ON SOME INFERENTIAL ASPECTS FOR TYPE-II AND PROGRESSIVE TYPE-II CENSORING

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By

WILLIAM VOLTERMAN, B.SC., M.SC.

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AUTHOR: William Volterman, B.SC. (McMaster University)

M.SC. (McMaster University)

SUPERVISOR: Professor N. Balakrishnan

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ABSTRACT: This thesis investigates nonparametric inference under multiple independent samples with various modes of censoring, and also presents results concerning Pitman Closeness under Progressive Type-II right censoring. For the nonparametric inference with multiple independent samples, the case of Type-II right censoring is first considered. Two extensions to this are then discussed: doubly Type-II censoring, and Progressive Type-II right censoring. We consider confidence intervals for quantiles, prediction intervals for order statistics from a future sample, and tolerance intervals for a population proportion. Benefits of using multiple samples over one sample are discussed. For each of these scenarios, we consider simulation as an alternative to exact calculations. In each case we illustrate the results with data from the literature. Furthermore, we consider two problems concerning Pitman Closeness and Progressive Type-II right censoring. We derive simple explicit formulae for the Pitman Closeness probabilities of the order statistics to population quantiles. Various tables are given to illustrate these results. We then use the Pitman Closeness measure as a criterion for determining the optimal censoring scheme for samples drawn from the exponential distribution. A general result is conjectured, and demonstrated in special cases.

KEY WORDS: Multiple independent samples, Type-II right censoring, Doubly Type-II censoring, Progressive Type-II right censoring, simulation, nonparametric, prediction intervals, tolerance intervals, confidence intervals, Pitman Closeness, optimality, Progressive censoring scheme, population quantiles

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Co-authorship and inclusion of previously published material

The material presented in this thesis was written primarily by myself.

Chapter 3 which discusses inference for type-II right censored samples from multiple independent samples is based on work which was co-written with my supervisor, Prof. N. Balakrishnan, for inclusion in a paper published in a refereed journal (Volterman and Balakrishnan, 2010).

Chapter 4 which discusses inference for doubly Type-II censored samples from multiple independent samples is based on work which was co-written with my supervisor, Prof. N. Balakrishnan, and Prof. E. Cramer (Professor, Institute of Statistics and Economics mathematics, RWTH Aachen), for inclusion in a paper submitted to a refereed journal.

Chapter 6 which discusses Pitman closeness for Progressively Type-II censored samples is based on work co-written with my supervisor, Prof. N. Balakrishnan, and Prof K. F. Davies (Assistant Professor, Department of Statistics, University of Manitoba), for inclusion in a paper published in a referred journal (Volterman et al., 2011).

Chapter 7 which discusses using Pitman closeness as a criterion for selecting optimal Progressive censoring schemes, is based on work co-written with my supervisor, Prof. N. Balakrishnan, and Prof. K. F. Davies for inclusion in a paper submitted to a referred journal.

For all of these papers, the programs were completely written by myself. Most derivations and all proofs were done by myself. The exception is for Chapter 7 where some, but not all, of the derivations were done in concert with the co-authors. Moreover, these chapters contain work not included in, and expanding on, the work in the submitted and published works.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

In lifetime and reliability analysis we are concerned with obtaining results which allow us to make inference about the processes or populations involved. Both cost and time may be factors that place constraints on the types of experimental designs that can be used. Thus, censoring can be used as a way to limit the time, cost, or a combination of both. This leads to the question of which designs are best to make inference given these constraints.

It can also be of interest to obtain more information from future independent samples. The question that arises now, is how to incorporate this new information. When we have multiple independent censored samples, one can always write the likelihood explicitly. However, this is not the case for multiple independent samples when we make no distributional assumptions.

With two independent samples it is known (see Balakrishnan et al., 2010b) how to make distribution free intervals for quantiles, tolerance intervals, and prediction intervals when both samples are Type-II right censored or progressively Type-II censored. The authors show that there are gains in the maximum coverage probabilities over the equivalent one sample scenario. Thus in some sense these designs are better.

Nonparametric inference for two independent samples of minimal repair systems is considered in Beutner and Cramer (2010). They have shown how to make prediction intervals for future samples conditional on surviving until some specified time. Again there are gains in some sense, over equivalent one sample scenarios.

We may ask what schemes for one or more samples would be best. Determination of optimal progressive censoring schemes has been considered for a variety of criteria with varying assumptions.

1.1 Order Statistics

Consider observing n independent observations $X_1, X_2, \ldots, X_n$. Placing the observations in ascending order, we have $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$, where $X_{i:n}$ is the i-th order statistic (OS).

Typically the n observations come from some common underlying distribution. We denote this common cumulative distribution function (CDF) as F_X. If the distribution is absolutely continuous, it has probability density function (PDF) f_X.

In this i.i.d. case the joint and marginal distributions of the order statistics have simple explicit formula involving the underlying distribution function and there exists a wide variety of literature on order statistics. Both Arnold et al. (1992) and David and Nagaraja (2003) provide an introduction to the topic.

It is well known that the joint density of n order statistics is

$$f^{X_{1:n},\dots,X_{n:n}}(x_1,\dots,x_n) = n!\prod_{j=1}^{n} f(x_j), \tag{1.1.1}$$

where $\xi_0 < x_1 \leq x_2 \leq \cdots \leq x_n < \xi_1$. Here, ξ_0 and ξ_1 represent the lower and upper endpoints of the distribution respectively; these may not be finite. For $1 \leq j_1 < j_2 \leq n$, the joint distribution of two order statistics is

$$\begin{aligned} f^{X_{j_1:n},X_{j_2:n}}(x_1,x_2) = {} & \frac{n!}{(j_1-1)!(j_2-j_1-1)!(n-j_2)!} \\ & \times F(x_1)^{j_1-1}\left[F(x_2)-F(x_1)\right]^{j_2-j_1-1}\left[1-F(x_2)\right]^{n-j_2} f(x_1)f(x_2), \end{aligned} \tag{1.1.2}$$

when $\xi_0 < x_{j_1} \leq x_{j_2} < \xi_1$. For $1 \leq j \leq n$, the marginal PDF and CDF of $X_{j:n}$ is known to be

$$f^{X_{j:n}}(x) = \frac{n!}{(j-1)!(n-j)!}F(x)^{j-1}\left[1-F(x)\right]^{n-j} f(x), \tag{1.1.3}$$

$$F^{X_{j:n}}(x) = \sum_{\ell=j}^{n}\binom{n}{\ell}F(x)^{\ell}\left[1-F(x)\right]^{n-\ell}, \tag{1.1.4}$$

respectively, where $\xi_0 < x < \xi_1$. Equation (1.1.4) applies when the underlying distribution is continuous, whereas equations (1.1.1) to (1.1.3) require absolute continuity. See Arnold et al. (1992) or David and Nagaraja (2003) for more about order statistics.

Group 1	0.31	0.66	1.54	1.70	1.82	1.89	2.17	2.24	4.03	9.99
Group 2	0.00	0.18	0.55	0.66	0.71	1.30	1.63	2.17	2.75	10.60
Group 3	0.49	0.64	0.82	0.93	1.08	1.99	2.06	2.15	2.57	4.75
Group 4	0.02	0.06	0.50	0.70	1.17	2.80	3.57	3.72	3.82	3.87
Group 5	0.20	0.78	0.80	1.08	1.13	2.44	3.17	5.55	6.63	8.11
Group 6	1.34	1.49	1.56	2.10	2.12	3.83	3.97	5.13	7.21	8.71

Table 1.1: Time to breakdown of insulating fluids

1.1.1 Pooled Order Statistics

Suppose we have B independent samples, upon combining the B samples and ordering them we have what we call the pooled order statistics. We shall denote the pooled order statistics as $Z_{(i)}$. Balakrishnan et al. (2010b) considered the pooled order statistics for Type-II right censored and progressively Type-II censored samples.

For complete i.i.d. samples the pooled order statistics are equivalent to order statistics from a large sample. This is of course the basis for taking a sample of size $n+1$ by obtaining an independent sample of size one and appending it to an existing sample of size n.

As it will become apparent later, under certain assumptions, these pooled order statistics are related to the usual order statistics from the underlying distribution.

1.1.2 Motivating Examples

As a motivating example consider the time to insulating fluid breakdown originally taken from Nelson (1982, Table 4.1, p. 462) as in Table 1.1.

This data set has been used repeatedly in the literature under various censoring schemes, particularly when the data is presumed to be exponentially distributed.

1.2 Types of censoring

Censoring of data can arise naturally due to the nature of the sampling or experimental design, or inherent structure of the situation. However, sometimes censoring can be exploited by experimenters as an efficient method of obtaining information with regards to cost and time. There are a number of censoring methods available to experimenters; below are a few of these which are commonly used in reliability and life testing.

1.2.1 Type-I Censoring

Consider a sample where we observe outcomes only in some specified interval (T_L, T_U), where $T_L < T_U$. Such an interval is to be known ahead of time. When an item fails in the given interval its time is observed exactly; if the item fails in the interval $(-\infty, T_L]$ or $[T_U, \infty)$, then only the interval that it fails in is known.

$$X_{1:n} \qquad X_{2:n} \quad X_{3:n} \qquad \cdots \qquad X_{i:n} \quad \Big|_{T_L} \longrightarrow \text{Censored}$$

Figure 1.1: Diagram of Type-I right censoring

In such a censoring scheme the number of observed failures is random. So if the upper and lower censoring bands are set too narrow, then an insufficient number of observations may be made.

Without distributional assumptions, no information about the distribution can be gleaned from such a Type-I sample outside the fixed interval. Thus unless one wishes to either make distributional assumptions about the population, or restrict their inference to the given region, Type-I censoring is not appropriate. However, it is widely used because the amount of time on test is bounded, and thus the cost for the experiment will be bounded.

1.2.2 Type-II Censoring

Type-II right censoring (herein referred to as Type-II censoring) is where the smallest r of n independent observations are observed. The number of observations r, is fixed before the experiment.

An experimenter would place n items on test, and after observing the first r failures, stop the test and the remaining $n - r$ items would be removed.

$$X_{1:n} \quad X_{2:n} \qquad \cdots \qquad X_{r-1:n} \qquad X_{r:n}^{\nearrow n-r}$$

Figure 1.2: Diagram of Type-II censoring

The advantage of such a scheme over Type-I censoring is that one knows exactly how many failures will be observed ahead of time; however, the time to test possibly unbounded, and could be on average much larger than in Type-I censoring.

Table 1.2 shows Type-II right censoring which had been introduced to the insulating fluid data in Table 1.1 by Balakrishnan et al. (2010b).

Group 1	0.31	0.66	1.54	1.70	1.82	1.89	2.17	2.24	4.03	$*$
Group 2	0.00	0.18	0.55	0.66	0.71	1.30	1.63	2.17	2.75	$*$
Group 3	0.49	0.64	0.82	0.93	1.08	1.99	2.06	2.15	2.57	$*$
Group 4	0.02	0.06	0.50	0.70	1.17	2.80	3.57	3.72	3.82	$*$
Group 5	0.20	0.78	0.80	1.08	1.13	2.44	3.17	5.55	$*$	$*$
Group 6	1.34	1.49	1.56	2.10	2.12	3.83	3.97	5.13	$*$	$*$

Table 1.2: Insulating fluid data - Type-II censoring

Group 1	$*$	$*$	1.54	1.70	1.82	1.89	2.17	2.24	4.03	$*$
Group 2	$*$	0.18	0.55	0.66	0.71	1.30	1.63	2.17	2.75	$*$
Group 3	$*$	0.64	0.82	0.93	1.08	1.99	2.06	2.15	2.57	$*$
Group 4	$*$	0.06	0.50	0.70	1.17	2.80	3.57	3.72	3.82	$*$
Group 5	$*$	0.78	0.80	1.08	1.13	2.44	3.17	5.55	$*$	$*$
Group 6	$*$	1.49	1.56	2.10	2.12	3.83	3.97	5.13	$*$	$*$

Table 1.3: Insulating fluid data - doubly Type-II censoring

Doubly Type-II censoring occurs when the smallest r^L, and largest r^U items are censored. In this case, the number of observed failures is $r = n - r^L - r^U$. It is clear that Type-II censoring is a special case of doubly Type-II censoring when $r^L = 0$ and $r^U = n - r$. Similarly, when $r^U = 0$ and $r^L = n - r$ then this is Type-II left censoring.

Table 1.3 shows the insulating fluid data with doubly Type-II data as introduced in Balakrishnan et al. (2004).

1.2.3 Progressive Type-II Right Censoring

Progressive Type-II right censoring (herein referred to as progressive Type-II censoring), is an extension of the Type-II censoring scheme mentioned prior. Place n items on a test. After the first failure $X^{\mathcal{R}}_{1:r:n}$, remove R_1 items randomly from the remaining $n-1$ items and then continue the test. After the next failure $X^{\mathcal{R}}_{2:r:n}$, remove R_2 items randomly from the remaining $n-2-R_1$ items and continue the test. One would continue in this manner until observing the final failure $X^{\mathcal{R}}_{r:r:n}$, and then the remaining R_r items are removed. The i-th progressive Type-II order statistic (PCOS) is denoted as $X^{\mathcal{R}}_{i:r:n}$ or $X_{i:r:n}$ when the scheme which generates the order statistic is unambiguous.

$$\overset{\nearrow R_1}{X_{1:r:n}} \qquad \overset{\nearrow R_2}{X_{2:r:n}} \qquad \cdots \qquad \overset{\nearrow R_r}{X_{r:r:n}}$$

Figure 1.3: Diagram of progressive Type-II censoring

We call $\mathcal{R} = (R_1, \ldots, R_r)$, the progressive Type-II censoring scheme. Much like Type-II right censoring, the censoring scheme $\mathcal{R}$ is fixed before the experiment.

It can be seen that Type-II censoring is a special case of progressive Type-II censoring, where the scheme is $\mathcal{R} = (0, ..., 0, n-r)$. Expressions and inference for Type-II censored samples are often much simpler than the more general progressive Type-II censored samples.

Given a censoring scheme $\mathcal{R}$ we can further define the following constants. Define $\gamma_1, \ldots, \gamma_r$ as $\gamma_\ell = \sum_{i=\ell}^{r}(R_i+1) = n-(\ell-1)-\sum_{i=1}^{\ell} R_i$ for $\ell = 1, \ldots, r$. In this context, γ_ℓ is the number of units remaining on test between the $(\ell-1)$-th and ℓ-th failures. We further define the constants $c_{\ell-1} = \prod_{i=1}^{\ell} \gamma_i$ and $a_i(\ell) = \prod_{\substack{k=1 \\ k\neq i}}^{\ell} \frac{1}{\gamma_k - \gamma_i}$.

With this in hand we can obtain the joint distribution of the PCOS as

$$f^{X^{\mathcal{R}}_{1:r:n},\ldots,X^{\mathcal{R}}_{r:r:n}}(x_1,\ldots,x_r) = c_{r-1}\prod_{\ell=1}^{r}\{1-F(x_\ell)\}^{R_\ell} f(x_\ell), \tag{1.2.1}$$

where $\xi_0 < x_1 \le x_2 \le \cdots \le x_r < \xi_1$. For $1 \le \ell_1 < \ell_2 \le r$, the joint distribution of two PCOS is

$$\begin{aligned} f^{X^{\mathcal{R}}_{\ell_1:r:n},X^{\mathcal{R}}_{\ell_2:r:n}}(x_{\ell_1},x_{\ell_2}) &= c_{\ell_2-1}\sum_{i=\ell_1+1}^{\ell_2}\left(a_i^{(\ell_1)}(\ell_2)\left[\frac{1-F(x_{\ell_2})}{1-F(x_{\ell_1})}\right]^{\gamma_i}\right) \\ &\quad\times\sum_{i=1}^{\ell_1}\left(a_i(\ell_1)(1-F(x_{\ell_1}))^{\gamma_i}\right)\frac{f(x_{\ell_1})}{1-F(x_{\ell_1})}\frac{f(x_{\ell_2})}{1-F(x_{\ell_2})}, \end{aligned} \tag{1.2.2}$$

when $\xi_0 < x_{\ell_1} \le x_{\ell_2} < \xi_1$. For $1 \le \ell \le r$, the marginal PDF and CDF of $X^{\mathcal{R}}_{\ell:r:n}$ is known to be

$$f^{X^{\mathcal{R}}_{\ell:r:n}} = c_{\ell-1}\sum_{i=1}^{\ell}a_i(\ell)\{1-F(x)\}^{\gamma_i-1}f(x), \tag{1.2.3}$$

$$F^{X^{\mathcal{R}}_{\ell:r:n}} = 1 - c_{\ell-1}\sum_{i=1}^{\ell}\frac{a_i(\ell)}{\gamma_i}\{1-F(x)\}^{\gamma_i} \tag{1.2.4}$$

respectively, where $\xi_0 < x < \xi_1$.

Note that equations (1.2.2)–(1.2.4) do not collapse to those in Section 1.1 in the special case of right censoring. However, these can be obtained from the previous results by appropriate expansions. Thus, we typically can consider results obtained with progressive censoring to provide alternate representations to those based on the usual order statistics.

For more general theory, and methods regarding progressive Type-II censoring, see Balakrishnan and Aggarwala (2000). Optimal progressive censoring schemes are discussed in Burkschat et al. (2006) and Burkschat (2007, 2008), for a general class of location-scale models.

There are extensions to progressive censoring allowing the number of items removed after the i-th failure, R_i, to be random. In one such extension from Cramer and Iliopoulos (2010), known as adaptive progressive Type-II censoring, R_i is random function of $R_1, \ldots, R_{i-1}$ and $X^{\mathcal{R}}_{1:r:n}, \ldots, X^{\mathcal{R}}_{i-1:r:n}$.

For illustrative purposes, we have introduced progressive Type-II censoring to the insulating fluid data. Table 1.4 is the insulating fluid data with the schemes $\mathcal{R}_1 = (2, 2, 3)$, $\mathcal{R}_2 = (6, 1, 0)$, $\mathcal{R}_3 = (0, 0, 7)$, and $\mathcal{R}_4 = (4, 0, 3)$ applied to each of the six samples. We include the censored items for comparisons sake.

1.3 Mixture Distributions

Mixture distributions naturally arise when a population can be divided into subpopulations (components), possibly with different distributions. The number of such components can be finite, countable, or uncountable. The idea of fitting mixture distributions goes as far back as Pearson (1894) who fit two normal distributions to a population of crabs; this provided evidence that there were two distinct subspecies of crabs.

Group 1	0.31	0.66	*	1.70	*	*	*	*	*	*
Group 2	0.00	0.18	0.55	*	*	*	*	*	*	*
Group 3	0.49	0.64	*	0.93	*	*	*	*	*	*
Group 4	0.02	0.06	0.50	*	*	*	*	*	*	*
Group 5	0.20	*	0.80	*	*	2.44	*	*	*	*
Group 6	1.34	1.49	1.56	*	*	*	*	*	*	*

(a) $\mathcal{R}_1 = (2, 2, 3)$

Group 1	0.31	*	1.54	*	*	*	2.17	*	*	*
Group 2	0.00	*	*	0.66	*	1.30	*	*	*	*
Group 3	0.49	*	*	*	1.08	*	*	*	2.57	*
Group 4	0.02	*	0.50	*	*	*	*	*	*	3.87
Group 5	0.20	*	0.80	*	*	*	*	*	6.63	*
Group 6	1.34	*	*	*	2.12	*	*	*	7.21	*

(b) $\mathcal{R}_2 = (6, 1, 0)$

Group 1	0.31	0.66	1.54	*	*	*	*	*	*	*
Group 2	0.00	0.18	0.55	*	*	*	*	*	*	*
Group 3	0.49	0.64	0.82	*	*	*	*	*	*	*
Group 4	0.02	0.06	0.50	*	*	*	*	*	*	*
Group 5	0.20	0.78	0.80	*	*	*	*	*	*	*
Group 6	1.34	1.49	1.56	*	*	*	*	*	*	*

(c) $\mathcal{R}_3 = (0, 0, 7)$

Group 1	0.31	*	1.54	*	*	1.89	*	*	*	*
Group 2	0.00	0.18	0.55	*	*	*	*	*	*	*
Group 3	0.49	*	0.82	0.93	*	*	*	*	*	*
Group 4	0.02	0.06	0.50	*	*	*	*	*	*	*
Group 5	0.20	*	*	1.08	1.13	*	*	*	*	*
Group 6	1.34	1.49	1.56	*	*	*	*	*	*	*

(d) $\mathcal{R}_4 = (4, 0, 3)$

Table 1.4: Insulating fluid data - Progressive Type-II censoring

We consider a finite mixture model with D components X_i, distribution functions F_i $(1 \leq i \leq D)$, and mixing weights $0 < \pi_i \leq 1$, subject to $\sum \pi_i = 1$. Mixtures with such weights are known as convex mixtures. The mixture distribution is represented as follows

$$X \stackrel{d}{=} \sum_{i=1}^{D} Y_i \, X_i, \tag{1.3.1}$$

where $Y = (Y_1, \ldots, Y_D)$ is a multinomial random variable of size one, and with success probabilities π_i; Y is independent of the underlying X_i's. Here $\stackrel{d}{=}$ is understood to be equality in distribution. Marginally, Y_i follows a Bernoulli distribution with $P(Y_i = 1) = \pi_i$.

The cumulative distribution function $F_X(\cdot)$ can be given as

$$F(t) = \sum_{i=1}^{D} \pi_i F_i(t), \tag{1.3.2}$$

which is a weighted sum of the D component distribution functions with $t \in \Re$. The mixing weight π_i represents the proportion of the total population from component i. If the random variables are absolutely continuous, then the mixture density exists and

$$f_X(t) = \sum_{i=1}^{D} \pi_i f_i(t). \tag{1.3.3}$$

It is easy to see that the distribution function $F_X(\cdot)$ and density function $f_X(\cdot)$ are valid distribution and density functions respectively. Sampling from such a distribution can be done in two stages. First generate one draw from the multinomial distribution with success vector $(\pi_1, \ldots, \pi_D)$; then given that $Y_i = 1$ ($Y_j = 0$, $j \neq i$), generate a single observation from the distribution of the i-th component X_i as x_i. Thus $X = x_i$ is the sampled value from X.

In some cases non-convex mixtures will still yield valid distributions, though interpretation and simulation may prove more difficult. For a practical example see

Jiang et al. (1999).

In Figures 1.4a and 1.4b, we can see the mixture/component densities and mixture/component CDF's. Here, there are two components, normally distributed, with means 10 and 13, variances 3 and 4, and mixing proportions 0.6 and 0.4.

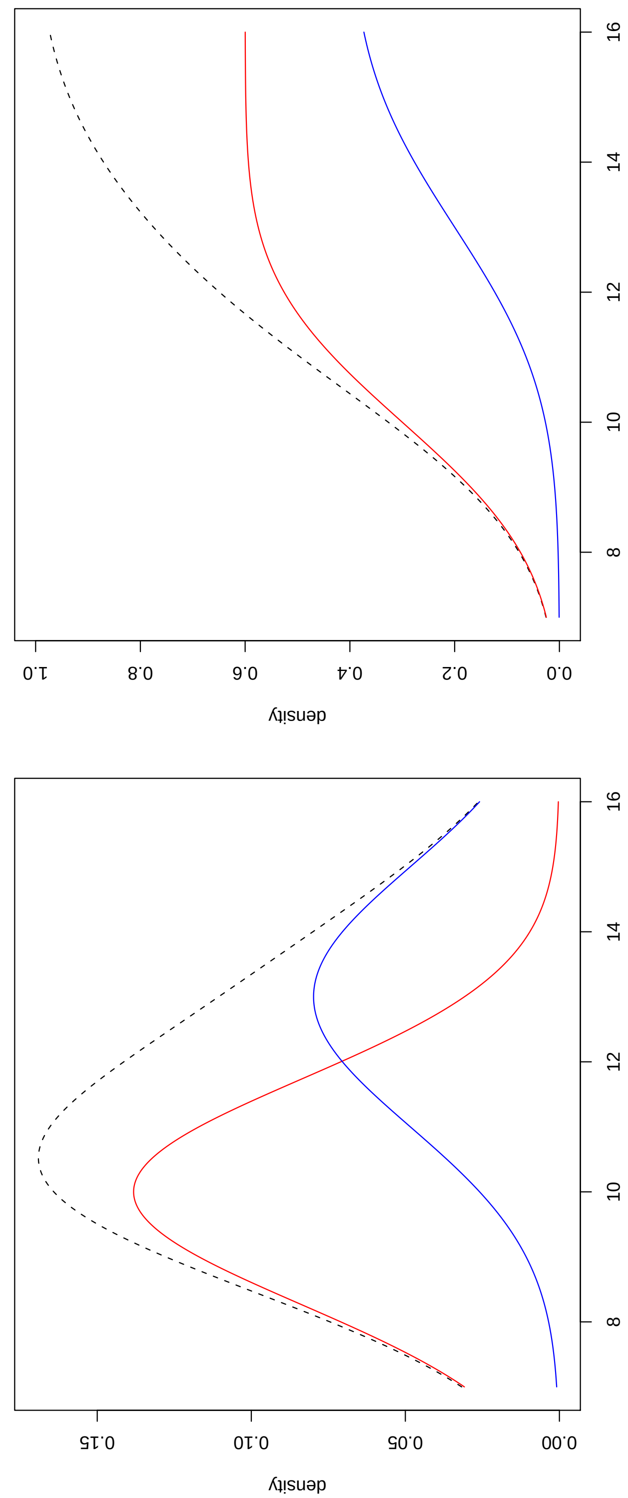
Mixture models can be useful as approximations to distributions as well. Kernel density estimators are a special case of mixture models used to estimate some population. In this case, the underlying component densities are usually identical up to a location parameter, and the mixing proportions are equal. And while mixture models naturally arise when there are known sub-populations, such models are useful to model multi-modal distributions even when no underlying sub-populations exist.

Whether one uses a kernel density or some other mixture model, often mixtures of normal distributions are used. For a more in depth discussion of mixture models, see McLachlan and Peel (2000).

1.4 Pitman Closeness

Pitman closeness (also known as Pitman nearness or Pitman's measure of closeness) has been presented as an alternative criterion when one is not concerned with the size of loss. The Pitman closeness (PC) probability is defined as follows.

Definition 1.4.1 *Given two estimators T_1 and T_2, and a population parameter θ,*



(a) Density (b) CDF

Figure 1.4: Mixture of two normal distributions

the PC probability of T_1 to θ relative to T_2 is

$$\pi_{T_1,T_2}(\theta) = P\left(|T_1 - \theta| < |T_2 - \theta|\right)$$

When $\pi_{T_1,T_2}(\theta) \geq 0.5$ we say T_1 is Pitman closer to θ than T_2.

Since θ is not usually known ahead of time, one may wish to determine which of T_1 and T_2 are better estimators for some $\theta \in \Omega$. Thus we have the following definition.

Definition 1.4.2 *Given T_1 and T_2, we say that T_1 is uniformly Pitman closer to θ than T_2 if $\forall\, \theta \in \Omega$, $\pi_{T_1,T_2}(\theta) \geq 0.5$, and $\pi_{T_1,T_2}(\theta) > 0.5$ for at least one $\theta \in \Omega$.*

See Keating et al. (1993) for a comprehensive discussion on Pitman closeness.

