viewpoint of classical field theory, where the classical field is the body's energy-momentum or matter tensor.

In Special Relativity a symmetric and divergence-free matter tensor combined with inertial frames is used to generate a kinematics for extended bodies; but I have shown that if the matter tensor also obeys the weak energy condition, then both types of massless spin radiation must have an infinite spatial extent in all Lorentz frames. This does not agree with the observation that finite light beams put a torque on crystals as they change their polarization while traversing the crystal.

In General Relativity I have suggested a kinematics analogous to that accepted in Special Relativity and applied it to the simplest non-trivial example of static, spherical stars. In essence one looks for special sets of vector fields whose matter currents are conserved. Such a set of ten vector fields defines a special frame and integrals of the conserved matter currents define ten momenta which give the kinematics. a simple application of de Rham cohomology theory shows that the conserved matter currents for isolated bodies will have mechanical potentials which enable the momenta to be found from flux integrals evaluated in the vacuum region surrounding the body.

These potentials contain the full Riemann curvature allowing a body's General Relativistic momenta to be determined by its vacuum gravitational field.

This approach has several important differences with previous attempts at a General Relativistic kinematics. By working directly with the matter tensor employed in Einstein's equations, it seems unnecessary to invent energy pseudo-tensors or other secondary objects to define momenta. By integrating matter currents which vanish in the vacuum, the momenta receive no contribution from vacuum regions. In this way one avoids the problem of motion without matter, which arises if the vacuum is endowed with momenta. By integrating divergence-free matter currents, one obtains conserved momenta for isolated bodies. The existence of divergence-free vector fields is a very weak condition, quite unlike the existence of metric symmetries, so that conserved momenta can be obtained in the absence of metric symmetries, as is explicitly done for six of the ten momenta for static spherical stars.

Although an example of this kinematics has been given for static spherical stars, much remains to be done. I have shown that there are many conserved matter currents for arbitrary bodies, but it is not yet known how to use Einstein's equations to single out the physically interesting ones for arbitrary space-times. Related to this problem is the question of the final form of the mechanical potential. Work in this area will shed some light on the special frames. Are they directly analogous to the inertial frames of flat space, but determined by the matter distribution? What is the group of transformations which links the special frames in arbitrary space-times? If it is not the Poincaré group, then would the group provide a richer physical structure?

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CHAPTER I

INTRODUCTION

The problem of motion is a venerable one and the major contributions to its understanding in this century have come from both Special and General Relativity. In the first modern kinematics the motion of an extended body is reduced to an equation for the line described by the body's mass centre which requires a Galilean frame for its definition. In the absence of external forces, Newton's first law requires the mass centre to travel a straight line in the Galilean frame. The advent of Special Relativity did not qualitatively change this view of motion. However, the view of both space-time and matter did change - the Galilean frame was exchanged for a Lorentz frame and the centre of mass became the centre of energy or centroid. As yet the problem of motion is unsolved for General Relativity, but it is the suggestion of this thesis that the same view of motion holds in General Relativity with Lorentz frames replaced by special frames determined by the matter distribution and the centroid defined as a moment of the matter tensor. This is a very difficult problem and although much work remains to be done, this thesis has made contributions towards its resolution.

The first of these is in the realm of Special Relativity and has to do with the frame dependence of the centroid. This work is presented in chapter III and has been published as "Minimum Size of Radiation with Spin"[1]. Previous treatments of Special Relativistic Kinematics[2,3] have suppressed the centroid's frame dependence by using only the rest frame centroid. The total angular momentum which has no Lorentz covariant splitting is usually split into a rotational part about the rest frame centroid and an orbital part due to the linear momentum acting at the rest frame centroid. Chapter III obtains a kinematics applicable to massless bodies or pulses of massless radiation as well as massive ones. Here the total angular momentum is split frame dependently into a spin about the frame's centroid and an orbital part due to the linear momentum acting at that frame's centroid. The important result which follows from this decomposition is that all the other centroids can be found simply from the body's spin and linear momentum, the centroid being unique when the spin vanishes. Applying this result to massless bodies with spin shows that they must have an infinite spatial extension. This is the massless case of the well known centroid theorem[2,4] for massive bodies.

With Special Relativistic Kinematics thus completed one can see how to write it invariantly and discover how much of it can be incorporated into General Relativity. This is accomplished in sections 2.3, 2.4 and 4.4. In Special Relativity the ten momenta one uses to describe the centroid are the integrals of matter currents arising from vector fields associated with the inertial frames[5]. The ten vector fields which give the matter currents are Killing and this ensures that the matter currents are conserved as are their momenta for isolated bodies. It is this reliance on metric symmetries and their Killing vector fields for conserved momenta which has hindered all previous attempts[6,7] at incorporating Special Relativistic ideas into General Relativity. When the kinematics is invariantly written one immediately sees that it is the conserved matter currents which are crucial to obtaining an equation of motion for the centroid and not the metric symmetries. This very fortunate circumstance creates the possibility of an analogous kinematics for General Relativity. In section 4.2 it is shown that the existence of a conserved matter current is an extremely weak condition quite unlike the existence of metric symmetries. Consequently one can inquire as to the existence in General Relativity of special coordinate frames with the property that their ten vector fields, analogous to the ten Killing vector fields of inertial frames, generate conserved matter currents. As yet this is an unsolved problem. However, section 4.5 gives an example of a fully General Relativistic special frame for static, spherical stars with a geodesic centroid and its momenta coincide with the Special Relativistic ones in the weak field limit. Since there may be many special frames for static, spherical stars, the frame was picked by the symmetries so that four of ten momenta would be the direct analogues of flat space momenta and arise from the metric symmetries. The matter was then restricted so that the remaining six momenta would be conserved even though they do not come from metric symmetries. Although the resulting matter is unphysical, this example shows that kinematically interesting conserved matter currents not due to metric symmetries and capable of giving geodesic centroid motion do exist in General Relativity. Many questions posed in section 5.2 remain to be answered, but a new approach has been opened to the problem of motion in General Relativity.

Another novel feature of this thesis is its use of two-forms whose exterior derivative is a matter current three-form. Such matter currents are automatically conserved and section 4.3 examines when such two-form potentials exist for conserved matter currents. The use of such potentials is more than just a calculational convenience to convert the three dimensional integrations for momenta to two dimensional ones. For under very general conditions discussed in section 5.1 these potentials can depend linearly on the full Riemann curvature as happens in section 4.5 for the Schwarzchild momenta. This circumstance permits the purely <u>mechanical</u> momenta to be derived from flux integrals of the vacuum <u>gravitational</u> field surrounding a body. This is a very strong version of the equivalence principle and provides another approach to the conserved matter currents of a special frame. Such curvature dependent potentials express the long range character of the gravitational field and should have more general applications in discussing singularities and the patching together of different solutions to Einstein's equations.

CHAPTER II

REVIEW OF KINEMATICS

2.1 Inertial Frames

Through Einstein's penetrating analysis[8] of mechanics, the existence of inertial frames appears somewhat mysterious, but absolutely essential to the problem of motion in flat space physics. Geometrically an inertial frame is a set of global coordinate functions x^μ whose vector fields

$$D\mu = \frac{\partial}{\partial x^\mu} \qquad (1a)$$

are orthonormal

$$\begin{aligned} g(D\mu, D\nu) &= -1 & \mu=\nu=0 \\ &= +1 & \mu=\nu\neq 0 \\ &= \;\;0 & \mu\neq\nu \end{aligned}$$

where g is the space-time metric[9]. Because of this an inertial frame has no gravitational forces

$$\Gamma^\mu_{\nu\lambda} = 0 \qquad \mu,\nu,\lambda\varepsilon(0,1,2,3)$$

and its basis vectors satisfy Killing's equation

$$\mathcal{L}_{D\mu} g = 0 \qquad (0,1,2,3) \; .$$

In any inertial frame, there are six more Killing vector fields with a simple coordinate representation, three rotational

$$\phi c = x^a \frac{\partial}{\partial x^b} - x^b \frac{\partial}{\partial x^a} \qquad (1b)$$

where a, b, c are a cylic permutation of (1,2,3), and three pseudo-rotational or boost vector fields

$$Ba = x^0 \frac{\partial}{\partial x^a} + x^a \frac{\partial}{\partial x^0} . \qquad (1c)$$

Inertial frames have many other nice properties, but the fact that the gravitational forces vanish and that the ten vector fields (1) are Killing is sufficient to generate a Special Relativistic Kinematics for appropriately described matter.

2.2 The Matter Tensor

The mechanical properties of matter are described with a second rank tensor T, called here the matter tensor. In general it has four properties:

1) $\forall$ vector fields X,Y $\quad T_p(X,Y) = 0$ iff there is no matter at point p

2) $\forall$ timelike vector fields W $\quad T(W,W) \geq 0$

3) $\forall$ vector fields X,Y $\quad T(X,Y) = T(Y,X)$

4) $\quad T^{\mu\nu}{}_{;\,\nu} = 0$

Property 1 says that the matter tensor vanishes where the matter does. Central to the problem of motion is the concept of an isolated body. Such a body is not contiguous with any other matter and is realized as a world tube of non-vanishing T surrounded by an immediate region of vacuum where T does vanish.

Property 2 is the weak energy condition and it assumes that the energy of a body is always positive. It is used in the next chapter to show that in Special Relativity a spinning body must have a minimum size.

Both property 3 that the matter tensor is symmetric and Property 4 that it is divergence-free are crucial to obtaining conserved momenta.

In General Relativity the matter tensor is assumed to have these four properties as well[10], but the last two obtain by virtue of the field equations

$$G = 8\pi T.$$

If a general relativistic energy could be defined for arbitrary physical manifolds, then property 2 would assure its positivity, as is explicitly done in Chapter IV for Schwarzchild manifolds.

2.3 Conserved Matter Currents

Any vector field K has a matter current J_K associated with it via the matter tensor by

$$J_K \equiv T(K,\cdot). \qquad (2)$$

Using the metric, the matter current may be regarded as either a vector field or a one-form. For the present J_K will be taken as a one-form having the coordinate expression

$$J_K = T_{\alpha\beta}K^{\beta}\omega^{\alpha}$$

where ω^{α} are the coordinate basis one forms.

For any compact three-surface Σ, a quantity of motion

$Q_K(\Sigma)$ can be defined as

$$Q_K(\Sigma) \equiv \int_\Sigma * J_K \qquad (3)$$

where the right hand side is the integral over Σ of the three-form that is dual to the one-form J_K. This quantity of motion can be thought of as the total amount of matter in Σ that is flowing along the vector field K.

Consider a compact four-dimensional region V^4 whose boundary ∂V^4 can be sliced by some spatial hyper-surface into two disjoint three-surfaces Σ_1, Σ_2

$$\partial V^4 = \Sigma_1 + \Sigma_2 \, . \qquad (4)$$

Then choosing ∂V^4 for Σ in (3) and noting that Σ_1 and Σ_2 have opposite orientations allows the use of (4) in (3) to give

$$Q_K(\partial V^4) + \int_{-\Sigma_1} * J_K = \int_{\Sigma_2} * J_K \, .$$

This says that the total amount of matter flowing along K into V^4 (in $-\Sigma_1$) may differ from the total amount of matter flowing along K out of V^4 (in Σ_2) by an amount $Q_K(\partial V^4)$.

The conserved matter currents arise from the vector fields V whose quantity of motion vanishes for any three-surface which bounds a compact four-surface.

$$Q_V(\partial V^4) \equiv 0 \quad ; \quad V \text{ compact } V^4 \qquad (5)$$

Stoke's theorem[27] states that the integral of the exterior derivative of a p-form ω over a compact p+1-surface Σ is the integral of the form over the surface's boundary:

$$\int_{\Sigma} d\omega = \int_{\partial\Sigma} \omega \quad .$$

Combining (3) and (5) and using this theorem shows that conserved matter currents arise from vector fields V satisfying

$$\int_{V^4} d{*}J_V = 0 \quad ; \quad \text{V compact } V^4$$

which will be true if and only if

$$d{*}J_V = 0 \ .$$

This is equivalent to

$$*d{*}J_V = 0 \qquad (6)$$

or in components

$$J_V{}^{\alpha}{}_{;\alpha} = 0 \ .$$

Thus a conserved matter current is divergence-free and in Chapter IV it is shown that regardless of metric symmetries every vector field can be rescaled to give a conserved matter current.

2.4 Special Relativistic Kinematics

The last three sections can be used to obtain a geodesic law of motion for isolated bodies in flat space, i.e. a Special Relativistic Kinematics.

Consider the divergence of a matter current J_K of an arbitrary vector field K

$$J_{K;\alpha}^{\alpha} = (T^{\alpha}{}_{\beta}K^{\beta})_{;\alpha} .$$

With properties 4 and 3 of the matter tensor, this becomes

$$J_K{}^{\alpha}{}_{;\alpha} = \frac{1}{2} T^{\alpha\beta}(K_{\beta;\alpha} + K_{\alpha;\beta}) . \qquad (7)$$

So a sufficient condition for J_K to be conserved is that

$$K_{\alpha;\beta} + K_{\beta;\alpha} = 0 .$$

which is just the component form of Killing's equation

$$\mathcal{L}_K g = 0$$

stating that K is a metric symmetry.

In flat space physics there is no connection between the matter and the space-time, so the oondition that K be a metric symmetry is also necessary for J_K to be conserved, in the following sense. Suppose J_V is conserved for some T, but V is not a metric symmetry

$$V_{\alpha;\beta} + V_{\beta;\alpha} \neq 0 . \qquad (8)$$

Then (7) becomes

$$T^{\alpha\beta}(V_{\alpha;\beta} + V_{\beta;\alpha}) = 0 . \qquad (9)$$

Now without altering the geometry or the value of (8), the matter T may be replaced with more arbitrary matter

$$T' = T+t .$$

So by (7) and (9) the divergence of J'_V is

$$J'^{\alpha}_{V;\alpha} = \frac{1}{2}\, t^{\alpha\beta}(V_{\alpha;\beta} + V_{\beta;\alpha})$$

which will not vanish unless V is Killing. Thus in flat space physics conserved matter currents for arbitrary matter distributions arise from Killing vector fields alone.

The Special Relativistic momenta are the ten quantities of motion

$$P_V = Q_V(\Sigma) \qquad (10)$$

where V is any of the ten Killing vector fields (1) associated with an inertial frame and Σ is some compact constant time surface of that frame[5,9]. For an isolated body a constant time surface will slice the body's world tube disjointly provided the body's motion is causal. Then to obtain momenta for the body Σ can be chosen as any portion of the constant time surface, whose boundary $\partial\Sigma$ lies in the immediate vacuum region surrounding the body.

Consider a compact four dimensional region V^4 which contains a segment of the isolated body's world tube and whose boundary ∂V^4 is made of three pieces

$$\partial V^4 = \Sigma - \Sigma_1 + C^3 \qquad (11)$$

where Σ_1 is arbitrary but has the same orientation as the constant time surface Σ and like Σ has its boundary in the vacuum, and C^3 is a three-dimensional "collar" surrounding the body's world tube and lying entirely in the vacuum. The quantities of motion for the surfaces in (11) are then from (5) and (10)

$$0 = P_V - Q_V(\Sigma_1) + Q_V(C^3) \ . \qquad (12)$$

Since the matter vanishes on C^3 so does its matter current and by (3)

$$Q_V(C^3) = 0$$

reducing (12) to

$$P_V = Q_V(\Sigma_1) \ . \qquad (13)$$

Thus the special relativistic momentum P_V for an isolated body can be found by integrating $*J_V$ over any compact three surface Σ_1 which has the same orientation as a constant time surface and has its boundary entirely in the vacuum. This important result permits the evaluation of the derivative

$$\frac{\partial Q_K(\Sigma)}{\partial x^0} = \lim_{\Delta t \to 0} \left[\frac{Q_K(\Sigma) - Q_K(\Sigma_1)}{\Delta t}\right] ,$$

where Σ is the constant time surface $x^0 = t$ and Σ_1 is $x^0 = t+\Delta t$. When the $Q_K(\Sigma)$ is a momentum for an isolated body, this derivative vanishes

$$\frac{\partial P_V}{\partial x^0} = 0 \ . \tag{14}$$

The kinematics is then obtained by introducing the centroid curve which is the world line of the centre of energy for the isolated body. This is done invariantly in Chapter IV and in the usual way[5] in Chapter III. The centroid curve is quite frame dependent; but because of (14), it is a line of constant slope in an inertial frame. Since the inertial coordinates have vanishing connection coefficients, such a line is geodesic. In this way an isolated body is found to obey a geodesic law of motion in flat space.

Before finishing this section it is worth showing the momenta, here defined invariantly, do in fact reduce to the usual non-invariant expressions. From (3) and (10) one has

$$P_V = \int_\Sigma \star J_V \tag{15}$$

where V is a member of (1) and Σ is an inertial constant time surface. In local coordinates with ω^α as basis one-forms, $\star J_V$ is

$$\star J_V = \frac{1}{6}\, T^\mu{}_\nu V^\nu \eta_{\mu\alpha\beta\gamma}\, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma$$

where η is the permutation tensor. Since Σ is a constant x^0 surface (15) reduces to

$$P_V = \int_\Sigma T^0{}_\beta V^\beta \quad . \tag{16}$$

When V is any of the $D\mu$

$$V^\beta = \delta^\beta_\mu$$

and (16) is

$$P_{D\mu} = \int_\Sigma T^0{}_\mu$$

which is the usual expression for the linear momenta P_μ. When V is any of the ϕc

$$V^\beta = x^a \delta^\beta_b - x^b \delta^\beta_a$$

the (16) is

$$P_{\phi c} = \int_\Sigma T^0{}_b x^a - T^0{}_a x^b$$

which is the usual definition of $J^a{}_b$, the spatial components of the so called angular momentum tensor. Similarly the three P_{Ba} are the components $J^0{}_a$. One can easily verify that P^μ and $J^{\mu\nu}$ transform invariantly under the Poincaré group which links the inertial frames.

2.5 Field Momenta in General Relativity

One very important difference between General Relativity and all other classical field theories is that the source-free fields in General Relativity are by definition matter-free. In other field theories there are two kinds of matter; that of ponderable bodies where the sources reside, and that of the source-free field. In Classical Electromagnetism the electromagnetic field has the Maxwell stress tensor for its matter tensor and the ponderable body whether charged or not has some other matter tensor. This distinction between field and matter appeared most acutely in the attempts to build a purely electromagnetic model of charge[11]. The Maxwell stress tensor for the charge had to be augmented by Poincaré stresses of completely unknown origin in order to prevent the charge from exploding. The all pervading character of the electromagnetic field made it seem somehow immaterial - even though the field has a well defined matter tensor. On the other hand General Relativity takes any matter tensor, whether it arises from a pure classical field or not, and with Einstein's equations uses the matter tensor as a source for space-time curvature. Thus unlike any other classical field theory General Relativity assumes a unified description of matter by means of the matter tensor. Consequently for matter-free regions the source-free gravitational fields occurring there are truly immaterial.

To gain some insight into General Relativity, the weak field approximation was done to make it look more like electromagnetism. However, the pure gravitational field, unlike the pure electromagnetic field, has no matter tensor associated with it and consequently no momenta. Unlike the electromagnetic case, no field momenta could be imparted to bodies and cause them to attract[12]. To avoid this problem various energy-momentum complexes were introduced[13]. Although they could be transformed away like the gravitational forces on a geodesic particle, these complexes were regarded as physical properties of the vacuum field. In this way vacuum regions were endowed with momenta and the physical problem transformed to motion without matter.

From a consideration of these complexes Komar[14] suggested that one might associate an invariant momentum P(X) with certain vector fields X by

$$P(X) = \frac{1}{8\pi}\int_{\Sigma} \frac{1}{6}(2X^{[\mu;\nu]}{}_{;\nu})\,\eta_{\mu\alpha\beta\gamma}\,\omega^{\alpha}\wedge\omega^{\beta}\wedge\omega^{\gamma} \qquad (17)$$

in the notation of this thesis. Using the Stoke's theorem and a result from appendix II this can be rewritten

$$P(X) = \frac{1}{8\pi}\int_{\partial\Sigma} *\,(X_{[\mu;\nu]}\,\omega^{\mu}\wedge\omega^{\nu}) \ .$$

Komar observed that in the Schwarschild geometry where $\partial\Sigma$ is a surface at infinity enclosing the star and X is the time-like Killing vector field $\frac{\partial}{\partial t}$ in the Schwarzchild coordinates then

$$P(\frac{\partial}{\partial t}) = M \ .$$

Moss and Davis[15] later pointed out that the integral would give the same result for a $\partial\Sigma$ enclosing the star anywhere in the vacuum, so that Komar's "energy" in this case was well localized to the star and not diffused through the vacuum as previously suspected.

Manoff[16] noticed a connection between Komar's momenta and the Ricci tensor. For any vector field on a manifold with no torsion one has

$$X^{\rho}{}_{;\alpha\beta} - X^{\rho}{}_{;\beta\alpha} = -\mathbb{R}^{\rho}{}_{\sigma\alpha\beta}X^{\sigma} \tag{18}$$

in local coordinates where $\mathbb{R}$ is the Riemann curvature tensor. On raising the index α and breaking the first term on the left hand side into symmetric and antisymmetric pieces one obtains

$$X^{[\rho;\alpha]}{}_{;\beta} + X^{(\rho;\alpha)}{}_{;\beta} - X^{\rho}{}_{;\beta}{}^{;\alpha} = -\mathbb{R}^{\rho}{}_{\sigma}{}^{\alpha}{}_{\beta}X^{\sigma} .$$

Interchanging ρ and α in the first two terms on the left hand side and tracing over ρ and β gives

$$-X^{[\alpha;\beta]}{}_{;\beta} + X^{(\alpha;\beta)}{}_{;\beta} - X^{\beta}{}_{;\beta}{}^{;\alpha} = R^{\alpha}{}_{\sigma}X^{\sigma} \tag{19}$$

where R is the Ricci tensor. From (18) the second term turns out to be

$$X^{(\alpha;\beta)}{}_{;\beta} = \frac{1}{2}X^{\alpha;\beta}{}_{;\beta} + \frac{1}{2}X^{\beta;\alpha}{}_{;\beta}$$

$$= \frac{1}{2}X^{\alpha;\beta}{}_{;\beta} + \frac{1}{2}X^{\beta}{}_{;\beta}{}^{;\alpha} + \frac{1}{2}R^{\alpha}{}_{\sigma}X^{\sigma} .$$

Substituting this result into (19), finally gives

$$2X^{[\mu;\nu]}{}_{;\nu} = - R^{\mu}{}_{\nu}X^{\nu} + X^{\mu;\nu}{}_{;\nu} - X^{\nu}{}_{;\nu}{}^{;\mu} \quad . \qquad (20)$$

Now if the arbitrary vector field X is a Killing vector field ζ, then the last term on the right hand side of (20) vanishes as Killing vector fields are divergence-free and the term beside it becomes

$$\zeta^{\mu;\nu}{}_{;\nu} = - R^{\mu}{}_{\nu}\zeta^{\nu} \ .$$

Thus the integrand of Komar's momenta for a Killing vector field is really minus twice the Ricci current of the Killing vector field

$$P(\zeta) = \frac{1}{8\pi} \int_{\Sigma} \frac{1}{6} \, (- \ 2R^{\mu}{}_{\nu}\zeta^{\nu})\eta_{\mu\alpha\beta\delta}\omega^{\alpha}\wedge\omega^{\beta}\wedge\omega^{\gamma} \qquad (21)$$

So that Komar's consideration[14] of energy momentum complexes actually led him to integrate Ricci currents of Killing vector fields rather than Einstein or matter currents. Since the Schwarzschild vacuum has a vanishing Ricci tensor, this explains Moss and Davis' observation[15] that the vacuum made no contribution to Komar's Schwarzchild integral.

Manoff[16] further notices that under certain restrictions Komar's "energy" for a time-like Killing vector field could be written as an integral of an Einstein or matter current.

Taking ζ in (21) as a time-like Killing coordinate vector field and Σ as a constant time surface of this coordinate allows the reduction of (21) to

$$P\left(\frac{\partial}{\partial x^0}\right) = \frac{1}{8\pi} \int_\Sigma - 2R^0{}_0 \quad . \tag{22}$$

On using the coordinate identity relating the Ricci tensor to the Einstein tensor G

$$R^\mu{}_\nu \equiv G^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu \left(G^0{}_0 + \sum_{a\neq 0} G^a{}_a\right)$$

for $R^0{}_0$ in the integrand, allows the "energy" to be written as

$$P\left(\frac{\partial}{\partial x^0}\right) = \frac{1}{8\pi}\int_\Sigma - G^0{}_0 + \frac{1}{8\pi}\int_\Sigma \sum_{a\neq 0} G^a{}_a \quad .$$

Apart from the minus sign the first integrand is the Einstein current of the Killing vector field $\frac{\partial}{\partial t}$. Thus if the sum of the pressures vanish in these coordinates

$$\sum_a G^a{}_a = 0 \qquad a \neq 0 \ .$$

Komar's "energy" is in fact minus the integral of the time-like Killing vector field's matter current. In flat-space the integral of the time-like Killing vector field's matter current is minus the energy and what is surprising about Manoff's observation is the amount of special pleading required to obtain M as the integral of $*J_{\frac{\partial}{\partial t}}$ in the Schwarzchild geometry. In fact when the matter tensor is trace-less, the Ricci tensor in (22) may be replaced by the Einstein tensor and since the right hand side of (22) is M for the Schwarzchild geometry, one has

$$\int_\Sigma * J_{\frac{\partial}{\partial t}} = -\frac{1}{2} M .$$

Thus in General Relativity one cannot even expect a metric symmetry to generate the appropriate momenta as shown by this example.

An important point of departure for this thesis from most previous work on the subject is its explicit attempt to find momenta as integrals of <u>matter</u> currents which receive no contribution from vacuum gravitational fields. This approach takes into account the special nature of the gravitational field and avoids the physical problem of motion without matter. The static, spherical example of Chapter IV has this property and was picked so that its energy would be the direct analogue of flat space energy, but with the correct value -M.

The crucial property of a Killing vector field is that it generates a <u>conserved</u> matter current. According to Trautman[6], one should not expect conserved matter currents in arbitrary space-times which do not have Killing vector fields. Returning to (20) and substituting the Einstein tensor and curvature scaler R for the Ricci tensor one obtains

$$G^\mu{}_\beta X^\beta = -2X^{[\mu;\nu]}{}_{;\nu} - \frac{1}{2} RX^\mu + X^{\mu;\nu}{}_{;\nu} - X^\nu{}_{;\nu}{}^{;\mu} .$$

With Einstein's equations the left hand side becomes the matter current and taking its divergence removes the first term on the right hand side leaving

$$*d*J_X = \frac{1}{8\pi}(-\frac{1}{2} RX^{\mu} + X^{\mu;\nu}{}_{;\nu} - X^{\nu}{}_{;\nu}{}^{;\mu})_{;\mu} .$$

It is not yet known which vector fields X cause this expression to vanish, but its vanishing is Manoff's necessary and sufficient condition[16] for a vector field to have a conserved matter current. The vanishing of a vector field's divergence is a rather weak condition, unlike the existence of a metric symmetry. In the fourth chapter it is shown that regardless of metric symmetries any smooth vector field with a non-vanishing matter current can be locally rescaled to have a conserved matter current. Contrary to expectation Killing vector fields are far from necessary for conserved matter currents, but the problem is to select the physically interesting ones from this multitude of non-trivial conserved matter currents.

2.~~6~~ 7 Equations of Motion in General Relativity

To reduce the motion of an extended body to a world line satisfying certain equations of motion appears to require privileged frames to define the momenta which characterize the line. These frames are somewhat less privileged than the inertial frames of flat space in that they are determined by Einstein's equations and are not independent of the matter distribution.

One such recent theory is that of Dixon[17,18] and it relies heavily on normal coordinates. For any point x and a time-like unit vector n at that point, he defines the momentum as

$$P^{\kappa}(x,n) = \int_{\Sigma} K_{\alpha}{}^{\kappa} T^{\alpha\beta} \eta_{\beta\mu\nu\rho} \omega^{\mu} \wedge \omega^{\nu} \wedge \omega^{\rho}$$

and the angular momentum as

$$S^{\kappa\lambda}(x,n) = \int_{\Sigma} A_{\alpha}^{\kappa\lambda} T^{\alpha\beta} \eta_{\beta\mu\nu\rho} \omega^{\mu} \wedge \omega^{\nu} \wedge \omega^{\rho}$$

where K_{α}^{κ} and $A_{\alpha}^{\kappa\lambda}$ are complicated "bitensor" functions of the DeWitt and Brehme two-point world function which requires Riemann normal coordinates for its definition. The integration is performed on a hypersurface that contains all the geodesics whose tangent vectors at x are perpendicular to n. The loose indices κ, λ in the integrand refer to the point x and merely reflect the integral's dependence on x. Now if the gravitational field is not too strong, one can find a unique time-like N satisfying

$$P^{\kappa}(x,N)N^{\lambda} - P^{\lambda}(x,N)N^{\kappa} = 0$$

provided $P^{\kappa}(x,N)$ is not null. With n determined in this way it is then claimed[19] that there is a unique world line of points z for which

$$P_{\kappa}(z,N)S^{\kappa\lambda}(z,N) = 0 .$$

Parameterizing this line by s which under more restrictions is the proper time along z, Dixon obtains the equations of motion

$$\frac{\delta}{ds} P^{\kappa} = \frac{1}{2} R^{\kappa}{}_{\lambda\mu\nu} \frac{dz^{\lambda}}{ds} S^{\mu\nu} + F^{\kappa}$$

$$\frac{\delta}{ds} S^{\kappa\lambda} = P^{\kappa} \frac{dz^{\lambda}}{ds} - P^{\lambda} \frac{dz^{\kappa}}{ds} + \theta^{\kappa\lambda}$$

where F and θ appearing as force and torque terms are really integrals of higher multipole moments of the matter tensor. For spaces of constant curvature, F and θ vanish by Einstein's equations and then the equations of motion reduce to the Matthisson-Papetrou equations for a dipole particle. It seems that this correspondence was a key factor in the particular choice of K and A in Dixon's definition of momenta[18].

Mathematically one might ask why it is necessary to introduce point dependent integrals to define momenta when one could more simply integrate matter current three forms. The drawback is that one does not yet know which matter currents should be used. Because the integrands for Dixon's momenta depend linearly on the matter tensor, this theory avoids endowing the vacuum with any momenta. Unfortunately the momenta are not necessarily conserved which Dixon[18] interprets as evidence for Bondi's gravitational induction. This interpretation does ascribe the change in "energy" of an isolated body to the vacuum gravitational fields, even though these fields have no "energy". The important distinction between such multipole moment kinematics and the theory envisioned here is that the momenta used in the thesis are integrals of conserved matter currents and so for an isolated body are themselves conserved. In this approach the motion of a body is reduced to the body's centroid curve whose equations of motion are found from the fact that the centroid curve is a line of constant slope in the privileged frame needed to define the momenta.

CHAPTER III

COMPLETION OF SPECIAL RELATIVISTIC KINEMATICS

3.1 Introduction

The centroid theorem[20-23] states that if a body with positive energy density is to have a certain mass and rest frame spin, then it must also have a minimum size, ensuring that no part of the body travels faster than light during the rotational motion. Here and throughout the word "spin" refers to classical rotational angular momentum.

The present work extends this purely kinematical result to massless bodies with well defined momentum, centroid and spins both parallel and perpendicular to the direction of motion. Intuitively one sees that a massless body is frozen in time, so any rotational angular momentum must be gained at the price of an infinite moment arm. This is the necessary reason for the "terms at infinity"[24,25] which give rise to the spin electromagnetic radiation.

It will be shown that radiation with only parallel spin must in any frame and at every instant occupy at least an entire spatial plane perpendicular to its direction of motion[26]. Though not proven here, this plane is the expected limit of the centroid disc for massive rotating bodies.

In a special frame where the radiation has only perpendicular spin, it will be shown that at every instant this sort of radiation must at least fill the interior of a spatial parabola whose latus rectum is parallel to the direction of motion. The shape and orientation of this region changes from frame to frame, but in any frame and at every instant this radiation must contain points infinitely distant in the direction of motion. Such radiation cannot be causally absorbed or emitted in a finite time, which is the classical analogue of Abbott's quantum field theoretic argument[28] against the existence of such particles. The author has not examined the case of rotating tachyons, but their "zero mass limit" should give rise to this radiation[29].

Any postulated radiation with conserved momentum, conserved non-zero spin, having a positive energy density and which is also finite in size would contradict the centroid theorem, so that all such radiation must be infinitely large.

At least one important conclusion can be drawn from this work. Classical Electromagnetism is an incomplete theory as it cannot explain Beth's observation[30] that light beams with finite cross section do possess angular momentum. Before it can be claimed that the properties of light may essentially be understood within the bounds of Classical Electromagnetism, this theory must be enriched by some new physical concept which can give finite light pulses an equivalent to spin.

The extension of the centroid theorem is divided into two parts. In the first section the centroid or centre of energy is used to define a spin that is appropriate to a general body. This spin contains the same information as the more usual intrinsic spin tensor[3] but the differences should be noted. The definitions employed here make sense for finite bodies, though they will apply to certain infinite systems also. In the last section the results of the first are used to construct minimum sizes for the two types of radiation which are found to to be infinite, contrary to expectation.

3.2 Spin as the Total Angular Momentum About a Centroid

A general body is to be described by a symmetric and divergence-free energy tensor, $T^{\mu\nu}$, defined over a flat space time. One then defines the body's total angular momentum by

$$J^{\mu\nu} \equiv \int_v x^{[\mu}T^{\nu]0} \qquad (2.1)$$

and its momentum by

$$P^{\mu} \equiv \int_v T^{\mu 0} \qquad (2.2)$$

where the integration is done on a constant time surface, v, of some inertial frame. These integrals are assumed to be finite. The properties of the $T^{\mu\nu}$ ensure the conservation of these quantities, provided certain integrals involving $T^{\mu\nu}$ vanish on the boundary of v. With a large enough volume these integrals will always be zero for a finite body, but this is not necessary for infinite systems [25].

The minimal desiderata of an isolated body are completed on introducing the centre of energy or centroid

$$c^{\mu} \equiv \frac{\int_v x^{\mu}T^{00}}{\int_v T^{00}} \qquad (2.3)$$

which is assumed to be well defined. At the heart of the theorem is the observation that if the energy density is everywhere positive, then the body must contain all possible centroids. Applying this definition to the conserved $J^{0\nu}$ one obtains Newton's first law [5]

$$c^{\mu} = \frac{P^{\mu}x^{0}}{P^{0}} + a^{\mu} \tag{2.4}$$

where the a^{μ} is the $x^{o} = 0$ centroid. These centroids always have the non-covariant form

$$a^{\mu} = (0,a^{i}) \tag{2.5}$$

To find the relation between this kind of centroid for two different frames it must be expressed generally in terms of quantities with known transformation rules. This is accomplished by substituting (2.4) via (2.3) into (2.1) for the $J^{0\nu}$ and with (2.2) one obtains the perspicuous relation

$$J^{0\nu} = a^{[0}P^{\nu]} \ . \tag{2.6}$$

Remembering (2.5) one can then solve for the a^{μ}, giving

$$a^{\mu} = -\frac{J^{0\mu}}{P^{0}} \ . \tag{2.7}$$

This equation determines the $x^{0}=0$ centroid for any frame, so writing $\bar{a}^{\bar{\mu}}$ for the $x^{\bar{0}}=0$ centroid of a boosted frame gives

$$\bar{a}^{\bar{\mu}} = -\frac{J^{\bar{0}\bar{\mu}}}{P^{\bar{0}}} \ . \tag{2.8}$$

Regarding the $\bar{a}^{\bar{\mu}}$ as a ~~point,~~ its coordinates can be found in the unboosted frame by applying the Lorentz transformation $\Lambda^{\bar{\mu}}_{\nu}$ which connects the two frames

$$\bar{a}^{\mu} = -\frac{\Lambda^{\bar{0}}_{\nu}J^{\nu\mu}}{\Lambda^{\bar{0}}_{\alpha}P^{\alpha}} \ . \tag{2.9}$$

The $\bar{a}^{\mu}$ can be spatially or temporally different from a^{μ}.

To see the differences clearly (2.6) must be used to motivate the definition of spin as

$$s^{\mu\nu} \equiv J^{\mu\nu} - a^{[\mu}P^{\nu]} \quad . \qquad (2.10)$$

In any frame this spin is the total angular momentum about a centroid for that frame. In the rest frame of a body it is the intrinsic spin tensor, but it does not transform like a tensor, because due to (2.6) it appears in every frame with the general form

$$s^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & s^z & -s^y \\ 0 & -s^z & 0 & s^x \\ 0 & s^y & -s^x & 0 \end{pmatrix} \qquad (2.11)$$

Unlike the intrinsic spin tensor it does not require a preferred frame for its definition. Within any frame it transforms as an axial 3-vector and is the remains of the total angular momentum after <u>all</u> the orbital angular momentum has been removed. The polarization vector

$$W_\mu \equiv \frac{1}{2}\, \varepsilon_{\mu\nu\rho\sigma}\, J^{\nu\rho}P^{\sigma} \qquad (2.12)$$

also removes all the orbital angular momentum. This allows the components of $s^{\mu\nu}$ to be found in any frame as

$$s^i = \frac{W^i}{P^0} \qquad (2.13)$$

which is obtained on evaluating the components of W^μ by using (2.10) and (2.11), and solving for the s^i.