These pairwise comparisons are typically how an estimator is chosen. However, in certain circumstances, the Pitman closeness may not be transitive. To some this is considered a severe issue. See Robert et al. (1993a) (with discussion in Blyth, 1993; Casella and Wells, 1993; Ghosh et al., 1993; Peddada, 1993; Rao, 1993; Robert et al., 1993b) for this and other criticisms.

Some of these considerations, such as transitivity are eliminated when considering ordered estimators, such as using order statistics as point estimators of quantiles. Much work has been done in this area recently (see for example Balakrishnan et al., 2009). Ahmadi and Balakrishnan (2009), Ahmadi and Balakrishnan (2011), Ahmadi and Balakrishnan (2010) consider a similar problem with record values, k-records, and upper-lower records respectively.

A further extension to the idea of Pitman closeness is the idea of simultaneous closeness.

Definition 1.4.3 *Given a class of estimators $\mathcal{T}$ of θ, then for every $T \in \mathcal{T}$ the Simultaneous Closeness Probability (SCP) is defined as follows.*

$$\pi_T(\theta) = P\left(|T - \theta| < \min_{T' \in \mathcal{T} \setminus T} |T' - \theta|\right)$$

The estimator chosen by the simultaneous closeness as in Definition 1.4.3 need not be the same as chosen by Definition 1.4.1. However, in the case of ordered estimators, with some conditions, they will be. Whether it is better to look at simultaneous comparisons or pairwise comparisons depends on the context of the problem, and so we do not discuss this issue in any detail.

1.5 Scope of Thesis

This thesis will investigate various inferential aspects for single and multiple samples under Type-II, and progressively Type-II right censoring. Throughout this thesis it will be assumed that the underlying distribution is continuous. Where specified, absolute continuity may also be assumed.

In Chapter 2 we describe nonparametric inference for a single sample based upon ordinary order statistics. Some methods for point estimation of quantiles are discussed; confidence intervals for quantiles, prediction intervals, and tolerance intervals are also mentioned. These nonparametric intervals will form the basis of the methods used in Chapters 3-5.

In Chapter 3, mixture representations for the marginal distribution of the pooled order statistics and joint distribution of two pooled order statistics are given. The joint distribution is briefly discussed along with some miscellaneous asymptotic properties of the pooled order statistics. We describe how to construct exact nonparametric inference in the pooled setting. Specifically, we discuss how to calculate coverage probabilities for confidence intervals for quantiles, prediction intervals, and tolerance intervals based on the pooled order statistics. The improvement over the single sample scenario is discussed, and the data in Table 1.2 is analyzed using these methods.

In Chapter 4, we extend the mixture representations from Chapter 3 to the case where the samples are doubly Type-II censored. A simple algorithm to obtain the necessary mixture weights is presented. We also provide comparisons of exact weights to simulated weights in terms of absolute and relative accuracy for a simple censoring scheme. The data in Table 1.3 is analyzed. In Chapter 5 we consider another extension to the Type-II censoring by considering progressively Type-II censoring. The representations here are different than those given in Chapters 3 and 4. We again consider simulation and analyze the data in Table 1.4.

In Chapter 6, we consider the Pitman closeness of a progressively censored order statistic to a population quantile. Some distribution-free results are given. In Chapter 7 we consider use Pitman closeness as a criterion to find an optimal censoring scheme for the exponential distribution. An algorithm is given, some general results are conjectured, and for some specific cases, demonstrated.

Finally in Chapter 8, we suggest directions for future research.

Chapter 2

Nonparametric Inference

The basis for nonparametric inference with multiple independent pooled samples, is nonparametric inference for a single sample. So consider a single i.i.d sample of size n from a continuous population with cumulative distribution function F. Intervals in the form $(X_{k_1:n}, X_{k_2:n})$ where $1 \leq k_1 < k_2 \leq n$, can be used as the basis for distribution free inference in the single sample case. These intervals can be used as confidence intervals for population quantiles, tolerance intervals, and prediction intervals for future samples.

2.1 Quantile Estimation

For a continuous distribution F, the quantile ξ_p is defined as $\inf_x F(x) \geq p$. Furthermore, all quantiles ξ_p for $0 < p < 1$ exist, and $\xi_{p_1} < \xi_{p_1}$ when $0 < p_1 < p_2 < 1$. Traditional point estimates for quantiles can be based on either a single order statistic such as $k = [np]$, $k = [(n+1)p]$, or $k = [np]+1$. One may also use a linear combination of two order statistics such as $gX_{k:n} + (1-g)X_{k+1:n}$, where $k + g = (n+1)p$ and $0 \leq g < 1$.

Davis and Harrell (1982) suggest an L-estimator based on the empirical distribution function. The Davis & Harrell estimator is $\mathrm{HD}_p = \sum_{i=1}^{n} W_{n,i} X_{i:n}$, with the weights $W_{n,i} = I_{i/n}\{p(n+1),(1-p)(n+1)\} - I_{(i-1)/n}\{p(n+1),(1-p)(n+1)\}$. Here, $I_x\{a,b\}$ represents the incomplete beta function. Huang (2001) suggests a similar estimator that is based instead upon the modified level crossing empirical distribution function. In many cases this modified HD estimator is more efficient than the original estimator.

Zielinski (2006) compares all of the previous quantile estimators among others and suggests using the local smoothing estimator. However, such an estimate is not distribution free; in this case one would use a single order statistic to achieve robustness. In this vein, Balakrishnan et al. (2010c) have looked at the best order statistic to estimate a quantile in terms of Pitman closeness. This method however is again not distribution free.

Distribution free interval estimation for quantiles is much simpler. It is clear that the number of items from an i.i.d sample of size n falling below the p-th quantile ξ_p is distributed as Binomial(n,p); so that

$$P(X_{k_1:n} \leq \xi_p \leq X_{k_2:n}) = \sum_{i=k_1}^{k_2-1} \binom{n}{i} p^i (1-p)^{n-i} \tag{2.1.1}$$

We can similarly obtain one-sided intervals as

$$P(X_{k:n} \leq \xi_p) = \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i} \tag{2.1.2}$$

$$P(\xi_p \leq X_{k:n}) = \sum_{i=0}^{k-1} \binom{n}{i} p^i (1-p)^{n-i} \tag{2.1.3}$$

This of course follows from equation (1.1.4).

One can improve upon these interval estimates under the assumption of symmetry. Breth (1982) suggests distribution free methods when the median is known or unknown. In the former case the improvements over previous methods are substantial. In the latter, gains may still be appreciable.

2.2 Tolerance Intervals

One may wish to have some interval that would contain at least some specified proportion (γ) of the population. Given γ and a desired level of confidence then

$$P(F(X_{k_2:n}) - F(X_{k_1:n}) \geq \gamma) = \sum_{i=0}^{k_2-k_1-1} \binom{n}{i} \gamma^i (1-\gamma)^{n-i} \tag{2.2.1}$$

This is based upon the assumption that the underlying random variable is continuous, so that $F(X) \sim \text{Unif}(0,1)$. Hence $F(X_{k_2:n}) - F(X_{k_1:n}) \stackrel{d}{=} U_{k_2:n} - U_{k_1:n} \stackrel{d}{=} U_{k_2-k_1:n}$, where $U_{k:n}$ is the k-th order statistic of a sample of size n from a standard uniform distribution. The latter distributional equality, is a property of uniform order statistics. The coverage probability of this interval depends only on the distance between the two order statistics.

For one-sided tolerance intervals we have

$$P(1 - F(X_{k:n}) \geq \gamma) = P(X_{k:n} \leq \xi_{1-\gamma}) \tag{2.2.2}$$

$$P(F(X_{k:n}) \geq \gamma) = P(X_{k:n} \geq \xi_{\gamma}) \tag{2.2.3}$$

which is equivalent to a one-sided confidence interval for the p-th quantile.

2.3 Prediction Intervals

It is often desirable to make predictions of order statistics from future samples. We may make prediction for either a specific order statistic, or at least a specified number from a future independent sample.

One-sided prediction intervals for a single order statistic is equivalent to exceedances (see Balakrishnan and Ng, 2006; David and Nagaraja, 2003; Gastwirth, 1968). Given a sample of size n_1, $(X_{i_1:n_1}, \infty)$ is a one-sided prediction interval for the i_2-th order statistic $W_{i_2:n_2}$, from a future sample of size n_2 with probability

$$g_{i_1,i_2} = P(X_{i_1:n_1} < W_{i_2:n_2}) = \sum_{i<i_2-1} \frac{\binom{n_2}{i}\binom{n_1}{i_1+i_2-i-1}}{\binom{n_1+n_2}{i_1+i_2-1}} \tag{2.3.1}$$

The probability for two-sided prediction intervals $(X_{i_1:n_1}, X_{i_1^*:n_1})$, is given by $g_{i_1,i_2} -$

$g_{i_1^*,i_2}$.

For prediction of future progressively type-II order statistics, Guilbaud (2001) expresses the marginal distribution of a progressively type-II censored order statistic as a mixture of typical order statistics. This mixture representation combined with (2.3.1) can be used to calculate prediction intervals for a specified order statistic from a future progressively type-II censored sample. Exceedances can similarly be obtained for the case of a usual order statistic and a PCOS directly (See Bairamov and Eryilmaz, 2006; Ng and Balakrishnan, 2005).

To obtain the probability of at least $\lambda > 0$ values from a future complete sample $\mathbf{W}$, consider the following. For $1 \leq i < j \leq n_2$ and $1 \leq k_1 < k_2 \leq n_1$, we have

$$\begin{aligned}
&P(\text{at least } \lambda\, W's \in (X_{k_1:n_1}, X_{k_2:n_1})) \\
&\qquad = \sum_{j=\lambda}^{n_2} \sum_{i=0}^{n_2-j} P(W_{i:n_2} < X_{k_1:n_1} < W_{i+1:n_2} < W_{i+j:n_2} < X_{k_2:n_1} < W_{i+j+1:n_2}) \\
&\qquad = \sum_{j=\lambda}^{n_2} \sum_{i=0}^{n_2-j} \frac{\binom{i+k_1-1}{i}\binom{j+k_2-k_1-1}{j}\binom{n_1+n_2-k_2-i-j}{n_2-i-j}}{\binom{n_1+n_2}{n_2}} \qquad (2.3.2)
\end{aligned}$$

To obtain the probabilities for a type-II censored sample we need only to modify the indices in the sums. If the new sample has r observed failures, then the probability need be separated into two sums reflecting whether the final observed value $W_{r:n_2}$ falls below or above $X_{k_2:n_2}$. The probability would be

$$
\begin{aligned}
&P(\text{at least } \lambda\ W's \in (X_{k_1:n_1}, X_{k_2:n_1})) \\
&= \sum_{j=\lambda}^{r-1} \sum_{i=0}^{r-j-1} P(W_{i:n_2} < X_{k_1:n_1} < W_{i+1:n_2} < W_{i+j:n_2} < X_{k_2:n_1} < W_{i+j+1:n_2}) \\
&\quad + \sum_{j=\lambda}^{r} \sum_{c=0}^{n_2-r} P(W_{r-j:n_2} < X_{k_1:n_1} < W_{r-j+1:n_2} < W_{r+c:n_2} < X_{k_2:n_1} < W_{r+c+1:n_2}) \\
&= \sum_{j=\lambda}^{r-1} \sum_{i=0}^{r-j-1} \frac{\binom{i+k_1-1}{i}\binom{j+k_2-k_1-1}{j}\binom{n_1+n_2-k_2-i-j}{n_2-i-j}}{\binom{n_1+n_2}{n_2}} \\
&\quad + \sum_{j=\lambda}^{r} \sum_{c=0}^{n_2-r} \frac{\binom{r-j+k_1-1}{r-j}\binom{j+c+k_2-k_1-1}{j+c}\binom{n_1+n_2-k_2-r-c}{n_2-r-c}}{\binom{n_1+n_2}{n_2}} \qquad (2.3.3)
\end{aligned}
$$

If $r = 1$, then the first term collapses leaving the second.

Chapter 3

Multiple Type-II Censored Samples

Consider estimating the p-th quantile ξ_p from a continuous distribution with the interval $(X_{k_1:n}, X_{k_2:n})$. If we do not wish to assume anything further, we can then find the coverage probability of this interval as in equation (2.1.1).

Suppose we take a sample of size $n = 10$. We can estimate the median with the interval $(X_{2:10}, X_{9:10})$, which has coverage probability

$$P(X_{2:10} < \xi_{0.5} < X_{9:10}) = \sum_{i=2}^{8} \binom{10}{i} \left[\frac{1}{2}\right]^{10} \approx 0.9785$$

If we wished to instead estimate an upper quantile, say $\xi_{0.9}$, the largest interval

$(X_{1:10}, X_{10:10})$ only has coverage probability

$$P(X_{1:10} < \xi_{0.9} < X_{10:10}) = \sum_{i=1}^{9} \binom{10}{i} \left[\frac{9}{10}\right]^{i} \left[\frac{1}{10}\right]^{n-i} \approx 0.6513$$

The median is the quantile which will have the highest coverage probability, and this nearly requires the complete sample. The 90-% quantile requires the whole range and can not achieve a reasonable coverage probability. It is evident that more information is needed.

With a censored sample, the problem is exasperated. If there is 30% right censoring, then the coverage probabilities for the quantiles $\xi_{0.5}$ and $\xi_{0.9}$, of the largest interval $(X_{1:10}, X_{7:10})$ are 82.71% and 7.02% respectively. One may wish to take additional future independent samples to obtain more favourable coverage probabilities.

However, this need not be the only reason one would wish to pool multiple type-II samples. It may be that a machine stresses items to failure, but can only place a certain number at a time. Multiple runs may then be done to fail a larger number of items. It will be shown later that it can be desirable to intentionally design an experiment with pooling in order to obtain better estimates of upper quantiles. Balakrishnan et al. (2010b) have considered inference for two independent type-II samples. In this chapter we will extend these results to B independent type-II samples.

Consider B independent type-II samples, where $X_{b,k:n_b}$ is the k-th order statistic from the b-th sample of size n_b. We have the condition that $1 \leq r_b \leq n_b$. When

$r_b = n_b$ the b-th sample is complete; and when $r_b < n_b$ the sample is type-II censored. Let $Z_{(i)}$ be the i-th $(1 \leq i \leq \dot{r})$ pooled order statistic from the pooled sample.

3.1 Distributional Representations

The multi-sample case is naturally more complex than its two-sample counterpart. Consider the marginal distribution of a pooled order statistic.

In the two-sample case, when the i-th pooled order statistic is conditioned to be from the first sample, then the number of observed items from the second sample above or below it are fixed. This is not generally true with 3 or more samples, though it can be in certain cases. In the multi-sample case, given some item from some sample being the i-th pooled order statistic, we can freely fix the number of items above and below the i-th pooled order statistic in at most $B-2$ samples.

As a result, we can expect the representations given here to be much more complex than in the two-sample case.

3.1.1 Marginal Distribution of a pooled OS

To obtain the marginal distribution of $Z_{(i)}$, we can partition the sample space to obtain the following

$$P(Z_{(i)} \leq \xi_p) = \sum_{b=1}^{B} \sum_{k=1}^{r_b} P(Z_{(i)} = X_{b,k:n_b} \leq \xi_p), \tag{3.1.1}$$

as the events, $Z_{(i)} = X_{b,k:n_b}$ are exhaustive, and are mutually exclusive with probability 1.

For any permutation of the samples that gives $Z_{(i)} = X_{b,k:n_b}$, let $\mathcal{A} = \{1, 2, \ldots, B\} \setminus b$. Some sample $b^o \in \mathcal{A}$ may have all of its observed values and some of the latent unobserved values below $X_{b,k:n_b}$. Namely $X_{b^o,r_{b^o}+c_j:n_{b^o}} < X_{b,k:n_b} < X_{b^o,r_{b^o}+c_j+1:n_{b^o}}$, for some $1 \leq c_j \leq n_{b^o} - r_{b^o}$. If this is the case, then for this permutation, $b^o \in \{b'\}$.

Otherwise the latent unobserved values all lie above $X_{b,k:n_b}$ and the observed values can be either above or below. Namely, $X_{b^o,c_j:n_{b^o}} < X_{b,k:n_b} < X_{b^o,c_j+1:n_{b^o}}$ for some $0 \leq c_j \leq r_{b^o}$. Thus these samples b^o are in $\{b''\}$.

For any permutation of the samples giving $Z_{(i)} = X_{b,k:n_b}$, we have a partition of $\mathcal{A}$ into $\{b'\}$, and $\{b''\}$. This is a valid partition of $\mathcal{A}$ iff $\dot{r}_{b'} + \dot{c}_{b''} = i - k$. When $\dot{r}_{\mathcal{A}} < i - k$ or when $i < k$, no partition leading to $Z_{(i)} = X_{b,k:n_b}$ exist.

$\{b''\}$ can be further subdivided into $\{b''_\beta\}$ and $\{b''_\alpha\}$. The former being "large" ($r_{b^o} \geq i - k - \dot{r}_{b'}$) or complete samples, and the latter being the "small", incomplete samples. All samples in $\{b''_\beta\}$ can be treated as one larger sample for computational purposes, reducing the dimension of the sums involved.

We can now define a weight W, as

$$W_{\{i\},\{h\},\{l\},\{j\}} = \frac{\binom{i-1}{h_1,\ldots,h_d,i-1-\sum h}\binom{n-i}{l_1,\ldots,l_d,n-i-\sum l}}{\binom{n}{j_1,\ldots,j_d,n-\sum j}}.$$

Thus, we have the following result.

Theorem 3.1.1 *For $i = 1, 2, \ldots, \dot{r}$, and $0 \leq p \leq 1$, we have*

$$P(Z_{(i)} \leq p) = \sum_{b=1}^{B} \sum_{k=1}^{r_b} \sum_{\sigma_{\{b'\}}} \sum_{\substack{c_j=1 \\ j\in\{b'\}}}^{n_j-r_j} \sum_{\substack{c_j=0 \\ j\in\{b''_\alpha\}}}^{r_j}\Big|_U F_{i+\dot{c}_{b'}:n}(p)$$

$$\times W_{i+\dot{c}_{b'},\left\{\substack{k-1 \\ \{c_j+r_j\},j\in\{b'\} \\ \{c_j\},j\in\{b''_\alpha\}}\right\},\left\{\substack{n_b-k \\ \{n_j-r_j-c_j\},j\in\{b'\} \\ \{n_j-c_j\},j\in\{b''_\alpha\}}\right\},\left\{\substack{n_b \\ \{n_j\},j\in\{b'\} \\ \{n_j\},j\in\{b''_\alpha\}}\right\}},$$

where

$$U = \begin{cases} \{c_j | \dot{c}_{b''_\alpha} = i-k-\dot{r}_{b'}\} & \textit{if } \{b''_\beta\} \textit{ is empty} \\ \{c_j | \dot{c}_{b''_\alpha} \leq i-k-\dot{r}_{b'}\} & \textit{if } \{b''_\beta\} \textit{ is non-empty and if } i-k-\dot{r}_{j'} \leq \dot{r}_{b''} \\ \{c_j | i-k-\dot{r}_{b'} \geq \dot{c}_{b''_\alpha} \geq i-k-\dot{r}_{b'}-\dot{r}_{b''_\beta}\} & \textit{if } \{b''_\beta\} \textit{ is non-empty and if } i-k-\dot{r}_{j'} > \dot{r}_{b''}. \end{cases}$$

Here $|U$ indicates that the multiple sum of c_j's in $\{b''_\alpha\}$ is restricted to the region given by U.

Proof: For convenience, let us assume $X \sim Unif(0,1)$, since otherwise a probability integral transformation can be performed to bring the problem to uniform setting. As a result, it is known that $X_{k:n} \sim Beta(k-1, n-k)$.

We have that, $P(Z_{(i)} \leq p) = \sum_{b=1}^{B} \sum_{k=1}^{r_b} P\left(X_{b,k:n_b} = Z_{(i)} \leq p\right)$. Let $\{b'\}$ be some subset of $\mathcal{A}$. If $i-k < \dot{r}_{b'}$ then $X_{b,k:n_b} = Z_{(i)}$ is impossible since there are more than i observed values up to and including $X_{b,k:n_b}$. If $\dot{r}_{\mathcal{A}} < i-k$, then there are insufficient observed items from the other $B-1$ samples for $X_{b,k:n_b}$ to be the i-th pooled value. Consequently, $\dot{r}_{b'} + \dot{c}_{b''} = i-k$ is necessary and sufficient for the probability to be non-zero, and so we obtain

$$P(Z_{(i)} \leq p) = \sum_{b=1}^{B} \sum_{k=1}^{r_b} \sum_{\sigma_{\{b'\}}} \sum_{\substack{c_j=1 \\ j\in\{b'\}}}^{n_j-r_j} \sum_{\substack{c_j \\ j\in\{b''\}}} \Big|_{\substack{\dot{c}_{b''}=i-k-\dot{r}_{b'} \\ 0\leq c_j\leq r_j}} P(X_{b,k:n_b} = Z_{(i)} \leq p, \cap_j \mathcal{B}_{b'}, \cap_j \mathcal{C}_{b''}),$$

where $\mathcal{B}_{b'} = \{X_{j,r_j+c_j:n_j} < X_{b,k:n_b} < X_{j,r_j+c_j+1:n_j}\}\ \forall j \in \{b'\}$, and $\mathcal{C}_{b''} = \{X_{j,c_j:n_j} < X_{b,k:n_b} < X_{j,c_j+1:n_j}\}\ \forall j \in \{b''\}$.

Let us now consider $P(X_{b,k:n_b} = Z_{(i)} \leq p, \cap_j \mathcal{B}_j, \cap_j \mathcal{C}_j)$. Marginally, $X_{b,k:n_b}$ is $Beta(k-1, n_b-k)$, each other (complete) sample conditioned on $X = x$ can be viewed as a binomial event with success probability x and failure probability $1-x$. So we then have,

$$\begin{aligned}
&P(X_{b,k:n_b} = Z_{(i)} \leq p, \cap_j \mathcal{B}_j, \cap_j \mathcal{C}_j) \\
&\quad= \int_0^p \frac{n_b!}{(k-1)!(n_b-k)!} x^{k-1}(1-x)^{n_b-k} \\
&\qquad\times \prod_{j\in\{b'\}} \binom{n_j}{r_j+c_j} x^{r_j+c_j}(1-x)^{n_j-r_j-c_j} \prod_{j\in\{b''\}} \binom{n_j}{c_j} x^{c_j}(1-x)^{n_j-c_j}\, dx \\
&\quad= \int_0^p x^{k-1+\dot{c}_{b''}+(\dot{r}_{b'}+\dot{c}_{b'})}(1-x)^{n_b-k+(n_{b''}-\dot{c}_{b''})+(n_{b'}-\dot{r}_{b'}-\dot{c}_{b'})}\, dx \\
&\qquad\times \frac{n_b!}{(k-1)!(n_b-k)!} \prod_{j\in\{b'\}} \binom{n_j}{r_j+c_j} \prod_{j\in\{b''\}} \binom{n_j}{c_j} \\
&\quad= F_{i+\dot{c}_{b'}:n}(p) \frac{(i-1+\dot{c}_{b'})!(n-i-\dot{c}_{b'})!n_b!}{n!(k-1)!(n_b-k)!} \prod_{j\in\{b'\}} \binom{n_j}{r_j+c_j} \prod_{j\in\{b''\}} \binom{n_j}{c_j} \\
&\quad= F_{i+\dot{c}_{b'}:n}(p) \frac{\binom{i-1+\dot{c}_{b'}}{k-1, \{c_j\}_{j\in\{b''\}}, \{r_j+c_j\}_{j\in\{b'\}}} \binom{n-i-\dot{c}_{b'}}{n_b-k, \{n_j-c_j\}_{j\in\{b''\}}, \{n_j-r_j-c_j\}_{j\in\{b'\}}}}{\binom{n}{n_b, \{n_j\}_{j\in\{b''\}}, \{n_j\}_{j\in\{b'\}}}}
\end{aligned}$$

$$= W_{i+\dot{c}_{b'}, \left\{ \substack{k-1 \\ \{c_j+r_j\}, j\in\{b'\} \\ \{c_j\}, j\in\{b''_\alpha\}} \right\}, \left\{ \substack{n_b-k \\ \{n_j-r_j-c_j\}, j\in\{b'\} \\ \{n_j-c_j\}, j\in\{b''_\alpha\}} \right\}, \left\{ \substack{n_b \\ \{n_j\}, j\in\{b'\} \\ \{n_j\}, j\in\{b''_\alpha\}} \right\}} F_{i+\dot{c}_{b'}:n}(p) \frac{\prod_{j\in\{b''_\beta\}} \binom{n_j}{c_j}}{\binom{n-n_b-n_{b'}-n_{b''_\alpha}}{i-k-\dot{r}_{b'}-\dot{c}_{b''_\alpha}}}.$$

The final term above is a multivariate hypergeometric probability; see Johnson et al. (1997) for relevant details on this distribution. We have three cases here to consider. Firstly, if $\{b''_\beta\}$ is empty, the final term is 1 and this is already in the form of Theorem 3.1.1. Secondly, if $\{b''_\beta\}$ is non-empty and $i - k - \dot{r}_{b'} \leq \dot{r}_{b''_\beta}$, then the final term can be summed out leaving the restriction $\dot{c}_{b''_\alpha} \leq i - k - \dot{r}_{b'}$. Finally, if $\{b''_\beta\}$ is non-empty but $i - k - \dot{r}_{b'} > \dot{r}_{b''_\beta}$, then $\{b''_\beta\}$ consists only of complete samples. In this case, they can be summed out but the restriction becomes $i - k - \dot{r}_{b'} \geq \dot{c}_{b''_\alpha} \geq i - k - \dot{r}_{b'} - \dot{r}_{b''_\beta}$. Hence, the Theorem. ∎

Remark 3.1.2 *If more than one sample is complete, then these samples can be combined into one sample for computational purposes. The will increase the efficiency of the calculations.*

Remark 3.1.3 *The pooled sample maximum $Z_{(\dot{r})}$ has an alternate representation based on the fact that $Z_{(\dot{r})} = \max_{1\leq b\leq B} X_{b,r_b:n_b}$. The CDF can be given as*

$$F_{Z_{(\dot{r})}}(t) = \prod_{b=1}^{B} F_{r_b:n_b}(t), \forall\, t \in \mathbb{R} \tag{3.1.2}$$

Remark 3.1.4 *The mixture weights are weighted hypergeometric probabilities, where the weight n_b/n is the probability that the $(i + \dot{c}_{b'})$-th value came from sample b.*

Remark 3.1.5 *Calculating the mixture probability for $X_{i:n}$ should be avoided as this is often the most computationally intensive portion.*

Corollary 3.1.6 *For $1 \leq i \leq \min_j r_j$, $Z_{(i)} \stackrel{d}{=} X_{i:n}$*

Proof: Since $i \leq \min_j r_j$ then for all b ($1 \leq b \leq B$) and k ($1 \leq k \leq r_b$), $i - k \leq i - 1 < \min_j r_j$, so that $\sigma_{\{b'\}} = \{\{\emptyset\}\}$. Thus all samples b^o ($b^o \neq b$) are in $\{b''_\beta\}$ as $r_{b^o} \geq \min_j r_j \geq i - k = i - k - r_{\{b'\}}$. Furthermore $P(Z_{(i)} = X_{b,k:n_b}) = 0$ for all $k > i$. Thus the mixture representation reduces to

$$\begin{aligned} P(Z_{(i)} \leq \xi_p) &= \sum_{b=1}^{B}\sum_{k=1}^{r_b} W_{i,k-1,n_b-k,n_b} F_{i:n}(\xi_p) \quad = F_{i:n}(\xi_p)\sum_{b=1}^{B}\sum_{k=1}^{i} \frac{\binom{i-1}{k-1}\binom{n-i}{n_b-k}}{\binom{n}{n_b}} \\ &= F_{i:n}(\xi_p)\sum_{b=1}^{B}\sum_{k=1}^{i} \frac{n_b}{n} \frac{\binom{i-1}{k-1}\binom{n-i}{n_b-k}}{\binom{n-1}{n_b-1}} = F_{i:n}(\xi_p)\sum_{b=1}^{B}\frac{n_b}{n}\sum_{k=1}^{i} \frac{\binom{i-1}{k-1}\binom{n-i}{n_b-k}}{\binom{n-1}{n_b-1}} \\ &= F_{i:n}(\xi_p)\sum_{b=1}^{B}\frac{n_b}{n}\cdot 1 = F_{i:n}(\xi_p). \end{aligned}$$ ∎

Corollary 3.1.7 *Given B independent type-II left censored samples, the marginal distribution of $Z_{(i)}$ has the mixture representation*

$$Z_{(i)} \stackrel{d}{=} \sum_{k=1}^{n} Q_{(\dot{r}-i+1)k} X_{n-k+1:n}$$

where Q is a multinomial random variable independent of the X's and with success vector (q_{ik}) as in Theorem 3.1.1, based on a type-II right censored sample with the same censoring scheme.