With this decomposition of the total angular momentum and the explicit form of the $\Lambda^{\bar{0}}_{\nu}$ [54] and by lapsing into obvious vector notation, (2.9), which relates the two centroids, can now be written as

$$\bar{a}^0 = \left(\frac{\beta\gamma\vec{n}\cdot\vec{a}}{\Lambda^{\bar{0}}_{\alpha}P^{\alpha}}\right)P^0$$

$$\vec{\bar{a}} = \vec{a} + \left(\frac{\beta\gamma\vec{n}\cdot\vec{a}}{\Lambda^{\bar{0}}_{\alpha}P^{\alpha}}\right)\vec{P} + \left(\frac{\beta\gamma}{\Lambda^{\bar{0}}_{\alpha}P^{\alpha}}\right)\vec{s}\times\vec{n} \qquad (2.14)$$

where β,γ are the usual boost parameters and $\vec{n}$ is the boost direction. The difference between the two centroids is transparent. The $\bar{a}^{\mu}$ has been shifted from the a^{μ} by a certain amount parallel to the momentum and by a purely spatial amount perpendicular to the spin vector. Since the world line c^{μ} is parallel to the momentum and passes through a^{μ}, a vanishing spin requires all possible centroids to lie on this line. A non-zero spin destroys the uniqueness of the c^{μ} and multiplies it into a bundle of world lines all parallel to the momentum which pierce the $x^0=0$ plane at

$$\vec{b} = \vec{a} + \left(\frac{\beta\gamma}{\Lambda^{\bar{0}}_{\alpha}P^{\alpha}}\right)\vec{s}\times\vec{n} . \qquad (2.15)$$

As remarked earlier a positive energy density forces the body to contain all these points. Thus from a knowledge of the body's momentum and polarization vector, (2.15) enables one to find the minimum size of a general body in an arbitrary frame. This equation makes use of all six independent quantities in the $J^{\mu\nu}$ and the four remaining quantities P^{μ} determine the direction of motion, so that no further independent information can be had from the Poincaré symmetry.

3.3 The Centroid Theorem for Radiation with Spin

The methods of the previous section are completely general, but since the massive case is well known it will not be treated. The massless case naturally subdivides into two types of radiation depending on whether the polarization vector is space-like or parallel to the momen~~tum.~~

3.3.1 $P^{\mu}P_{\mu} = 0$, $W^{\mu}W_{\mu} > 0$

For this radiation a frame can always be found where the momentum and polarization vectors have the values

$$P^{\mu} = E(1,1,0,0) \qquad (3.1.1)$$

$$W^{\mu} = Es(0,0,0,1). \qquad (3.1.2)$$

On taking the coordinate origin of the frame at the $x^0=0$ centroid for this frame, (2.15) simplifies to

$$\vec{b} = \left(-\frac{\beta n_y}{1-\beta n_x}, \frac{\beta n_x}{1-\beta n_x}, 0\right)\frac{s}{E} \qquad (3.1.3)$$

so that in the $x^0=0$ surface all the possible centroid world lines occupy a two dimensional region which is perpendicular to the spin vector and contains the direction of motion. Since the choice of the constant time surface was arbitrary, this region is the same at any instant. The actual region is determined by the values of β and $\vec{n}$ which describe permissable Lorentz boosts. There are many boosts which will shift the centroid by the same amount; but if it is claimed that a certain point is in the region, then at least one boost which does move the centroid to that point must be exhibited. Since there are only two boost directions present in (3.1.3), one need only consider boosts in their plane and a single parameter θ suffices to specify the boost direction. Expressing the region in polar coordinates (R,ϕ), now allows equation (3.13) to be written as

$$(R\cos\phi,\ R\sin\phi,\ 0) = \left(-\frac{\beta\sin\theta}{1-\beta\cos\theta},\ \frac{\beta\cos\theta}{1-\beta\cos\theta},\ 0\right)\frac{s}{E} \qquad (3.1.4)$$

where $\sin\theta$ is n_y and $\cos\theta$ is n_x. By taking ratios of components from both sides of this equation one has

$$\cot\phi = -\tan\theta \qquad (3.1.5)$$

which is always true provided

$$\theta = \phi + \frac{3\pi}{2} \ . \qquad (3.1.6)$$

Substituting this value for θ back into (3.14) and solving for β yields

$$\beta = \frac{R}{R\sin\phi + \frac{s}{E}} \ . \qquad (3.1.7)$$

Although (3.1.6) shows the centroid can be moved in all directions perpendicular to the spin vector, the restriction that β be less than one, allows it to be shifted only certain distances. On rewriting (3.17) as

$$\beta^2 = \frac{R^2(\cos^2\phi + \sin^2\phi)}{(R\sin\phi + \frac{s}{E})^2} , \qquad (3.1.8)$$

and reverting to rectangular coordinates after applying the inequality for β, gives

$$y > \frac{1}{2}(\frac{E}{s})\, x^2 - \frac{1}{2}(\frac{s}{E}) . \qquad (3.1.9)$$

Thus any point in this parabolic disc is a possible centroid and the assumption of positive energy density requires the radiation to be at least as large as the disc at any instant in this frame. The radiation is travelling in the x-direction and as the parabola opens out there will be points in the disc that have an indefinitely large separation in the direction of motion. Consideration of the Lorentz invariant projection of this separation on the momentum vector shows the property of an infinite size in the direction of motion to persist in all physical frames.

3.3.2 $P^\mu P_\mu = 0,\ W^\mu || P^\mu$

The spin of this radiation is a Lorentz invariant and in any frame the momentum and polarization vectors can have the values

$$P^\mu = E(1,1,0,0) \qquad (3.2.1)$$

$$W^\mu = Es(1,1,0,0) \qquad (3.2.2)$$

Again setting the coordinate origin at the $x^0=0$ centroid for this frame, (2.15) works out to

$$\vec{b} = (0, -\frac{\beta n_z}{1-\beta n_x}, \frac{\beta n_y}{1-\beta n_x})\frac{s}{E} . \qquad (3.2.3)$$

Unfortunately all three boost directions appear here, so to find which boosts shift the centroid whither, a spherical parameterization of the boost direction is introduced

$$\begin{aligned} n_x &\equiv \cos\rho \\ n_y &\equiv \sin\rho\cos\theta \\ n_z &\equiv \sin\rho\sin\theta \end{aligned} \qquad (3.2.4)$$

where ρ lies between 0 and π, and θ between 0 and 2π. Using polar coordinates for the y-z plane then gives

$$(0, R\cos\phi, R\sin\phi) = (0, -\frac{\beta\sin\rho\sin\theta}{1-\beta\cos\rho}, \frac{\beta\sin\rho\cos\theta}{1-\beta\cos\rho})\frac{s}{E} \qquad (3.2.5)$$

As before, this implies

$$\theta = \phi + \frac{3\pi}{2} \qquad (3.2.6)$$

so that the centroid may be moved from the origin in any direction to the y-z plane. Putting this value of θ into (3.2.5) yields

$$R = \frac{\beta\sin\rho}{1-\beta\cos\rho} \qquad (3.2.7)$$

allowing the centroid to be sent any distance in the y-z plane.

To see this consider (β,ρ) as polar coordinates of some unphysical boost space with rectangular coordinates (η,ζ).

The physically possible boosts then correspond to the interior points of the unit semidisc in the upper half of this boost plane. On the other hand (3.2.7) describes lines in the boost plane with negative slope R and a constant η intercept of one. Such lines even with arbitrarily large values of R always intersect the semidisc of physical bosts, so in fact there are boosts which can move the centroid to any point in the plane perpendicular to the direction of motion at $x^0=0$. Since the choice of frames merely alters the value of E, this is true for the $x^0=0$ surface of all frames. Finally since the choice of constant time surface is arbitrary, the assumption of positive energy density demands that radiation with parallel spin must in all frames and at all times occupy at least an entire plane that is normal to the direction of motion.

3.4 Conclusions

The work presented in section 2 permits the extension of the centroid theorem to general bodies in arbitrary frames and in section 3 the theorem is given for massless bodies. The result that these bodies must be infinitely large is purely kinemetical and rests on three assumptions: Poincaré symmetry of space-time; the body has a positive energy density, conserved momentum and angular momentum; all the angular momentum comes from the integrated antisymmetric moment of the momentum density. The free fields for radiation pulses in both Classical Electromagnetism and the linear approximation to General Relativity satisfy these assumptions. Consequently such radiation having parallel spin must also have an infinite size perpendicular to its direction of motion. This is not observed for light.

In 1936 R. A. Beth[30] found a torque due to the change in polarization of a finite light beam. The idea of the experiment was to pass light through a quartz plate and measure the resultant torque as the light changed its polarization. Assuming the conservation of angular momentum at the crystal's boundary, Beth concluded that he was observing torques due to changes in the angular momentum of the electromagnetic field. The measured torque agreed well with the calculated torque per unit area exerted on an infinite plate by plane waves. The theorem clarifies the theoretical need for infinite plates and infinite waves, but they do not correspond to the experimental situation. It is natural to introduce plane waves for the resultant simplicity of calculation, yet in the end one must be able to properly superpose them and recover the realistic situation. Any possible superposition satisfying the above assumptions and still maintaining a spin will also maintain at least an infinite plane of radiation. This means that Classical Electromagnetism is incomplete as it cannot possibly produce the observed localised light beams which have a polarization dependent angular momentum.

The Neoclassical position conjectures the possibility of recovering the basic facts of electrodynamics without resorting to field quantization[31] and several provisional theories have been suggested[32]. Unless a Neoclassical theory alters the connection between mechanical quantities and the field quantities in Classical Electromagnetism (e.g. permitting singularities) it cannot hope to recover the basic fact of localised light beams possessing angular momentum. On the other hand, the difficulties involved with a relativistic position operator for massless particles[33] leave open the question of whether field quantization will avoid or accommodate this purely classical theorem.

CHAPTER IV

MOMENTA WITHOUT SYMMETRIES

4.1 Introduction

This work sketches a General Relativistic Kinematics of extended bodies based on conserved matter currents, which is analogous to Special Relativistic Kinematics. In the last section ten General Relativistic momenta are calculated for certain static, spherical stars by curvature dependent flux integrals evaluated in the vacuum close to the star. These momenta agree with the appropriate flat space limit[34] and the example is the only one where time symmetry generates the correct energy. Such stars cannot be perfect fluids, but their centroids are geodesic .

There is a further analogy between Classical Electromagnetic theory and the approach taken here for conserved matter currents and their momenta. The conserved matter currents are found from a matter tensor which satisfies Einstein's equations, so that the matter is the source of the vacuum gravitational field. Although the momenta may be found with flux integrals of the vacuum field, the momenta truly belong to the sources of these fields. This is in direct analogy with Classical Electromagnetism where one can find the charge with a flux integral of the electromagnetic field but the charge is a property of the source for the electromagnetic field. Because of this the exterior Schwarzchild solution has no momenta, while the interior solution does - it is the star which has the momenta and not the vacuum.

It has been the practice in relativity to use two "energy tensors". One, the matter tensor, is used to solve the field equations and the other object, usually the pseudo-energy tensor, is used to find the "momenta" which are usually non-zero for vacuum regions. This work suggests that only the matter tensor is necessary to obtain the momenta needed for kinematics. The logical economy of such a proposal is attractive, and if successful, the physical problem of "momenta" without matter would be avoided.

The basic ingredient of a kinematical theory is a body of finite spatial extent surrounded by vacuum. In both Special and General Relativity such bodies are described by a matter tensor which is non-zero inside a world tube and vanishes in the immediate regions outside the tube. Only after the kinematics of such isolated bodes are understood, should one try to incorporate the infinite matter fields or the dynamics due to the presence of "external" matter fields. The goal of a kinematical theory would be to provide conserved quantities describing the body and associating a geodesic with its motion. The next section shows that regardless of metric symmetries there is an infinity of non-trivial conserved matter currents. It is not yet known how to reduce this plethora of conserved quantities to the physically interesting ones which can associate geodesic motion with an arbitrary isolated body. However, this work does provide some direction to the task.

The Einstein, Infeld, Hoffmann result that Einstein's equations force a particle to obey certain laws of motion points to a link between a body's vacuum gravitational field and the mechanical quantities needed for its kinematics. In the linearized theory where Special Relativistic Kinematics apply, Misner et al. have given this link explicitly. Their notation and conventions[35] will be used throughout this chapter. In the flat space tensor theory of gravity[36] the gravitational field is described by a fourth rank tensor H with the same symmetries as the curvature tensor and is regarded as a linearized relic of real curvature in a fully General Relativistic problem. In this theory the matter tensor T obeys the field equation[37]

$$H^{\mu\alpha\nu\beta}{}_{,\alpha\beta} = 16\pi T^{\mu\nu} ,$$

and the flat space momentum and total angular momentum are respectively[37]

$$P^{\mu} = \frac{1}{16\pi} \oint H^{\mu\alpha 0j}{}_{,\alpha} d^2S_j ,$$

$$J^{\mu\nu} = \frac{1}{16\pi} \oint (x^{\mu}H^{\nu\alpha 0j}{}_{,\alpha} - x^{\nu}H^{\mu\nu 0j}{}_{,\alpha} + H^{\mu j0\nu} - H^{\nu j0\mu})d^2S_j ,$$

where the x^μ are coordinate functions of an inertial frame whose constant time surfaces contain the closed two-surfaces of integration situated in the vacuum surrounding the isolated body. It is a remarkable property of the linearized theory that the purely mechanical quantities P^μ, $J^{\mu\nu}$ are determined by the vacuum gravitational field of the body. This appealing feature may also be shared by a General Relativistic Kinematics where the General Relativistic momenta would be calculated from analogous flux integrals involving the full vacuum curvature, as is explicitly done for the example in the last section.

The third section examines the question of when the usual three surface integral for a conserved quantity can be replaced by a flux integral of some potential whose existence is guaranteed. This is basically a topological matter[38-40] and physicists are probably more familiar with an analogous problem in electromagnetic theory concerning the existence of a vector potential for the electromagnetic field tensor. Provided the region of interest V^4 in the space-time manifold has no three-dimensional holes characterised by the third deRham cohomology $H^3(V^4,\mathbb{R})$, a mechanical potential will always exist in V^4 for a conserved matter current. A globally defined mechanical potential may not exist for spatially closed universes and so to obtain conserved quantities for an entire universe of this sort, a three-surface integration may have to be performed[41]. However, the usual kinematical situation is that the region V^4, containing the body and its surrounding vacuum, has no missing points and can be covered by a single coordinate system. In this case a mechanical potential will always exist throughout V^4 for any conserved matter current and the conserved quantities can be obtained from flux integrals evaluated in the vacuum. Although the potential's existence is guaranteed, one is at a loss for its explicit form until the matter currents for which it is needed are known precisely.

The fourth section rewrites Special Relativistic Kinematics in an invariant fashion, basing it solely on the existence of conserved matter currents and to that extent independent of curvature or the laws governing the matter tensor. This provides the scafolding for a General Relativistic Kinematics and shows how seven of the ten momenta determine a frame dependent, but fully General Relativistic centroid curve for the body. In Special Relativistic Kinematics all the centroid curves are geodesics and a General Relativistic Kinematics would be established when Einstein's equations could be used to single out those physically interesting conserved matter currents whose momenta give a geodesic centroid curve.

To further demonstrate the possibility of a General Relativistic Kinematics within the framework developed here, a non-trivial example is presented in the last section. Here static, spherically symmetric bodies are used, but these symmetries are a two edged sword. On the one hand, they make the calculations tractable, and provide four of ten conserved matter currents which determine the remaining matter currents that should be conserved. On the other hand, they so severely restrict the geometry it is surprising that there is enough freedom left to construct a kinematics. Strictly speaking these symmetries are unphysical, but they are an approximation to a physical situation and perhaps this is why they do permit a kinematics under certain restrictions. Six of the ten momenta do not arise from metric symmetries and this feature of momenta without symmetries is crucial to any General Relativistic Kinematics based on conserved matter currents. In this example, the mechanical potential is particularly powerful since its vacuum flux integral avoids integrating the arbitrary functions in the matter tensor for the static, spherical bodies. Although the potentials are found by an ad hoc method, this method may have some relevance to the wider problem of finding the physically interesting conserved matter currents for an arbitrary body.

The approach taken here has several important differences with previous work on this topic. By using matter currents depending linearly on the matter tensor, the conserved quantities receive no contributions from the vacuum where the matter tensor vanishes. Thus no energy is ascribed to a vacuum region containing source-free gravitational fields. In Synge's point dependent, though fully General Relativistic theory[42] "curvature currents" are integrated to obtain conserved quantities and this can give "energy" to Ricci flat regions. Dixon also gives a point dependent General Relativistic theory[18] which integrates matter currents to obtain "momenta", but these currents are not necessarily conserved and so neither are the "momenta". This circumstance permits an isolated body to acquire "mass" from the surrounding vacuum. However, the abundance of conserved currents shown here may make it possible to find preferred points for which Dixon's matter currents will be conserved. In that case, one would be closer to the sort of kinematics envisioned here. Finally there are the linearized theories[36,43] which are really flat space theories; though when the curved space-time is asymptotically Minkowskian, Misner et.al. ascribe Special Relativistic momenta to the entire curved universe. The theory sketched here does not require the linearized theory or asymptotic flatness.

4.2 Conserved Currents

By describing matter with a smooth second rank tensor T, one can immediately introduce a matter current J_V associated with any vector field V. In local coordinates the components of J_V are

$$J^{\alpha}_{V} = T^{\alpha}{}_{\beta} V^{\beta} \tag{1}$$

and the matter current is conserved when it satisfies

$$J^{\alpha}_{V;\alpha} = 0 \ . \tag{2}$$

In both General[9] and Special[5] Relativistic Field Theories where the matter tensor obeys

$$T^{\mu\nu} = T^{\nu\mu} \ , \quad T^{\mu\nu}{}_{;\mu} = 0 \tag{3}$$

a <u>sufficient</u> condition for (2) is that the vector field V generate a metric symmetry and thereby satisfy Killing's equations

$$V_{\alpha;\beta} + V_{\beta;\alpha} = 0 \ . \tag{4}$$

In flat space there are enough of these Killing vector fields to erect a kinematical theory based on their conserved matter currents[36,43,5] and no conditions weaker than (3) and (4) are needed. However; in General Relativity where the matter affects the geometry, Killing vector fields are quite unphysical as a mere speck of dust being out of place will destroy the metric symmetry. Consequently a General Relativistic Kinematics based on conserved matter currents requires a much weaker condition than (4) for (2). The close connection between symmetries and conserved currents given by Nöther's theorem for Lagrangian theories[43,5] has led to the feeling that symmetries are really needed for momenta[6]. On the other hand it is well known[44] that a matter current will yield a conserved quantity for some region of space-time provided (2) holds in that region, irrespective of any metric symmetries. Thus if conserved currents were sufficiently common, there may well be a General Relativistic Kinematics founded on them.

In fact every smooth vector field K can be rescaled to

$$\bar{K} \equiv fK \quad , \tag{5}$$

with the function f chosen so that $\bar{K}$ is divergence-free

$$\bar{K}^{\alpha}{}_{;\alpha} = 0 \; . \tag{6}$$

To see this, consider another local coordinate expression for (6)

$$\frac{1}{\sqrt{g}} (\sqrt{g}\, \bar{K}^{\alpha})_{,\alpha} = 0, \tag{7}$$

where g is the absolute value of the metric's determinant in these coordinates. Substituting (5) into (7) gives a differential equation for f

$$(\sqrt{g}\ f\ K^{\alpha})_{,\alpha} = 0\ . \tag{8}$$

Passing to local coordinates where

$$K = \delta^{\bar{\alpha}}_{\bar{0}} \frac{\partial}{\partial x^{\bar{\alpha}}}$$

(8) becomes

$$\frac{\partial}{\partial x^{\bar{0}}} (\sqrt{\bar{g}}\ f) = 0\ .$$

So the rescaled vector field

$$\bar{K} = \frac{1}{\sqrt{\bar{g}}} K$$

where $\bar{g}$ is the absolute value of the metric's determinant in a frame where K is a coordinate field, satisfies (6).

This trick can be played on any non-zero matter current J_V and because of the linearity of T the $\sqrt{\bar{g}}^{\,-1}$ can be pushed through to rescale V. It should be realized that Einstein's equations must be solved to obtain this scale factor. Thus every smooth vector field V on a space-time manifold with a matter tensor gives rise to a conserved matter current $J_{\bar{V}}$ where

$$\bar{V} = \frac{1}{\sqrt{g}} V$$

and this $\sqrt{g}$ is evaluated in a coordinate frame where J_V is a coordinate vector field. Consequently there is an infinity of conserved matter currents regardless of metric symmetries. Presumably there will be enough conserved matter currents to build a General Relativistic Kinematics.

4.3 Conserved Quantities and de Rham Cohomology

A conserved quantity comes from integrating the three-form $*J_V$ dual to the conserved J_V. In local coordinates with ω^α as basis one-forms it is given by

$$*J_V = \frac{1}{6} J_V^\alpha \eta_{\alpha\beta\gamma\delta} \omega^\beta \wedge \omega^\gamma \wedge \omega^\delta \tag{9}$$

where η is the permutation tensor. Now (2) is a local coordinate condition equivalent to $*J_V$ being closed

$$d*J_V = 0 . \tag{10}$$

So every conserved matter current has a closed three-form associated with it. The conserved quantity generated by the conserved J_V on a closed and bounded three-surface Σ is defined by

$$P_V \equiv \int_\Sigma *J_V . \tag{11}$$

The physically interesting P_V arise when Σ is a spatial hypersurface; but because $*J_V$ is closed, an application[44] of Stokes theorem shows that any two Σ's with the same orientation and non-zero boundary yield the same P_V.

If $*J_V$ were globally exact

$$*J_V = d\theta_V , \tag{12}$$

where θ_V is a two-form; then (10) would be automatically satisfied and Stoke's theorem could be used on (11), allowing

$$P_V = \int_{\partial\Sigma} \theta_V \tag{13}$$

where $\partial\Sigma$ is the boundary of the compact Σ in (11). Since not all closed forms are exact, the definition (11) is more general than the flux integral (13).

It turns out that the existence of closed forms which are not exact hinges on the topology of the manifold[38-40]. If the region of physical interest on the manifold is smoothly contractable (homotopic) to a point, then all closed forms in this region are exact. This version of Poincaré's Lemma covers almost all physical situations and de Rham cohomology theory treats the odd exceptions. The theory employs equivalence classes of closed p-forms called de Rham cohomology classes and two closed p-forms are in the same class when they differ by an exact p-form. The pth de Rham cohomology classes for a differentiable manifold M form a vector space over the reals which is denoted by $H^p(M,\mathbb{R})$. All closed p-forms being exact on M is equivalent to

$$H^p(M,\mathbb{R}) = 0,$$

as the class of exact p-forms is the zero element of the vector space. The de Rham isomorphism theorem then shows that for $H^p(M,\mathbb{R})$ to be non-zero the manifold must have compact p-dimensional surfaces which have zero boundary and are not the boundary of compact p+1 dimensional surfaces. Such surfaces are said to enclose p-dimensional holes and may be created by removing points from the manifold, rendering compact regions non-compact; or by having the manifold close on itself. In the latter case, the p-dimensional hole is deprived of any locality. The hallmark of a p-dimensional hole is the existence of a closed p-form α and a compact p-surface c with no boundary, such that

$$\int_c \alpha \neq 0. \tag{14}$$

So a sufficient condition for (12) is that the manifold have no three dimensional holes

$$H^3(M,\mathbb{R}) = 0\ . \tag{15}$$

The only manifolds seriously considered by physicists which do not satisfy (15) are spatially closed universes such as those with topology $\mathbb{R}\times S^3$ or wormholes with $\mathbb{R}\times S^1\times S^2$ topology. In such situations one may have to abandon the flux integral (13) and use the definition (11) to calculate a conserved quantity for the entire universe[41].

However, the problem of Kinematics is essentially a local one dealing with isolated bodies of finite spatial extent surrounded by vacuum. The world tubes of such bodies can always be sliced into two disjoint pieces by a spatial hypersurface V^3 which is non-compact. This surface can be regarded as a three-dimensional submanifold of M and since it is non-compact, its top cohomology vanishes

$$H^3(V^3, \mathbb{R}) = 0 \ . \tag{16}$$

Thus on V^3 a potential θ_V must exist for any closed $*J_V$ and the flux integral (13) can be used to find P_V for any compact Σ contained in V^3. Of course one wants to use the same θ_V throughout an entire four-dimensional submanifold V^4 containing the matter-filled world tubes. Another version of Poincaré's lemma states that for an n-dimensional manifold $\mathbb{R}\times N$

$$H^{n-1}(\mathbb{R}\times N, \mathbb{R}) = H^{n-1}(N, \mathbb{R}) \ , \tag{17}$$

where N is any n-1 dimensional manifold. So if V^4 is $\mathbb{R}\times V^3$, one has

$$H^3(V^4, \mathbb{R}) = 0, \tag{18}$$

by combining (16) and (17). Physically the demand that V^4 have the topology $\mathbb{R}\times V^3$ means that the spatial slices through the body's world tube have the same topology at different times. By this physically mild assumption, (18) guarantees the existence of a potential θ_V for every conserved J_V throughout the four-dimensional V^4 containing the isolated matter, ensuring the equivalence of (11) and (13) in V^4.

A potential is only unique up to an exact two-form. Any θ_V' given by

$$\theta_V' = \theta_V + d\chi \tag{19}$$

where χ is an arbitrary differentiable one-form, yields the same $*J_V$ and P_V as θ_V in (12) and (13) respectively. Also since $*J_V$ is proportional to T, it vanishes where T does, and here (12) becomes

$$d\theta_V\Big|_{T=0} = 0 \ . \tag{20}$$

So the potential is closed in matter-free regions.

Consider the usual physical situation where the region V^4 containing the tube of an isolated body and its surrounding vacuum is homotopic to a point so that

$$H^p(V^4, \mathbb{R}) = 0 \quad , \quad p = 1,2,3,4 \ .$$

Suppose there is a conserved matter current, which gives a non-zero P_V calculated from either (11) or (13) for a compact Σ in V^4 which slices the body's world tube into two disjoint pieces and has its boundary $\partial\Sigma$ lying entirely in the vacuum. It is obvious that the conserved current's potential cannot vanish throughout the vacuum, as there its integral over $\partial\Sigma$ gives the non-zero P_V

$$P_V = \int_{\partial\Sigma} \theta_V \neq 0 \ . \tag{21}$$

cutting the matter-filled region out of V^4 while leaving the rest of V^4 intact would not alter (21). But then from (20), θ_V would be closed on the matter-less remains of V^4 and (21) then fits the criterion (14) for the existence of a two-dimensional hole. This hole cannot be repaired by merely filling in the missing points. Just the right amount of matter needed to smooth the potentials must be returned as well. So the vacuum potentials characterize the matter in a similar fashion to the vacuum gravitational field and it would be no accident if an explicit expression for a potential contained the gravitational field.

4.4 General Relativistic Kinematics

The work of the last two sections has shown that there are many conserved matter currents, irrespective of metric symmetries, and that flux integrals evaluated in the vacuum exist for conserved quantities associated with the conserved matter currents of an isolated system. This gives some reason to hope that there ~~will~~ be sufficient conserved currents to describe the system's motion by momenta which depend on the vacuum gravitational field surrounding the system. Assuming the existence of several conserved currents enables Special Relativistic Kinematics to be mimicked in the following manner.

Every coordinate system x^α on any space-time has a set of ten associated vector fields

$$D\mu = \frac{\partial}{\partial x^{\mu}} \qquad \mu\varepsilon(0,1,2,3)$$

$$\phi c = x^{a}\frac{\partial}{\partial x^{b}} - x^{b}\frac{\partial}{\partial x^{a}} \qquad (a;b,c) \text{ cyclic permutation of } (1,2,3) \tag{22}$$

$$Ba = x^{0}\frac{\partial}{\partial x^{a}} + x^{a}\frac{\partial}{\partial x^{0}} \qquad a\varepsilon(1,2,3)$$

which generate the Poincaré Lie algebra by taking the commutator of any two of them for their Lie bracket. A frame will be called special when each of the vector fields (22) generate a conserved matter current by (1), and when the x^{0} coordinate singled out by the algebra is time-like. The inertial frames of Special Relativistic Field theories are special because their vector fields (22) are all Killing vector fields, but there are many special frames which are not inertial. Apart from vacuum universes where every frame with a time-like coordinate is trivially special, their existence in General Relativity is an unsolved problem. Given one special frame x^{α}; then irrespective of the matter tensor, there are many linearly related ones. The frame $x^{\bar{\alpha}}$ given by

$$x^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}{}_{\beta}x^{\beta} + a^{\bar{\alpha}}$$

Where $\Lambda^{\bar{\alpha}}{}_{\beta}$ is any constant matrix and the $a^{\bar{\alpha}}$ are constants, also has conserved matter currents for its associated vector fields (22). The additional requirement that $x^{\bar{0}}$ is time-like restricts the constants $\Lambda^{\bar{\alpha}}{}_{\beta}$ somewhat, but these linearly related special frames do have a subset of Poincaré related special frames where the $\Lambda^{\bar{\alpha}}{}_{\beta}$ are Lorentz matrices. When the matter tensor is taken into acount there may be additional nonlinear transformations which give rise to whole new families of linearly related special frames.

Suppose that a space-time region containing the world tube of an isolated body and its surrounding vacuum has no holes and admits a special frame. Since special frames are local objects, the frame is only required to be special for the isolated body, and it need not remain so for distant matter-filled regions. Provided the body's motion is causal, the requirement that the x^0 coordinate is time-like ensures the existence of compact spatial hypersurfaces Σ which slice the body's world tube disjointly and have their boundary $\partial\Sigma$ lying entirely in the vacuum. Familiar integration arguments[45] applied to conserved quantities P_V calculated on such surfaces Σ yield

$$\frac{dP_V}{dx^0} = 0 \ , \qquad (23)$$

and when the V are from (22) for the special frame, the P_V will be called momenta.

The matter tensor is assumed to obey the weak energy condition

$$T(U,U) \geq 0,$$

where U is any time-like vector field. This energy condition and the metric convention force

$$P_{D0} < 0 \ , \qquad (24)$$

as can be seen by using a constant x^0 surface in (11) for P_{D0}.

The ten momenta enable the body's motion to be reduced to the centroid curve for the special frame. This curve comes from the four vector fields

$$K\mu = x^{\mu} \frac{\partial}{\partial x^0} ,$$

whose matter currents $J_{K\mu}$ are not conserved. Consequently the integrals of $*J_{K\mu}$ are strongly surface dependant. The coordinates of a frame's centroid curve are defined by the four functions

$$C\mu(x^0) = \frac{\int_{\bar{\Sigma}} *J_{K\mu}}{P_{D0}} \tag{25}$$

where $\bar{\Sigma}$ is a constant x^0 surface whose boundary lies in the vacuum. Because $\bar{\Sigma}$ is a constant x^0 surface and the integrand is linear in x^{μ}, $C0$ reduces to

$$C0 = x^0 . \tag{26a}$$

To find the remaining functions, the P_{Ba} momenta are used. Explicitly from (1), (11) and (22) they are

$$P_{Ba} = \int_{\Sigma} *T(\cdot, x^0 \frac{\partial}{\partial x^a} + x^a \frac{\partial}{\partial x^0}) . \tag{27}$$

A possible choice for this Σ is $\bar{\Sigma}$ of (15). So with $\bar{\Sigma}$ in (27) and by the linearity of its integrand, (27) simplifies to

$$P_{Ba} = P_{Da}x^0 + P_{D0}Ca , \tag{28}$$

on using (1), (11), (22) and (25). Because of (24), (28) can be solved for the Ca giving

$$Ca = -\left(\frac{P_{Da}}{P_{D0}}\right)x^0 + \frac{P_{Ba}}{P_{D0}} \quad . \tag{26b}$$

Since these momenta obey (23), the x^0 dependence of $C\mu$ is explicitly given in (26).

The frame dependence of the $C\mu$ in (25) is quite complicated, but within a Poincaré family of special frames, its frame dependence is identical to its behavior for inertial frames[1]. In Special Relativistic Kinematics, the inertial frames are distinguished from all other families of special frames, by being the family of special frames whose coordinates are orthonormal, ensuring that the centroid curve (26) is geodesic. Unfortunately for General Relativistic Kinematics sufficiently little is known about special frames, that it is unclear whether one should look for a linearly related family of special frames whose generic tube of centroid curves[1] are all geodesic or contain a geodesic, etc. The only apparent salvation to this problem is a weaker sufficiency condition for (2) than (4) which relies more heavily on Einstein's equations than (3). As an indication that such a condition does in fact exist, the next section presents non-trivial special frames for static, spherically symmetric stars and calculates their General Relativistic momenta.

4.5 Schwarzchild Momenta

In this section the previous considerations are applied to static, spherically symmetric space-time regions with the Schwarzchild vacuum solution holding outside the matter's world tube which has no event horizon. First a special frame is found with the help of the metric symmetries. Mechanical potentials whose exterior derivatives give the matter current forms throughout the frame are then found (Appendix III). The existence of these curvature dependent potentials is very important as they allow the purely mechanical momenta to be found from flux integrals of the vacuum gravitational field.

The symmetries are quite restrictive and their presence alone permits the quantities of interest to be written in the following manner. All the calculations are done in the curvature coordinates[46] with

$$ds^2 = -e^{\gamma}dt^2 + e^{\alpha}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \,, \qquad (29)$$

where γ and α are arbitrary functions only of r. The full curvature is given by $\mathcal{G}$, the "double dual" of the Riemann tensor, and its non-vanishing components are

$$\mathcal{G}^{tr}{}_{tr} = \frac{e^{-\alpha}-1}{r^2} \,, \quad \mathcal{G}^{t\theta}{}_{t\theta} = \mathcal{G}^{t\phi}{}_{t\phi} = -\frac{1}{2r} e^{-\alpha}\alpha_{,r} \,,$$

$$\mathcal{G}^{\theta\phi}{}_{\theta\phi} = \frac{1}{2} e^{-\alpha}(\gamma_{,rr}+\frac{1}{2}(\gamma_{,r})^2-\frac{1}{2}\alpha_{,r}\gamma_{,r}) \,, \quad \mathcal{G}^{r\theta}{}_{r\theta} = \mathcal{G}^{r\phi}{}_{r\phi} = \frac{1}{2r} e^{-\alpha}\gamma_{,r} \,, \qquad (30)$$

As can be found from the Riemann tensor given by Synge[46].

The Einstein tensor has the non-vanishing components

$$G^t{}_t = \frac{1}{r^2}\,(e^{-\alpha}[1-r\alpha_{,r}]-1)\ , \ G^r{}_r = \frac{1}{r^2}\,(e^{-\alpha}[1+r\gamma_{,r}]-1),$$
$$G^\theta{}_\theta = G^\phi{}_\phi = \frac{1}{2}\,e^{-\alpha}(\gamma_{,rr} + \frac{1}{2}\,(\gamma_{,r})^2 + \frac{1}{r}\,\gamma_{,r} - \frac{1}{r}\,\alpha_{,r} - \frac{1}{2}\,\alpha_{,r}\gamma_{,r})\ . \quad (31)$$

Since the static, spherical body has no event horizon, these coordinates may be used throughout the region containing the body and its surrounding vacuum with the usual caveat for spherical coordinates. Applying Einstein's equations

$$G = 8\pi T \quad , \quad (32)$$

to the vacuum region gives the exterior Schwarzchild solution:

$$e^\gamma = e^{-\alpha} = 1 - \frac{2M}{r}\ ,$$
$$\mathcal{G}^{tr}{}_{tr} = \mathcal{G}^{\theta\phi}{}_{\theta\phi} = -\frac{2M}{r^3}\ ,\ \mathcal{G}^{t\theta}{}_{t\theta} = \mathcal{G}^{t\phi}{}_{t\phi} = \mathcal{G}^{r\theta}{}_{r\theta} = \mathcal{G}^{r\phi}{}_{r\phi} = \frac{M}{r^3}, \quad (33)$$
$$G = 0\ ,$$

where M is a positive number, corresponding to the mass of the star in the weak field approximation.

The special frame is to be determined by a specific matter tensor which will satisfy some equation of state relating the independent α and γ. It is doubtful that a special frame will exist regardless of the equation of state and depending on the relation between α and γ different frames will be special. To exhibit a special frame the symmetries can be used to generate four of the ten conserved currents. The four vector fields of (22) which are Killing are then Dt, ϕa. This does not determine the frame, but one such frame is the rectangular version of the curvature coordinates

$$t = t,\ x = r\sin\theta\cos\phi \quad , \quad y = r\sin\theta\sin\phi \quad , z = r\cos\theta \ . \qquad (34)$$

Because of the spherical symmetry, it suffices to check only two of the remaining six vector fields and see whether they give rise to conserved matter currents. Picking the two vector fields as

$$Dz = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} , \qquad (35)$$

$$Bz = r\cos\theta \frac{\partial}{\partial t} + t\cos\theta \frac{\partial}{\partial r} - t \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} , \qquad (36)$$

a short calculation using (32), (31), (1) and the connection coefficients yields (Appendix I)

$$J_{Dz}{}^{\mu}{}_{;\mu} = \frac{1}{16\pi r^2} \cos\theta(e^{-\alpha}-1)(\alpha_{,r}+\gamma_{,r})$$

$$J_{Bz}{}^{\mu}{}_{;\mu} = \frac{t}{16\pi r^2} \cos\theta(e^{-\alpha}-1)(\alpha_{,r}+\gamma_{,r}) \ .$$

Since Dt is time-like, the vanishing of these divergences would make (34) a special frame. This can only happen if the further restriction

$$\alpha_{,r} + \gamma_{,r} = 0 \qquad (37)$$

is placed on the geometry. All matter currents are trivially conserved in the vacuum and (33) guarantees (37) in the vacuum regions, so (37) is really a restriction on the matter constituting the static, spherical star. Thus for matter with (37) as its equation of state, (34) is a non-trivial special frame.