Proof: Since $\{X_{b,k:n_b}; 1 \leq b \leq B, n_b - r_b + 1 \leq k \leq n_b\}$ is a collection of B left censored samples from $F_X(x)$, then for $\tilde{X} = -X$, $\{\tilde{X}_{b,k:n_b}; 1 \leq b \leq B, 1 \leq k \leq r_b\}$ is a collection of right censored samples with distribution function $F_{\tilde{X}}(x) = 1 - F_X(-x)$. Here the pooled order statistic have the property that $\tilde{Z}_{(\dot{r}-i+1)} = -Z_{(i)}$ and so

$$\begin{aligned} P(Z_{(i)} \leq \xi_p) &= P(\tilde{Z}_{(\dot{r}-i+1)} \geq -\xi_p) \\ &= \sum_{k=1}^{n} q_{(\dot{r}-i+1)k} P(\tilde{X}_{k:n} \geq -\xi_p) = \sum_{k=1}^{n} q_{(\dot{r}-i+1)k} P(X_{n-k+1:n} \leq \xi_p). \end{aligned}$$

The second equality being the application of Theorem 3.1.1 ■

We can similarly obtain results for type-II left censored samples, for the joint distribution of two or more pooled order statistics (Theorem 3.1.8, Propositions 3.1.10, 3.1.12, and 3.1.14) as done in Corollary 3.1.7.

3.1.2 Joint Distribution of two pooled OS

To obtain the joint probability of $P(Z_{(i_1)} \leq p_1, Z_{(i_2)} \leq p_2)$ for $1 \leq i_1 < i_2 \leq \dot{r}$, we proceed in a similar fashion as in Theorem 3.1.1. For $p_1 < p_2$, we are interested in

$$\begin{aligned} P(Z_{(i_1)} \leq p_1, Z_{(i_2)} \leq p_2) = &\sum_{b=1}^{B} \sum_{1 \leq k_1 < k_2 \leq r_b} P(X_{b,k_1:n_b} = Z_{(i_1)} \leq p_1, X_{b,k_2:n_b} = Z_{(i_2)} \leq p_2) \\ &+ \sum_{b^o \neq b} \sum_{b=1}^{B} \sum_{k_1=1}^{r_{b^o}} \sum_{k_2=1}^{r_b} P(X_{b^o,k_1:n_{b^o}} = Z_{(i_1)} \leq p_1, X_{b,k_2:n_b} = Z_{(i_2)} \leq p_2), \end{aligned} \quad (3.1.3)$$

of which the two terms in either summand must be treated separately albeit in a similar fashion. When $p_1 \geq p_2$ then the joint distribution reduces to $P(Z_{(i_2)} \leq p_2)$, the marginal distribution of $Z_{(i_2)}$.

Again we introduce similar notation as used in the previous section. Here the b-th sample, may or may not be the same as the b^o-th sample, so $\mathcal{A} = \{1, 2, \ldots, B\} \setminus \{b \bigcup b^o\}$. We define $\{b'\}$ as samples in $\mathcal{A}$ such that all the observed and some unobserved values from these samples are below $Z_{(i_2)}$. Furthermore, we define a $\{b'_1\}$ as a subset of $\{b'\}$ such that all the observed values fall below $Z_{(i_1)}$. Thus $\{b'_2\} = \{b'\} \setminus \{b'_1\}$ has the final observed value between $Z_{(i_1)}$ and $Z_{(i_2)}$.

Similarly $\{b''\}$ is the complement of $\{b'\}$ in $\mathcal{A}$. $\{b''_\beta\}$ is similarly defined as samples b^1 in $\{b''\}$ such that $\dot{r}_{b^1} \geq i - k - \dot{r}_{b'}$.

We again define $\sigma_{\{b'\}}$ as the collection of all valid $\{b'\}$. $\sigma_{\{b'_1\}}$ is the collection of valid $\{b'_1\} \in \{b'\}$.

Again we define a weight $\mathcal{W}$, as

$$\mathcal{W}_{\{i\},\{h\},\{m\},\{l\},\{j\}} = \frac{\binom{i_1-1}{h_1,\ldots,h_d,i_1-1-\sum h}\binom{i_2-i_1-1}{m_1,\ldots,m_d,i_2-i_1-1-\sum m}\binom{n-i_2}{l_1,\ldots,l_d,n-i_2-\sum l}}{\binom{n}{j_1,\ldots,j_d,n-\sum j}}.$$

Then we have the following Theorem.

Theorem 3.1.8 *For $i_1 = 1, 2, \ldots, \dot{r} - 1$, $i_1 < i_2 \leq \dot{r}$, and $0 \leq p_1 < p_2 \leq 1$, the first term on the RHS of (3.1.3) is*

$$\sum_{\sigma_{\{b'\}}} \sum_{\sigma_{\{b'_1\}}} \sum_{\substack{\{c_{j_1}, c_{j_2}\} \\ j \in \{b'_1\}}} \sum_{\substack{\{c_{j_1}, c_{j_2}\} \\ j \in \{b'_2\}}} \sum_{\substack{\{c_{j_1}, c_{j_2}\} \\ j \in \{b''_\alpha\}}} \Bigg|_{U_1} \mathcal{W}_{\{\cdot,\ldots,\cdot\}} F_{i_1 + \dot{c}_{j_1,b'_1}, i_2 + \dot{c}_{j_1,b'_1} + \dot{c}_{j_2,b'_1} + \dot{c}_{j_2,b'_2} : n}(p_1, p_2),$$

where

$$\{i\} = \left\{ \begin{matrix} i_1+\dot{c}_{j_1,b'_1} \\ i_2-i_1+\dot{c}_{j_2,b'_1}+\dot{c}_{j_2,b'_2} \end{matrix} \right\},$$

$$\{h\} = \left\{ \begin{matrix} k_1-1 \\ \{r_j+c_{j_1}\},j\in\{b'_1\} \\ \{c_{j_1}\},j\in\{b'_2\} \\ \{c_{j_1}\},j\in\{b''_\alpha\} \end{matrix} \right\}, \qquad \{m\} = \left\{ \begin{matrix} k_2-k_1-1 \\ \{c_{j_2}\},j\in\{b'_1\} \\ \{r_j+c_{j_2}-c_{j_1}\},j\in\{b'_2\} \\ \{c_{j_2}\},j\in\{b''_\alpha\} \end{matrix} \right\},$$

$$\{l\} = \left\{ \begin{matrix} n_b-k_2 \\ \{n_j-r_j-c_{j_1}-c_{j_2}\},j\in\{b'_1\} \\ \{n_j-r_j-c_{j_2}\},j\in\{b'_2\} \\ \{n_j-c_{j_1}-c_{j_2}\},j\in\{b''_\alpha\} \end{matrix} \right\}, \qquad \{j\} = \left\{ \begin{matrix} n_b \\ \{n_j\},j\in\{b'_1\} \\ \{n_j\},j\in\{b'_2\} \\ \{n_j\},j\in\{b''_\alpha\} \end{matrix} \right\};$$

and the second term on the RHS of (3.1.3) is

$$\sum_{c_b=0}^{k_2-1} \sum_{c_{b^o}=0}^{r_{b^o}-k_1} \sum_{\sigma_{\{b'\}}} \sum_{\sigma_{\{b'_1\}}} \sum_{\substack{\{c_{j_1},c_{j_2}\} \\ j\in\{b'_1\}}} \sum_{\substack{\{c_{j_1},c_{j_2}\} \\ j\in\{b'_2\}}} \sum_{\substack{\{c_{j_1},c_{j_2}\} \\ j\in\{b''_\alpha\}}}\Bigg|_{U_2} \mathcal{W}^1_{\{\cdot,\ldots,\cdot\}} F_{i_1+\dot{c}_{j_1,b'_1},i_2+\dot{c}_{j_1,b'_1}+\dot{c}_{j_2,b'_1}+\dot{c}_{j_2,b'_2}:n}(p_1,p_2)$$

$$+ \mathbb{1}_{1\leq n_{b^o}-r_{b^o}} \sum_{c_b=0}^{k_2-1} \sum_{c_{b^o}=1}^{n_{b^o}-r_{b^o}} \sum_{\sigma_{\{b'\}}} \sum_{\sigma_{\{b'_1\}}} \sum_{\substack{\{c_{j_1},c_{j_2}\} \\ j\in\{b'_1\}}} \sum_{\substack{\{c_{j_1},c_{j_2}\} \\ j\in\{b'_2\}}} \sum_{\substack{\{c_{j_1},c_{j_2}\} \\ j\in\{b''_\alpha\}}}\Bigg|_{U_3}$$

$$\mathcal{W}^2_{\{\cdot,\ldots,\cdot\}} F_{i_1+\dot{c}_{j_1,b'_1},i_2+c_{b^o}+\dot{c}_{j_1,b'_1}+\dot{c}_{j_2,b'_1}+\dot{c}_{j_2,b'_2}:n}(p_1,p_2),$$

where for $\mathcal{W}^1$, we have

$$\{i\} = \left\{ \begin{matrix} i_1+\dot{c}_{j_1,b'_1} \\ i_2-i_1+\dot{c}_{j_2,b'_1}+\dot{c}_{j_2,b'_2} \end{matrix} \right\},$$

$$\{h\} = \left\{ \begin{matrix} k_1-1 \\ c_b \\ \{r_j+c_{j_1}\},j\in\{b'_1\} \\ \{c_{j_1}\},j\in\{b'_2\} \\ \{c_{j_1}\},j\in\{b''_\alpha\} \end{matrix} \right\}, \qquad \{m\} = \left\{ \begin{matrix} c_{b^o} \\ k_2-c_b-1 \\ \{c_{j_2}\},j\in\{b'_1\} \\ \{r_j-c_{j_1}+c_{j_2}\},j\in\{b'_2\} \\ \{c_{j_2}\},j\in\{b''_\alpha\} \end{matrix} \right\},$$

$$\{l\} = \left\{ \begin{matrix} n_{b^o}-k_1-c_{b^o} \\ n_b-k_2 \\ \{n_j-r_j-c_{j_1}-c_{j_2}\},j\in\{b'_1\} \\ \{n_j-r_j-c_{j_2}\},j\in\{b'_2\} \\ \{n_j-c_{j_1}-c_{j_2}\},j\in\{b''_\alpha\} \end{matrix} \right\}, \qquad \{j\} = \left\{ \begin{matrix} n_{b^o} \\ n_b \\ n_{b^o} \\ \{n_j\},j\in\{b'_1\} \\ \{n_j\},j\in\{b'_2\} \\ \{n_j\},j\in\{b''_\alpha\} \end{matrix} \right\},$$

and for $\mathcal{W}^2$, we have

$$\{i\} = \left\{\begin{matrix} i_1 + \dot{c}_{j_1, b'_1} \\ i_2 - i_1 + c_{b^o} + \dot{c}_{j_2, b'_1} + \dot{c}_{j_2, b'_2} \end{matrix}\right\},$$

$$\{h\} = \left\{\begin{matrix} k_1 - 1 \\ c_b \\ \{r_j + c_{j_1}\}, j \in \{b'_1\} \\ \{c_{j_1}\}, j \in \{b'_2\} \\ \{c_{j_1}\}, j \in \{b''_\alpha\} \end{matrix}\right\}, \qquad \{m\} = \left\{\begin{matrix} r_{b^o} + c_{b^o} - k_1 \\ k_2 - c_b - 1 \\ \{c_{j_2}\}, j \in \{b'_1\} \\ \{r_j - c_{j_1} + c_{j_2}\}, j \in \{b'_2\} \\ \{c_{j_2}\}, j \in \{b''_\alpha\} \end{matrix}\right\},$$

$$\{l\} = \left\{\begin{matrix} n_{b^o} - r_{b^o} - c_{b^o} \\ n_b - k_2 \\ \{n_j - r_j - c_{j_1} - c_{j_2}\}, j \in \{b'_1\} \\ \{n_j - r_j - c_{j_2}\}, j \in \{b'_2\} \\ \{n_j - c_{j_1} - c_{j_2}\}, j \in \{b''_\alpha\} \end{matrix}\right\}, \qquad \{j\} = \left\{\begin{matrix} n_b \\ n_{b^o} \\ \{n_j\}, j \in \{b'_1\} \\ \{n_j\}, j \in \{b'_2\} \\ \{n_j\}, j \in \{b''_\alpha\} \end{matrix}\right\}.$$

In the above expressions,

$$U_1 = \begin{cases} \{(c_{j_1,b''_\alpha}, c_{j_2,b''_\alpha}) | \dot{c}_{j_1,b''_\alpha} = i_1 - k_1 - \dot{r}_{b'_1} - \dot{c}_{j_1,b'_2}, \dot{c}_{j_2,b''_\alpha} = i_2 - k_2 - \dot{r}_{b'_2} + \dot{c}_{j_1,b'_2} - i_1 + k_1\} \\ \qquad \text{if } \{b''_\beta\} \text{ is empty,} \\ \{(c_{j_1,b''_\alpha}, c_{j_2,b''_\alpha}) | \dot{c}_{j_1,b''_\alpha} \leq i_1 - k_1 - \dot{r}_{b'_1} - \dot{c}_{j_1,b'_2}, \dot{c}_{j_2,b''_\alpha} \leq i_2 - k_2 - \dot{r}_{b'_2} + \dot{c}_{j_1,b'_2} - i_1 + k_1\} \\ \qquad \text{if } \{b''_\beta\} \text{ is non-empty with } \dot{r}_{b''_\beta} \geq i_2 - k_2 - \dot{r}_{b'}, \\ \{(c_{j_1,b''_\alpha}, c_{j_2,b''_\alpha}) | \dot{c}_{j_1,b''_\alpha} \leq i_1 - k_1 - \dot{r}_{b'_1} - \dot{c}_{j_1,b'_2}, \dot{c}_{j_2,b''_\alpha} \leq i_2 - k_2 - \dot{r}_{b'_2} + \dot{c}_{j_1,b'_2} - i_1 + k_1, \\ \quad \dot{c}_{j_1,b''_\alpha} + \dot{c}_{j_2,b''_\alpha} \geq i_2 - k_2 - \dot{r}_{b'} - \dot{r}_{b''_\beta}\} \\ \qquad \text{if } \{b''_\beta\} \text{ is non-empty with } \dot{r}_{b''_\beta} < i_2 - k_2 - \dot{r}_{b'}, \end{cases}$$

$$U_2 = \begin{cases} \{(c_{j_1,b''_\alpha}, c_{j_2,b''_\alpha}) | \dot{c}_{j_1,b''_\alpha} = i_1 - k_1 - \dot{r}_{b'_1} - \dot{c}_{j_1,b'_2} - c_b, \dot{c}_{j_2,b''_\alpha} = i_2 - i_1 - k_2 + c_b - \dot{r}_{b'_2} + \dot{c}_{j_1,b'_2} - c_{b^o}\} \\ \qquad \text{if } \{b''_\beta\} \text{ is empty,} \\ \{(c_{j_1,b''_\alpha}, c_{j_2,b''_\alpha}) | \dot{c}_{j_1,b''_\alpha} \leq i_1 - k_1 - \dot{r}_{b'_1} - \dot{c}_{j_1,b'_2} - c_b, \dot{c}_{j_2,b''_\alpha} \leq i_2 - i_1 - k_2 + c_b - \dot{r}_{b'_2} + \dot{c}_{j_1,b'_2} - c_{b^o}\} \\ \qquad \text{if } \{b''_\beta\} \text{ is non-empty with } \dot{r}_{b''_\beta} \geq i_2 - k_2 - k_1 - c_{b^o} - \dot{r}_{b'}, \\ \{(c_{j_1,b''_\alpha}, c_{j_2,b''_\alpha}) | \dot{c}_{j_1,b''_\alpha} \leq i_1 - k_1 - \dot{r}_{b'_1} - \dot{c}_{j_1,b'_2} - c_b, \dot{c}_{j_2,b''_\alpha} \leq i_2 - i_1 - k_2 + c_b - \dot{r}_{b'_2} + \dot{c}_{j_1,b'_2} - c_{b^o} \\ \quad c_{j_1,b''_\alpha} + c_{j_2,b''_\alpha} \geq i_2 - k_2 - k_1 - c_{b^o} - \dot{r}_{b'}\} \\ \qquad \text{if } \{b''_\beta\} \text{ is non-empty with } \dot{r}_{b''_\beta} < i_2 - k_2 - k_1 - c_{b^o} - \dot{r}_{b'}, \end{cases}$$

$$U_3 = \begin{cases} \{(c_{j_1,b''_\alpha}, c_{j_2,b''_\alpha}) | \dot{c}_{j_1,b''_\alpha} = i_1 - k_1 - \dot{r}_{b'_1} - \dot{c}_{j_1,b'_2} - c_b, \dot{c}_{j_2,b''_\alpha} = i_2 - i_1 - k_2 + c_b - \dot{r}_{b'_2} + \dot{c}_{j_1,b'_2} - r_{b^o} + k_1\} \\ \qquad\qquad \text{if } \{b''_\beta\} \text{ is empty,} \\ \{(c_{j_1,b''_\alpha}, c_{j_2,b''_\alpha}) | \dot{c}_{j_1,b''_\alpha} \le i_1 - k_1 - \dot{r}_{b'_1} - \dot{c}_{j_1,b'_2} - c_b, \dot{c}_{j_2,b''_\alpha} \le i_2 - i_1 - k_2 + c_b - \dot{r}_{b'_2} + \dot{c}_{j_1,b'_2} - r_{b^o} + k_1\} \\ \qquad\qquad \text{if } \{b''_\beta\} \text{ is non-empty with } \dot{r}_{b''_\beta} \ge i_2 - k_2 - r_{b^o} - \dot{r}_{b'}, \\ \{(c_{j_1,b''_\alpha}, c_{j_2,b''_\alpha}) | \dot{c}_{j_1,b''_\alpha} \le i_1 - k_1 - \dot{r}_{b'_1} - \dot{c}_{j_1,b'_2} - c_b, \dot{c}_{j_2,b''_\alpha} \le i_2 - i_1 - k_2 + c_b - \dot{r}_{b'_2} + \dot{c}_{j_1,b'_2} - r_{b^o} + k_1 \\ \quad c_{j_1,b''_\alpha} + c_{j_2,b''_\alpha} \ge i_2 - k_2 - r_{b^o} - \dot{r}_{b'}\} \\ \qquad\qquad \text{if } \{b''_\beta\} \text{ is non-empty with } \dot{r}_{b''_\beta} < i_2 - k_2 - r_{b^o} - \dot{r}_{b'}. \end{cases}$$

Here $|U_1$, $|U_2$, and $|U_3$ imply a constraint to the multiple sum of c_{j_1}'s and c_{j_2}'s in $\{b''_\alpha\}$ to the described region.

Proof: We will restrict attention to the first summand in equation (3.1.3). The other summand is done similarly.

Again we consider only the case where $X \sim Unif(0,1)$ wlog. The joint distribution of two uniform OS is $f^{X_{i_1:n}, X_{i_2:n}}(x_1, x_2) = \frac{n!}{(i_1-1)!(i_2-i_1-1)!(n-i_2)!} x_1^{i_1-1}[x_2 - x_2]^{i_2-i_1-1}(1-x_2)^{n-i_2}$ for $0 < x_1 < x_2 < 1$. We have $P(Z_{(i_1)} \le p_1, Z_{(i_2)} \le p_2) = \sum_{b=1}^{B} \sum_{1 \le k_1 < k_2} P(Z_{(i_1)} = X_{b,k_1:n_b} \le p_1, Z_{(i_2)} = X_{b,k_2:n_b} \le p_2)$.

Clearly if $i_2 - i_1 < k_2 - k_1$ then $X_{b,k_1:n_b}$ and $X_{b,k_2:n_b}$ can not simultaneously be $Z_{(i_1)}$ and $Z_{(i_2)}$ respectively as they are too far apart in terms of indices. We also require that $\dot{r}_{\mathcal{A}} \ge i_2 - k_2 \ge 0$ as in Theorem 3.1.1. Thus necessary and sufficient condition for $P(X_{b,k_1:n_b} = Z_{(i_1)}, X_{b,k_2:n_b} = Z_{(i_2)}) > 0$ are $0 \le i_2 - k_2 \le \dot{r}_{\mathcal{A}}$ and $1 \le k_2 - k_1 \le i_2 - i_1$, so we obtain the following,

$$P(Z_{(i_1)} \le p_1, Z_{(i_2)} \le p_2) = \sum_{b=1}^{B} \sum_{1 \le k_1 < k_2} \sum_{\sigma_{\{b'\}}} \sum_{\sigma_{\{b'_1\}}} \sum_{\substack{\{c_{j_1}, c_{j_2}\} \\ j \in \{b'_1\}}} \sum_{\substack{\{c_{j_1}, c_{j_2}\} \\ j \in \{b'_2\}}} \sum_{\substack{\{c_{j_1}, c_{j_2}\} \\ j \in \{b''_\alpha\}}} \Big|_{U_1}$$

$$P(Z_{(i_1)} = X_{b,k_1:n_b} \le p_1, Z_{(i_2)} = X_{b,k_2:n_b} \le p_2, \cap \mathcal{B}_{\{b'_1\}}, \cap \mathcal{B}_{\{b'_2\}}, \cap \mathcal{C}_{\{b''\}}),$$

where

$$\mathcal{B}_{\{b'_1\}} = \begin{cases} X_{j,r_j+c_{j_1}:n_j} < X_{b,k_1:n_b} < X_{j,r_j+c_{j_1}+1:n_j} \\ X_{j,r_j+c_{j_1}+c_{j_2}:n_j} < X_{b,k_2:n_b} < X_{j,r_j+c_{j_1}+c_{j_2}+1:n_j} \end{cases}$$

$$0 \le c_{j_1} \le n_j - r_j \qquad 0 \le c_{j_2} \le n_j - r_j - c_{j_1} \qquad c_{j_1} + c_{j_2} \ge 1$$

$$\mathcal{B}_{\{b'_2\}} = \begin{cases} X_{j,c_{j_1}:n_j} < X_{b,k_1:n_b} < X_{j,c_{j_1}+1:n_j} \\ X_{j,r_j+c_{j_2}:n_j} < X_{b,k_2,:n_b} < X_{j,r_j,c_{j_2}+1:n_j} \end{cases}$$

$$0 \le c_{j_1} < r_j \qquad 1 \le c_{j_2} \le n_j - r_j$$

$$\mathcal{C}_{\{b''\}} = \begin{cases} X_{j,c_{j_1}:n_b} < X_{b,k_1:n_b} < X_{j,c_{j_1}+1:n_b} \\ X_{j,c_{j_1}+c_{j_2}:n_b} < X_{b,k_2:n_b} < X_{j,c_{j_1}+c_{j_2}+1:n_b} \end{cases}$$

$$0 \le c_{j_1} \qquad 0 \le c_{j_2} \qquad 0 \le c_{j_1} + c_{j_2} \le r.$$

Concerning ourselves with the probability in the summand we have

$$P(Z_{(i_1)} = X_{b,k_1:n_b} \le p_1, Z_{(i_2)} = X_{b,k_2:n_b} \le p_2, \cap \mathcal{B}_{\{b'_1\}}, \cap \mathcal{B}_{\{b'_2\}}, \cap \mathcal{C}_{\{b''\}})$$

$$= \int_0^{p_1} \int_u^{p_2} \frac{n_b!}{(k_1-1)!(k_2-k_1-1)!(n_b-k_2)!} u^{k_1-1} (v-u)^{k_2-k_1-1} (1-v)^{n_b-k_2}$$

$$\prod_{j\in\{b_1'\}} \binom{n_j}{r_j + c_{j_1}, c_{j_2}, n - r_j - c_{j_1} - c_{j_2}} u^{r_j+c_{j_1}} (v-u)^{c_{j_2}} (1-v)^{n-r_j-c_{j_1}-c_{j_2}}$$

$$\prod_{j\in\{b_2'\}} \binom{n_j}{c_{j_1}, r_j - c_{j_1} + c_{j_2}, n - r_j - c_{j_2}} u^{c_{j_1}} (v-u)^{r_j-c_{j_1}+c_{j_2}} (1-v)^{n-r_j-c_{j_2}}$$

$$\prod_{j\in\{b''\}} \binom{n_j}{c_{j_1}, c_{j_2}, n - c_{j_1} - c_{j_2}} u^{c_{j_1}} (v-u)^{c_{j_2}} (1-v)^{n-c_{j_1}-c_{j_2}} dvdu$$

$$= \frac{n!}{(i_1 - 1 + \dot{c}_{j_1,\{b_1'\}})!(i_2 - i_1 - 1 + \dot{c}_{j_2,\{b_1'\}} + \dot{c}_{j_2,\{b_2'\}})!(n - i_2 - \dot{c}_{j_1,\{b_1'\}} - \dot{c}_{j_2,\{b_1'\}} - \dot{c}_{j_2,\{b_2'\}})!}$$

$$\int_0^{p_1}\int_u^{p_2} u^{i_1-1+\dot{c}_{j_1,\{b_1'\}}} (v-u)^{i_2-i_1-1+\dot{c}_{j_2,\{b_1'\}}+\dot{c}_{j_2,\{b_2'\}}} (1-v)^{n-i_2-\dot{c}_{j_1,\{b_1'\}}-\dot{c}_{j_2,\{b_1'\}}-\dot{c}_{j_2,\{b_2'\}}} dvdu$$

$$\frac{\prod_{j\in\{b_\beta''\}} \binom{n_j}{c_{j_1}, c_{j_2}, n-c_{j_1}-c_{j_2}}}{\binom{n-n_b-n_{\{b'\}}-n_{\{b_\alpha''\}}}{i_2-k_2-\dot{r}_{\{b'\}}-\dot{c}_{\{b_\alpha''\}}}} \times \text{ Constant},$$

wherein the integral is the joint CDF of two OS from a sample of size n. The constant can be seen to be the weight $\mathcal{W}^1$, and with U_1 being the same restriction as in Theorem 3.1.1 which follows from the definition of $\{b_\beta''\}$. ∎

Remark 3.1.9 *The number of mixture terms required is strictly less than $\binom{n}{2}$, so the number of terms required for storage is at most $O(n^2)$. There are a total of $\binom{\dot{r}}{2}$ of these. So the total storage to calculate all terms is $O(n^4)$. However this bound is not sharp and in likely much lower.*

3.1.3 Joint Distribution of pooled OS

There are many ways to obtain the mixture distribution of the joint pooled order statistics. Firstly one can represent the joint distribution of the $\dot{r}$ pooled order statistics as a mixture of the joint distribution of $\dot{r}$ order statistics from a sample of size

n.

Proposition 3.1.10 *The joint distribution of the pooled order statistics, can be represented as a mixture of joint distributions of size $\dot{r}$, of a subset of the usual order statistics from a sample of size n. For all $0 < \xi_{p_1} < \cdots < \xi_{p_{\dot{r}}} < 1$ we have*

$$F_{Z_{(1)},\ldots,Z_{(\dot{r})}}(\xi_{p_1},\ldots,\xi_{p_{\dot{r}}}) = \sum_{1\leq i_1<i_2<\ldots<i_{\dot{r}}\leq n} q_{k_1,\ldots,k_{\dot{r}}} F_{X_{i_1:n},\ldots,X_{i_{\dot{r}}:n}}(\xi_{p_1},\ldots,\xi_{p_{\dot{r}}}).$$

Proof: Consider any one of the $\binom{n}{n_1,\ldots,n_B}$ permutations of the B samples, with respect to both the observable items, and latent unobserved items. Label the observable failures $X_{b,k:n_b}$ ($1 \leq b \leq B$ and $1 \leq k \leq r_b$) as $Z_{(1)}$ through $Z_{(\dot{r})}$. Count the number of unobservable failures before the l-th pooled order statistic and call it j'_l ($1 \leq l \leq \dot{r}$). Then conditioned on this permutation, the i-th pooled order statistic is the $i + j'_i$-th out of the n total items. This yields a vector $(1 + j'_1, \ldots, i + j'_i, \ldots, \dot{r} + j'_{\dot{r}})$ of strictly increasing values, as the sequence of j-primes is non-decreasing by construction. Since there are at most $n - \dot{r}$ items unobserved, and since $j'_i \geq 0$ for all i, this is an increasing subset of size $\dot{r}$, from $\{1, 2, \ldots, B\}$.

We can then partition the outcome space into all possible combinations that yield identical vectors $(1 + j'_1, \ldots, i + j'_i, \ldots, \dot{r} + j'_{\dot{r}})$. Conditioned on being in this group of permutations, then the observed pooled order statistics are always the $1 + j'_1$-st $\ldots$ $i + j'_i$-th $\ldots$ and $\dot{r} + j'_{\dot{r}}$)-th typical order statistics of a sample of size n.

The mixture weight q is the number of unique permutations of all the observed and unobserved items that yield the vector $(1+j_1', \ldots, i+j_i', \ldots, \dot{r}+j_{\dot{r}}')$, divided by the total number of permutations. ∎

Remark 3.1.11 *If the observed fraction* $\frac{\dot{r}}{n} \to \varpi$ *for some* $0 \leq \varpi < 1$*, then the number of mixture terms required is at most* $\binom{n}{r} = \binom{n}{[n\varpi]} \geq \left(\frac{n}{[n\varpi]}\right)^{n\varpi} \sim \left(\frac{1}{\varpi}\right)^{n\varpi}$*. Similarly* $\binom{n}{r} \leq \binom{n}{[n/2]} \sim \frac{2^n}{\sqrt{n}} \geq 2^{n-\delta}$ *for all* $\delta > 0$*. So an upper bound for the number of terms required for storage is between* $O\left(\left(\frac{1}{\varpi^{\varpi}}\right)^n\right)$ *and* $O(2^n)$*. Thus storage is at most exponential growth.*

The upper bound of $O((2-\delta)^n)$ for all $\delta > 0$ can be obtained. Consider B samples of size 2, each with one observation so that $n = 2B$. To determine the number of possible combinations we consider permutations, we consider two additional restrictions $X_{1,1:2} < X_{2,1:2} < \cdots < X_{B,1:2}$ and $X_{1,2:2} < X_{2,2:2} < \cdots < X_{B,2:2}$. That is both the observed variables are ordered and the unobserved latent variables are similarly ordered. The first is justified because the samples are identical. The second because the ordering of the latent variables does not change the distribution, only the number above or below an observed item.

It is clear that this is the number of Dyck words of length $n = 2B$. Thus the possible number of orderings is the B-th Catalan number, $C_B = \frac{1}{B+1}\binom{2B}{B}$. Catalan numbers satisfy the recurrence relation $(B+2)C_{B+1} = 2(2B+1)C_B$, so that $\frac{C_{B+1}}{C_B} = 2\frac{2B+1}{B+2} \to 4$ as $B \to \infty$. Thus for all $\delta > 0$, the storage is $O((4-\delta)^B)$, or equivalently $O((2-\delta)^n)$).

We can obtain a different representation based on the joint distribution of progressively type-II censored samples.

Proposition 3.1.12 *The joint distribution of the pooled order statistics, can be represented as a mixture of progressively censored samples with number of observations $\dot{r}$ and total sample size n. For all $0 < \xi_{p_1} < \cdots < \xi_{p_{\dot{r}}} < 1$ we have*

$$F_{Z_{(1)},\ldots,Z_{(\dot{r})}}(\xi_{p_1},\ldots,\xi_{p_{\dot{r}}}) = \sum_{\substack{(j_1,\ldots,j_B)\in\\ \sigma\{1,\ldots,B\}}} \sum_{\substack{0\le i_2\le r_{j_2}+i_3-1\\ \vdots \\ 0\le i_{B-1}\le r_{j_{B-1}}+i_B-1\\ 0\le i_B\le r_{j_B}-1}} q^{i_1,\ldots,i_B}_{j_1,\ldots,j_B} F^{\vec{R}}_{\vec{T}}(\xi_{p_1},\ldots,\xi_{p_{\dot{r}}})$$

Where $\vec{T}^{\vec{R}} = \{T_{1:\dot{r}:n},\ldots,T_{\dot{r}:\dot{r}:n}\}$ are the $\dot{r}$ observations from a progressively type-II censored sample of size n. The censoring scheme $\vec{R}$ is

$$\vec{R} = (\underbrace{0,\ldots,0}_{r_{j_1}+i_2-1 \ \textit{times}}, n_{j_1}-r_{j_1}, \underbrace{0,\ldots,0}_{r_{j_2}+i_3-i_2-1 \ \textit{times}}, n_{j_2}-r_{j_2},\ldots,\underbrace{0,\ldots,0}_{r_{j_B}-i_B-1 \ \textit{times}}, n_{j_B}-r_{j_B}).$$

Proof: Consider any permutation of the observed items and latent unobserved items, and label the observed items as the $\dot{r}$ pooled order statistics.

Furthermore, given this permutation, $(j_1,\ldots,j_B)$ is some (not necessarily increasing) permutation of $\{1,2,\ldots,B\}$, such that $X_{j_1,r_{j_1}:n_{j_1}} < X_{j_2,r_{j_2}:n_{j_2}} < \ldots < X_{j_B,r_{j_B}:n_{j_B}}$. Let j'_l $(1 \le l \le B)$ be the number of observed values from all samples before the l-th sample.

Conditioning only on the events $X_{j_1,r_{j_1}:n_{j_1}} < X_{j_2,r_{j_2}:n_{j_2}} < \ldots < X_{j_B,r_{j_B}:n_{j_B}}$, and the vector of j-primes, we consider sequentially the latent unobserved items. After observing $r_{j_1}+j_1'$ failures we remove $n_{j_1}-r_{j_1}$ items. These will be selected uniformly, and without replacement, amongst the remaining items due to independence amongst samples, and that all (unordered) items not yet removed are conditionally i.i.d.

We continue recursively, so that after observing $X_{j_l,r_{j_l}:n_{j_1}}$ we remove $n_{j_l}-r_{j_l}$ items uniformly amongst the remaining items. We continue this until the last observed failure $X_{j_{\dot{r}}}, r_{j_{\dot{r}}}:n_{j_{\dot{r}}}$, where the final $n_{j_{\dot{r}}} - r_{j_{\dot{r}}}$ items are removed.

As described this is a progressively type-II censored sample. It is clear that the collection of permutations of all items as described above is exactly the same as all possible permutations that result from a progressively type-II censored sample by construction. Thus the Theorem. ∎

Consider the previous example with B samples, each with one observed value, and one censored value. The representation in Proposition 3.1.12 is simply a single progressively censored sample. The censoring scheme is $\vec{R} = (1, \ldots, 1)$, which is of length B.

In general if all samples have one observed failure, and R_1 unobserved failures, the joint distribution will be a single progressively censored sample with scheme $\vec{R} = (R_1, \ldots, R_1)$. In this sense we can consider a progressively censored sample with an identical number of removals at each step, to be a pooling of several type-II censored samples.

Given some ordering of the final observable failure in B samples we can determine the number of unique progressive type-II schemes as in Proposition 3.1.12 recursively.

$$
\begin{array}{llll}
\xleftarrow{\hspace{2cm}} X_{j_1,r_{j_1}:n_{j_1}} & & & \\
\xleftarrow{\hspace{3.5cm}} X_{j_2,r_{j_2}:n_{j_2}} & & & \\
\vdots \qquad\qquad \vdots \qquad\qquad \ddots & & & \\
\xleftarrow{\hspace{6cm}} X_{j_B,r_{j_B}:n_{j_B}} & & &
\end{array}
$$

Given the ordering $X_{j_1,r_{j_1}:n_{j_1}} < X_{j_2,r_{j_2}:n_{j_2}} < \cdots < X_{j_B,r_{j_B}:n_{j_B}}$, we determine the number of unique schemes recursively as follows. With one sample, there is trivially one scheme only. With two samples, we can have i_2 observed failures from the second sample fall below $X_{j_1,r_{j_1}:n_{j_1}}$ $(0 \leq i_2 \leq r_{j_2} - 1)$. So thus there are r_{j_2} schemes. Given $B-1$ samples we can add one sample and place i_{j_B} observed failures from that sample in the $B-1$-st group. Leading to the recursive sum

$$
\begin{aligned}
S\left(B, r_{j_1}, \ldots, r_{j_B}, \vec{j}_B\right) &= \sum_{i_B=0}^{r_{j_B}-1} \sum_{i_{B-1}=0}^{r_{j_{B-1}}-1+i_B} \cdots \sum_{i_3=0}^{r_{j_3}-1+i_4} \sum_{i_2=0}^{r_{j_2}-1+i_3} 1 \\
&= \sum_{i_B=0}^{r_{j_B}-1} S\left(B-1, r_{j_1}, \ldots, r_{j_{B-2}}, r_{j_{B-1}} + i_B, \vec{j}_{B-1}\right).
\end{aligned}
$$

Remark 3.1.13 *In the special case where $r_b = r_1$ and $n_b - r_b = n_1 - r_1 > 0$ for all $1 \leq b \leq B$ we can solve the recursion exactly. In this case the number of unique schemes is $S(B,r) = \frac{1}{(r-1)B+1}\binom{Br}{B}$.*

We can also consider a more direct method of obtaining the joint distribution. Firstly, consider the joint distribution of the un-pooled statistics. Under the independence assumption between samples this is given as

$$\begin{aligned} f_{1,\ldots,\dot{r}}(x_{1,1},\ldots,x_{b,r_b}) &= \prod_{b=1}^{B} f_{X_{b,1:n_b},\ldots,X_{b,r_b:n_b}}(x_{b,1},\ldots,x_{b,r_b}) \\ &= \prod_{b=1}^{B}[1-F_X(x_{b,r_b})]^{n_b-r_b}\prod_{j=1}^{r_b} f_X(x_{b,j}) \end{aligned}$$

when $x_{b,1} < \cdots < x_{b,r_b}, b = 1, 2, \ldots B$. We then have the following proposition.

Proposition 3.1.14 *The pooled order statistics have the joint distribution as*

$$f_{Z_{(1)},\ldots,Z_{(\dot{r})}}(z_1,\ldots,z_{\dot{r}}) = \sum_{\pi\in\mathfrak{S}} f_{1,\ldots,\dot{r}}(z_{\pi(1)},\ldots,z_{\pi(\dot{r})}), z_1 < \cdots < z_{\dot{r}},$$

where $\mathfrak{S}$ represents all permutations π on $(1, 2, \ldots, \dot{r})$ that respect the ordering within each independent sample.

As given, Proposition 3.1.14 is not a mixture distribution as the summand will not integrate to 1. However, it is clear that each term in the summand is a distinct permutation of the original data and thus Proposition 3.1.14 can be rewritten as a mixture distribution, where the component distributions are the pooled distribution given some particular ordering of the original data, and the weight is the probability of that ordering.

Given the representations in Propositions 3.1.10-3.1.14, we can obtain the joint distribution of any number of pooled order statistics.

For two pooled order statistics, if we were to use Proposition 3.1.10, we would obtain the same representation as Theorem 3.1.8. In the case of Proposition 3.1.12, we would obtain a mixture representation involving the joint distribution of two progressively censored order statistics. If one were to apply the results of Guilbaud (2004), we could turn this representation into the same as one would obtain with Proposition 3.1.10.

In the case of Proposition 3.1.14, to obtain the marginal or joint k-variate distributions, one needs to integrate out the appropriate pooled OS. However, the distribution will not be in terms of order statistics, but more general distributions for which the properties are not well known. Moreover, one needs all possible permutations of which there are $\binom{\dot{r}}{r_1,\ldots,r_B}$. Consequently, it does not seem reasonable to use this representation for the inference as discussed in Section 3.2.

3.2 Inference

Given the distributional representations in the previous section, one is able to obtain nonparametric confidence intervals for quantiles, tolerance, intervals, and prediction intervals.

The marginal distribution of $Z_{(i)}$ as given in Theorem 3.1.1, will be used in confidence intervals for quantiles, and one-sided tolerance and prediction intervals. Two sided tolerance or prediction intervals require pairwise joint distributions as given in Theorem 3.1.8.

One can also use Propositions 3.1.10 or 3.1.12 to obtain the marginal or joint distributions. These have the advantage of being easily able to obtain multiple marginal and pairwise joint distributions simultaneously. Whereas with the representation in Theorems 3.1.1 and 3.1.8, one would need to obtain a mixture representation for each marginal or pairwise joint distribution individually.

For smaller sample sizes, the representations from the Propositions 3.1.10 or 3.1.12 may be more desirable. However as seen before, the storage space grows exponentially. In the case of Proposition 3.1.12, it may grow much faster than this. Storage is not the only issue however. Equal computational precision must be used for all weights, as they each will be used for different pairwise joint distribution. With only a modest number of samples and small number of observed values, there can be hundreds of thousands of necessary progressively type-II censored distributions.

So considering this, the representation in Theorem 3.1.8 may be best used when there are 3 or more samples. And either the representation in Theorem 3.1.8 or Proposition 3.1.12 in the two-sample case.

3.2.1 Confidence Intervals for Quantiles

One-sided coverage probabilities are very easily calculated combining the mixture representation as in 3.1.1, and one-sided confidence intervals for an order statistic as in equation (2.1.2). Thus, the one-sided coverage probability is calculated as

$$\begin{aligned} P(Z_{(i)} \leq \xi_p) &= \sum_{k=0}^{n} q_{ik} P(X_{k:n} \leq \xi_p) \\ &= \sum_{k=0}^{n} q_{ik} \sum_{l=k}^{n} \binom{n}{l} p^l (1-p)^{n-l}. \end{aligned} \tag{3.2.1}$$

The two-sided interval $(Z_{(i_1)}, Z_{(i_2)})$, for $1 \leq i_1 < i_2 \leq \dot{r}$, has coverage probability

$$P(Z_{(i_1)} \leq \xi_p \leq Z_{(i_2)}) = P(Z_{(i_1)} \leq \xi_p) - P(Z_{(i_2)} \leq \xi_p).$$

The two terms on the right hand side are one-sided coverage probabilities which can be calculated as in equation (3.2.1).

To compare the coverage probabilities between various schemes, Balakrishnan et al. (2010b) suggest the Standardized Maximum Coverage Probability (SCMP). We define this as

$$SCMP_{\sigma;r,n} = \frac{P(Z_{(\dot{r})} > p)}{EZ_{(\dot{r})}}.$$

Here σ represents the pooling design. The SCMP compares the highest possible coverage probability for any quantile and accounts for the cost in terms of increase time to test.

The alternate representation as in equation (3.1.2) can be used for any distribution without calculating the mixture distribution as in Theorem 3.1.1. However, for simplicity, we will use the standard uniform distribution when making comparisons.

Results for different distributions will be similar.

Using the sampling design from Table 1.2[1], we consider the following two scenarios. This censoring scheme will be referred to as the "Base" scheme.

First consider the gain in SCMP if we were to have observed an individual complete sample. Samples 1-4, or samples 5-6 would yield the same results, so we only consider sample 1, or sample 5 being complete. If we look at the first plot in Figure 3.1(a), we see the SCMP for the three cases, higher is better. The second plot shows the gains over the base case, in this regard, lower is better. Clearly there is a modest rise in SCMP. Observing all failures in sample 5 is mildly better than sample 1. However, this it because two more items are observed, as only the maximum observation in each sample can affect the SCMP.

We can see that observing even one additional failure, can have appreciable gains on the SCMP.

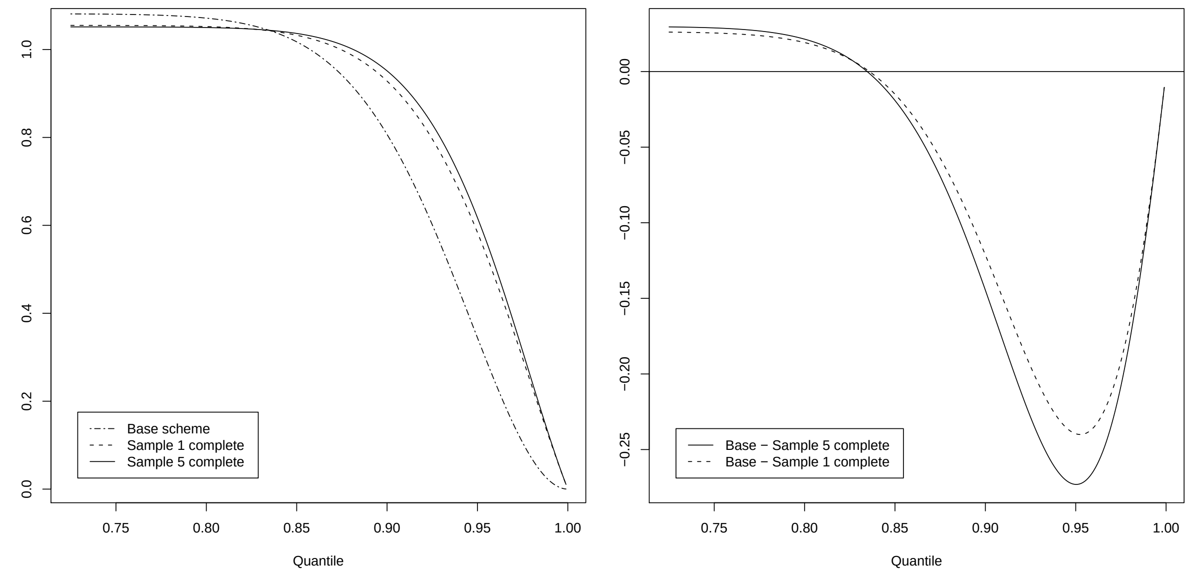
Next we consider the gain of the base case to the equivalent one-sample schemes[2] and two-sample schemes[3]. Again the first plot in Figure 3.1(b) represents the SCMP for all three schemes. Here the gains are very noticeable. In the second plot we can see that the two-sample scheme has a marked improvement, but the base scheme has a very large gain. Furthermore, the peak gain is at a higher level.

We can see in this specific case that there is a significant gain in the maximum coverage probability after accounting for the expected time to test.

[1] $\vec{n} = (10, 10, 10, 10, 10, 10), \vec{r} = (9, 9, 9, 9, 8, 8)$
[2] $n = 60, r = 52$
[3] $\vec{n} = (30, 30), \vec{r} = (26, 26)$

Figure 3.1: Gains in maximum coverage probabilities for upper quantiles

(a) SMCP for the original 6 sample (base) censoring scheme, SMCP for scheme with Sample 1 (or 2-4) complete, SMCP for scheme with Sample 5 (or 6) complete, and of the difference of the base case over each.



(b) SMCP for equivalent proportion, one-sample scheme (r=52,n=60), two-sample scheme, base case, and the difference of one-sample case over the base case and the one-sample case over the two-sample case.

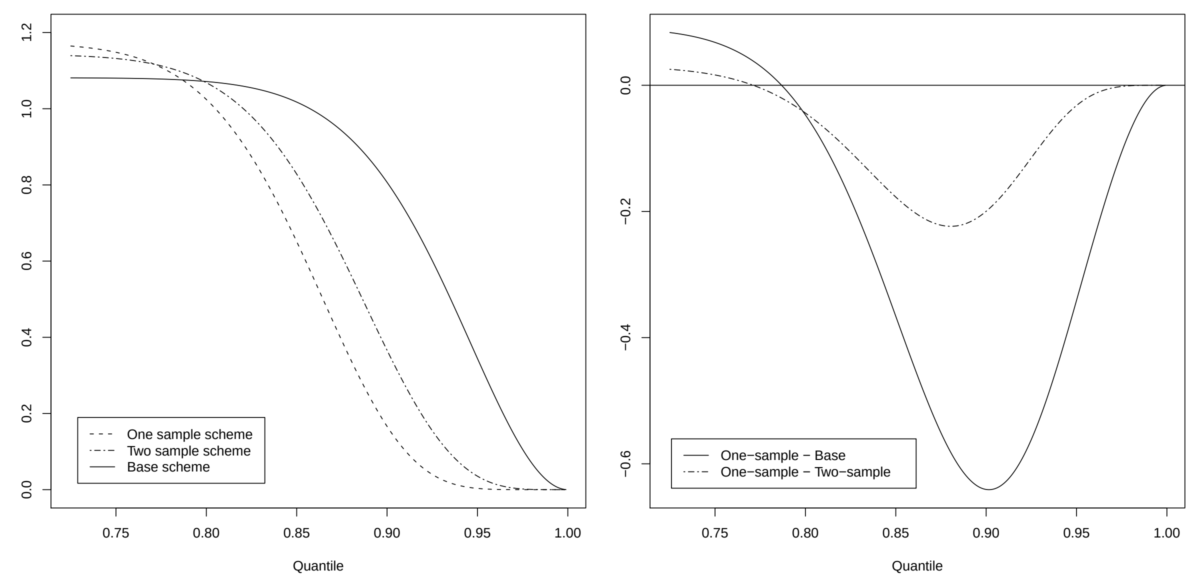


Table 3.1 shows the MCP for various upper quantiles, and various censoring schemes. The "pooled" censoring scheme is based on pooling the first two samples, first three samples, and so on up to all six samples. The "normal" censoring scheme is a single type-II censored sample with an equivalent amount of overall censoring as the pooled sample. While all confidence levels are somewhat low (as compared to the typical 95% desired), we can notice some interesting points. One can see that

n, $\dot{r}$	type	0.70	0.75	0.80	0.85	0.90	0.95	0.975	0.99
20, 18	Normal	0.965	0.909	0.794	0.595	0.323	0.075	0.013	0.001
	Pooled	0.978	0.940	0.859	0.704	0.458	0.165	0.049	0.009
30, 27	Normal	0.991	0.963	0.877	0.678	0.353	0.061	0.006	<0.001
	Pooled	0.997	0.985	0.947	0.839	0.601	0.237	0.072	0.013
40, 36	Normal	0.997	0.984	0.924	0.737	0.371	0.048	0.003	<0.001
	Pooled	0.999	0.996	0.980	0.912	0.706	0.303	0.095	0.017
50, 44	Normal	0.998	0.981	0.897	0.639	0.230	0.012	<0.001	<0.001
	Pooled	0.999	0.998	0.986	0.928	0.727	0.311	0.096	0.017
60, 52	Normal	0.998	0.979	0.873	0.555	0.142	0.003	<0.001	<0.001
	Pooled	0.999	0.999	0.991	0.941	0.746	0.318	0.098	0.017

Table 3.1: Coverage Probabilities of $(-\infty, X_{\dot{r}:n})$ vs. $(-\infty, Z_{(\dot{r})})$

the MCP for the pooled scheme is higher as expected. Furthermore, we can notice that the pooled sample MCP is monotonic in the number of samples. Clearly adding an additional sample can not reduce the coverage probability. However in the one-sample case, a larger sample reduces the variability of the sample maximum around the limiting quantile ($\pi_b = \lim r_b/n_b$). So that the MCP goes to 1 if the quantile is below π_b, and 0 if the quantile is above π_b.

Minimal Width Confidence Intervals

Since we are working in a distribution free setting, we can not ascertain a priori the minimal width interval $(Z_{(i_1)}, Z_{(i_2)})$. However we can consider the shortest interval in terms of indices, namely minimizing $i_2 - i_1$. The problem is to find indices i_1 and i_2 such that $P(Z_{(i_1)} \leq \xi_p \leq Z_{(i_2)}) \geq \alpha$ and is of minimal width. Typically one would choose $\alpha = 0.95$; however, due to the discrete nature of the problem exactly $100\alpha\%$ is likely impossible.

As the mixture representations can become computationally intensive for larger sample sizes, or for a large number of samples, Balakrishnan et al. (2010a) suggest a branching algorithm to reduce the number of computations necessary. Unless otherwise stated, in this paper all confidence, prediction, and tolerance intervals are of "minimal-width".

3.2.2 Tolerance Intervals

As stated in Section 2.2, one-sided tolerance intervals are equivalent to confidence intervals for a population quantile. So the intervals $(-\infty, Z_{(i)})$ and $(Z_{(i)}, \infty)$, which have probability $P(F(Z_{(i)}) \geq \gamma)$ and $P(F(Z_{(i)}) \leq 1-\gamma)$ respectively, can be rewritten as $P(Z_{(i)} \geq \xi_\gamma)$ and $P(Z_{(i)} \leq \xi_{1-\gamma})$.