Using (11), (29), (31) and (32), the conserved quantity due to the Killing vector field $\frac{\partial}{\partial t}$ is

$$P_{\frac{\partial}{\partial t}} = \frac{1}{2}\int_0^R e^{\frac{1}{2}(\alpha+\gamma)} (re^{-\alpha}-r)_{,r}dr$$

which on integrating by parts with (33) can be written as

$$P_{\frac{\partial}{\partial t}} = -M + \frac{1}{4}\int_0^R - r(e^{-\alpha}-1)e^{\frac{1}{2}(\alpha+\gamma)}(\alpha_{,r}+\gamma_{,r})dr \ .$$

The energy condition forces the integrand to be positive[55]; but with (24) and to agree with the Newtonian limit, the star's energy must be -M. This shows that even when Killing vector fields do exist in General Relativity they cannot be relied on to give the physically expected conserved quantities. However, (37) is the necessary and sufficient condition for the General Relativistic energy of a static spherical star to be the direct analogue of flat space energy. It should be pointed out that (37) is a strong condition and is not satisfied by any isolated static spherical perfect fluid[56].

Although one can now find the P_{Da} and $P_{\phi a}$ by inspection, it is an important matter of principle to be able to find curvature dependent mechanical potentials for all ten conserved matter currents. The existence of curvature dependent potentials shows the link between a body's gravitational field and its momenta enabling its kinematic quantities to be found from its gravitational field. Hopefully if one knew how to obtain geodesic motion from conserved matter currents for arbitrary space-times one might be able to define the momenta with the appropriate curvature dependent potentials. Until constructive existence theorems are found for conserved matter currents in general space-times, explicit forms for the potential will have to be found by ad hoc procedures.

One such method sufficient for the present purposes, but possibly having a wider applicability, begins by considering the two-form β_v whose dual is the potential

$$\theta_v = *\beta_v \ . \tag{38}$$

Then (13) becomes

$$P_v = \int_{\partial\Sigma} *\beta_v \ . \tag{39}$$

Using (1), (32) and the fact that G is the trace of $\mathcal{G}$, (11) can be written as

$$P_v = \frac{1}{8\pi} \int_{\Sigma} *(\mathcal{G}^{\mu}{}_{\nu\rho\lambda} \langle\omega^{\rho}, \frac{\partial}{\partial x^{\mu}}\rangle V^{\nu} \omega^{\lambda}) \ , \tag{40}$$

where $\langle \omega^\rho, \frac{\partial}{\partial x^\mu} \rangle$ is the trace. Comparing (39) and (40), it appears that a local coordinate expression for β_V, depending on the full curvature and linear in V, might be found; if ω^ρ could be somehow freed from the trace, while maintaining the relation between the integrands

$$*J_V = d*\beta_V \,, \tag{41}$$

obtained by combining (38) and (12). The simplest such candidate for β_V is

$$\beta_V \equiv \frac{1}{16\pi} Y^\mu V^\nu \mathcal{E}_{\mu\nu\rho\lambda} \omega^\rho \wedge \omega^\lambda \,, \tag{42}$$

where Y is any vector field ensuring (41). On substituting (1), (32), (42) into (41) and taking the dual of the resulting equation, further simplifications using the Bianchi identities and symmetries of $\mathcal{E}$ show that (41) has the local coordinate equivalent (see Appendix II)

$$G^\lambda{}_\mu V^\mu = (V^\mu Y^\nu)_{;\rho} \mathcal{E}^{\rho\lambda}{}_{\mu\nu} \,. \tag{43}$$

So given a G, V, $\mathcal{E}$ and the connection coefficients, a solution of (43) for Y guarantees not only that J_V is conserved, but also, that its potential is the dual of (42). If there is one solution of (43) for Y there will be many, because of the guage freedom (19) in the potential. On the other hand, it is not yet known that a solution to (43) for Y exists for every conserved current J_V. Work in this direction may have some bearing on the constructive existence theorems for conserved matter currents needed for the sort of kinematics envisioned here for arbitrary space-times.

Proceeding towards a solution of (43), the diagonal form of G and $\mathfrak{G}$ allows its reduction to

$$G^{\lambda}{}_{\lambda}V^{\lambda} = 2[(V^{[t}Y^{\lambda]})_{;t}\mathfrak{G}^{t\lambda}{}_{t\lambda} + (V^{[r}Y^{\lambda]})_{;r}\mathfrak{G}^{r\lambda}{}_{r\lambda}$$

$$+ (V^{[\theta}Y^{\lambda]})_{;\theta}\mathfrak{G}^{\theta\lambda}{}_{\theta\lambda} + (V^{[\phi}Y^{\lambda]})_{;\phi}\mathfrak{G}^{\phi\lambda}{}_{\phi\lambda}] \; , \qquad (44)$$

where there is no sum on repeated indices and indices with square brackets are antisymmetrized. A short calculation involving the connection coefficients for a diagonal metric yields

$$(V^{[\rho}Y^{\lambda]})_{;\rho} = (V^{[\rho}Y^{\lambda]})_{,\rho} + V^{[\rho}Y^{\lambda]}\frac{\partial}{\partial x^{\rho}}\ell n\sqrt{|g_{\lambda\lambda}g_{\rho\rho}|}$$

$$+ \sum_{\substack{a\neq\rho \\ \neq\lambda}} Y^{[\lambda}V^{a]}\frac{\partial}{\partial x^{a}}\ell n\sqrt{|g_{\rho\rho}|} \; , \qquad (45)$$

where the only sum is explicit in the last term. Then substituting (29), (30), (31), (45) into (44) and simplifying gives the four partial differential equations equivalent to (43)

$$V^{t}\left(\frac{1-e^{\alpha}}{r} - \alpha_{,r}\right) = 2\left(\frac{1-e^{\alpha}}{r}\right)(V^{[r}Y^{t]})_{,r} - \alpha_{,r}(V^{[\theta}Y^{t]})_{,\theta} - \alpha_{,r}(V^{[\phi}Y^{t]})_{,\phi} +$$

$$- \alpha_{,r}\cot\theta V^{[\theta}Y^{t]} + \frac{1}{r}(\gamma_{,r} - \alpha_{,r} - (\alpha_{,r}+\gamma_{,r})e^{\alpha})V^{[r}Y^{t]} \qquad (46a)$$

$$V^r\left(\frac{1-e^\alpha}{r}+\gamma_{,r}\right) = 2\left(\frac{1-e^\alpha}{r}\right)(V^{[t}Y^{r]})_{,t} + \gamma_{,r}(V^{[\theta}Y^{r]})_{,\theta} + \gamma_{,r}(V^{[\phi}Y^{r]})_{,\phi} +$$

$$+ \gamma_{,r}\cot\theta V^{[\theta}Y^{r]} \qquad (46b)$$

$$\frac{1}{2} V^\theta\left(\gamma_{,rr} + \frac{1}{2}(\gamma_{,r})^2 + \frac{1}{r}\gamma_{,r} - \frac{1}{r}\alpha_{,r} - \frac{1}{2}\alpha_{,r}\gamma_{,r}\right) =$$

$$-\frac{\alpha_{,r}}{r}(V^{[t}Y^{\theta]})_{,t} + \frac{\gamma_{,r}}{r}(V^{[r}Y^{\theta]})_{,r} + \left(\gamma_{,rr}+\frac{1}{2}(\gamma_{,r})^2 - \frac{1}{2}\alpha_{,r}\gamma_{,r}\right)(V^{[\phi}Y^{\theta]})_{,\phi} -$$

$$+ \frac{1}{r}\left(\gamma_{,rr}+\frac{1}{2}(\gamma_{,r})^2 + \frac{1}{r}\gamma_{,r} - \frac{1}{2}\alpha_{,r}\gamma_{,r}\right)V^{[r}Y^{\theta]} \qquad (46c)$$

$$\frac{1}{2} V^\phi\left(\gamma_{,rr} + \frac{1}{2}(\gamma_{,r})^2 + \frac{1}{r}\gamma_{,r} - \frac{1}{r}\alpha_{,r} - \frac{1}{2}\alpha_{,r}\gamma_{,r}\right) =$$

$$-\frac{\alpha_{,r}}{r}(V^{[t}Y^{\phi]})_{,t} + \frac{\gamma_{,r}}{r}(V^{[r}Y^{\phi]})_{,r} + \left(\gamma_{,rr}+\frac{1}{2}(\gamma_{,r})^2 - \frac{1}{2}\alpha_{,r}\gamma_{,r}\right)(V^{[\theta}Y^{\phi]})_{,\theta}$$

$$+ \frac{1}{r}\left(\gamma_{,rr}+\frac{1}{2}(\gamma_{,r})^2 + \frac{1}{r}\gamma_{,r} - \frac{1}{2}\alpha_{,r}\gamma_{,r}\right)V^{[r}Y^{\phi]} +$$

$$+ \cot\theta\left(\gamma_{,rr}+\frac{1}{2}(\gamma_{,r})^2 - \frac{1}{2}\alpha_{,r}\gamma_{,r}\right)V^{[\theta}Y^{\phi]} . \qquad (46d)$$

The restriction (37) has not been applied to (46) as it results in no great simplification. So given V, any solution to (46) for Y will generate a potential from (42) and (38) ensuring the conservation of J_V in an arbitrary static, spherical space-time.

Only the potentials for matter currents of the vector fields

$$Dt = \frac{\partial}{\partial t} , \qquad (47)$$

$$\phi z = \frac{\partial}{\partial \phi} , \qquad (48)$$

and Dz and Bz given in (35), (36) with the condition (37), need be found from (46). This is because the spherical symmetry forces

$$P_{\phi z} = P_{\phi y} = P_{\phi x} \quad , \tag{49}$$

$$P_{Dz} = P_{Dy} = P_{Dx} \quad , \tag{50}$$

$$P_{Bz} = P_{By} = P_{Bx} \quad . \tag{51}$$

Starting with Dt and substituting (47) for V into (46) yields

$$-\frac{e^{\alpha}-1}{r} - \alpha_{,r} = \left(\frac{e^{\alpha}-1}{r}\right)Y^{r}{}_{,r} + \frac{1}{2r}\left(\alpha_{,r}-\gamma_{,r} + (\alpha_{,r}+\gamma_{,r})e^{\alpha}\right)Y^{r} +$$

$$+ \frac{1}{2}\alpha_{,r}\left(Y^{\theta}{}_{,\theta} + Y^{\phi}{}_{,\phi} + \cot\theta Y^{\theta}\right) ,$$

$$0 = Y^{r}{}_{,t} = Y^{\theta}{}_{,t} = Y^{\phi}{}_{,t} .$$

By matching coefficients, one finds the simple solution

$$Y^{t} = Y^{\phi} = 0 \;,\; Y^{r} = -r \;,\; Y^{\theta} = \frac{1}{\alpha_{,r}}(\alpha_{,r}+\gamma_{,r})(1-e^{\alpha})\cot\theta \; . \tag{52}$$

Substituting (52) into (42), and the result into (38), gives θ_{Dt} after some simplification with (30), as

$$\theta_{Dt} = -\frac{1}{16\pi} e^{\frac{\gamma+\alpha}{2}}(\alpha_{,r}+\gamma_{,r})(e^{-\alpha}-1)r\cos\theta\,\omega^{r}\wedge\omega^{\phi} +$$

$$+ \frac{1}{8\pi} e^{\frac{\alpha+\gamma}{2}}(e^{-\alpha}-1)r\sin\theta\,\omega^{\theta}\wedge\omega^{\phi} \quad .$$

Although Y^{θ} is badly behaved when $\sin\theta$ or $\alpha_{,r}$ vanish, the offending factors are vitiated by the $\sqrt{g}$ in η and by ϵ respectively in the calculation of θ_{Dt}. Unfortunately this θ_{Dt} is not defined at $\theta = 0$ or π as its $r\phi$ component does not vanish there and ω^{ϕ} is undefined there. However, imposing (37) removes the problem, leaving

$$\theta_{Dt} = \frac{1}{8\pi} e^{\frac{\alpha+\gamma}{2}} (e^{-\alpha}-1) r\sin\theta\omega^{\theta}\wedge\omega^{\phi} . \qquad (53)$$

It is surprising that this simple ad hoc method of finding potentials requires (37) to obtain a curvature dependent potential ensuring

$$P_{Dt} = -M . \qquad (54)$$

Consider the vector field

$$D\bar{t} = e^{\frac{1}{2}(\alpha+\gamma)} \frac{\partial}{\partial t}$$

which although not a Killing vector field does have a conserved matter current. Using (11), (29), (31), (32) and (33) its conserved quantity is

$$P_{D\bar{t}} = \frac{1}{2} \int_0^R (re^{-\alpha}-r)_{,r} dr = -M$$

for all static spherical isolated bodies[51]. Applying the same method to find its curvature dependent potential yields

$$y^t = y^{\theta} = y^{\phi} = 0 \quad , \quad y^r = -r$$

$$\theta_{D\bar{t}} = \frac{1}{8\pi} (e^{-\alpha}-1) r\sin\theta\omega^{\theta}\wedge\omega^{\phi} .$$

Remarkably the vector field Y agrees with the remains of (52) after imposing (37). Although $\theta_{D\bar{t}}$ differs from (53) only the metric symmetries[52] have been used to find it. Thus the exterior solution either encloses a mass M or a two dimensional hole. This has nothing to do with coordinate singularities and happens because the potential $\theta_{D\bar{t}}$ is closed, but not exact, in the absence of matter. The important fact that $\theta_{D\bar{t}}$ is curvature dependent will be used again in the next chapter.

When ϕz from (48) is put in (46), it reduces to

$$O = Y^t{}_{,\phi} = Y^r{}_{,\phi} = Y^\theta{}_{,\phi} \ ,$$

$$\gamma_{,rr} + \frac{1}{2}(\gamma_{,r})^2 + \frac{1}{r}\gamma_{,r} - \frac{1}{r}\alpha_{,r} - \frac{1}{2}\alpha_{,r}\gamma_{,r} = \frac{\alpha_{,r}}{r} Y^t{}_{,t} - \frac{\gamma_{,r}}{r} Y^r{}_{,r} +$$

$$+ (\gamma_{,rr} + \frac{1}{2}(\gamma_{,r})^2 - \frac{1}{2}\alpha_{,r}\gamma_{,r})(Y^\theta{}_{,\theta} + \cot\theta Y^\theta) +$$

$$- \frac{1}{r}(\gamma_{,rr} + \frac{1}{2}(\gamma_{,r})^2 + \frac{\gamma_{,r}}{r} - \frac{1}{2}\alpha_{,r}\gamma_{,r}) Y^r \ ,$$

and matching coefficients gives the simple solution

$$Y^t = -t\,\frac{(\alpha_{,r}+\gamma_{,r})}{\alpha_{,r}} \ , \ Y^r = -r \ , \ Y^\theta = Y^\phi = 0 \ .$$

With these values for Y and (30), the potential found from (42) and (38) is:

$$\theta_{\phi z} = \frac{1}{16\pi} e^{\frac{\gamma-\alpha}{2}} \gamma_{,r} r^2 \sin\theta\,\omega^t \wedge \omega^\theta + \frac{1}{16\pi} e^{\frac{\gamma-\alpha}{2}} (\alpha_{,r}+\gamma_{,r}) tr\sin\theta\,\omega^r \wedge \omega^\theta \ .$$

Here the potential is well defined and the matter has only been restricted by the metric symmetries. The integral of this potential over any constant r,t surface S obviously vanishes, turning (49) into

$$O = P_{\phi z} = P_{\phi y} = P_{\phi x} \ . \qquad (55)$$

So an arbitrary, static, spherical star has no General Relativistic angular momentum about the origin in agreement with the linearized theory.

Continuing with Dz and substituting (35) into (46) gives

$$0 = 2(1-e^{\alpha})\,\frac{\cos\theta}{r}\,Y^{t}{}_{,r} + \alpha_{,r}\,\frac{\sin\theta}{r}\,Y^{t}{}_{,\theta} + (\alpha_{,r}+\gamma_{,r})(1-e^{\alpha})\,\frac{\cos\theta}{r}\,Y^{t} \qquad (56)$$

$$-2\cos\theta\left(\gamma_{,r} + \frac{(1-e^{\alpha})}{r}\right) = 2(1-e^{\alpha})\,\frac{\cos\theta}{r}\,Y^{t}{}_{,t} +$$

$$+\,\gamma_{,r}\,\frac{\sin\theta}{r}\,Y^{r}{}_{,\theta} + \gamma_{,r}\cos\theta\,(Y^{\theta}{}_{,\theta} + Y^{\phi}{}_{,\phi}) +$$

$$+\,2\gamma_{,r}\,\frac{\cos\theta}{r}\,Y^{r} + \gamma_{,r}\,\frac{\cos 2\theta}{\sin\theta}\,Y^{\theta} \qquad (57b)$$

$$-\,\frac{\sin\theta}{r}\left(\gamma_{,rr} + \frac{1}{2}(\gamma_{,r})^{2} + \frac{1}{r}\gamma_{,r} - \frac{1}{r}\alpha_{,r} - \frac{1}{2}\alpha_{,r}\gamma_{,r}\right)$$

$$= -\,\alpha_{,r}\,\frac{\sin\theta}{r^{2}}\,Y^{t}{}_{,t} + \gamma_{,r}\,\frac{\sin\theta}{r^{2}}\,Y^{r}{}_{,r} + \gamma_{,r}\,\frac{\cos\theta}{r}\,Y^{\theta}{}_{,r} +$$

$$+\,\frac{\sin\theta}{r^{2}}\left(\gamma_{,rr} + \frac{1}{2}(\gamma_{,r})^{2} - \frac{1}{2}\alpha_{,r}\gamma_{,r}\right)(rY^{\phi}{}_{,\phi} + Y^{r}) +$$

$$+\left(\gamma_{,rr} + \frac{1}{2}(\gamma_{,r})^{2} + \frac{1}{r}\gamma_{,r} - \frac{1}{2}\alpha_{,r}\gamma_{,r}\right)\frac{\cos\theta}{r}\,Y^{\theta} \qquad (57c)$$

$$0 = \gamma_{,r}\,\frac{\cos\theta}{r^{2}}\,(rY^{\phi}{}_{,r} + Y^{\phi}) - \frac{1}{r}\left(\gamma_{,rr} + \frac{1}{2}(\gamma_{,r})^{2} +\right.$$

$$\left.-\,\frac{1}{2}\alpha_{,r}\gamma_{,r}\right)(\sin\theta\,Y^{\phi}{}_{,\theta} + \cos\theta\,Y^{\phi}) \quad . \qquad (57d)$$

There can be no solution to these equations without imposing

(37). Invoking (37), merely changes (56) to

$$0 = 2(1-e^{\alpha}) \frac{\cos\theta}{r} Y^{t}{}_{,r} + \alpha_{,r} \frac{\sin\theta}{r} Y^{t}{}_{,\theta}. \qquad (57a)$$

Now the four equations (57) have the simple solution

$$Y^{t} = -t \ , \ Y^{r} = -r \ , \ Y^{\theta} = Y^{\phi} = 0.$$

These values of Y lead to the well-behaved potential

$$\theta_{Dz} = \frac{1}{16\pi} e^{\frac{\gamma-\alpha}{2}} \gamma_{,r} r\sin^{2}\theta\omega^{t}\wedge\omega^{\phi} + \frac{1}{16\pi} e^{\frac{\gamma-\alpha}{2}} \alpha_{,r} t\sin^{2}\theta\omega^{r}\wedge\omega^{\phi} +$$

$$- \frac{1}{16\pi} e^{\frac{\alpha+\gamma}{2}} (e^{-\alpha}-1) t\sin 2\theta\omega^{\theta}\wedge\omega^{\phi} \ ,$$

giving a momentum

$$P_{Dz} = \frac{Mt}{8\pi r} \int_{S} \sin 2\theta\omega^{\theta}\wedge\omega^{\phi} = 0.$$

So (50) becomes

$$0 = P_{Dz} = P_{Dy} = P_{Dx} \ , \qquad (58)$$

and the General Relativistic linear momentum for a static, spherical source subject to (37) agrees with the corresponding flat space limit.

Finally Bz in (36) is substituted into (46) giving

$$2\cos\theta(1-e^{\alpha}-r\alpha_{,r}) = \frac{2\cos\theta}{r}(1-e^{\alpha})(tY^{t}{}_{,r}-rY^{r}{}_{,r}-Y^{r}) + \alpha_{,r}\frac{t\sin\theta}{r}Y^{t}{}_{,\theta} +$$

$$+r\alpha_{,r}\cos\theta(Y^{\theta}{}_{,\theta}+Y^{\phi}{}_{,\phi}+2\cot 2\theta Y^{\theta})+2\cos\theta\alpha_{,r}Y^{r} +$$

$$+(\alpha_{,r}+\gamma_{,r})(1-e^{\alpha})\frac{\cos\theta}{r}(tY^{t}-rY^{r}) \qquad (59)$$

$$2\frac{\cos\theta}{r}(1-e^{\alpha}+r\gamma_{,r})t = -2(1-e^{\alpha})\frac{\cos\theta}{r}(tY^{t}{}_{,t}+Y^{t}-rY^{r}{}_{,t}) +$$

$$-\gamma_{,r}\frac{t}{r}(\sin\theta Y^{r}{}_{,\theta}+2\cos\theta Y^{r}) +$$

$$-\gamma_{,r}t\cos\theta(Y^{\theta}{}_{,\theta}+Y^{\phi}{}_{,\phi}) + 2\gamma_{,r}\cot 2\theta Y^{\theta} \qquad (60b)$$

$$-t\sin\theta(\gamma_{,rr}+\frac{1}{2}(\gamma_{,r})^{2}+\frac{1}{r}\gamma_{,r}-\frac{1}{r}\alpha_{,r}-\frac{1}{2}\alpha_{,r}\gamma_{,r}) =$$

$$-\alpha_{,r}\frac{\sin\theta}{r}(tY^{t}{}_{,t}+Y^{t})-\cos\theta(r\alpha_{,r}Y^{\theta}{}_{,t}-t\gamma_{,r}Y^{\theta}{}_{,r})+\gamma_{,r}\frac{t\sin\theta}{r}Y^{r}{}_{,r} +$$

$$+t\sin\theta(\gamma_{,rr}+\frac{1}{2}(\gamma_{,r})^{2}-\frac{1}{2}\alpha_{,r}\gamma_{,r})(Y^{\phi}{}_{,\phi}+\frac{1}{r}Y^{r}) +$$

$$+t\cos\theta(\gamma_{,rr}+\frac{1}{2}(\gamma_{,r})^{2}+\frac{1}{r}\gamma_{,r}-\frac{1}{2}\alpha_{,r}\gamma_{,r})Y^{\theta} \qquad (60c)$$

$$0 = -\alpha_{,r}r\cos\theta Y^{\phi}{}_{,t} + \gamma_{,r}t\cos\theta(Y^{\phi}{}_{,r}+\frac{1}{r}Y^{\phi})-t(\gamma_{,rr}+\frac{1}{2}(\gamma_{,r})^{2} +$$

$$-\frac{1}{2}\alpha_{,r}\gamma_{,r})(\sin\theta Y^{\phi}{}_{,\theta}+\cos\theta Y^{\phi}) \qquad (60d)$$

Again the imposition of (37) merely removes a term from (59) leaving

$$2\cos\theta(1-e^{\alpha} - r\alpha_{,r}) = \frac{2\cos\theta}{r}(1-e^{\alpha})(tY^{t}{}_{,r} - rY^{r}{}_{,r} - Y^{r}) + \alpha_{,r}\frac{t}{r}\sin\theta Y^{t}{}_{,\theta} +$$

$$r\alpha_{,r}\cos\theta(Y^{\theta}{}_{,\theta} + 2\cot 2\theta Y^{\theta} + Y^{\phi}{}_{,\phi}) + 2\cos\theta\alpha_{,r}Y^{r} \qquad (60a)$$

The equations (60) now have the solution

$$Y^{t} = -\frac{t^2+r^2}{2t}, \quad Y^{r} = -r, \quad Y^{\theta} = Y^{\phi} = 0,$$

which gives rise to the potential

$$\theta_{Bz} = \frac{1}{16\pi}e^{\frac{\gamma-\alpha}{2}}\gamma_{,r}tr\sin^2\theta\omega^{t}\wedge\omega^{\phi} + \frac{1}{32\pi}e^{\frac{\gamma-\alpha}{2}}\alpha_{,r}(t^2+r^2)\sin^2\theta\omega^{r}\wedge\omega^{\phi} +$$

$$-\frac{1}{32\pi}e^{\frac{\alpha+\gamma}{2}}(e^{-\alpha}-1)(t^2-r^2)\sin 2\theta\omega^{\theta}\wedge\omega^{\phi},$$

and momentum

$$P_{Bz} = \frac{M}{16\pi r}(t^2-r^2)\int_S \sin 2\theta\omega^{\theta}\wedge\omega^{\phi} = 0.$$

Combining this result with (51) yields

$$0 = P_{Bz} = P_{By} = P_{Bx}. \qquad (61)$$

The role of these "boost momenta" is to specify the position of the centroid at t=0, as can be seen from (26). They also permit a useful decomposition of the total angular momentum into a rotational and orbital part, which describes the frame dependence of the centroid curve within a Poincaré family of special frames[1].

The vanishing spatial and boost momenta of this special frame give its centroid curve (26) the simple form

$$C\mu = (t,0,0,0) \quad . \qquad (62)$$

Since both the angular momenta (55) and boost momenta (61) are zero, (62) is a unique curve for this Poincaré family of special frames[1]. It is very satisfying that the centroid (62) coincides with the r=0 centre of symmetry curve. The spherical symmetry forces this r=0 curve to be geodesic. For the rate of change of its tangent vector U may be written as

$$\nabla_U U = \lambda U + W \; ,$$

where λ is some function and W is perpendicular to U. The spherical symmetry requires that all vectors perpendicular to U be equivalent, so W must vanish leaving

$$\nabla_U U = \lambda U \; ,$$

which is just the geodesic equation for the r=0 curve. Thus if the Schwarzchild time t that parameterizes the centroid (62) were linearly related to the proper time of the central geodesic, one would have a unique geodesic centroid for this Poincaré family of special frames.

Using only spherical symmetry and the field equations Synge[47] gives the relation between the Schwarzchild time t and the proper time t' of the central geodesic in the notation of this paper as

$$t = \int \exp\left[\frac{1}{2}\int_0^{\chi(t')} (\alpha_{,r}+\gamma_{,r})\,dr\right]dt' \qquad (63)$$

where $\chi(t')$ describes the boundary of the star. Remarkably on imposing (37), (63) reduces to

$$t = t' \qquad (64)$$

and (37) is both necessary and sufficient for the Schwarzchild time to be the proper time of the central geodesic. Thus condition (37) which was necessary and sufficient to obtain the special frame also assures that the centroid (62) is geodesic.

Although geodesic motion has been shown for this Poincaré family of special frames, (64) shows how unsatisfactory (37) is from a physical point of view. Condition (37) is logically equivalent to (64) and (64), by equating the proper time on the central geodesic to that on a geodesic at infinity, is also equating the gravitational potential g_{tt} at these places

$$g_{tt}\Big|_{r=0} = \lim_{r\to\infty} g_{tt} = -1 \ .$$

Thus a photon emitted from the centre of a star satisfying (37) would not show any gravitational red shift - in fact such a photon observed in the Schwarzchild vacuum would appear blue shifted. Similarly any test body falling radially through the star could not penetrate r=0 unless it had an initial radial velocity at infinity.

It was symmetries which motivated the choice of (34) as a candidate for a special frame and it seems quite likely that there will be many other frames not linearly related to (34) which will be special for equations of state different than (37). Of considerable interest is whether there are special frames which can give geodesic motion for all isolated static spherical perfect fluids. Although four of the ten momenta for (34) arise from metric symmetries, six do not. This feature of momenta without symmetries is crucial to a kinematics based on conserved matter currents. In fact it was shown that only under (37) does the time translation symmetry generate the expected energy. So for both arbitrary and symmetric space-times Killing vector fields cannot be relied upon to generate the correct momenta. To link the purely mechanical momenta to the gravitational field requires curvature dependent potentials and the simple method embodied in (43) may be of more general use. Oddly enough the vector field Y almost always has the form $(0,-r,0,0)$. Understanding this feature and why the non-Killing vector field $D\bar{t}$ generates the correct energy may shed light on the special frames for perfect fluids or the kinematics of other space-times. Nevertheless a non-trivial special frame with curvature dependent potentials and geodesic centroids has been shown to exist and only further research can answer the questions raised here.

CHAPTER V

OUTLOOK

5.1 Curvature Dependent Potentials

This section elaborates some of the features of the potentials introduced in the last chapter. There it was shown that a mechanical potential would exist for every conserved matter current in a physically reasonable space-time. It was also suggested that the potential ought to be linear in the full Riemann curvature and this was shown to be the case for the kinematically useful matter currents for the static spherical stars. As yet one does not know how to characterise the kinematically useful matter currents for arbitrary space-time, but one might hope that a better understanding of the potentials would help in this matter.

It is not clear whether or not the requirement for the potential to be linear in the full curvature would eliminate too many conserved currents. The following argument shows that the previously assumed structure of the potential is rather general. Let $*J$ be an exact three-form so that

$$*J = d*\alpha \tag{1}$$

where α is dual to some potential two-form. Passing to local coordinates with basis forms ω^{α}, one asks if there is a tensor A satisfying

$$*J = d*(\tfrac{1}{2} A^{\alpha\beta} \mathcal{C}_{\alpha\beta\mu\nu} \omega^{\mu} \wedge \omega^{\nu}) \tag{2}$$

giving $*J$ a potential linear in the full curvature $\mathcal{C}$. Equating (1) and (2) and noting that the indices occur in antisymmetric blocks allows the result to be written in terms of 6 vectors and 6×6 matrices as

$$A^{A} \mathcal{C}_{AB} = \alpha_{B} \ . \tag{3}$$

If det $\mathcal{C}_{AB}$ does not vanish, the $\mathcal{C}_{AB}$ has an inverse $\tilde{\mathcal{C}}^{BC}$

$$\mathcal{C}_{AB} \tilde{\mathcal{C}}^{BC} = \delta_{A}^{C} \ .$$

Using this inverse allows the solution to (3) as

$$A^{C} = \alpha_{B} \tilde{\mathcal{C}}^{BC} \ . \tag{4}$$

Thus a sufficient condition on the space-time that all conserved matter currents have potentials linear in the full curvature is that det $\mathcal{C}_{AB}$ not vanish. This is a very weak condition, but its violation poses some interesting questions.

Before examining them it is worth looking at another restriction imposed on the potential in the previous chapter. For mathematical simplicity it was suggested that the potential be linear in the vector field V which generated the conserved matter current. This is accomplished by merely setting

$$A^{\alpha\beta} = B^{\alpha\beta}{}_{\gamma} V^{\gamma} \tag{5}$$

where there is considerable freedom in the choice of B. To see this go to the frame where V is a basis vector, say

$$V = \frac{\partial}{\partial x^0} .$$

Then in this frame set

$$B^{\alpha\beta}_{\;\;0} = A^{\alpha\beta}$$

$$B^{\alpha\beta}_{\;\;a} \text{ arbitrary} \qquad a \neq 0 .$$

So B is unique up to any third rank tensor K satisfying

$$K^{\alpha\beta}{}_{\gamma} V^{\gamma} = 0 . \tag{6}$$

Thus combining (5) and (2), every conserved matter current has a potential of the form

$$*J_V = d*(\tfrac{1}{2} V^{\gamma} B^{\alpha\beta}{}_{\gamma} \epsilon_{\alpha\beta\mu\nu} \omega^{\mu} \wedge \omega^{\nu})$$

provided det ϵ_{AB} is non-vanishing. Finally the ad hoc form used for the potentials in section 4.5

$$*J_V = d*(\tfrac{1}{2} V^{\alpha} Y^{\beta} \epsilon_{\alpha\beta\mu\nu} \omega^{\mu} \wedge \omega^{\nu})$$

can be obtained, if V is an eigen-vector of B

$$V^{\gamma} B^{\alpha\beta}{}_{\gamma} = V^{\alpha} Y^{\beta} \tag{7}$$

Whether there is enough freedom in the gauge or in B due to (6), to always obtain (7) is not yet known.

Physically it is not clear what the vanishing of det ϵ_{AB} means; however, before dismissing it as a non-generic condition, it must be remembered that the existence of a vacuum region (Ricci flatness) is also non-generic but necessary for isolated bodies. Thus if the space-time could always be changed a little bit to prevent the vanishing of det ϵ_{AB} without altering the vacuum regions one would have a good argument for always being able to find curvature dependent potentials.

The most drastic way of permitting the vanishing of det ϵ_{AB} is to give the space-time Riemann flat regions. Although this contradicts the idea that the gravitational field should reach out everywhere, Riemann flat regions are not prohibited by causality, properties of the matter tensor or the field equations[53]. In fact a solution due to Ehlers and Kundt which is a Ricci flat region propagating through a Riemann flat space is used to discuss exact gravitational waves[48]. It is accomplished by allowing the metric to be non-analytic at the boundary between the two regions and perhaps this is an argument for demanding analyticity.

Using C^{∞} functions such as partitions of unity[49] instead of the more usually required C^3 or C^4 functions permits one to patch together two metrices $\bar{g},\tilde{g}$ each describing a separate region, into a single metric g for the manifold by

$$g = \lambda\bar{g} + (1-\lambda)\tilde{g} \quad 0 \leq \lambda \leq 1$$

6

where λ is a partition of unity equal to 1 in the region where $\bar{g}$ holds and passing smoothly to zero in the region where $\tilde{g}$ applies. The boundary region where λ is between zero and one may be made as small as one pleases, but of course cannot vanish. The metrics $\bar{g}$ and $\tilde{g}$ may be taken as solutions to Einstein's equations and then if there was just the right amount of physical matter in the boundary region so that g was a solution to the field equations there as well, the patching would be complete. Although it may not always be possible to patch two given metrics together with physical matter, the possibility is raised of gravitationally hiding large amounts of matter in this way.

The first thing one might try is to construct a static spherical shell of matter for the boundary region enclosing an exterior Schwarzchild solution with curvature parameter M and having Riemann flat space-time outside this boundary shell. Thus the shell and the star giving the Schwarzchild solution would be gravitationally invisible in the flat region. However, consideration of the system's General Relativistic energy shows that the shell must have an energy opposite in sign to that of the star, so that the matter in the shell must violate the energy condition. To see this note that because the system is static and spherical all the machinery of the previous chapter for the mechanical potential of the time-like $D\bar{t}$ vector field's matter current remains valid. Even though det G_{AB} is zero in the Riemann flat region the mechanical potential is still linear in the curvature by construction and so the potential vanishes there too. Thus the energy of the whole system calculated from a flux integral in the flat region vanishes. On the other hand the energy of the star calculated from a flux integral in the exterior Schwarzchild region is $-M$ which is negative by convention. To find the energy of the shell one merely subtracts the interior integral from the exterior one to obtain $+M$ which can never be obtained by integrating a negative $T^0{}_0$ through the shell.

Another example due to Jessup[50] takes a static, spherical metric which is not analytic on a constant r surface in curvature coordinates so that the space is flat outside and inside the metric functions approach the flat values smoothly. For any choice of such an interior metric and there are many, one simply evaluates the Einstein tensor (4.5.31) to check for physicality. Although his choice gave a physical matter tensor for some distance into the "star", it had a singularity at the origin. Probably one can pick a metric with a physical matter tensor arbitrarily close to the origin, but the singularity is absolutely necessary for the following reason. Again using the curvature dependent potential one tries to calculate the "star's" energy with a flux integral in the flat vacuum - the energy is then zero. However, integrating the negative $T^0{}_0$ through the star must give some number less than zero. The contradiction can only be resolved if the compact three-surface of integration through the star has a disjoint boundary - one piece in the flat vacuum giving no contribution to the energy and one piece inside the star covering the singularity and giving the total energy.

These two examples show the power and utility of curvature dependent mechanical potentials. Without invoking analyticity they can express the long range character of the gravitational field and their mere existence forces certain pathologies such as energy condition violation and singularities in odd situations. For this reason alone and apart from any kinematical desiderata, curvature dependent potentials should become an important feature in General Relativity.

5.2 Questions

The purpose of this section is to concisely review the work of previous chapters pointing out some of the questions which will have to be answered to obtain a General Relativistic kinematics for arbitrary space-times with the assumptions made here.

Assumption I: The conserved quantities of kinematic interest arise as integrals of conserved matter currents from a matter tensor which satisfies Einstein's equations

As pointed out in section 2.5, this assumption takes into account the special nature of the gravitational field and avoids the problem of motion without matter. Thus one would like to know if there is any simple geometric characterization of the vector fields X satisfying Manoff's condition

$$(X^{\mu;\nu}{}_{;\nu} - X^{\nu}{}_{;\nu}{}^{;\mu} - \frac{1}{2} RX^{\mu})_{;\mu} = 0 \qquad (8)$$

so that J_X is conserved.