For a two-sided tolerance interval of the form $(Z_{(i_1)}, Z_{(i_2)})$, for $1 \leq i_1 < i_2 \leq \dot{r}$

such that we wish to contain $100\gamma\%$ of the population, the coverage probability is

$$\begin{aligned}
P(F(Z_{(i_2)}) - F(Z_{(i_1)}) \geq \gamma) &= \sum_{i_1 \leq l_1} \sum_{i_2+i_1-l_1 \leq l_2} q_{i_1 l_1}^{i_2 l_2} P(F(X_{l_2:n}) - F(X_{l_1:n}) \geq \gamma) \\
&= \sum_{i_1 \leq l_1} \sum_{i_2+i_1-l_1 \leq l_2} q_{i_1 l_1}^{i_2 l_2} P(F(X_{l_2-l_1:n}) \geq \gamma) \\
&= \sum_{i_1 \leq l_1} \sum_{i_2+i_1-l_1 \leq l_2} q_{i_1 l_1}^{i_2 l_2} \sum_{l=0}^{l_2-l_1-1} \binom{n}{l} \gamma^l (1-\gamma)^{n-l}.
\end{aligned}$$

Here, $q_{i_1 l_1}^{i_2 l_2}$ is the mixture weight from Theorem 3.1.8. The latter two equalities follow from equation (2.2.1).

3.2.3 Prediction Intervals

We can consider nonparametric prediction intervals for failures from future independent samples. We denote the future sample as $\mathbf{W} = \{W_{1:l:T}, \ldots, W_{t:t:T}\}$, and which may be a complete sample, type-II censored sample, or progressively type-II censored sample.

We can obtain the exceedance probability for a single order statistics $W_{l:t:T}$ $(1 \leq l \leq t)$, as

$$P(Z_{(i)} \leq W_{l:t:T}) = \sum_{k=1}^{n} q_{ik} P(X_{k:n} \leq W_{l:t:T}).$$

Here, q_{ik} represents the mixture weights from Theorem 3.1.1.

In the case of a type-II censored sample or complete sample we can express this further as

$$P(Z_{(i)} \leq W_{l:T}) = \sum_{k=1}^{n} q_{ik} \sum_{\kappa \leq l-1} \frac{\binom{T}{\kappa}\binom{n}{k+l-\kappa-1}}{\binom{n+T}{k+l-1}}. \tag{3.2.2}$$

When **W** is a progressively type-II censored sample we can use the mixture representation from Guilbaud (2001) or Guilbaud (2004) to represent $W_{l:t:T}$ as a mixture of the usual order statistics. We then obtain the following.

$$\begin{aligned} P(Z_{(i)} \leq W_{l:t:T}) &= \sum_{k=1}^{n} q_{ik} P(X_{k:n} \leq W_{l:t:T}) \\ &= \sum_{k=1}^{n} q_{ik} \sum_{k'=0}^{T} q'_{k'lT} P(X_{k:n} \leq W'_{k':T}) \\ &= \sum_{k=1}^{n} q_{ik} \sum_{k'=0}^{T} q'_{k'lT} \sum_{\kappa \leq k'-1} \frac{\binom{T}{\kappa}\binom{n}{k+k'-\kappa-1}}{\binom{n+T}{k+k'-1}}. \end{aligned} \tag{3.2.3}$$

Where $q'_{k'lT}$ are the mixture weights representing $W_{l:t:T}$ as a mixture of the usual order statistics. Rather than represent $W_{l:t:T}$ as a mixture of regular order statistics, the probability $P(X_{k:n} \leq W_{l:t:T})$ can alternatively be calculated for example, as an exceedance probability between and order statistic and a progressive Type-II order statistic (see Bairamov and Eryilmaz, 2006).

In a similar manner we can also calculate the following

$$P(W_{l:t:T} \leq Z_{(i)}) = 1 - P(Z_{(i)} \leq W_{l:t:T}),$$

$$P(Z_{(i_1)} \leq W_{l:t:T} \leq Z_{(i_2)}) = P(Z_{(i_1)} \leq W_{l:t:T}) - P(Z_{(i_2)} \leq W_{l:t:T}).$$

For obtaining the probability of at least λ values from **W** fall in-between two pooled order statistics, let us consider the following. For $1 \leq i_1 < i_2 \leq \dot{r}$ and $1 \leq l_1 < l_2 < T$,

$$P(\text{at least } \lambda \ \mathbf{W}\text{'s} \in (Z_{(i_1)}, Z_{(i_2)})) = \sum_{k_1=1}^{n-1} \sum_{k_2=k_1+1}^{n} q_{i_1 k_1}^{i_2 k_2} P(\text{at least } \lambda \ \mathbf{W}\text{'s} \in (X_{k_1:n}, X_{k_2:n})).$$

When $\mathbf{W}$ is a complete sample, using equation (2.3.2), we can express this as

$$\begin{aligned} P(\text{at least } \lambda \ \mathbf{W}\text{'s} \in (Z_{(i_1)}, Z_{(i_2)})) &= \sum_{k_1=1}^{\dot{r}-1} \sum_{k_2=k_1+1}^{\dot{r}} q_{i_1 k_1}^{i_2 k_2} P(\text{at least } \lambda \ \mathbf{W}\text{'s} \in (X_{k_1:n}, X_{k_2:n})) \\ &= \sum_{k_1=1}^{\dot{r}-1} \sum_{k_2=k_1+1}^{\dot{r}} q_{i_1 k_1}^{i_2 k_2} \sum_{j=\lambda}^{T} \sum_{i=0}^{m-j} \frac{\binom{i+k_1-1}{i}\binom{j+k_2-k_1-1}{j}\binom{T+n-k_2-i-j}{T-i-j}}{\binom{T+n}{n}}. \end{aligned} \tag{3.2.4}$$

If one wishes a particular group of order statistics from $\mathbf{W}$ to be in this interval, simply extract the appropriate portion from the inner summand.

When $\mathbf{W}$ is a type-II censored sample, using equation (2.3.3) instead of equation (2.3.2) as above, will yield the desired results.

Finally, when $\mathbf{W}$ is a progressively type-II censored sample, we can express the joint distribution of $\{W_{i:t:T}, W_{i+1:t:T}, W_{i+j:t:T}, W_{i+j+1:t:T}\}$ as a mixture of (1,2,3, or 4) the usual order statistics depending on i and j. Alternatively one can use the results of Ng and Balakrishnan (2005).

3.2.4 Miscellaneous results

Here we present some results regarding the asymptotic nature of pooled order statistics. However, we first need to define what we mean by asymptotic means as either the sample sizes n_b or the number of samples B, can become large.

The former case includes the typical one-sample scenario. We can obtain the following result.

Proposition 3.2.1 *Suppose that B is fixed, and let us define $\pi_b = \lim_{n_b \to \infty} \frac{r_b}{n_b}$ and $0 < \pi^* = \max_b \pi_b < 1$. Then as $n_b \to \infty$, $\forall j$, we have*

$$F(Z_{(\dot{r})}) \xrightarrow{p} \pi^*.$$

Proof: For every sample, we note that $X_{r_b:n_b} \xrightarrow{p} \xi_{\pi_b}$ as $n_b \to \infty$. Thus for every $\delta > 0$ and $\epsilon > 0$, $\exists$ an N_b which depends on ϵ and δ, such that when $n_b > N_b$, $P(|X_{r_b:n_b} - \xi_{\pi_b}| < \delta) \geq 1 - \epsilon$.

We then consider the distribution of $Z_{(\dot{r})}$. For all $n_b > \max_b N_b$, we have

$$\begin{aligned} P(Z_{(\dot{r})} \leq \xi_{\pi^*} + \delta) &= \prod_{b=1}^{B} P(X_{b,r_b:n_b} \leq \xi_{\pi^*} + \delta) \\ &\geq \prod_{b=1}^{B} P(|X_{r_b:n_b} - \xi_{\pi_b}| < \delta) \geq \prod_{b=1}^{B} (1-\epsilon) = (1-\epsilon)^B. \end{aligned}$$

Consider partitioning all B samples as follows, samples b^o such that $\pi_{b^o} = \pi^*$, and all other samples b^a such that $\pi_{b^a} < \pi_*$. Then consider the following

$$\begin{aligned}
P(Z_{(\dot{r})} \leq \xi_{\pi^*} - \delta) &= \prod_{b=1}^{B} P(X_{b,r_b:n_b} \leq \xi_{\pi^*} - \delta) \\
&= \prod_{b^a} P(X_{b^a,r_{b^a}:n_{b^a}} \leq \xi_{\pi^*} - \delta) \prod_{b^o} P(X_{b^o,r_{b^o}:n_{b^o}} \leq \xi_{\pi^*} - \delta) \\
&\leq \prod_{b^o} P(X_{b^o,r_{b^o}:n_{b^o}} \leq \xi_{\pi^*} - \delta) \leq \prod_{b^o} \epsilon \leq \epsilon.
\end{aligned}$$

Finally we have that $P(|Z_{(\dot{r})} - \xi_{\pi^*}| \leq \delta) \geq (1-\epsilon)^B - \epsilon$. So $F(Z_{(\dot{r})}) \xrightarrow{p} \pi^*$. ∎

The theorem can easily be extended to the cases $\pi^* = 0$ and $\pi^* = 1$.

We are primarily interested with the case where the number of samples becomes large. So we obtain a similar result.

Proposition 3.2.2 *Given B samples with $1 \leq n_b \leq M < \infty$ for some $M \geq 1$ for all j, then as $B \to \infty$ we have*

$$F(Z_{(\dot{r})}) \xrightarrow{p} 1.$$

Proof: Define $X^* = \max_b X_{b,1:n_b}$. Clearly, $Z_{(\dot{r})} \geq X^*$ surely, as the latter is a subset of the observed values. Thus trivially, $X^* \leq_{st} Z_{(\dot{r})}$ (where $\leq_{st}$ is a stochastic ordering such that $X \leq_{st} Y \Leftrightarrow P(X > x) \leq P(Y > x)$).

Define $\tilde{X}$ as follows. For each sample b with size strictly smaller than M we observe an additional $M - n_b$ items and append it to the b-th sample. Define $\tilde{X}$ as the minimum of the new samples. Thus $\tilde{X} \leq X^*$ surely by construction, then $\tilde{X} \leq_{st} X^*$. Finally we have $\tilde{X} \leq_{st} Z_{(\dot{r})}$ by transitivity. Now clearly $\tilde{X}$ is the minimum of B i.i.d., random variables with distribution function $F_{\tilde{X}}(x) = 1 - (1 - F_X(x))^M$, with same support as X. Thus as $B \to \infty$, $\tilde{F}(\tilde{X}) \xrightarrow{p} 1$. So $P(\tilde{F}(\tilde{X}) > 1 - \epsilon) \leq P(F(Z_{(\dot{r})}) > 1 - \epsilon) \leq 1$ and by the squeeze theorem $P(F(Z_{(\dot{r})}) > 1 - \epsilon) \to 1$ as desired. ∎

Remark 3.2.3 *An alternate proof of Proposition 3.2.2 can be given, using the representation of $Z_{(\dot{r})}$ as in equation (3.1.2).*

Proposition 3.2.2 suggests that additional samples will enable the estimation of any quantile, with any desired precision, given that we are able to take enough samples of bounded size. Whereas Proposition 3.2.1 suggests that in a fixed sample scenario there is an upper limit to quantiles which we wish to estimate and obtain meaningful confidence.

There is however a cost in terms of the intervals we make. Since pooling spreads out the sample, at larger sizes (i.e., as $n \to \infty$) the confidence intervals for quantiles may be narrower in the fixed sample case as opposed to the increasing sample case.

Another issue one needs to consider is that convergence in probability of the pooled order statistics to a constant, which may not be guaranteed for some sampling schemes. Consider the following example, we will also assume a uniform distribution for simplicity as a probability integral transformation can give us the uniform distribution. Observe B samples, and in the b-th sample, observe only the first failure of b items. The distribution function for the maximum can be represented as in equation

(3.1.2).

$$P(Z_{(\dot{r})} \leq \epsilon) = P(Z_{(B)} \leq \epsilon) = \prod_{b=1}^{B} P(X_{b,1:b} \leq \epsilon) = \prod_{b=1}^{B} (1 - P(X_{b,1:b} \geq \epsilon))$$
$$= \prod_{b=1}^{B} (1 - (1-\epsilon)^b) \xrightarrow{B \to \infty} \prod_{b=1}^{\infty} (1 - (1-\epsilon)^b) = (1-\epsilon)_\infty$$

Where $(q)_\infty$ is the q-pochhammer function. See Andrews et al. (1999) for more information on the q-pochhammer function.

Thus some care must be taken when designing experiments using multiple type-II censored samples.

Bounding the sample size is sufficient though not a necessary condition to ensure that $Z_{\dot{r}} \to 1$ in probability.

3.3 Motivating Example Revisited

In this section we consider the motivating example from Table 1.2.

Using all six samples, the confidence interval for the 80% quantile is $[2.75, 5.55]$, which has 95.9% confidence. Given the original complete data as in Table 1.1, and treating it as one sample with equivalent proportion of censoring the best possible interval would include the whole range of the data. The interval would be $[0, 4.75]$ and with confidence level 87.3%. If we wished to obtain a 95% confidence interval for this quantile under a one-sample scenario, we would have had to observe two more additional failures of the total 60 units. The resultant confidence interval would be

	95% Pred. Int.		Max 2-sided 1 width int.			Max 1-sided 1 width int.		
l	L (i)	U (j)	L (i)	U ($i+1$)	Prob	L (i)	U ($i+1$)	Prob
1	$F^{-1}(0)$	0.82 (17)	0.00 (1)	0.02 (2)	0.124	$F^{-1}(0)$	0.00 (1)	0.143
2	0.00 (1)	1.63 (28)	0.31 (6)	0.49 (7)	0.060			
3	0.02 (2)	2.06 (33)	0.70 (13)	0.71 (14)	0.047			
4	0.18 (4)	2.24 (39)	1.08 (20)	1.13 (21)	0.042			
5	0.50 (8)	3.17 (44)	1.56 (27)	1.63 (28)	0.040			
6	0.80 (16)	4.03 (50)	2.10 (34)	2.12 (35)	0.040			
7	1.13 (21)	5.55 (52)	2.57 (41)	2.75 (42)	0.045			
8	1.63 (28)	$F^{-1}(1)$	3.97 (49)	4.03 (50)	0.062			
9	2.12 (35)	$F^{-1}(1)$	5.13 (51)	5.55 (52)	0.124	5.55 (52)	$F^{-1}(1)$	0.183
10	2.80 (43)	$F^{-1}(1)$	5.13 (51)	5.55 (52)	0.153	5.55 (52)	$F^{-1}(1)$	0.501

Table 3.2: Prediction intervals for individual order statistics $W_{l:10}$

$[2.75, 5.55]$ and would have confidence of 95.8%. The total time to test would have been the same, but we would have had to fail two more items with no additional benefit.

This suggests that when the limiting issue is number of items failed, pooling samples can be beneficial.

Balakrishnan et al. (2001) develops exact inference in the case or multiple progressively type-II censored samples from the exponential distribution. It is well known that the MLE of the scale parameter ϑ, is $\vartheta^* = (1/\dot{r}) \sum_{b=1}^{B} [\sum_{k=1}^{r_b} X_{b,k:n_b} + (n_b - r_b) X_{b,r_b:n_b}]$. Thus, the MLE of the quantile ξ_p is $\hat{\xi}_p = -\vartheta^* \ln(1-p)$. ϑ^* is the sum of independent increments which are exponential, so that the normalized sum, $2\dot{r}\vartheta^*/\vartheta$ has χ^2 distribution with $2\dot{r}$ degrees of freedom.

Thus using all six samples we can obtain a two-sided 95.9% confidence interval for the 80% quantile as $[3.01, 5.31]$. The point estimate is 3.92. This is narrower than the nonparametric confidence interval, as expected, but the difference is not immense.

In Table 3.2 we can see various prediction intervals for all order statistics from a future independent sample of size 10. The number in the parenthesis next to the time to failure, represents which pooled order statistic it is.

The first part of the table gives the 95% prediction interval. In the cases of many of the extreme order statistics, these are one-sided intervals.

The second part gives the two-sided interval of width one, that has the highest confidence level, and its corresponding probability. One can note that the extreme order statistics will likely be pushed towards the boundaries, so that the confidence level is higher for the one-width interval.

The third part of the table gives the one-sided one-width interval with the highest confidence when it is higher than the two-sided, one-width interval in the second part. The corresponding probability is included as well. One can note that if we added a complete sample of size 10, the largest value has greater than a 50% chance of being the maximum as we have censored all the previous sample maximums. We can also note that there are several one-sided intervals, particularly in the upper extremes. If we wish to reduce this width, one would need to take additional samples.

The one-width prediction intervals are of interest for making point estimates. One could use any point in the interval as a point estimate; a common choice would be the mean. This can be done likewise for confidence intervals for quantile. Considering Table 3.2, we could use 2.11 as a point estimate for the 6-th order statistic of a future sample of size 10. The center of the 95% prediction interval is 2.41, much higher.

Chapter 4

Multiple Doubly Type-II Censored Samples

In this chapter we consider a natural extension of the scenario considered in Chapter 3 to the case of multiple independent doubly Type-II censored samples. We denote again n_b the sample size of the b-th sample, where $1 \leq b \leq B$, and r_b to be the number of observed failures. We denote r_b^U to be the number of right (or upper) censored items, and r_b^L to be the number of left (or LOWER) censored items. These are all non-negative integers and $r_b^U + r_b + r_b^L = n_b$.

When $r_b^L = 0$ for $b = 1, \ldots, B$, then this becomes the case of Type-II right censoring considered previously in Chapter 3. Given the mixture representations in this chapter, inference for multiple doubly Type-II censored samples is the same as given in Section 3.2.

4.1 Distributional Representations

We first discuss the simpler case of two samples which was not considered in Balakrishnan et al. (2010b). The representations are much simpler for the same reasons as in Type-II right censoring. That is, given that one item is the i-th pooled OS, then the number of observed failures from the other sample above and below $Z_{(i)}$ is fixed.

4.1.1 Two-Samples

Suppose we have two independent doubly Type-II censored samples. For simplicity, we will order the samples by observed size, i.e., $1 \leq r_1 \leq r_2$. Let us denote

$$w_{i,k}^{b} = \frac{\binom{i-1}{k-1}\binom{n-i}{n_b-k}}{\binom{n}{n_b}} = \frac{n_b}{n}\frac{\binom{n_b-1}{k-1}\binom{n-n_b}{i-k}}{\binom{n-1}{i-1}}; \tag{4.1.1}$$

here, $b = 1, 2$ indexes the samples. If $b = 1$, then the numerator of the first part counts the number of orderings of all items such that $X_{2,i-k:n_2} < X_{1,k:n_1} < X_{2,i-k+1:n_2}$. We again use the convention that $X_{b,0:n_b} = \xi_0$ and $X_{b,n_b+1:n_b} = \xi_1$.

Marginal Distribution of a pooled OS

Consider the marginal distribution $P(Z_{(i)} \leq \xi_p)$, where ξ_p is the p-th population quantile. Then, we have the following result.

Proposition 4.1.1 *For any $0 < p < 1$ we have*

$$P(Z_{(i)} \leq \xi_p) = \sum_{j=1}^{i} \left[w^1_{i+r_1^L+r_2^L, r_1^L+j} + w^2_{i+r_1^L+r_2^L, r_2^L+j} \right] F_{i+r_1^L+r_2^L:n}(\xi_p)$$
$$+ \sum_{j=0}^{r_2^L-1} w^1_{i+r_1^L+j, i+r_1^L} F_{i+r_1^L+j:n}(\xi_p) + \sum_{j=0}^{r_1^L-1} w^2_{i+r_2^L+j, i+r_2^L} F_{i+r_2^L+j:n}(\xi_p)$$

when $1 \leq i \leq r_1$,

$$P(Z_{(i)} \leq \xi_p) = \left[\sum_{j=1}^{r_1} w^1_{i+r_1^L+r_2^L, r_1^L+j} + \sum_{j=i-r_1}^{i} w^2_{i+r_1^L+r_2^L, r_2^L+j} \right] F_{i+r_1^L+r_2^L:n}(\xi_p)$$
$$+ \sum_{j=1}^{r_1^U} w^2_{i+r_1^L+r_2^L+j, i-r_1+r_2^L} F_{i+r_1^L+r_2^L+j:n}(\xi_p) + \sum_{j=0}^{r_1^L-1} w^2_{i+r_2^L+j, i+r_2^L} F_{i+r_2^L+j:n}(\xi_p)$$

when $r_1 < i \leq r_2$, and

$$P(Z_{(i)} \leq \xi_p) = \left[\sum_{j=i-r_2}^{r_1} w^1_{i+r_1^L+r_2^L, r_1^L+j} + \sum_{j=i-r_1}^{r_2} w^2_{i+r_1^L+r_2^L, r_2^L+j} \right] F_{i+r_1^L+r_2^L:n}(\xi_p)$$
$$+ \sum_{j=1}^{r_2^U} w^1_{i+r_1^L+r_2^L+j, i-r_2+r_1^L} F_{i+r_1^L+r_2^L+j:n}(\xi_p) + \sum_{j=1}^{r_1^U} w^2_{i+r_1^L+r_2^L+j, i-r_1+r_2^L} F_{i+r_1^L+r_2^L+j:n}(\xi_p)$$

when $r_2 < i \leq r_1 + r_2$.

Contrasting Corollary 3.1.6, it is evident that there is no i such that the pooled order statistic $Z_{(i)}$ is equal in distribution to a single order statistic when both samples have left and right censoring.

4.1.2 Multiple Samples

We now consider the case of multiple independent samples.

Marginal Distribution of a pooled OS

As with equation (3.1.1), we can write the distribution function of $Z_{(i)}$ as follows,

$$P(Z_{(i)} \leq \xi_p) = \sum_{b=1}^{B} \sum_{k=r_b^L+1}^{r_b^L+r^b} P(Z_{(i)} = X_{b,k:n_b} \leq \xi_p). \tag{4.1.2}$$

For any permutation of all the items and from all samples such that $Z_{(i)} = X_{b,k:n_b}$, some samples will have all of the observed and some right censored items below $Z_{(i)}$. For this permutation, we say that these samples are in $\{b'_U\}$. Conversely, some samples will have all observed and some left censored items above $Z_{(i)}$ (these samples are in $\{b'_L\}$). The remaining samples would be have all left/right censored samples below/above $Z_{(i)}$ (these samples are in $\{b''\}$).

In this way, we have assigned groups for each permutation of the n observed and unobserved failure times, based on the position of the left/right censored items relative to $Z_{(i)} = X_{b,k:n_b}$.

Let us define the weight function $\mathcal{W}$ as

$$W_{b,k,\{b'\},\mathcal{C},\ell} = \frac{\frac{n_b}{n}\binom{n_b-1}{k-1}}{\binom{n-1}{\ell-1}} \prod_{j\in\{b'_U\}} \binom{n_j}{r_j^L + r_j + c_j^U} \prod_{j\in\{b'_L\}} \binom{n_j}{c_j^L} \prod_{j\in\{b''\}} \binom{n_j}{r_j^L + c_j},$$

which in the appropriate context is a weighted multivariate hypergeometric probability and can be re-written so it appears in Section 3.1. This can be seen to collapse to the same weight function as in Section 3.1 when $r_b^L = 0$ for all b.

Then, we have the following result.

Theorem 4.1.2 *For any $1 \leq i \leq \dot{r}$ and $0 < p < 1$, the marginal distribution of $Z_{(i)}$ can be expressed as*

$$P(Z_{(i)} \leq \xi_p) = \sum_{b=1}^{B} \sum_{k=r_b^L+1}^{r_b^L+r_b} \sum_{\sigma_{\{b'\}}} \sum_{\mathcal{C}} W_{b,k,\{b'\},\mathcal{C},\ell}\ F_{\ell:n}(\xi_p)$$

where

$$\ell = i + r_b^L + \dot{r}^L_{\{b'_U\}} + \dot{c}^U_{\{b'_U\}} + \dot{c}^L_{\{b'_L\}} + \dot{r}^L_{\{b''\}}$$

and

$$\mathcal{C} = \left\{ \begin{array}{c} (c_1, \ldots, c_{b-1}, c_{b+1}, \ldots, c_B) : \dot{c}_{\{b''\}} = i - k + r_b^L - \dot{r}_{\{b'_U\}} \\ 1 \leq c_j \leq r_j^U \ \forall\ j \in \{b'_U\},\ 0 \leq c_j \leq r_j^L\ \forall j\ \in \{b'_L\},\ 0 \leq c_j \leq r_j\ \forall j\ \in \{b''\} \end{array} \right\}.$$

For proof we refer to the proof of Theorem 3.1.1. The Theorem follows from a similar argument.

When considering the marginal distribution of $Z_{(i)}$, necessary and sufficient conditions for $\{b'_L\}$ and $\{b'_U\}$ to be valid are that $0 \leq i - k + r_b^L - \dot{r}_{\{b'_U\}} \leq \dot{r}_{\{b''\}}$. When $\dot{r}_{\mathcal{A}} < i - k + r_b^L$, no valid partition of $\mathcal{A}$ will exist, and so $P(X_{b,k:n_b} = Z_{(i)}) = 0$ in this case.

Remark 4.1.3 *Alternate simpler representations for $Z_{(1)}$, $Z_{(\dot{r})}$ are as follows*

$$P(Z_{(1)} \leq \xi_p) = 1 - \prod_{b=1}^{B}(1 - F_{r_b^L+1:n_b}(\xi_p)) \qquad P(Z_{(\dot{r})} \leq \xi_p) = \prod_{b=1}^{B} F_{r_b^L+r_b:n_b}(\xi_p)$$

These alternate representations become useful when calculating the maximum coverage probability for some given scheme.

Remark 4.1.4 *Instead of Theorem 4.1.2, one can use the results of Theorem 3.1.1 (along with Corollary 3.1.7) when $1 \leq i \leq \min_b r_b$ and $\dot{r} - \min_b r_b << i \leq \dot{r}$, as the sets $\{b'_U\}$ and $\{b'_L\}$ respectively will necessarily be empty.*

It may be more practical for large i to obtain the mixture distribution of $Z_{(i)}$ analogous to what is done in Corollary 3.1.7. Namely, reversing the schemes so that left and right censoring switch, and considering the mixture distribution of $Z_{(\dot{r}-i+1)}$, then reversing appropriately again. This should prove to be more efficient in terms of computation. A similar idea will work with the joint distribution of two pooled OS.

Joint Distribution of two pooled OS

To obtain a similar result for the bivariate distribution, we can again partition the sample space (up to a set of measure 1) as

$$P(Z_{(i_1)} \leq p_1, Z_{(i_2)} \leq p_2) = \sum_{b=1}^{B} \sum_{r_b^L+1 \leq k_1 < k_2 \leq r_b^L + r_b} P(X_{b,k_1:n_b} = Z_{(i_1)} \leq p_1, X_{b,k_2:n_b} = Z_{(i_2)} \leq p_2)$$
$$+ \sum_{b^o \neq b} \sum_{b=1}^{B} \sum_{k_1=r_{b^o}^L+1}^{r_{b^o}^L + r_{b^o}} \sum_{k_2=r_b^L+1}^{r_b^L + r_b} P(X_{b^o,k_1:n_{b^o}} = Z_{(i_1)} \leq p_1, X_{b,k_2:n_b} = Z_{(i_2)} \leq p_2)$$

for any $1 \leq i_1 < i_2 \leq \dot{r}$ and $0 < p_1 < p_2 < 1$. When $p_1 \geq p_2$ the bivariate distribution reduces to the marginal distribution of $Z_{(i_2)}$.

For any permutation of the items such that $Z_{(i_1)} = X_{b^o,k_1:n_{b^o}}$ and $Z_{(i_2)} = X_{b,k_2:n_b}$, we can define $\{b'_L\}/\{b'_U\}$ as in the marginal case using $Z_{(i_2)}/Z_{(i_1)}$ as the cutoff instead. We require extra condition that all right/left censored items fall above/below $Z_{(i_1)}/Z_{(i_2)}$. Then define $\{b'_{L1}\}$ and $\{b'_{U1}\}$ as the subsets of $\{b'_L\}$ and $\{b'_U\}$, such that the first/final observed value falls above/below $Z_{(i_1)}/Z_{(i_2)}$.

Additionally, however, some samples may have left censored items above $Z_{(i_1)}$ and right censored items below $Z_{(i_2)}$. So, these samples can be labeled as $\{b'_{UL}\}$. The remaining samples are again labeled as $\{b''\}$.

For brevity, we define $\mathcal{W}$ as

$$\mathcal{W}_{\ell_1,\ell_2,\{h\},\{l\},\{m\}} = \frac{\binom{\ell_1-1}{h_1,\ldots,h_B}\binom{\ell_2-\ell_1-1}{l_1,\ldots,l_B}\binom{n-\ell_2}{m_1,\ldots,m_B}}{\binom{n}{n_1,\ldots,n_B}}.$$

Then, we have the following result.

Theorem 4.1.5 *For any $1 \leq i_1 < i_2 \leq \dot{r}$ and $0 < p_1 < p_2 < 1$, we have the joint distribution of $Z_{(i_1)}$ and $Z_{(i_2)}$ as follows:*

$$
\begin{aligned}
P(Z_{(i_1)} \leq \xi_{p_1}, Z_{(i_2)} \leq \xi_{p_2}) = & \sum_{b=1}^{B} \sum_{k_1=r_b^L+1}^{r_b^L+r_b-1} \sum_{k_2=k_1+1}^{r_b^L+r_b} \sum_{\sigma_{\{b'\}}} \sum_{\mathcal{C}_1} \mathcal{W}^1_{\ell_1^1,\ell_2^1,\ldots} F_{l_1^1,l_2^1:n}(\xi_{p_1},\xi_{p_2}) \\
& + \sum_{b=1}^{B} \sum_{b^o \neq b} \sum_{k_1=r_{b^o}^L+1}^{r_{b^o}^L+r_{b^o}} \sum_{k_2=r_b^L+1}^{r_b^L+r_b} \sum_{\sigma_{\{b'\}}} \sum_{\mathcal{C}_2} \mathcal{W}^2_{\ell_1^2,\ell_2^2,\ldots} F_{l_2^1,l_2^2:n}(\xi_{p_1},\xi_{p_2}),
\end{aligned}
$$

where

$$
\begin{aligned}
\ell_1^1 &= i_1 + r_b^L + \dot{r}^L_{\{b'_U\}} + \dot{c}^U_{j_1,\{b'_{U1}\}} + \dot{c}^L_{j_1,\{b'_{UL}\}} + \dot{r}^L_{\{b'_L\}} - \dot{c}^L_{j_1,\{b'_{L1}\}} - \dot{c}^L_{j_2,\{b'_L\}} + \dot{r}^L_{\{b''\}}, \\
\ell_2^1 &= i_2 + r_b^L + \dot{r}^L_{\{b'_U\}} + \dot{c}^U_{j_1,\{b'_{U1}\}} + \dot{c}^U_{j_2,\{b'_U\}} + \dot{r}^L_{\{b'_{UL}\}} + \dot{r}^U_{\{b'_{UL}\}} + \dot{c}^U_{j_2,\{b'_{UL}\}} + \dot{r}^L_{\{b'_L\}} - \dot{c}_{j_1,\{b'_L\}} + \dot{r}_{\{b''\}}, \\
\ell_1^2 &= \ell_1^1 + r_{b^o}^L + \min(0, c_b - r_b^L), \\
\ell_2^2 &= \ell_2^1 + \max(r_{b^o}^L, c_{b^o} - r_{b^o} + k_1)
\end{aligned}
$$

and the constraint for the sum is

$$
\mathcal{C}_1 = \left\{
\begin{array}{l}
\{(c_{j_1}, c_{j_2})\}_{j \neq b} : 0 \leq c_b \leq k_2 - 1,\ 0 \leq c_b \leq n_b - k_1, \\
\quad \dot{c}_{j_1,\{b''\}} = i_1 - (k_1 - r_b^L) - \dot{r}_{\{b'_{U1}\}} - \dot{c}_{j_1,\{b'_{U2}\}}, \\
\quad \dot{c}_{j_2,\{b''\}} = (i_2 - i_1) - (k_2 - k_1) - \dot{r}_{\{b'_{UL}\}} - (\dot{r}_{\{b'_{U2}\}} - \dot{c}_{j_1,\{b'_{U2}\}}) - (\dot{r}_{\{b'_{L2}\}} - \dot{c}_{j_1,\{b'_{L2}\}}), \\
\ldots
\end{array}
\right\},
$$

$$
\mathcal{C}_2 = \left\{
\begin{array}{l}
\{(c_{j_1}, c_{j_2})\}_{j \neq b, b^o} : 0 \leq c_b \leq k_2 - 1,\ 0 \leq c_{b^o} \leq n_{b^o} - k_1 \\
\quad \dot{c}_{j_1,\{b''\}} = i_1 - (k_1 - r_{b^o}^L) - \max(0, c_b - r_b^L) - \dot{r}_{\{b'_{U1}\}} - \dot{c}_{j_1,\{b'_{U2}\}}, \\
\quad \dot{c}_{j_2,\{b''\}} = (i_2 - i_1) - [k_2 - r_b^L - \max(0, c_b - r_b^L)] - \min(c_{b^o}, r_{b^o} - k_1 + r_{b^o}^L) - \dot{r}_{\{b'_{UL}\}} \\
\qquad\qquad -(\dot{r}_{\{b'_{U2}\}} - \dot{c}_{j_1,\{b'_{U2}\}}) - (\dot{r}_{\{b'_{L2}\}} - \dot{c}_{j_1,\{b'_{L2}\}}), \ldots
\end{array}
\right\},
$$

with the restrictions common to both C's being

$$\cdots = \left\{ \begin{array}{l} 0 \le c^U_{j_1} \le r^U_j,\ 0 \le c^U_{j_2} \le r^U_j,\ 1 \le c^U_{j_1} + c^U_{j_2} \le r^U_j\ \forall j \in \{b'_{U1}\}, \\ 0 \le c_{j_1} \le r_j - 1,\ 1 \le c^U_{j_2} \le r^U_j\ \forall\, j \in \{b'_{U2}\}, \\ 0 \le c^L_{j_1} \le r^L_j,\ 0 \le c^L_{j_2} \le r^L_j,\ 1 \le c^L_{j_1} + c^L_{j_2} \le r^L_j\ \forall j \in \{b'_{L1}\}, \\ 0 \le c_{j_1} \le r_j - 1,\ 1 \le c^L_{j_2} \le r^L_j\ \forall\, j \in \{b'_{L2}\}, \\ 0 \le c^L_{j_1} \le r^L_j - 1,\ 0 \le c^U_{j,2} \le r^U_j - 1\ \forall\, j \in \{b'_{UL}\},\ \ 0 \le c_{j_1} + c_{j_2} \le r_j\ \forall j \in \{b''\} \end{array} \right\}.$$

The remaining arguments for $\mathcal{W}^1_{\{\ldots\}}$ *are*

$$\{h\} = \left\{ \begin{array}{c} k_1 - 1 \\ \{r^L_j + r_j + c^U_{j_1}\}, j \in \{b'_{U1}\} \\ \{r^L_j + c_{j_1}\}, j \in \{b'_{U2}\} \\ \{c^L_{j_1}\}, j \in \{b'_{UL}\} \\ \{r^L_j - c^L_{j_1} - c^L_{j_2}\}, j \in \{b'_{L1}\} \\ \{r^L_j - c^L_{j_2}\}, j \in \{b'_{L2}\} \\ \{r^L_j + c_{j_1}\}, j \in \{b''\} \end{array} \right\},\ \{l\} = \left\{ \begin{array}{c} k_2 - k_1 - 1 \\ \{c^U_{j_2}\}, j \in \{b'_{U1}\} \\ \{r_j - c_{j_1} + c^U_{j_2}\}, j \in \{b'_{U2}\} \\ \{n_j - c^L_{j_1} - c^U_{j_2}\}, j \in \{b'_{UL}\} \\ \{c^L_{j_2}\}, j \in \{b'_{L1}\} \\ \{r_j - c_{j_1} + c^L_{j_2}\}, j \in \{b'_{L2}\} \\ \{c_{j_2}\}, j \in \{b''\} \end{array} \right\},\ \{m\} = \left\{ \begin{array}{c} n_b - k_2 \\ \{r^U_j - c^U_{j_1} - c^U_{j_2}\}, j \in \{b'_{U1}\} \\ \{r^U_j - c^U_{j_2}\}, j \in \{b'_{U2}\} \\ \{c^U_{j_2}\}, j \in \{b'_{UL}\} \\ \{r_j + r^U_j + c^L_{j_1}\}, j \in \{b'_{L1}\} \\ \{r^U_j + c_{j_1}\}, j \in \{b'_{L2}\} \\ \{r^U_j + r_j - c_{j_1} - c_{j_2}\}, j \in \{b''\} \end{array} \right\},$$

and for $\mathcal{W}^2_{\{\ldots\}}$

$$\{h\} = \left\{ \begin{array}{c} k_1 - 1 \\ c_b \\ \{r^L_j + r_j + c^U_{j_1}\}, j \in \{b'_{U1}\} \\ \{r^L_j + c_{j_1}\}, j \in \{b'_{U2}\} \\ \{c^L_{j_1}\}, j \in \{b'_{UL}\} \\ \{r^L_j - c^L_{j_1} - c^L_{j_2}\}, j \in \{b'_{L1}\} \\ \{r^L_j - c^L_{j_2}\}, j \in \{b'_{L2}\} \\ \{r^L_j + c_{j_1}\}, j \in \{b''\} \end{array} \right\},\ \{l\} = \left\{ \begin{array}{c} c_{b^o} \\ k_2 - c_b - 1 \\ \{c^U_{j_2}\}, j \in \{b'_{U1}\} \\ \{r_j - c_{j_1} + c^U_{j_2}\}, j \in \{b'_{U2}\} \\ \{n_j - c^L_{j_1} - c^U_{j_2}\}, j \in \{b'_{UL}\} \\ \{c^L_{j_2}\}, j \in \{b'_{L1}\} \\ \{r_j - c_{j_1} + c^L_{j_2}\}, j \in \{b'_{L2}\} \\ \{c_{j_2}\}, j \in \{b''\} \end{array} \right\},\ \{m\} = \left\{ \begin{array}{c} n_{b^o} - k_1 - c_{b^o} \\ n_b - k_2 \\ \{r^U_j - c^U_{j_1} - c^U_{j_2}\}, j \in \{b'_{U1}\} \\ \{r^U_j - c^U_{j_2}\}, j \in \{b'_{U2}\} \\ \{c^U_{j_2}\}, j \in \{b'_{UL}\} \\ \{r_j + r^U_j + c^L_{j_1}\}, j \in \{b'_{L1}\} \\ \{r^U_j + c_{j_1}\}, j \in \{b'_{L2}\} \\ \{r^U_j + r_j - c_{j_1} - c_{j_2}\}, j \in \{b''\} \end{array} \right\}.$$

We again refer to Section 3 for the proof.

For the necessary and sufficient conditions for each partition to be valid, we need to consider the cases $b^o = b$ and $b^o \neq b$ separately.

When $b = b^o$, necessary and sufficient conditions for $\{b'_L\}$, $\{b'_{L1}\}$, $\{b'_U\}$, $\{b'_{U1}\}$, and

$\{b'_{UL}\}$ to be valid are as follows:

$$\begin{aligned}
0 &\leq i_1 - (k_1 - r_b^L) - \dot{r}_{\{b'_{U1}\}} - \mathcal{C}_{U2}, \\
0 &\leq (i_2 - i_1) - (k_2 - k_1) - \dot{r}_{\{b'_{UL}\}} - (\dot{r}_{\{b'_{U2}\}} - \mathcal{C}_{U2}) - (\dot{r}_{\{b'_{U2}\}} - \mathcal{C}_{L2}), \\
\dot{r}_{\{b''\}} &\geq i_2 - (k_2 - r_b^L) - \dot{r}_{\{b'_U\}} - \dot{r}_{\{b'_{UL}\}} - (\dot{r}_{\{b'_{U2}\}} - \mathcal{C}_{L2})
\end{aligned}$$

for some $0 \leq \mathcal{C}_{U2} \leq \dot{r}_{\{b'_{U2}\}}-$ size$\{b'_{U2}\}$ and $0 \leq \mathcal{C}_{L2} \leq \dot{r}_{\{b'_{L2}\}}$ - size$\{b'_{L2}\}$. When

$$0 \leq i_1 - k_1 + r_b^L, \qquad 0 \leq (i_2 - i_1) - (k_2 - k_1), \qquad i_2 - k_2 + r_b^L \leq \dot{r}_{\mathcal{A}},$$

such a partition is guaranteed to exist; if one of those conditions fail, then $P(Z_{(i_1)} = X_{b,k_1:n_b}, Z_{(i_2)} = X_{b,k_2:n_b}) = 0$.

When $b \neq b^o$, necessary and sufficient conditions for $\{b'_L\}$, $\{b'_{L1}\}$, $\{b'_U\}$, $\{b'_{U1}\}$, and $\{b'_{UL}\}$ to be valid are as follows:

$$\begin{aligned}
0 &\leq i_1 - k_1 + r_{b^o}^L - \mathcal{C}_b + r_b^L - \dot{r}_{\{b'_{U1}\}} - \dot{r}_{\{b'_{U2}\}} + \mathcal{C}_{U2}, \\
0 &\leq i_2 - i_1 - k_2 + \mathcal{C}_b - \dot{r}_{\{b'_{UL}\}} - \mathcal{C}_{U2} - \mathcal{C}_{L2}, \\
\dot{r}_{\{b''\}} &\geq i_2 - k_2 - k_1 + r_b^L + r_{b^o}^L - \dot{r}_{\{b'_U\}} - \dot{r}_{\{b'_{UL}\}} - \mathcal{C}_{L2}
\end{aligned}$$

for some $r_b^L \leq \mathcal{C}_b \leq r_b^L + r_b - 1$, $0 \leq \mathcal{C}_{U2} \leq \dot{r}_{\{b'_{U2}\}}-$ size$\{b'_{U2}\}$ and $0 \leq \mathcal{C}_{L2} \leq \dot{r}_{\{b'_{L2}\}}-$

size$\{b'_{L2}\}$. When

$$0 \leq i_1 - k_1 + r^L_{b^o} - \mathcal{C}_b \qquad 0 \leq (i_2 - i_1) - (k_2 - r^L_b - \mathcal{C}_b) \qquad i_2 - k_2 + r^L_b - r_{b^o} \leq \dot{r}_{\mathcal{A}}$$

for some $0 \leq \mathcal{C}_b \leq k_2 - r^L_b - 1$, such a partition is guaranteed to exist; if one of these conditions fail, then $P(Z_{(i_1)} = X_{b^o,k_1:n_{b^o}}, Z_{(i_2)} = X_{b,k_2:n_b}) = 0$.

Joint Distribution of pooled OS

The joint distribution of the pooled OS can be given as in Proposition 3.1.10 or 3.1.14, but not Proposition 3.1.12. These representations can also be used to obtain the joint distribution of any number of pooled OS. However, this will typically not be computationally feasible, except when pooling only a few samples.

4.2 Computational Algorithm

Here, we will present a basic algorithm for the calculation of the marginal mixture weights and then demonstrate it with a simple example.

Figure 4.1 shows the simple algorithm. Here k++ and b++ indicate incrementing by 1. First one sorts by increasing observation sizes (i.e., sort by r_b), and then within this by lower censoring, and then upper censoring. This groups identical samples and allows terminating earlier without checking all possible $\{b'_L\}$ or $\{b'_U\}$.

For the purpose of this algorithm, incrementing $\{b'_U\}$ means choosing its successor in any way such that $\dot{r}_{\{b'_{U,old}\}} \leq \dot{r}_{\{b'_{U,new}\}}$, with equality necessary if there is some other set of samples which have not been used, such that the previous equality holds. Otherwise, choose any set of samples such that $\dot{r}_{\{b'_{U,new}\}}$ is a minimum and the above inequality holds. $\{b'_L\}$ is chosen in the same way from $\mathcal{A} \setminus \{b'_U\}$.

While not shown in Figure 4.1, one can reduce the number of iterations by weighting. This is done by counting the number of exchangeable subsets. As an example, assume B samples are identical. If the size of $\{b'_U\}$ is l, there are $\binom{B-1}{l}$ exchangeable subsets in $\sigma_{\{b'\}}$ that are exchangeable with size l. One would then need to calculate only the weights with the first l samples in $\mathcal{A}$, then multiply each weight $\mathcal{W}$ by $\binom{B-1}{l}$. Other similar improvements can be made, but we describe only the simple algorithm in Figure 4.1 for clarity.

4.2.1 An Example

Consider the following simple example. Suppose we have three identical doubly Type-II censored samples with $n_b = 4$, $r_b^L = 1$, $r_b^U = 1$, and $r_b = 2$ for $b = 1, 2, 3$. Let us consider the marginal distribution of the third pooled order statistic $Z_{(3)}$. Below, we represent a censored item with $\circ$ for ease in notation.

Table 4.1 shows the partitions that would appear as in the algorithm applied to this sample, along with the class of permutations. Although we could omit the exchangeable samples by including extra weights, everything is shown for the sake of clarity. The order statistic and the mixture probabilities corresponding to that group

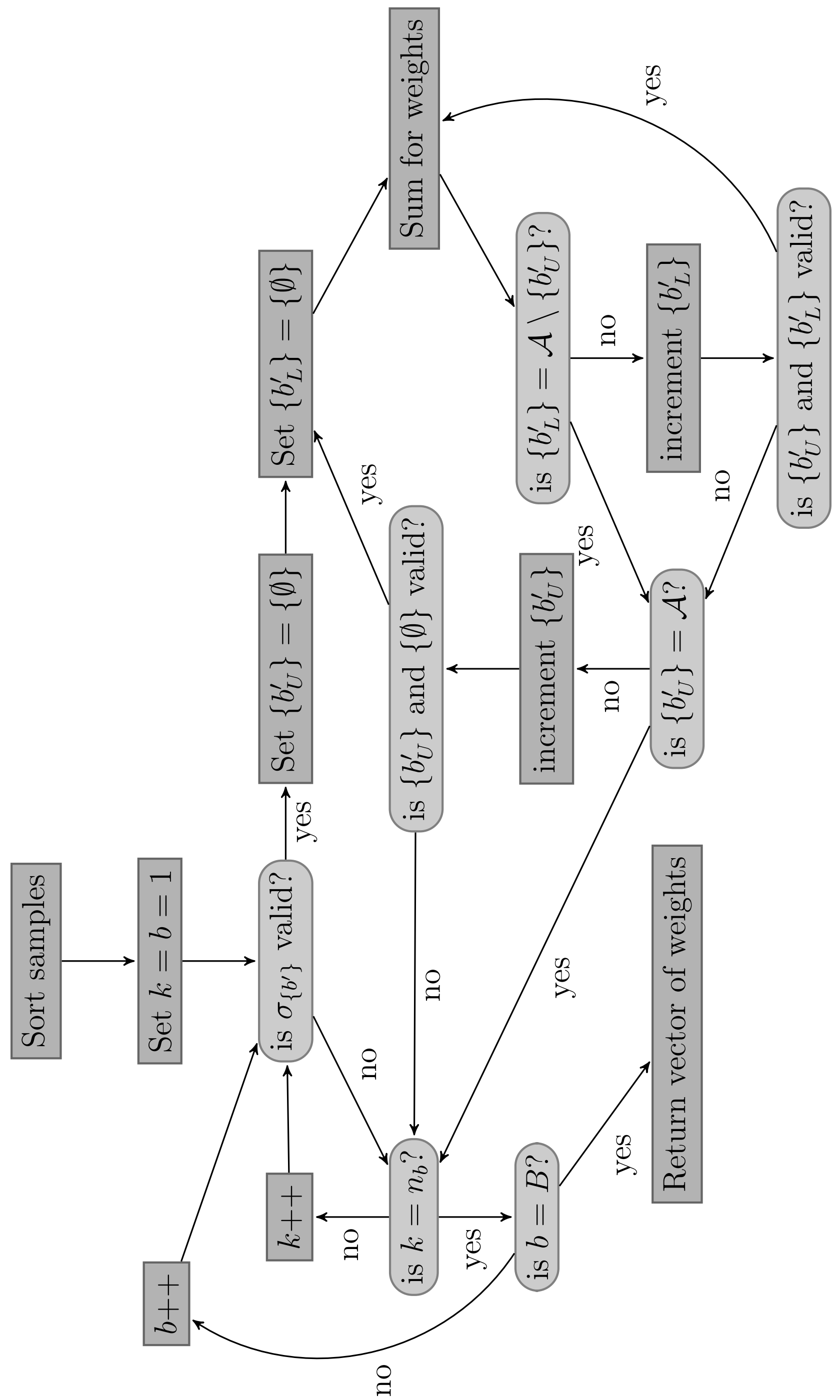


Figure 4.1: A simple algorithm for obtaining the mixture weights in the marginal distribution of $Z_{(i)}$

of permutations is given as well.

Since all samples are identical, we only need to consider $b = 1$ for the outermost sum. Since $0 \leq i - k + r_1^L = 3 - 2 + 1 = 2 \leq \dot{r}_{\mathcal{A}}$, $P(Z_{(3)} = X_{1,2:4}) > 0$, we proceed as in Table 4.1a.

Next, since $0 \leq i - k + r_1^L = 3 - 3 + 1 = 1 \leq \dot{r}_{\mathcal{A}}$, $P(Z_{(3)} = X_{1,3:4}) > 0$, we proceed as in Table 4.1b.

Finally, we add up the probabilities that the 3-rd pooled order statistic is the 5-th, 6-th, or 7-th out of 12 as given in the first and second columns; then we multiply these by 3 since the samples have identical censoring schemes. In this way, the mixture weights are finally found to be $(0.18182, 0.76623, 0.05195)$, corresponding to the mixture representation $Z_{(3)} \stackrel{d}{=} 0.18182X_{5:12} + 0.76623X_{6:12} + 0.05195X_{7:12}$.

4.3 Simulation Results

A large number of samples, high censoring, or large samples, can lead to the computations being extremely computationally intensive and also demanding heavy memory usage. Furthermore, when the samples are not exchangeable, then the computations can not be simplified in any meaningful way. In such situations, one may instead simulate the mixture weights. Obtaining the weights in this way has two advantages. Direct simulation can allow the calculation of all mixture weights for all marginal and bivariate distributions of interest simultaneously. Furthermore, when calculating coverage probabilities, errors are dampened if an increase/decrease at the i-th weight

Prob.	OS	$\{b'_U\}$	$\{b'_L\}$	$\circ$	$X_{1,2:4}$	$X_{1,3:4}$ $\circ$
0.03463	6:12	$\{\emptyset\}$	$\{\emptyset\}$	$\circ$ $X_{2,2:4}$ $X_{2,3:4}$ $\circ$		$\circ$ $X_{3,2:4}$ $X_{3,3:4}$ $\circ$
0.07792	6:12			$\circ$ $X_{2,2:4}$ $\circ$ $X_{3,2:4}$		$X_{2,3:4}$ $\circ$ $X_{3,3:4}$ $\circ$
0.03463	6:12			$\circ$ $\circ$ $X_{3,2:4}$ $X_{3,3:4}$		$X_{2,2:4}$ $X_{2,3:4}$ $\circ$ $\circ$
0.01212	5:12		$\{2\}$	$\circ$ $X_{3,2:4}$ $X_{3,3:4}$		$\circ$ $X_{2,2:4}$ $X_{2,3:4}$ $\circ$ $\circ$
0.01212	5:12		$\{3\}$	$\circ$ $X_{2,2:4}$ $X_{2,3:4}$		$\circ$ $\circ$ $X_{3,2:4}$ $X_{3,3:4}$ $\circ$
0.00866	7:12	$\{2\}$	$\{\emptyset\}$	$\circ$ $X_{2,2:4}$ $X_{2,3:4}$ $\circ$ $\circ$		$X_{3,2:4}$ $X_{3,3:4}$ $\circ$
0.00216	6:12		$\{3\}$	$\circ$ $X_{2,2:4}$ $X_{2,3:4}$ $\circ$		$\circ$ $X_{3,2:4}$ $X_{3,3:4}$ $\circ$
0.00866	7:12	$\{3\}$	$\{\emptyset\}$	$\circ$ $\circ$ $X_{3,2:4}$ $X_{3,3:4}$ $\circ$		$X_{2,2:4}$ $X_{2,3:4}$ $\circ$
0.00216	6:12		$\{2\}$	$\circ$ $X_{3,2:4}$ $X_{3,3:4}$ $\circ$		$\circ$ $X_{2,2:4}$ $X_{2,3:4}$ $\circ$

(a) $b = 1$, $k = 2$, i.e., $Z_{(3)} = X_{1,2:4}$

Prob.	OS	$\{b'_U\}$	$\{b'_L\}$	$\circ$ $X_{1,3:4}$	$X_{1,3:4}$	$\circ$
0.05195	6:12	$\{\emptyset\}$	$\{\emptyset\}$	$\circ$ $X_{2,2:4}$ $\circ$		$X_{2,3:4}$ $\circ$ $X_{3,2:4}$ $X_{3,3:4}$ $\circ$
0.05195	6:12			$\circ$ $\circ$ $X_{3,2:4}$		$X_{2,2:4}$ $X_{2,3:4}$ $\circ$ $X_{3,3:4}$ $\circ$
0.01818	5:12		$\{2\}$	$\circ$ $X_{3,2:4}$		$\circ$ $X_{2,2:4}$ $X_{2,3:4}$ $\circ$ $X_{3,3:4}$ $\circ$
0.01818	5:12		$\{3\}$	$\circ$ $X_{2,2:4}$		$X_{2,3:4}$ $\circ$ $\circ$ $X_{3,2:4}$ $X_{3,3:4}$ $\circ$

(b) $b = 1$, $k = 3$, i.e., $Z_{(3)} = X_{1,3:4}$

Table 4.1: Algorithm as applied to the example in Section 4.2.1

is moved to a "nearby" weight.