Assumption II: The interesting conserved matter currents should have mechanical potentials which depend linearly on the full Riemann curvature and the vector field which generates the current

In section 4.3 it was observed that a potential will always exist provided $H^3(M,\mathbb{R})$ vanishes for the manifold and in 5.1 that one can always find a potential linear in the full curvature, provided det ϵ_{AB} does not vanish. As remarked in section 4.1 this would give a very strong form of the "equivalence principle" as the purely mechanical properties of a body would be determined by the body's vacuum gravitational field. Although Komar's potential (section 2.5) for the Ricci current of a Killing vector field is well known, one does not even know yet what the potential is for the Einstein current of a Killing vector field, let alone the vector fields satisfying (8).

Assumption III: The kinematics requires a special coordinate system x^{μ} determined by the matter distribution by demanding the four basis vectors $D\mu$ to obey (8). Causality requirements are satisfied if D0 is time-like.

This assumption contains a version of "Mach's principle" in that if the matter distribution changes, then so must the special frame in order that the basis vectors still satisfy (8). Although these basis vectors would generate conserved linear momentum it is not yet known under which conditions one can find four commuting vector fields

$$[D\mu, D\nu] = 0 \tag{9}$$

which satisfy (8). The imposition of kinematic constraints such as (9) (and those below) restricts the vector fields for which one wants curvature dependent potentials and these constraints will probably figure significantly in obtaining the final form of the mechanical potentials. Demanding that D0 be time-like

$$g(D0, D0) \leq 0 \tag{10}$$

ensures that constant x^0 surfaces suitable for integration will exist for isolated bodies enjoying causal motion. As mentioned in section 4.4, the weak energy condition and the metric convention force the General Relativistic energy P_{D0} to be negative.

Assumption IV: The equations of motion for an extended body are the equations of the centroid world line and when the three Boost vectors

$$Ba = x^a D0 + x^0 Da \quad a \neq 0 \tag{11}$$

for the special frame satisfy (8), the centroid world line has a constant slope in the special frame (cf. eq. 4.4.26)

As pointed out in Chapter 3 and section 4.4 the weak energy condition ensures that a convex body contain the centroid. But of all the world lines inside a body's world tube why pick this one? At this point one is beginning to pay dearly for the lack of knowledge about conserved matter currents and the special frames. The reasons for picking it are essentially that it is most simply defined; it works in flat space theories; and until further research can prove otherwise, it is a physically reasonable candidate. It should be realised that conserved angular momenta $P_{\phi a}$ are not needed to obtain the linear character of the centroid, as only the $P_{D\mu}$ and P_{Ba} figure in the derivation of (4.4.26). From the $P_{D\mu}$ and the centroid one can obtain a conserved orbital angular momentum as was done explicitly for Special Relativistic kinematics in Chapter 3. Also as shown in Chapter 3 the kinematic use of conserved angular momentum $P_{\phi a}$ is that it allows the definition of a conserved spin or rotational angular momentum which specifies the non-uniqueness of the centroid (3.2.14). The frame dependence of the centroid intimately links the question of its uniqueness to the uniqueness of special frames. One would expect the transformations between the special frames to form a group and the smallest group containing both displacements and boosts is the Poincaré group.

Assumption V: The special frame's rotation vectors

$$\phi c = x^a Db - x^b Da \tag{12}$$

where (a,b,c) is a cyclic permutation of (1,2,3), also satisfy (8).

This completes the desiderata for special frames as outlined in section 4.4 and ensures that the Poincaré group is a subgroup of the transformation group relating special frames. Although a non-trivial example of special frames has been shown to exist (section 4.5), general questions of their existence, let alone uniqueness, require further research. It should be pointed out that a General Relativistic Kinematics inherits the Special Relativistic problem of massless radiation treated in Chapter 3, if the transformation group between special frames contains the Poincaré group. For if a massless body with spin satisfying Einstein's equations admits a Poincaré family of special frames, then the coordinate values for the possible centroids in a constant time surface are unbounded (section 3.3). Even so, one cannot help wondering whether the added richness of the transformation group between special frames (if it is larger than the Poincaré group) could account for the observed angular momentum equivalent of light.

Assumption VI: Geodesic postulate

For the sake of Newton's first law one would like an extended body to travel a geodesic. As mentioned in section 4.4 one does not yet know enough about special frames to demand that all centroids be geodesics or that the tube of centroids contain a geodesic. Remarkably in the example of section 4.5 the same condition (4.5.37) which gave the existence of special frames also ensured that the centroid was a geodesic. It might turn out that there are many special frames which are not linearly related to one another and that only one or some of these would have geodesic centroids. In this regard one might note the simplifications which result if one looks for divergence-free special frames whose basis vectors satisfy

$$\mathrm{Div} D\mu = 0 . \tag{14}$$

Inertial frames have this constant volume character and it is a very weak condition. Any divergence-free vector field U can be rescaled to another divergence-free vector field fU if and only if the function f satisfies

$$\nabla_U f = 0 .$$

This fact ensures that B_a (11) and ϕ_a (12) are divergence-free if the $D\mu$ satisfy (14). For divergence-free vector fields U Manoff's condition (8) simplifies to

$$\mathrm{Div} \Box U = \frac{1}{2} \nabla_u R \tag{15}$$

where $\Box$ is the D'Alembertian operator. The special frame found in section 4.5 does satisfy (14) and in fact the condition (4.5.37) on the matter which made it special is equivalent to (14).

APPENDIX I

CALCULATION OF $J^{\mu}_{Dz,\mu}$ and $J^{\mu}_{Bz;\mu}$

First the divergence of an arbitrary matter current is simplified.

$$J^{\mu}_{V;\mu} = (T^{\mu}{}_{\nu}V^{\nu})_{;\mu} \qquad \text{by } J^{\mu}_{V} \equiv T^{\mu}{}_{\nu}V^{\nu}$$

$$= T^{\mu}{}_{\nu}V^{\nu}{}_{;\mu} \qquad \text{by } T^{\mu}{}_{\nu;\mu} = 0$$

$$= T^{\mu}{}_{\nu}(V^{\mu}{}_{,\mu} + \Gamma^{\nu}_{\alpha\mu}V^{\alpha}) \quad \text{by introducing connection coefficient } \Gamma$$

$$= \sum_{\mu} T^{\mu}{}_{\mu}(V^{\mu}{}_{,\mu} + \Gamma^{\mu}_{\alpha\ \mu}V^{\alpha}) \quad \text{by diagonality of } T$$

$$J^{\mu}_{V;\mu} = \sum_{\mu} T^{\mu}{}_{\mu}(V^{\mu}{}_{,\mu} + \frac{\partial}{\partial x^{\alpha}}(\ln\sqrt{|g_{\mu\mu}|})\ V^{\alpha}) \quad \text{by diagonality of } g.$$

Substitute $Dz = \cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}$ for V into $J^{\mu}_{V;\mu}$

$$J^{\mu}_{Dz;\mu} = T^{\theta}{}_{\theta}(-\frac{\cos\theta}{r}) + \sum_{\mu} T^{\mu}{}_{\mu}\ [\frac{\partial}{\partial r}(\ln\sqrt{|g_{\mu\mu}|})\cos\theta - \frac{\partial}{\partial\theta}(\ln\sqrt{|g_{\mu\mu}|})\ \frac{\sin\theta}{r}]$$

On using $ds^2 = -e^{\gamma}dt^2 + e^{\alpha}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$, one has

$$J^{\mu}_{Dz;\mu} = -T^{\theta}{}_{\theta}\ \frac{\cos\theta}{r} + \frac{1}{2}(T^{t}{}_{t}\gamma_{,r} + T^{r}{}_{r}\alpha_{,r} + T^{\theta}{}_{\theta}\ \frac{2}{r} + T^{\phi}{}_{\phi}\ \frac{2}{r})\cos\theta +$$

$$- T^{\phi}{}_{\phi}(\frac{\cos\theta}{\sin\theta})(\frac{\sin\theta}{r})$$

$$= \frac{1}{2}(T^{t}{}_{t}\gamma_{,r} + T^{r}{}_{r}\alpha_{,r})\cos\theta\ .$$

With Einstein's equations, $T^t{}_t = \frac{1}{8\pi} G^t{}_t = \frac{1}{8\pi r^2}(e^{-\alpha}[1-r\alpha_{,r}]-1)$ and $T^r{}_r = \frac{1}{8\pi} G^r{}_r = \frac{1}{8\pi r^2}(e^{-\alpha}[1+r\gamma_{,r}]-1)$, $J^{\mu}_{Dz;\mu}$ can now be written

$$J^{\mu}_{Dz;\mu} = \frac{\cos\theta}{16\pi r^2}[(e^{-\alpha}[1-r\alpha_{,r}]-1)\gamma_{,r} + (e^{-\alpha}[1+r\gamma_{,r}]-1)\alpha_{,r}]$$

$$J^{\mu}_{Dz;\mu} = \frac{\cos\theta}{16\pi r^2}(e^{-\alpha}-1)(\alpha_{,r}+\gamma_{,r}),$$

Finally to calculate $J^{\mu}_{Bz;\mu}$,

$$Bz = r\cos\theta \frac{\partial}{\partial t} + t\cos\theta \frac{\partial}{\partial r} - t\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

is substituted into $J_{V;\mu}{}^{\mu}$, yielding

$$J^{\mu}_{Bz;\mu} = T^{\theta}{}_{\theta}\left(-\frac{t\cos\theta}{r}\right) + \sum_{\mu} T^{\mu}{}_{\mu}\left[\frac{\partial}{\partial t}(\ln\sqrt{|g_{\mu\mu}|})\, r\cos\theta + \right.$$

$$\left. + \frac{\partial}{\partial r}(\ln\sqrt{|g_{\mu\mu}|})\, t\cos\theta - \frac{\partial}{\partial\theta}(\ln\sqrt{|g_{\mu\mu}|})\frac{t\sin\theta}{r}\right].$$

The above expression for ds^2 permits the reduction of $J^{\mu}_{Bz;\mu}$ to

$$J^{\mu}_{Bz;\mu} = \frac{1}{2}(T^t{}_t\gamma_{,r} + T^r{}_r\alpha_{,r})\,t\cos\theta .$$

and using Einstein's equations as before gives

$$J^{\mu}_{Bz;\mu} = \frac{t\cos\theta}{16\pi r^2}(e^{-\alpha}-1)(\alpha_{,r}+\gamma_{,r}) .$$

APPENDIX II

The equivalence of $G^{\lambda}{}_{\mu}V^{\mu} = (V^{\mu}Y^{\nu})_{;\rho}\,\varepsilon^{\rho\lambda}{}_{\mu\nu}$ to $*J_{V} = d{*}\beta_{V}$

The equivalence rests on a local coordinate expression for $*d*\alpha$ where α is any two-form

$$\alpha = \frac{1}{2}\,\alpha_{\mu\nu}\omega^{\mu}\wedge\omega^{\nu} \quad ,$$

in local coordinates. Applying $*d*$ to α gives

$$*d*\alpha = *d*[\tfrac{1}{2}\,\alpha_{\mu\nu}\omega^{\mu}\wedge\omega^{\nu}]$$

$$= *d[\tfrac{1}{2}(\tfrac{1}{2}\,\alpha^{\mu\nu})\eta_{\mu\nu\alpha\beta}\omega^{\alpha}\wedge\omega^{\beta}] \quad \text{by the local coordinate action of } *$$

$$= \frac{1}{4}\,*[(\alpha^{\mu\nu}{}_{;\gamma}\omega^{\gamma}\eta_{\mu\nu\alpha\beta}+\alpha^{\mu\nu}\eta_{\mu\nu\alpha\beta;\gamma}\omega^{\gamma})\wedge\omega^{\alpha}\wedge\omega^{\beta}] \ .$$

As α and η are tensor quantities, the product rule requires d to act as a covariant derivative. Since the volume element is parallel transported, the covariant derivative of η vanishes, leaving

$$*d*\alpha = \frac{1}{4}*[\alpha^{\mu\nu}{}_{;\gamma}\eta_{\mu\nu\alpha\beta}\omega^{\alpha}\wedge\omega^{\beta}\wedge\omega^{\gamma}]$$

$$= \frac{1}{4}*[\alpha^{\mu\nu}{}_{;\varepsilon}\eta_{\mu\nu\alpha\beta}\delta^{\varepsilon}_{\gamma}\omega^{\alpha}\wedge\omega^{\beta}\wedge\omega^{\gamma}] \quad \text{by merely inserting } \delta^{\varepsilon}_{\gamma}$$

$$= \frac{1}{4}*[\alpha^{\mu\nu}{}_{;\varepsilon}\eta_{\mu\nu[\alpha\beta}\delta^{\varepsilon}_{\gamma]}\omega^{\alpha}\wedge\omega^{\beta}\wedge\omega^{\gamma}] \quad \text{by the antisymmetry of three forms}$$

$$= \frac{1}{24}*[\alpha^{\mu\nu}{}_{;\varepsilon}(\eta_{\mu\nu\alpha\beta}\delta^{\varepsilon}_{\gamma}+\eta_{\mu\nu\gamma\alpha}\delta^{\varepsilon}_{\beta}+\eta_{\mu\nu\beta\gamma}\delta^{\varepsilon}_{\alpha}-\eta_{\mu\nu\beta\alpha}\delta^{\varepsilon}_{\gamma}-\eta_{\mu\nu\gamma\beta}\delta^{\varepsilon}_{\alpha} +$$

$$- \eta_{\mu\nu\alpha\gamma}\delta^{\varepsilon}_{\beta})\omega^{\alpha}\wedge\omega^{\beta}\wedge\omega^{\gamma})]$$

$$= \frac{1}{12}{*}[\alpha^{\mu\nu}{}_{;\varepsilon}(\eta_{\mu\nu\alpha\beta}\delta^{\varepsilon}_{\gamma}+\eta_{\mu\nu\gamma\alpha}\delta^{\varepsilon}_{\beta}+\eta_{\mu\nu\beta\gamma}\delta^{\varepsilon}_{\alpha})\omega^{\alpha}\wedge\omega^{\beta}\wedge\omega^{\gamma}] \quad \text{by the antisymmetry of } \eta$$

$$= \frac{1}{12}\,\alpha^{\mu\nu}{}_{;\varepsilon}(\eta_{\mu\nu}{}^{\alpha\beta}g^{\gamma\varepsilon}+\eta_{\mu\nu}{}^{\gamma\alpha}g^{\beta\varepsilon}+\eta_{\mu\nu}{}^{\beta\gamma}g^{\alpha\varepsilon})\eta_{\alpha\beta\gamma\delta}\omega^{\delta} \quad \text{by the action of } *$$

$$= \frac{1}{12}\,\alpha_{\mu\nu;\varepsilon}(\eta^{\mu\nu\alpha\beta}\eta_{\gamma\delta\alpha\beta}g^{\gamma\varepsilon}+\eta^{\mu\nu\gamma\alpha}\eta_{\beta\delta\gamma\alpha}g^{\beta\varepsilon}+\eta^{\mu\nu\beta\gamma}\eta_{\alpha\delta\beta\gamma}g^{\alpha\varepsilon})\omega^{\delta}$$

$$= -\frac{1}{6}\,\alpha_{\mu\nu;\varepsilon}(\delta^{\mu\nu}_{\gamma\delta}g^{\gamma\varepsilon}+\delta^{\mu\nu}_{\beta\delta}g^{\beta\varepsilon}+\delta^{\mu\nu}_{\alpha\delta}g^{\alpha\varepsilon})\omega^{\delta} \quad \text{by } \delta^{\rho\lambda}_{\mu\nu} \equiv -\frac{1}{2}\,\eta^{\rho\lambda\alpha\beta}\eta_{\mu\nu\alpha\beta}\,.$$

$$= -\frac{1}{3}\,(\alpha_{\gamma\delta;\varepsilon}g^{\gamma\varepsilon}+\alpha_{\beta\delta;\varepsilon}g^{\beta\varepsilon}+\alpha_{\alpha\delta;\varepsilon}g^{\alpha\varepsilon})\omega^{\delta} \quad \text{by } \delta^{\rho\lambda}_{\mu\nu} = \delta^{\rho}_{\mu}\delta^{\lambda}_{\nu}-\delta^{\rho}_{\nu}\delta^{\lambda}_{\mu}$$

$$= -\,\alpha^{\varepsilon}{}_{\delta;\varepsilon}\omega^{\delta}$$

So $*d*\alpha$ has the local coordinate expression

$$*d*\alpha = -\alpha^{\beta}{}_{\lambda;\beta}\omega^{\lambda}\,.$$

Taking duals of $*J_v = d*\beta_v$ gives

$$J_v^b = *d*\beta_v\ ,$$

where J_v^b is the one-form associated with the matter current. With the local coordinate expressions

$$J_v^b = T_{\lambda\mu}V^{\mu}\omega^{\lambda}\ ,$$

and

$$\beta_v = \frac{1}{16\pi}\,Y^{\mu}V^{\nu}\epsilon_{\mu\nu\rho\lambda}\omega^{\rho}\wedge\omega^{\lambda}\ .$$

The equation for J_v^b becomes

$$T_{\lambda\mu}V^{\mu}\omega^{\lambda} = \frac{1}{8\pi} *d*[\frac{1}{2} Y^{\mu}V^{\nu}\mathcal{E}_{\mu\nu\rho\lambda}\omega^{\rho}\wedge\omega^{\lambda}].$$

Using Einstein's equations and the above result for the action of $*d*$ allows

$$G_{\lambda\mu}V^{\mu} = -(Y^{\mu}V^{\nu}\mathcal{E}_{\mu\nu}{}^{\rho}{}_{\lambda})_{;\rho}.$$

Raising the index λ and using the block symmetry of $\mathcal{E}$ gives

$$G^{\lambda}{}_{\mu}V^{\mu} = -(Y^{\mu}V^{\nu})_{;\rho}\mathcal{E}^{\rho\lambda}{}_{\mu\nu} - Y^{\mu}V^{\nu}(\mathcal{E}^{\rho\lambda}{}_{\mu\nu})_{;\rho} .$$

The second term on the right hand side vanishes by Bianchi's second identity and the antisymmetry of $\mathcal{E}$ finally permits

$$G^{\lambda}{}_{\mu}V^{\mu} = (V^{\mu}Y^{\nu})_{;\rho}\mathcal{E}^{\rho\lambda}{}_{\mu\nu} .$$