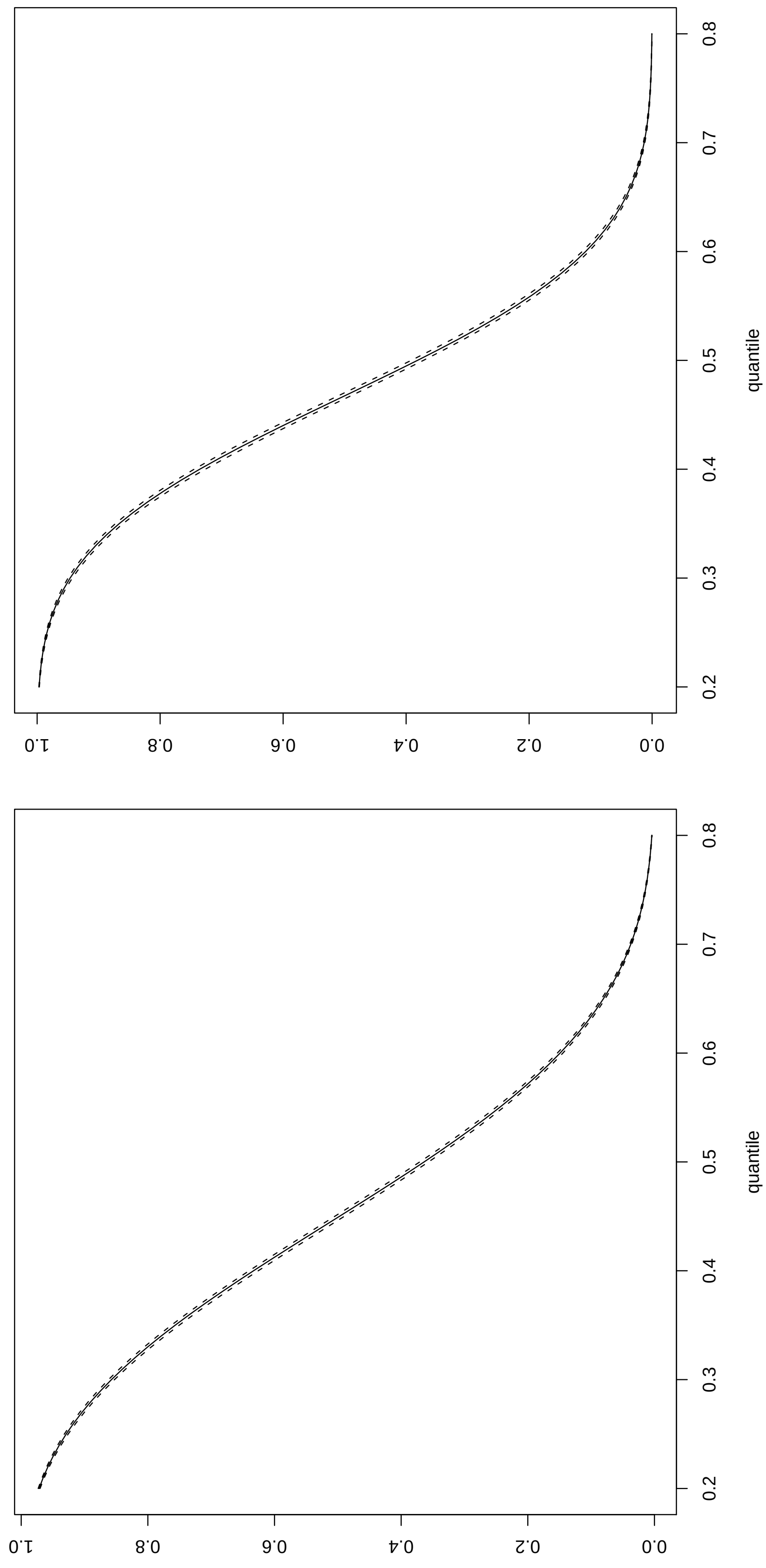
In Figure 4.2, we compare the exact coverage probability for $(-\infty, Z_{(3)})$ for the quantiles $0.2 < p < 0.8$ to the mean of 1000 simulations where the coverage probability is exact given estimated weights. The weights are estimated with a simulation size of 1000.

This is done for the following two schemes; $n_b = 4/8$, $r_b^L = 1/3$, $r_b^U = 1/3$, and $r_b = 2/2$ for $b = 1, 2, 3$. Figures 4.2a and 4.2b show the exact coverage probabilities. The first is the example from Section 4.2.1, and the second is the same but with increased upper and lower censoring.

Figures 4.2c and 4.2d show the signed absolute error (SAE), while Figures 4.2e and 4.2f show the signed relative error (SRE). The dashed lines in each plot contains simulated 98% confidence bands, whereas the solid line is the exact coverage probability. The mean line and 95% bands were indistinguishable from the exact line.

If we focus on Figures 4.2c and 4.2d, we can see that the SAE is small particularly in the first scheme. In the second scheme, the SAE's are larger, but still comparatively small as evidenced by Figure 4.2b. Here, the solid line represents the difference between the exact and mean estimated coverage probabilities. In each case, the bias is found to be negligible.

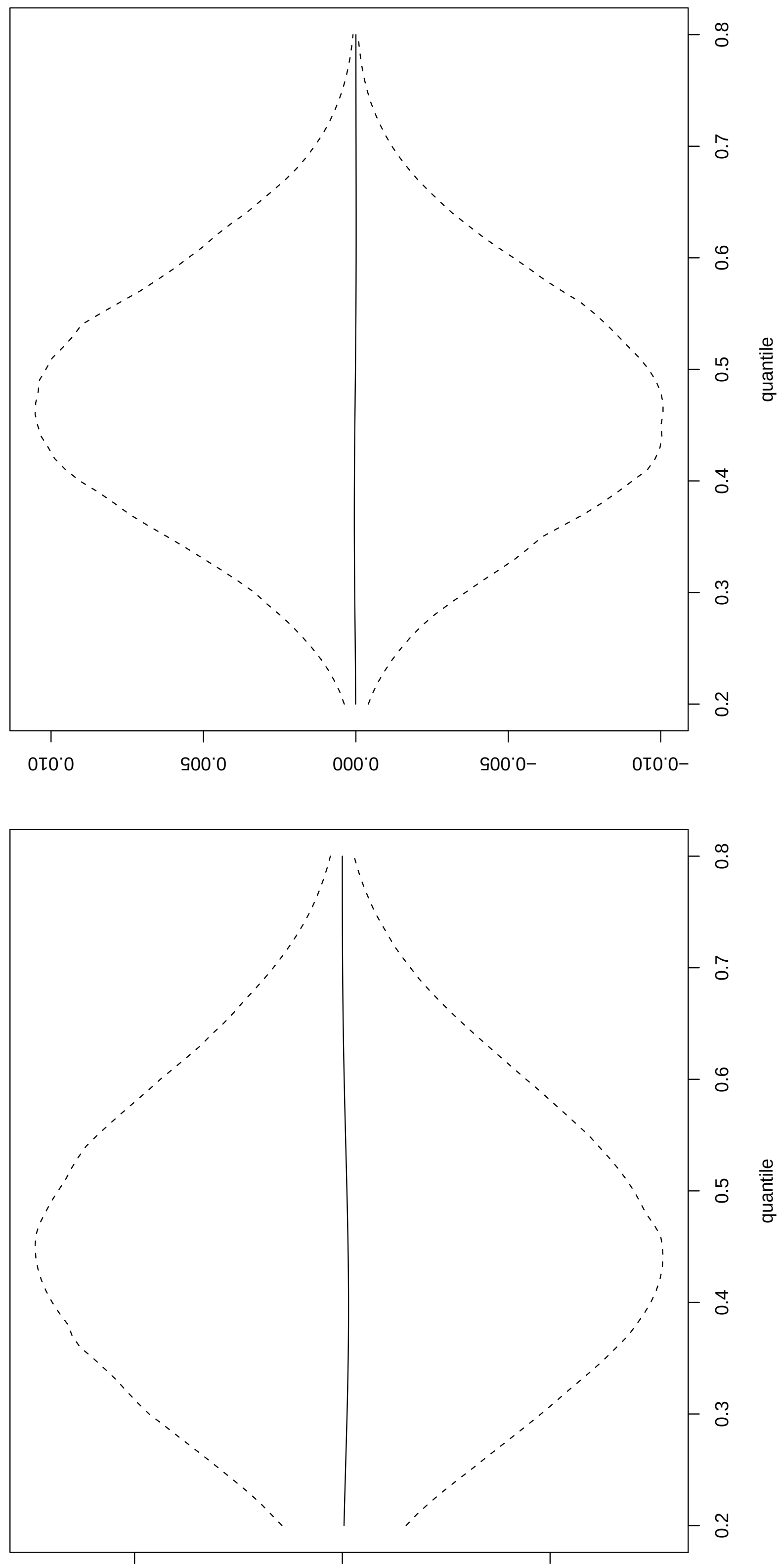
From Figures 4.2e and 4.2f, we notice that the SRE's can be larger; however, this occurs primarily at upper quantiles where the cost of error is lower. Again, the solid line represents the difference between the exact and mean estimated coverage



(a) Coverage Probability: $n_b = 4, r_b^L = r_b^U = 1$

(b) Coverage Probability: $n_b = 8, r_b^L = r_b^U = 3$

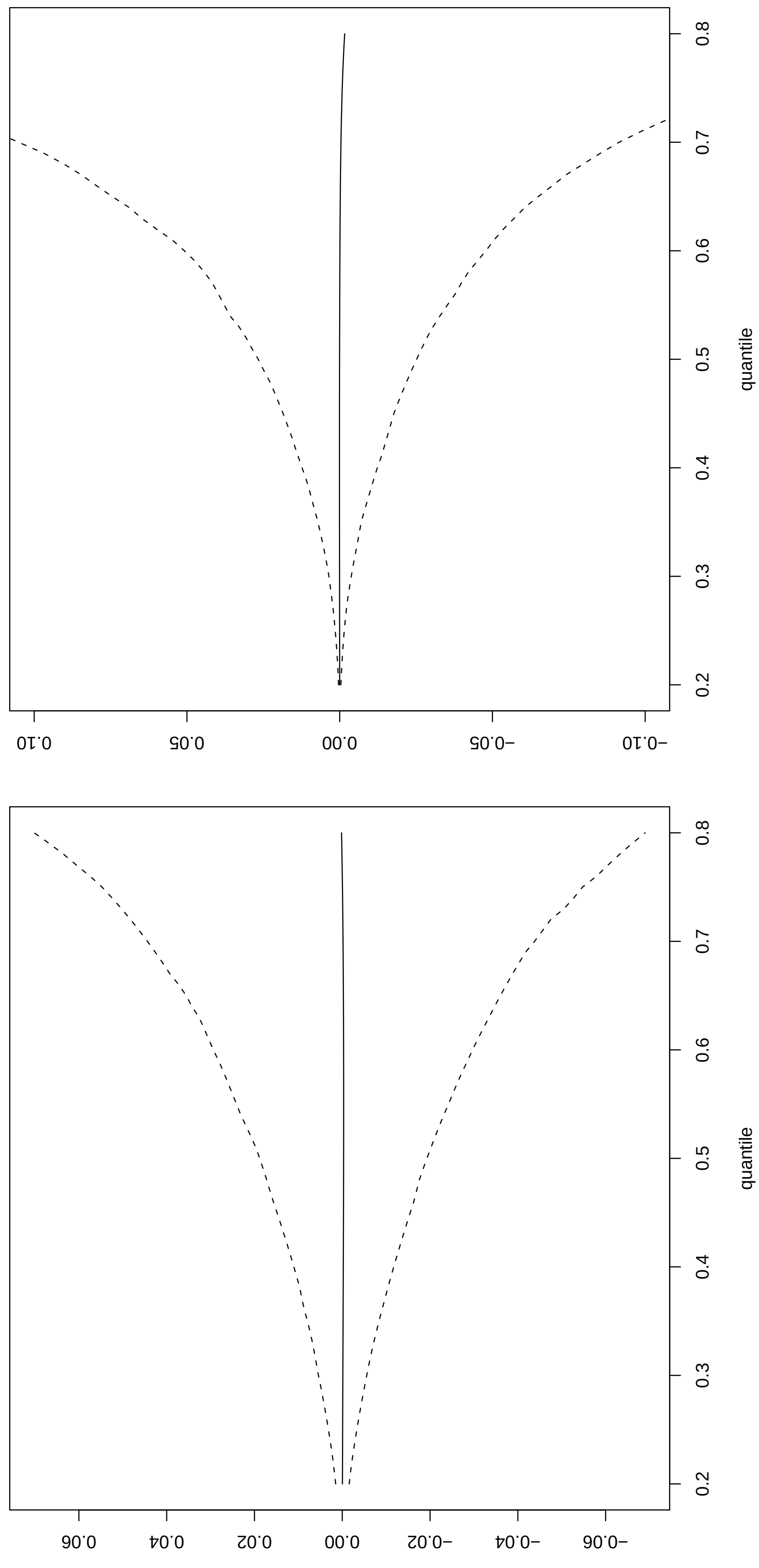
Figure 4.2: Exact coverage probability with simulated 98% bands for $Z_{(3)}$



(c) SAE:$n_b = 4, r_b^L = r_b^U = 1$

(d) SAE:$n_b = 8, r_b^L = r_b^U = 3$

Figure 4.2: SAE with simulated 98% bands for $Z_{(3)}$



(e) SRE:$n_b = 4, r_b^L = r_b^U = 1$

(f) SRE:$n_b = 8, r_b^L = r_b^U = 3$

Figure 4.2: SRE with simulated 98% bands for $Z_{(3)}$

probabilities.

4.4 Motivating Example Revisited

Consider Table 1.3, which gives the failure times of insulating fluid in minutes while under high stress. These data have been analyzed by Balakrishnan et al. (2004) and Balakrishnan and Lin (2005) by assuming an exponential distribution with and without a threshold parameter. We provide these parametric results as a comparison to the nonparametric methods given here.

We obtained the mixture weights from 100000 simulations. For each quantile presented in Table 4.2, we determine the two-sided minimal width interval as described in Balakrishnan et al. (2010a). If more than one interval is of minimal width, the one with the largest coverage probability is chosen. For all quantiles considered, first the intervals chose are with confidence at least 94.75%. If none exist the level is brought to 70%. Again if none exist the largest two-sided interval is chosen. These intervals are then compared to 95% confidence intervals based on the BLUE and MLE of a one-parameter exponential distribution. Note that while the nonparametric and BLUE intervals are exact, the MLE interval is not. Since we are comparing to the exponential distribution, we also show the $1 - 1/e$-th quantile which is the scale parameter of a single parameter exponential.

The BLUE and MLE of the scale parameter ϑ are $\vartheta^* = 2.432$ and $\hat{\vartheta} = 2.240$, respectively. Both these estimates interestingly fall inside the nonparametric interval

	Nonparametric											Exponential			
	Multi Sample					Single Sample						BLUE		MLE	
p	C.P.	i_1	i_2	$Z_{(i_1)}$	$Z_{(i_2)}$	C.P.	i_1	i_2	$X_{i_1:n}$	$X_{i_2:n}$	LOW	UPP	LOW	UPP
0.05	0.369	1	45	0.06	5.55	0.010	8	52	0.50	4.75	0.097	0.167	0.084	0.146
0.10	0.798	1	45	0.06	5.55	0.248	8	52	0.50	4.75	0.199	0.343	0.172	0.300
0.15	0.951	1	10	0.06	0.80	0.695	8	52	0.50	4.75	0.306	0.529	0.265	0.463
0.20	0.967	1	12	0.06	0.93	0.933	8	52	0.50	4.75	0.421	0.727	0.364	0.636
0.25	0.954	3	15	0.50	1.13	0.949	9	22	0.55	1.17	0.542	0.937	0.469	0.820
0.30	0.948	5	18	0.64	1.49	0.951	12	26	0.66	1.54	0.673	1.162	0.581	1.016
0.35	0.951	8	22	0.71	1.70	0.959	14	29	0.71	1.70	0.812	1.403	0.702	1.227
0.40	0.957	10	25	0.80	1.99	0.953	17	32	0.82	1.99	0.963	1.664	0.833	1.455
0.45	0.952	13	28	1.08	2.12	0.949	20	35	1.08	2.12	1.127	1.947	0.974	1.703
0.50	0.951	16	31	1.17	2.17	0.948	23	38	1.30	2.17	1.307	2.258	1.130	1.975
0.55	0.953	19	34	1.54	2.57	0.949	26	41	1.54	2.57	1.506	2.601	1.301	2.275
0.60	0.959	22	37	1.70	3.17	0.953	29	44	1.70	3.17	1.728	2.985	1.493	2.610
$1-1/e$	0.950	24	38	1.89	3.57	0.956	31	46	1.89	3.72	1.885	3.257	1.630	2.849
0.65	0.954	25	39	1.99	3.72	0.959	32	47	1.99	3.82	1.979	3.419	1.711	2.991
0.70	0.953	28	41	2.12	3.83	0.951	35	49	2.12	3.87	2.270	3.922	1.962	3.430
0.75	0.959	31	43	2.17	4.03	0.949	39	52	2.24	4.75	2.614	4.515	2.259	3.949
0.80	0.959	35	45	2.75	5.55	0.873	8	52	0.50	4.75	3.035	5.242	2.623	4.585
0.85	0.941	1	45	0.06	5.55	0.555	8	52	0.50	4.75	3.577	6.179	3.092	5.405
0.90	0.746	1	45	0.06	5.55	0.142	8	52	0.50	4.75	4.341	7.500	3.753	6.560
0.95	0.319	1	45	0.06	5.55	0.003	8	52	0.50	4.75	5.648	9.758	4.883	8.535

Table 4.2: Two-sided confidence intervals for ξ_p, at various p, for the insulating fluid data in Table 1.3

for $\xi_{1-1/e}$. Though the BLUE and MLE intervals are shorter as one would expect, they are not significantly shorter than the nonparametric intervals.

For the sake of comparison, we also show the equivalent censoring scheme for a single doubly Type-II censored sample; that is $n = 60$, $r^L = 7$, and $r^U = 8$. The intervals have comparable confidence levels at the central quantiles, but as in Chapter 3, the extreme quantiles have much lower coverage probabilities. Moreover the length seems to be of comparable length. In terms of indices, the pooled intervals are the same size or smaller. This seems to be compatible with the results of Ozturk and Deshpande (2006).

Chapter 5

Multiple Progressively Type-II Right Censored Samples

In this chapter we consider an extension to Chapter 3 for multiple independent progressively Type-II right censored samples. The representations here are distinct from those given prior due to the different representation of PCOS. That is, in the case of Type-II censoring, these representations do not collapse to those considered in Chapter 3.

5.1 Distributional Representations

We consider here, various representations for the distributions of the pooled PCOS. There are a variety of representations of which some are more practical to use than others.

5.1.1 Marginal Distribution of a pooled OS

As in Chapters 3 and 4 we can partition the sample space to obtain

$$P(Z_{(i)} \le \xi_p) = \sum_{b=1}^{B} \sum_{k_b=1}^{r_b} P(Z_{(i)} = X^{\mathcal{R}^{(b)}}_{k_b:r_b:n_b} \le \xi_p). \tag{5.1.1}$$

For each probability on the right we can follow along the same lines as before and obtain the following result.

Theorem 5.1.1 *For any $1 \le i \le \dot{r}$, and $0 < p < 1$ the marginal distribution of $Z_{(i)}$ is given by,*

$$P(Z_{(i)} \le \xi_p) = \sum_{b=1}^{B} \sum_{k_b=1} \sum_{\mathcal{K}_{b,k_b}} \sum_{S \in \mathcal{P}(\beta)} (-1)^{|S|} \sum_{\mathcal{L}} A^{(\mathcal{L})} C^{(\mathcal{L})} P(X^{\mathcal{R}^{(\mathcal{L})}}_{k_b:r_b:n^{(\mathcal{L})}} \le \xi_p)$$

Where,

$$\mathcal{K}_{b,k_b} = \left\{ (k_j)_{\substack{j=1\\ j\neq b}}^{B} : \sum_{j=1}^{B} = k_j = i, 0 \le k_j \le r_j \right\}, \qquad n^{(\mathcal{L})} = n_b + \sum_{j\in S} \gamma^{(j)}_{\ell_j} + \sum_{j \in \alpha} \gamma^{(j)}_{\ell_j}.$$

$$\mathcal{L} = \left\{ (\ell_j)_{j \in S \bigcup \alpha} : \begin{array}{ll} 1 \le \ell_j \le k_j + 1, & j \in \alpha \\ 1 \le \ell_j \le r_j, & j \in S \end{array} \right\}, \quad \gamma^{(\mathcal{L})}_{\ell_b} = \gamma^{(b)}_{\ell_b} + \sum_{j \in S} \gamma^{(j)}_{\ell_j} + \sum_{j\in\alpha} \gamma^{(j)}_{\ell_j},$$

$$A^{(\mathcal{L})} = \prod_{j\in\alpha} a^{(j)}_{\ell_j}(k_j+1) \prod_{j \in S} \frac{a^{(j)}_{\ell_j}(r_j)}{\gamma^{(j)}_{\ell_j}}, \qquad C^{(\mathcal{L})} = \frac{c^{(j)}_{k_b-1}}{c^{(\mathcal{L})}_{k_b-1}} \prod_{j\in\alpha} c^{(j)}_{k_j-1} \prod_{j\in S} c^{(j)}_{r_j-1},$$

$$\mathcal{R}^{(\mathcal{L})} = \left(R^{(b)}_1, R^{(b)}_2, \ldots, R^{(b)}_{k_b-1}, \gamma^{(\mathcal{L})} - 1 \right),$$

and $c^{\mathcal{L}}_{\ell-1}$, $\ell = 1, \ldots, k_b$, is generated by $\mathcal{R}^{(\mathcal{L})}$.

Proof: We consider the uniform distribution wlog. Rewriting the probability again conditioning on the pooled order statistic being any given PCOS from any sample we have,

$$
\begin{aligned}
P(Z_{(i)} \leq p) &= \sum_{b=1}^{B} \sum_{k_b=1}^{r_b} P\left(Z_{(i)} \leq p, Z_{(i)} = X^{\mathcal{R}^{(b)}}_{k_b:r_b:n_b}\right) \\
&= \sum_{b=1}^{B} \sum_{k_b=1}^{r_b} \sum_{\mathcal{K}_{b,k_b}} P\left(X^{\mathcal{R}^{(b)}}_{k_b:r_b:n_b} \leq p, \bigcap_{\substack{j=1 \\ j\neq b}}^{B} \left\{X^{\mathcal{R}^{(j)}}_{k_j:r_j:n_j} < X^{\mathcal{R}^{(b)}}_{k_b:r_b:n_b} < X^{\mathcal{R}^{(j)}}_{k_j+1:r_j:n_j}\right\}\right) \\
&= \sum_{b=1}^{B} \sum_{k_b=1}^{r_b} \sum_{\mathcal{K}_{b,k_b}} \int_0^p f^{X^{\mathcal{R}^{(b)}}_{k_b:r_b:n_b}}(x) \prod_{\substack{j=1 \\ j\neq b}}^{B} \left[F^{X^{\mathcal{R}^{(j)}}_{k_j:r_j:n_j}}(x) - F^{X^{\mathcal{R}^{(j)}}_{k_j+1:r_j:n_j}}(x)\right] dx,
\end{aligned}
$$

where $\mathcal{K}_{b,k_b}$ is a partition of the event that $X^{\mathcal{R}^{(b)}}_{k_b:r_b:n_b} = Z_{(i)}$.

From Kamps and Cramer (2001) we have

$$
F^{X^{\mathcal{R}^{(j)}}_{k_j:r_j:n_j}}(x) - F^{X^{\mathcal{R}^{(j)}}_{k_j+1:r_j:n_j}}(x) = \frac{1}{\gamma^{(j)}_{k_j+1}} f^{X^{\mathcal{R}^{(j)}}_{k_j+1:r_j:n_j}}(x) \frac{1-F(x)}{f(x)}, \qquad f(x) \neq 0,
$$

for $k_j = 1, \ldots, r_j - 1$. When $k_j = 0$ we have it to be true following the convention that $F^{X^{\mathcal{R}^{(j)}}_{0:r_j:n_j}}(x) = 1, \forall\ x > -\infty$. If $k_j = r_j$ then $F^{X^{\mathcal{R}^{(j)}}_{k_j:r_j:n_j}}(x) - F^{X^{\mathcal{R}^{(j)}}_{k_j+1:r_j:n_j}}(x) = F^{X^{\mathcal{R}^{(j)}}_{k_j:r_j:n_j}}(x)$ assuming the convention that $F^{X^{\mathcal{R}^{(j)}}_{r_j+1:r_j:n_j}}(x) = 0, \forall\ x < \infty$.

So for every partition in $\mathcal{K}_{b,k_b}$ we have

$$
P(Z_{(i)} \leq p) = \sum_{b=1}^{B} \sum_{k_b=1}^{r_b} \sum_{\mathcal{K}_{b,k_b}} \int_0^p f^{X^{\mathcal{R}^{(b)}}_{k_b:r_b:n_b}}(x)
$$

$$\times \prod_{\substack{j=1\\ j\neq b}}^{B} \left\{ \begin{array}{lll} \frac{1}{\gamma^{(j)}_{k_j+1}} f^{X^{\mathcal{R}(j)}_{k_j+1:r_j:n_j}}(x)(1-x) & \text{if } k_j = 0,1,\ldots,r_j-1 & (\alpha) \\ F^{X^{\mathcal{R}(j)}_{r_j:r_j:n_j}}(x) & \text{if } k_j = r_j & (\beta) \end{array} \right. dx$$

$$= \sum_{b=1}^{B}\sum_{k_b=1}^{r_b}\sum_{\mathcal{K}_{b,k_b}} \int_0^p \left(c^{(b)}_{k_b-1} \sum_{\ell_b=1}^{k_b} a^{(b)}_{\ell_b}(k_b)(1-x)^{\gamma^{(b)}_{\ell_b}-1} \right)$$

$$\times \prod_{j\in\alpha} \left(\frac{(1-x)}{\gamma^{(j)}_{k_j+1}} c^{(j)}_{k_j} \sum_{\ell_j=1}^{k_j+1} a^{(j)}_{\ell_j}(k_j)(1-x)^{\gamma^{(j)}_{\ell_j}-1} \right) \prod_{j\in\beta} \left(1 - c^{(j)}_{r_j-1} \sum_{\ell_j=1}^{r_j} \frac{a^{(j)}_{\ell_j}(k_j)}{\gamma^{(j)}_{\ell_j}} (1-x)^{\gamma^{(j)}_{\ell_j}} \right)$$

$$= \sum_{b=1}^{B}\sum_{k_b=1}^{r_b}\sum_{\mathcal{K}_{b,k_b}} \int_0^p \left(c^{(b)}_{k_b-1} \sum_{\ell_b=1}^{k_b} a^{(b)}_{\ell_b}(k_b)(1-x)^{\gamma^{(b)}_{\ell_b}-1} \right)$$

$$\times \prod_{j\in\alpha} \left(c^{(j)}_{k_j-1} \sum_{\ell_j=1}^{k_j+1} a^{(j)}_{\ell_j}(k_j)(1-x)^{\gamma^{(j)}_{\ell_j}} \right) \sum_{S\in\mathcal{P}(\beta)} (-1)^{|S|} \prod_{j\in S} \left(c^{(j)}_{r_j-1} \sum_{\ell_j=1}^{r_j} \frac{a^{(j)}_{\ell_j}(r_j)}{\gamma^{(j)}_{\ell_j}} (1-x)^{\gamma^{(j)}_{\ell_j}} \right)$$

$$= \sum_{b=1}^{B}\sum_{k_b=1}^{r_b}\sum_{\mathcal{K}_{b,k_b}} \sum_{S\in\mathcal{P}(\beta)} (-1)^{|S|} \sum_{\mathcal{L}} C^{(\mathcal{L})} A^{(\mathcal{L})} \int_0^p c^{(\mathcal{L})}_{k_b-1} \sum_{\ell_b=1}^{k_b} a^{(b)}_{\ell_b}(k_b)(1-x)^{\gamma^{(\mathcal{L})}_{\ell_b}-1}.$$

The last equality follows from collecting terms appropriately and expanding the summations to form $\sum_{\mathcal{L}}$. The integral above is the CDF of $X^{\mathcal{R}^{\mathcal{L}}}_{k_b:r_b:n^{(\mathcal{L})}}$, so the theorem follows. ∎

Remark 5.1.2 *It follows that the distribution of $Z_{(1)}$ is the minima of a sample of size n, but for $1 < i \leq \dot{r}$, the mixtures may not be convex let alone have a simple interpretation.*

That is, unlike Chapters 3 and 4, the weights do not arise as counting marbles in bins given by above and below $Z_{(i)}$.