APPENDIX III

CALCULATIONS SHOWING $d\theta_V = *\check{J}_V$

a) $V = Dt = \frac{\partial}{\partial t} \Rightarrow J_{Dt} = T^{\alpha}{}_{t} \frac{\partial}{\partial x^{\alpha}}$

$$*J_{Dt} = \frac{1}{6} T^{\alpha}{}_{t}\eta_{\alpha\beta\gamma\delta}\omega^{\beta}\wedge\omega^{\gamma}\wedge\omega^{\delta}$$

$$= \frac{1}{6} T^{t}{}_{t}\eta_{t\beta\gamma\delta}\omega^{\beta}\wedge\omega^{\gamma}\wedge\omega^{\delta} \quad \text{as T is diagonal from field equations}$$

$$= T^{t}{}_{t} \sqrt{g}\, \omega^{r}\wedge\omega^{\theta}\wedge\omega^{\phi} \quad \text{by definition of } \eta$$

$$= T^{t}{}_{t}\, e^{\frac{\gamma+\alpha}{2}} r^{2}\sin\theta\omega^{r}\wedge\omega^{\theta}\wedge\omega^{\phi} \quad \text{by substitution of } \sqrt{g}$$

$$= \frac{1}{8\pi} G^{t}{}_{t}\, e^{\frac{\gamma+\alpha}{2}} r^{2}\sin\theta\omega^{r}\wedge\omega^{\theta}\wedge\omega^{\phi} \quad \text{by field equations}$$

$$= \frac{1}{8\pi} (e^{-\alpha}[1-r\alpha_{,r}]-1)e^{\frac{\gamma+\alpha}{2}} \sin\theta\omega^{r}\wedge\omega^{\theta}\wedge\omega^{\phi} \quad \text{by substitution of } G^{t}{}_{t}$$

$$\theta_{Dt} = -\frac{1}{16\pi} e^{\frac{\gamma+\alpha}{2}} (\alpha_{,r}+\gamma_{,r})(e^{-\alpha}-1)r\cos\theta\omega^{r}\wedge\omega^{\phi} + \frac{1}{8\pi} e^{\frac{\gamma+\alpha}{2}}(e^{-\alpha}-1)r\sin\theta\omega^{\theta}\wedge\omega^{\phi}$$

$$d\theta_{Dt} = \frac{1}{16\pi} e^{\frac{\gamma+\alpha}{2}} (\alpha_{,r}+\gamma_{,r})(e^{-\alpha}-1)r\sin\theta\omega^{\theta}\wedge\omega^{r}\wedge\omega^{\phi} +$$

$$+\frac{1}{8\pi} [e^{\frac{\gamma+\alpha}{2}} \frac{(\gamma_{,r}+\alpha_{,r})}{2} (e^{-\alpha}-1)r + e^{\frac{\gamma+\alpha}{2}} e^{-\alpha}(-\alpha_{,r})r + e^{\frac{\gamma+\alpha}{2}} (e^{-\alpha}-1)]\sin\theta\omega^{r}\wedge\omega^{\theta}\wedge\omega^{\phi}$$

$$= \frac{1}{16\pi} e^{\frac{\gamma+\alpha}{2}} (\alpha_{,r}+\gamma_{,r})(e^{-\alpha}-1)r\sin\theta(\omega^{\theta}\wedge\omega^{r}\wedge\omega^{\phi} + \omega^{r}\wedge\omega^{\theta}\wedge\omega^{\phi}) +$$

$$+\frac{1}{8\pi} (e^{-\alpha}[1-r\alpha_{,r}]-1)e^{\frac{\gamma+\alpha}{2}} \sin\theta\omega^{r}\wedge\omega^{\theta}\wedge\omega^{\phi}$$

$$= *J_{Dt}$$

b) $V = \phi z = \frac{\partial}{\partial\phi} \Rightarrow J_{\phi z} = T^{\alpha}{}_{\phi} \frac{\partial}{\partial x^{\alpha}}$

$$*J_{\phi z} = \frac{1}{6} T^{\alpha}{}_{\phi} \eta_{\alpha\beta\gamma\delta} \omega^{\beta} \wedge \omega^{\gamma} \wedge \omega^{\delta}$$

$$= \frac{1}{6} T^{\phi}{}_{\phi} \eta_{\phi\beta\gamma\delta} \omega^{\beta} \wedge \omega^{\gamma} \wedge \omega^{\delta}$$

$$= -T^{\phi}{}_{\phi} \sqrt{g}\, \omega^{t} \wedge \omega^{r} \wedge \omega^{\theta}$$

$$= -\frac{1}{16\pi} e^{-\alpha}(\gamma_{,rr} + \frac{1}{2}(\gamma_{,r})^{2} + \frac{1}{r}\gamma_{,r} - \frac{1}{r}\alpha_{,r} - \frac{1}{2}\alpha_{,r}\gamma_{,r}) e^{\frac{\gamma+\alpha}{2}} r^{2}\sin\theta\, \omega^{t} \wedge \omega^{r} \wedge \omega^{\theta}$$

$$\theta_{\phi z} = \frac{1}{16\pi} e^{\frac{\gamma-\alpha}{2}} \gamma_{,r} r^{2}\sin\theta\, \omega^{t} \wedge \omega^{\theta} + \frac{1}{16\pi} e^{\frac{\gamma-\alpha}{2}} (\alpha_{,r}+\gamma_{,r}) tr\sin\theta\, \omega^{r} \wedge \omega^{\theta}$$

$$d\theta_{\phi z} = \frac{1}{16\pi} [e^{\frac{\gamma-\alpha}{2}} \frac{(\gamma_{,r}-\alpha_{,r})}{2} \gamma_{,r} r^{2} + e^{\frac{\gamma-\alpha}{2}} \gamma_{,rr} r^{2} +$$

$$+ e^{\frac{\gamma-\alpha}{2}} \gamma_{,r}(2r)] \sin\theta\, \omega^{r} \wedge \omega^{t} \wedge \omega^{\theta} +$$

$$+ \frac{1}{16\pi} e^{\frac{\gamma-\alpha}{2}} (\alpha_{,r}+\gamma_{,r}) r\sin\theta\, \omega^{t} \wedge \omega^{r} \wedge \omega^{\theta}$$

$$= -\frac{1}{16\pi} e^{\frac{\gamma-\alpha}{2}} r^{2}\sin\theta [\frac{1}{2}(\gamma_{,r})^{2} - \frac{1}{2}\alpha_{,r}\gamma_{,r} + \gamma_{,rr} + \frac{2}{r}\gamma_{,r} - \frac{1}{r}\alpha_{,r} - \frac{1}{r}\gamma_{,r}] \omega^{t} \wedge \omega^{r} \wedge \omega^{\theta}$$

$$= -\frac{1}{16\pi} e^{-\alpha}(\gamma_{,rr} + \frac{1}{2}(\gamma_{,r})^{2} + \frac{1}{r}\gamma_{,r} - \frac{1}{r}\alpha_{,r} - \frac{1}{2}\alpha_{,r}\gamma_{,r}) e^{\frac{\gamma+\alpha}{2}} r^{2}\sin\theta\, \omega^{t} \wedge \omega^{r} \wedge \omega^{\theta}$$

$$= *J_{\phi z}$$

c) $V = D_{z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \Rightarrow J_{Dz} = (T^{\alpha}{}_{r}\cos\theta - T^{\alpha}{}_{\theta}\frac{\sin\theta}{r}) \frac{\partial}{\partial x^{\alpha}}$

$$*J_{Dz} = \frac{1}{6} (T^{\alpha}{}_{r}\cos\theta - T^{\alpha}{}_{\theta}\frac{\sin\theta}{r}) \eta_{\alpha\beta\gamma\delta} \omega^{\beta} \wedge \omega^{\gamma} \wedge \omega^{\delta}$$

$$= \frac{1}{6} (T^{r}{}_{r}\cos\theta\, \eta_{r\beta\gamma\delta} - T^{\theta}{}_{\theta}\frac{\sin\theta}{r} \eta_{\theta\beta\gamma\delta}) \omega^{\beta} \wedge \omega^{\gamma} \wedge \omega^{\delta}$$

$$= -T^{r}{}_{r}\cos\theta\sqrt{g}\, \omega^{t} \wedge \omega^{\theta} \wedge \omega^{\phi} - T^{\theta}{}_{\theta}\frac{\sin\theta}{r}\sqrt{g}\, \omega^{t} \wedge \omega^{r} \wedge \omega^{\phi}$$

$$*J_{Dz} = -\frac{1}{16\pi}\,[e^{-\alpha}(1+r\gamma_{,r})-1]e^{\frac{\alpha+\gamma}{2}}\,\sin 2\theta\,\omega^t\wedge\omega^\theta\wedge\omega^\phi\;+$$

$$-\frac{1}{16\pi}\,e^{-\alpha}(\gamma_{,rr}+\frac{1}{2}(\gamma_{,r})^2+\frac{1}{r}\,\gamma_{,r}-\frac{1}{r}\,\alpha_{,r}-\frac{1}{2}\,\alpha_{,r}\gamma_{,r})e^{\frac{\gamma+\alpha}{2}}\,r\sin^2\theta\,\omega^t\wedge\omega^r\wedge\omega^\phi$$

$$\theta_{Dz} = \frac{1}{16\pi}\,e^{\frac{\gamma-\alpha}{2}}\,\gamma_{,r}r\sin^2\theta\,\omega^t\wedge\omega^\phi + \frac{1}{16\pi}\,e^{\frac{\gamma-\alpha}{2}}\,\alpha_{,r}t\sin^2\theta\,\omega^r\wedge\omega^\phi\;+$$

$$-\frac{1}{16\pi}\,e^{\frac{\alpha+\gamma}{2}}\,(e^{-\alpha}-1)\,t\sin 2\theta\,\omega^\theta\wedge\omega^\phi$$

$$d\theta_{Dz} = \frac{1}{16\pi}\,[e^{\frac{\gamma-\alpha}{2}}\,\frac{(\gamma_{,r}-\alpha_{,r})}{2}\,\gamma_{,r}r + e^{\frac{\gamma-\alpha}{2}}\,\gamma_{,rr}r + e^{\frac{\gamma-\alpha}{2}}\,\gamma_{,r}]\sin^2\theta\,\omega^r\wedge\omega^t\wedge\omega^\phi\;+$$

$$+\frac{1}{16\pi}\,e^{\frac{\gamma-\alpha}{2}}\,\gamma_{,r}r\sin 2\theta\,\omega^\theta\wedge\omega^t\wedge\omega^\phi + \frac{1}{16\pi}\,e^{\frac{\gamma-\alpha}{2}}\,\alpha_{,r}\sin^2\theta\,\omega^t\wedge\omega^r\wedge\omega^\phi\;+$$

$$+\frac{1}{16\pi}\,e^{\frac{\gamma-\alpha}{2}}\,\alpha_{,r}t\sin 2\theta\,\omega^\theta\wedge\omega^r\wedge\omega^\phi - \frac{1}{16\pi}\,e^{\frac{\alpha+\gamma}{2}}\,(e^{-\alpha}-1)\sin 2\theta\,\omega^t\wedge\omega^\theta\wedge\omega^\phi\;+$$

$$-\frac{1}{16\pi}\,[e^{\frac{\alpha+\gamma}{2}}\,\frac{(\alpha_{,r}+\gamma_{,r})}{2}\,(e^{-\alpha}-1) - e^{\frac{\alpha+\gamma}{2}}\,e^{-\alpha}\alpha_{,r}]\,t\sin 2\theta\,\omega^r\wedge\omega^\theta\wedge\omega^\phi$$

$$d\theta_{Dz} = -\frac{1}{16\pi}\,e^{-\alpha}(\frac{1}{2}(\gamma_{,r})^2-\frac{1}{2}\,\alpha_{,r}\gamma_{,r}+\gamma_{,rr}+\frac{1}{r}\,\gamma_{,r}-\frac{1}{r}\,\alpha_{,r})e^{\frac{\gamma+\alpha}{2}}\,r\sin^2\theta\,\omega^t\wedge\omega^r\wedge\omega^\phi$$

$$-\frac{1}{16\pi}\,[e^{-\alpha}(1+r\gamma_{,r})-1]e^{\frac{\gamma+\alpha}{2}}\,\sin 2\theta\,\omega^t\wedge\omega^\theta\wedge\omega^\phi\;+$$

$$-\frac{1}{32\pi}\,(\alpha_{,r}+\gamma_{,r})(e^{-\alpha}-1)e^{\frac{\alpha+\gamma}{2}}\,t\sin 2\theta\,\omega^r\wedge\omega^\theta\wedge\omega^\phi$$

$$d\theta_{Dz} = *J_{Dz} - \frac{1}{32\pi}\,(\alpha_{,r}+\gamma_{,r})(e^{-\alpha}-1)e^{\frac{\alpha+\gamma}{2}}\,t\sin 2\theta\,\omega^r\wedge\omega^\theta\wedge\omega^\phi$$

But $d{*}J_{Dz} = 0$; if and only if, $\alpha_{,r}+\gamma_{,r} = 0$.

d) $V = Bz = r\cos\theta \frac{\partial}{\partial t} + t\cos\theta \frac{\partial}{\partial r} - t\frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \Rightarrow$

$$J_{Bz} = (T^{\alpha}_{t} r\cos\theta + T^{\alpha}{}_{r} t\cos\theta - T^{\alpha}{}_{\theta} t\frac{\sin\theta}{r})\frac{\partial}{\partial x^{\alpha}}$$

$$*J_{Bz} = \frac{1}{6}(T^{t}_{t} r\cos\theta\eta_{t\beta\gamma\delta} + T^{r}{}_{r} t\cos\theta\eta_{r\beta\gamma\delta} - T^{\theta}{}_{\theta} t\frac{\sin\theta}{r}\eta_{\theta\beta\gamma\delta})\omega^{\beta}\wedge\omega^{\gamma}\wedge\omega^{\delta}$$

$$= T^{t}{}_{t} r\cos\theta\sqrt{g}\,\omega^{r}\wedge\omega^{\theta}\wedge\omega^{\phi} - T^{r}{}_{r} t\cos\theta\sqrt{g}\,\omega^{t}\wedge\omega^{\theta}\wedge\omega^{\phi} - T^{\theta}{}_{\theta} t\frac{\sin\theta}{r}\sqrt{g}\,\omega^{t}\wedge\omega^{r}\wedge\omega^{\phi}$$

$$= \frac{1}{16\pi}(e^{-\alpha}[1-r\alpha_{,r}]-1)e^{\frac{\gamma+\alpha}{2}} r\sin 2\theta\omega^{r}\wedge\omega^{\theta}\wedge\omega^{\phi} +$$

$$- \frac{1}{16\pi}(e^{-\alpha}[1+r\gamma_{,r}]-1)e^{\frac{\gamma+\alpha}{2}} t\sin 2\theta\omega^{t}\wedge\omega^{\theta}\wedge\omega^{\phi} +$$

$$- \frac{1}{16\pi}e^{-\alpha}(\gamma_{,rr} + \frac{1}{2}(\gamma_{,r})^{2} + \frac{1}{r}\gamma_{,r} - \frac{1}{r}\alpha_{,r} - \frac{1}{2}\alpha_{,r}\gamma_{,r})e^{\frac{\alpha+\gamma}{2}} tr\sin^{2}\theta\omega^{t}\wedge\omega^{r}\wedge\omega^{\phi}$$

$$\theta_{Bz} = \frac{1}{16\pi}e^{\frac{\gamma-\alpha}{2}}\gamma_{,r} tr\sin^{2}\theta\omega^{t}\wedge\omega^{\phi} + \frac{1}{32\pi}e^{\frac{\gamma-\alpha}{2}}\alpha_{,r}(t^{2}+r^{2})\sin^{2}\theta\omega^{r}\wedge\omega^{\phi} +$$

$$- \frac{1}{32\pi}e^{\frac{\alpha+\gamma}{2}}(e^{-\alpha}-1)(t^{2}-r^{2})\sin 2\theta\omega^{\theta}\wedge\omega^{\phi}$$

$$d\theta_{Bz} = \frac{1}{16\pi}[e^{\frac{\gamma-\alpha}{2}}\frac{(\gamma_{,r}-\alpha_{,r})}{2}\gamma_{,r} r + e^{\frac{\gamma-\alpha}{2}} r\gamma_{,rr} + e^{\frac{\gamma-\alpha}{2}}\gamma_{,r}] t\sin^{2}\theta\omega^{r}\wedge\omega^{t}\wedge\omega^{\phi} +$$

$$+ \frac{1}{16\pi}e^{\frac{\gamma-\alpha}{2}}\gamma_{,r} tr\sin 2\theta\omega^{\theta}\wedge\omega^{t}\wedge\omega^{\phi} + \frac{1}{16\pi}e^{\frac{\gamma-\alpha}{2}}\alpha_{,r} t\sin^{2}\theta\omega^{t}\wedge\omega^{r}\wedge\omega^{\phi} +$$

$$+ \frac{1}{32\pi}e^{\frac{\gamma-\alpha}{2}}\alpha_{,r}(t^{2}+r^{2})\sin 2\theta\omega^{\theta}\wedge\omega^{r}\wedge\omega^{\phi} - \frac{1}{16\pi}e^{\frac{\alpha+\gamma}{2}}(e^{-\alpha}-1)t\sin 2\theta\omega^{t}\wedge\omega^{\theta}\wedge\omega^{\phi} +$$

$$- \frac{1}{32\pi}[e^{\frac{\alpha+\gamma}{2}}\frac{(\alpha_{,r}+\gamma_{,r})}{2}(e^{-\alpha}-1)(t^{2}-r^{2}) - e^{\frac{\alpha+\gamma}{2}}e^{-\alpha}\alpha_{,r}(t^{2}-r^{2}) +$$

$$- 2e^{\frac{\alpha+\gamma}{2}}(e^{-\alpha}-1)r]\sin 2\theta\omega^{r}\wedge\omega^{\theta}\wedge\omega^{\phi}$$

$$d\theta_{Bz} = -\frac{1}{16\pi} e^{-\alpha}\left(\frac{1}{2}(\gamma_{,r})^2 - \frac{1}{2}\alpha_{,r}\gamma_{,r}+\gamma_{,rr}+\frac{1}{r}\gamma_{,r}-\frac{1}{r}\alpha_{,r}\right)e^{\frac{\gamma+\alpha}{2}} tr\sin^2\theta\,\omega^t\wedge\omega^r\wedge\omega^\phi +$$

$$-\frac{1}{16\pi}\left(e^{-\alpha}[1+r\gamma_{,r}]-1\right)e^{\frac{\gamma+\alpha}{2}} t\sin 2\theta\,\omega^t\wedge\omega^\theta\wedge\omega^\phi +$$

$$+\left[-\frac{1}{64\pi}(\alpha_{,r}+\gamma_{,r})e^{\frac{\alpha+\gamma}{2}}(e^{-\alpha}-1)(t^2-r^2)+\frac{1}{32\pi}\alpha_{,r}e^{\frac{\gamma-\alpha}{2}}(t^2-r^2-t^2-r^2) + \right.$$

$$\left.+\frac{1}{16\pi}e^{\frac{\alpha+\gamma}{2}}(e^{-\alpha}-1)r\right]\sin 2\theta\,\omega^r\wedge\omega^\theta\wedge\omega^\phi$$

$$d\theta_{Bz} = *J_{Bz} - \frac{1}{64\pi}(\alpha_{,r}+\gamma_{,r})e^{\frac{\alpha+\gamma}{2}}(e^{-\alpha}-1)(t^2-r^2)\sin 2\theta\,\omega^r\wedge\omega^\theta\wedge\omega^\phi .$$

But $d{*}J_{Bz} = 0$; if and only if $\alpha_{,r}+\gamma_{,r} = 0$.

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