Remark 5.1.3 *The distribution of the maxima $Z_{(\dot{r})}$ can be evaluated as in equation (3.1.2), which will be significantly more convenient for maximal coverage probabilities.*

We can also obtain the marginal distribution of the pooled order statistics as a mixture of regular OS, or a mixture of PCOS. The first could be obtained in theory by expressing each sample as a mixture of regular OS and pooling them, and applying ideas similar to those in Chapters 3 and 4. However, except for the smallest possible sample sizes and number of samples, this is likely impossible. Simulation would be direct and efficient however, and inference can thus be given simply in a similar manner to Section 3.2.

Representing the marginal distribution as a mixture of PCOS, would also be computationally difficult, and if desired, could be simulated. However, this will likely be less efficient than the representations involving the usual OS.

5.1.2 Joint Distribution of pooled OS

The joint distribution can be represented in multiple ways as in Chapter 3.

Since the PCOS are themselves a mixture of regular OS, we can write the joint distribution of the pooled PCOS as a mixture of regular OS. Namely, as subsets of the regular OS of size $\dot{r}$ from a sample of size n as in Proposition 3.1.10.

Similarly, we can write the joint distribution as a mixture of progressively censored samples. In particular, given some ordering all $\dot{r}$ observations, the censoring scheme would be $\tilde{\mathcal{R}}$ where $\tilde{R}_i = R^{(b)}_{k_b}$ if $Z_{(i)} = X^{\mathcal{R}^{(b)}}_{k_b:r_b:n_b}$ as in Proposition 3.1.12.

These have the same setbacks as mentioned in Chapters 3 and 4.

5.2 Simulation

As mentioned in Section 4.3, when the number of observations or number of samples are large, it may be more practical to obtain the mixture weights by simulation; there are various ways this can be done.

As discussed earlier, one can represent the joint distribution of the pooled PCOS as a mixture of regular OS. Given this, it is clear that we can represent the marginal distribution of the pooled OS as a mixture of the usual OS. Simulating these weights is trivial.

The marginal distribution can also be simulated by marginal PCOS. This is quite simple to simulate as well, but is substantially more work as i increases. This is a consequence of the discussion of storage space in Section 3.1.3. In particular when $R_1^{(b)} = \cdots = R_{r-1}^{(b)} \neq R_r^{(b)}$ is a common scheme for all samples, then Remark 3.1.13 gives the exact number of component distributions for $i = \dot{r}$. For any $1 \leq i < \dot{r}$, the number of component distributions would be a subset of that case. For the censoring scheme as applied to the Nelson data in Table 1.4, there are 1428 schemes, which is quite large considering how small $\dot{r}$ is.

However, if one wishes to obtain the mixture weights as given in Theorem 5.1.1 through simulation, it is not clear how to do this.

5.3 Motivating Example Revisited

We consider the insulating fluid data as in Table 1.4 where we introduced progressive Type-II censoring. In Table 5.1 we can see the coverage probability for all schemes considered. For comparison, an exponential interval based on the BLUE is given which has the same confidence level. The results of this are not too surprising. Progressive Type-II censoring tends to favour sampling the smallest order statistics. This becomes more pronounced when censoring is primarily right censoring. Scheme $\mathcal{R}_3$ is the extreme example of this. Schemes with moderate left censoring, $\mathcal{R}_2$ and $\mathcal{R}_4$ fare the best. If we compare these results to the ones in Table 3.1, we can see that scheme $\mathcal{R}_2$ improves on the coverage probability even when the Type-II has 52 observed failures. The primary difference between the Type-II example from Chapter 3 and the example considered here, would be expected length of the resulting intervals. One would expect both of these to improve had we observed 52 items instead of the 18 considered.

In Table 5.2 we then consider the expected length of the nonparametric interval assuming a standard exponential, uniform$(0, 1)$, and standard logistic distribution.

We note for lower quantiles, the schemes with more right censoring have shorter lengths. The increase in coverage probability for left censoring is thus countered by the increase in expected length.

5.4 Miscellaneous comments

Table 5.2 compares the coverage probabilities and expected lengths for the pooled scheme and single sample scheme which is generated by appending the 6 schemes in the pooled scheme. Unlike Chapter 3, there seems to be no guarantee that the maximal coverage probability for the pooled scheme is higher than the single sample (see 5.2b and 5.2c). Furthermore, there is no consistent relationship in terms of expected length.

However, there are many one sample schemes that could be considered as alternatives to a given multiple sample scheme, particularly when the censoring schemes for each sample are unique. As a result, we can not make any general conclusion with regards to expected length of the intervals or coverage probability.

We may however look at extreme scenarios for each such as left censoring. Consider $\mathcal{R}_5 = \mathcal{R}_5^{(b)} = (7, 0, 0)$ for the (balanced size) pooled scenario, and $\mathcal{R}_5^s = (42, 0, \ldots, 0)$ for the single sample scenario. In such a case then $Z_{18}^{\mathcal{R}_5} \leq_{st} Z_{18}^{\mathcal{R}_5^s}$, and so the maximal coverage probability for $\mathcal{R}_5^s$ will be better than for $\mathcal{R}_5$. The opposite extreme is that of Chapter 3, for which the stochastic ordering was reversed. This would suggest that pooling will be best in terms of coverage probability when there tends to be more right censoring, and worse when there is more left censoring.

In Section 3.2.4 we discussed some issues with regards to convergence of the maximal OS, in particular Proposition 3.2.2. For any given censoring scheme $\mathcal{R}$, we have $X_{i:r:n}^{\mathcal{S}} \leq_{st} X_{i:r:n}^{\mathcal{R}}$, where $\mathcal{S}$ is the right censoring scheme with the same number of units and observed failures as $\mathcal{R}$. Thus, bounding the number of items on test is a sufficient condition for the maximum to to converge to the upper end point of the distribution when pooling multiple progressively Type-II censored samples.

	$\mathcal{R}_1 = (2,2,3)$							$\mathcal{R}_2 = (6,1,0)$						
		Nonparametric				Exponential			Nonparametric				Exponential	
p	C.P.	i_1	i_2	$Z_{(i_1)}$	$Z_{(i_2)}$	LOW	UPP	C.P.	i_1	i_2	$Z_{(i_1)}$	$Z_{(i_2)}$	LOW	UPP
0.05	0.9497	1	8	0.00	0.50	0.058	0.220	0.9497	1	7	0.00	0.66	0.034	0.130
0.10	0.9612	2	10	0.02	0.64	0.115	0.475	0.9479	2	8	0.02	0.80	0.070	0.265
0.15	0.9651	4	13	0.18	0.93	0.175	0.746	0.9702	3	10	0.20	1.30	0.101	0.452
0.20	0.9616	6	15	0.31	1.49	0.243	1.007	0.9632	4	11	0.31	1.34	0.143	0.597
0.25	0.9674	7	16	0.49	1.56	0.308	1.336	0.9626	6	13	0.50	2.12	0.184	0.768
0.30	0.9580	9	17	0.55	1.70	0.393	1.584	0.9622	6	13	0.50	2.12	0.229	0.951
0.35	0.9544	11	18	0.66	2.44	0.479	1.885	0.9665	7	14	0.66	2.17	0.272	1.173
0.40	0.9478	9	18	0.55	2.44	0.577	2.181	0.9700	8	15	0.80	2.57	0.319	1.418
0.45	0.8742	1	18	0.00	2.44	0.758	2.153	0.9701	9	16	1.08	3.87	0.373	1.660
0.50	0.7566	1	18	0.00	2.44	0.977	2.161	0.9681	9	16	1.08	3.87	0.436	1.903
0.55	0.6049	1	18	0.00	2.44	1.239	2.207	0.9778	10	17	1.30	6.63	0.484	2.333
0.60	0.4413	1	18	0.00	2.44	1.545	2.297	0.9540	11	17	1.34	6.63	0.600	2.358
$1-1/e$	0.3415	1	18	0.00	2.44	1.765	2.382	0.9527	12	18	1.54	7.21	0.657	2.560
0.65	0.2904	1	18	0.00	2.44	1.896	2.439	0.9625	12	18	1.54	7.21	0.672	2.802
0.70	0.1699	1	18	0.00	2.44	2.289	2.647	0.9707	12	18	1.54	7.21	0.750	3.355
0.75	0.0863	1	18	0.00	2.44	2.731	2.939	0.9534	12	18	1.54	7.21	0.910	3.559
0.80	0.0364	1	18	0.00	2.44	3.238	3.339	0.9076	1	18	0.00	7.21	1.150	3.632
0.85	0.0117	1	18	0.00	2.44	3.856	3.895	0.8186	1	18	0.00	7.21	1.498	3.722
0.90	0.0023	1	18	0.00	2.44	4.699	4.709	0.6635	1	18	0.00	7.21	2.034	3.907
0.95	0.0002	1	18	0.00	2.44	6.120	6.120	0.4067	1	18	0.00	7.21	3.024	4.343

	$\mathcal{R}_3 = (0,0,7)$							$\mathcal{R}_4 = (4,0,3)$						
		Nonparametric				Exponential			Nonparametric				Exponential	
p	C.P.	i_1	i_2	$Z_{(i_1)}$	$Z_{(i_2)}$	LOW	UPP	C.P.	i_1	i_2	$Z_{(i_1)}$	$Z_{(i_2)}$	LOW	UPP
0.05	0.9515	1	9	0.00	0.55	0.065	0.253	0.9516	1	8	0.00	0.50	0.047	0.183
0.10	0.9620	2	11	0.02	0.66	0.131	0.543	0.9720	2	10	0.02	0.82	0.091	0.413
0.15	0.9618	4	14	0.18	0.82	0.202	0.837	0.9525	4	12	0.18	1.08	0.149	0.580
0.20	0.9593	6	16	0.31	1.49	0.279	1.136	0.9707	5	14	0.20	1.34	0.194	0.868
0.25	0.9593	9	18	0.55	1.56	0.360	1.465	0.9646	7	16	0.49	1.54	0.255	1.083
0.30	0.9447	1	18	0.00	1.56	0.462	1.718	0.9738	8	17	0.50	1.56	0.307	1.415
0.35	0.8378	1	18	0.00	1.56	0.647	1.677	0.9585	10	18	0.82	1.89	0.389	1.576
0.40	0.6664	1	18	0.00	1.56	0.872	1.682	0.9500	11	18	0.93	1.89	0.472	1.808
0.45	0.4668	1	18	0.00	1.56	1.136	1.733	0.9322	1	18	0.00	1.89	0.573	1.999
0.50	0.2863	1	18	0.00	1.56	1.430	1.833	0.8503	1	18	0.00	1.89	0.741	1.979
0.55	0.1535	1	18	0.00	1.56	1.743	1.988	0.7260	1	18	0.00	1.89	0.946	1.990
0.60	0.0715	1	18	0.00	1.56	2.071	2.201	0.5702	1	18	0.00	1.89	1.191	2.035
$1-1/e$	0.0404	1	18	0.00	1.56	2.290	2.370	0.4635	1	18	0.00	1.89	1.372	2.086
0.65	0.0286	1	18	0.00	1.56	2.416	2.476	0.4050	1	18	0.00	1.89	1.481	2.123
0.70	0.0095	1	18	0.00	1.56	2.793	2.816	0.2555	1	18	0.00	1.89	1.815	2.263
0.75	0.0025	1	18	0.00	1.56	3.226	3.233	0.1394	1	18	0.00	1.89	2.195	2.472
0.80	0.0005	1	18	0.00	1.56	3.749	3.750	0.0629	1	18	0.00	1.89	2.632	2.777
0.85	0.0001	1	18	0.00	1.56	4.419	4.420	0.0215	1	18	0.00	1.89	3.157	3.216
0.90	0.0000	1	18	0.00	1.56	5.364	5.364	0.0045	1	18	0.00	1.89	3.860	3.875
0.95	0.0000	1	18	0.00	1.56	6.979	6.979	0.0003	1	18	0.00	1.89	5.031	5.032

Table 5.1: Two-sided confidence intervals for ξ_p, at various p, for the insulating fluid data in Table 1.4

	Multiple Samples ($\mathcal{R}_1$)						Single Sample ($\mathcal{R}_1^s$)					
p	C.P.	i_1	i_2	Exp	Unif	Log	C.P.	i_1	i_2	Exp	Unif	Log
0.05	0.9497	1	8	0.149	0.134	2.882	0.9501	1	8	0.152	0.137	2.904
0.10	0.9612	2	10	0.190	0.166	2.196	0.9649	2	10	0.197	0.171	2.230
0.15	0.9651	4	13	0.270	0.218	1.784	0.9680	4	13	0.287	0.229	1.836
0.20	0.9616	6	15	0.346	0.257	1.615	0.9641	5	14	0.325	0.249	1.727
0.25	0.9674	7	16	0.409	0.286	1.620	0.9736	7	16	0.449	0.307	1.699
0.30	0.9580	9	17	0.483	0.307	1.526	0.9638	9	17	0.535	0.330	1.612
0.35	0.9544	11	18	0.667	0.360	1.637	0.9491	11	18	0.715	0.374	1.691
0.40	0.9478	9	18	0.733	0.411	1.972	0.9570	11	18	0.715	0.374	1.691
0.45	0.8742	1	18	0.909	0.569	5.033	0.9104	1	18	0.967	0.590	5.128
0.50	0.7566	1	18	0.909	0.569	5.033	0.8141	1	18	0.967	0.590	5.128
0.55	0.6049	1	18	0.909	0.569	5.033	0.6780	1	18	0.967	0.590	5.128
0.60	0.4413	1	18	0.909	0.569	5.033	0.5175	1	18	0.967	0.590	5.128
0.65	0.2904	1	18	0.909	0.569	5.033	0.3565	1	18	0.967	0.590	5.128
0.70	0.1699	1	18	0.909	0.569	5.033	0.2177	1	18	0.967	0.590	5.128
0.75	0.0863	1	18	0.909	0.569	5.033	0.1148	1	18	0.967	0.590	5.128
0.80	0.0364	1	18	0.909	0.569	5.033	0.0499	1	18	0.967	0.590	5.128
0.85	0.0117	1	18	0.909	0.569	5.033	0.0163	1	18	0.967	0.590	5.128
0.90	0.0023	1	18	0.909	0.569	5.033	0.0032	1	18	0.967	0.590	5.128
0.95	0.0002	1	18	0.909	0.569	5.033	0.0002	1	18	0.967	0.590	5.128

(a) Schemes $\mathcal{R}_1 = (2, 2, 3)$ and $\mathcal{R}_1^s = (2, 2, 3, \ldots, 2, 2, 3)$

	Multiple Samples ($\mathcal{R}_2$)						Single Sample ($\mathcal{R}_2^s$)					
p	C.P.	i_1	i_2	Exp	Unif	Log	C.P.	i_1	i_2	Exp	Unif	Log
0.05	0.9497	1	7	0.174	0.155	3.022	0.9511	1	8	0.161	0.145	2.963
0.10	0.9479	2	8	0.208	0.178	2.227	0.9667	2	10	0.208	0.179	2.250
0.15	0.9702	3	10	0.328	0.258	2.190	0.9526	4	12	0.260	0.209	1.713
0.20	0.9632	4	11	0.396	0.293	2.039	0.9725	5	14	0.363	0.271	1.797
0.25	0.9626	6	13	0.584	0.369	1.952	0.9567	7	15	0.405	0.281	1.565
0.30	0.9622	6	13	0.584	0.369	1.952	0.9605	9	17	0.776	0.415	1.967
0.35	0.9665	7	14	0.718	0.408	1.981	0.9630	10	17	0.743	0.389	1.797
0.40	0.9700	8	15	0.897	0.445	2.067	0.9794	11	18	1.700	0.555	2.900
0.45	0.9701	9	16	1.154	0.481	2.239	0.9757	12	18	1.652	0.521	2.718
0.50	0.9681	9	16	1.154	0.481	2.239	0.9647	13	18	1.602	0.486	2.549
0.55	0.9778	10	17	1.570	0.513	2.577	0.9477	13	18	1.602	0.486	2.549
0.60	0.9540	11	17	1.476	0.455	2.306	0.9074	1	18	1.971	0.785	6.413
0.65	0.9625	12	18	2.364	0.478	3.134	0.8462	1	18	1.971	0.785	6.413
0.70	0.9707	12	18	2.364	0.478	3.134	0.7659	1	18	1.971	0.785	6.413
0.75	0.9534	12	18	2.364	0.478	3.134	0.6675	1	18	1.971	0.785	6.413
0.80	0.9076	1	18	2.939	0.897	7.524	0.5534	1	18	1.971	0.785	6.413
0.85	0.8186	1	18	2.939	0.897	7.524	0.4264	1	18	1.971	0.785	6.413
0.90	0.6635	1	18	2.939	0.897	7.524	0.2898	1	18	1.971	0.785	6.413
0.95	0.4067	1	18	2.939	0.897	7.524	0.1467	1	18	1.971	0.785	6.413

(b) Schemes $\mathcal{R}_2 = (6, 1, 0)$ and $\mathcal{R}_2^s = (6, 1, 0, \ldots, 6, 1, 0)$

Table 5.2: Comparison of expected length and coverage probability for multiple samples ($\mathcal{R}$) and a single sample ($\mathcal{R}^s$)

	Multiple Samples ($\mathcal{R}_3$)						Single Sample ($\mathcal{R}_3^s$)					
p	C.P.	i_1	i_2	Exp	Unif	Log	C.P.	i_1	i_2	Exp	Unif	Log
0.05	0.9515	1	9	0.150	0.136	2.897	0.9492	1	8	0.146	0.132	2.865
0.10	0.9620	2	11	0.181	0.159	2.168	0.9610	2	10	0.189	0.165	2.203
0.15	0.9618	4	14	0.241	0.197	1.722	0.9672	4	13	0.272	0.219	1.804
0.20	0.9593	6	16	0.302	0.231	1.558	0.9648	6	15	0.339	0.253	1.616
0.25	0.9593	9	18	0.445	0.296	1.549	0.9681	7	16	0.414	0.289	1.643
0.30	0.9447	1	18	0.595	0.431	4.446	0.9580	9	17	0.473	0.304	1.523
0.35	0.8378	1	18	0.595	0.431	4.446	0.9602	10	18	0.565	0.336	1.582
0.40	0.6664	1	18	0.595	0.431	4.446	0.9061	1	18	0.770	0.517	4.802
0.45	0.4668	1	18	0.595	0.431	4.446	0.7877	1	18	0.770	0.517	4.802
0.50	0.2863	1	18	0.595	0.431	4.446	0.6188	1	18	0.770	0.517	4.802
0.55	0.1535	1	18	0.595	0.431	4.446	0.4286	1	18	0.770	0.517	4.802
0.60	0.0715	1	18	0.595	0.431	4.446	0.2561	1	18	0.770	0.517	4.802
0.65	0.0286	1	18	0.595	0.431	4.446	0.1286	1	18	0.770	0.517	4.802
0.70	0.0095	1	18	0.595	0.431	4.446	0.0525	1	18	0.770	0.517	4.802
0.75	0.0025	1	18	0.595	0.431	4.446	0.0164	1	18	0.770	0.517	4.802
0.80	0.0005	1	18	0.595	0.431	4.446	0.0036	1	18	0.770	0.517	4.802
0.85	0.0001	1	18	0.595	0.431	4.446	0.0005	1	18	0.770	0.517	4.802
0.90	0.0000	1	18	0.595	0.431	4.446	0.0000	1	18	0.770	0.517	4.802
0.95	0.0000	1	18	0.595	0.431	4.446	0.0000	1	18	0.770	0.517	4.802

(c) Schemes $\mathcal{R}_3 = (0,0,7)$ and $\mathcal{R}_3^s = (0,0,7,\ldots,0,0,7)$

	Multiple Samples ($\mathcal{R}_4$)						Single Sample ($\mathcal{R}_4^s$)					
p	C.P.	i_1	i_2	Exp	Unif	Log	C.P.	i_1	i_2	Exp	Unif	Log
0.05	0.9516	1	8	0.169	0.151	3.006	0.9505	1	8	0.155	0.140	2.924
0.10	0.9720	2	10	0.222	0.190	2.328	0.9648	2	10	0.200	0.173	2.227
0.15	0.9525	4	12	0.268	0.215	1.730	0.9483	4	12	0.243	0.198	1.672
0.20	0.9707	5	14	0.360	0.269	1.776	0.9675	5	14	0.337	0.256	1.749
0.25	0.9646	7	16	0.479	0.319	1.701	0.9744	7	16	0.461	0.313	1.717
0.30	0.9738	8	17	0.590	0.360	1.773	0.9658	9	17	0.603	0.357	1.721
0.35	0.9585	10	18	0.769	0.398	1.815	0.9475	10	17	0.570	0.331	1.545
0.40	0.9500	11	18	0.728	0.367	1.639	0.9650	11	18	0.780	0.393	1.775
0.45	0.9322	1	18	1.009	0.605	5.196	0.9350	1	18	1.038	0.614	5.237
0.50	0.8503	1	18	1.009	0.605	5.196	0.8587	1	18	1.038	0.614	5.237
0.55	0.7260	1	18	1.009	0.605	5.196	0.7433	1	18	1.038	0.614	5.237
0.60	0.5702	1	18	1.009	0.605	5.196	0.5969	1	18	1.038	0.614	5.237
0.65	0.4050	1	18	1.009	0.605	5.196	0.4376	1	18	1.038	0.614	5.237
0.70	0.2555	1	18	1.009	0.605	5.196	0.2874	1	18	1.038	0.614	5.237
0.75	0.1394	1	18	1.009	0.605	5.196	0.1645	1	18	1.038	0.614	5.237
0.80	0.0629	1	18	1.009	0.605	5.196	0.0782	1	18	1.038	0.614	5.237
0.85	0.0215	1	18	1.009	0.605	5.196	0.0282	1	18	1.038	0.614	5.237
0.90	0.0045	1	18	1.009	0.605	5.196	0.0063	1	18	1.038	0.614	5.237
0.95	0.0003	1	18	1.009	0.605	5.196	0.0004	1	18	1.038	0.614	5.237

(d) Schemes $\mathcal{R}_4 = (4,0,3)$ and $\mathcal{R}_4^s = (4,0,3,\ldots,4,0,3)$

Table 5.2: Comparison of expected length and coverage probability for multiple samples ($\mathcal{R}$) and a single sample ($\mathcal{R}^s$)

Chapter 6

Pitman Closeness of PCOS to Quantiles

In this chapter we consider the problem of choosing a best estimator of a population quantile ξ_p, among the PCOS from a progressively Type-II censored sample, $X^{\mathcal{R}}_{1:r:n}, \ldots, X^{\mathcal{R}}_{r:r:n}$. We are interested in obtaining the SCP for the various PCOS as it encapsulates all the relevant information in the pairwise comparisons between all PCOS.

6.1 Simultaneous Closeness

We can define the SCP for the ℓ-th PCOS as follows.

Definition 6.1.1 *For a Progressively Type-II censored sample,* $X^{\mathcal{R}}_{1:r:n}, \ldots, X^{\mathcal{R}}_{r:r:n}$ *the*

SCP $\pi_\ell^{\mathcal{R}}(\xi_p)$ *is defined as*

$$\pi_\ell^{\mathcal{R}}(\xi_p) = P\left(|X_{\ell:r:n}^{\mathcal{R}} - \xi_p| < \min_{1\le j\le r, j\ne \ell} |X_{j:r:n}^{\mathcal{R}} - \xi_p|\right).$$

This region can be collapsed into simpler regions. Let us define the PC probability between two PCOS as follows.

Definition 6.1.2 *For a Progressively Type-II censored sample,* $X_{1:r:n}^{\mathcal{R}}, \ldots, X_{r:r:n}^{\mathcal{R}}$ *the pairwise PC probability* $\pi_{\ell_1,\ell_2}^{\mathcal{R}}(\xi_p)$ *where* $\ell_1 \neq \ell_2$*, is defined as*

$$\pi_{\ell_1,\ell_2}^{\mathcal{R}}(\xi_p) = P\left(|X_{\ell_1:r:n}^{\mathcal{R}} - \xi_p| < |X_{\ell_2:r:n}^{\mathcal{R}} - \xi_p|\right).$$

However, as it will be shown later, it is sufficient to consider consecutive order statistics. To obtain the pairwise PC probability of consecutive PCOS we need the joint distribution of successive PCOS. From equation (1.2.2), we can see that this is given by

$$f^{X_{\ell:r:n}^{\mathcal{R}}, X_{\ell+1:r:n}^{\mathcal{R}}}(x_\ell, x_{\ell+1}) = \left[c_\ell \sum_{i=1}^{\ell} a_i(\ell)\{1 - F(x_\ell)\}^{\gamma_i - \gamma_{\ell+1} - 1} f(x_\ell)\right] f(x_{\ell+1})\{1 - F(x_{\ell+1})\}^{\gamma_{\ell+1}-1}, \tag{6.1.1}$$

when $\xi_0 < x_\ell \le x_{\ell+1} < \xi_1$. We thus have the following Lemma.

Lemma 6.1.3 *Given a Progressively Type-II censored sample from an absolutely continuous distribution with PDF* $f(x)$ *and CDF* $F(x)$*, then for* $\ell = 1, 2, \ldots, r-1$ *and* *any fixed quantile ξ_p $(0 < p < 1)$, the probability that $X^{\mathcal{R}}_{\ell+1:r:n}$ is Pitman closer to ξ_p than $X^{\mathcal{R}}_{\ell:r:n}$ is given by*

$$\pi^{\mathcal{R}}_{\ell+1,\ell}(\xi_p) = F^{X^{\mathcal{R}}_{\ell:r:n}}(\xi_p) - c_{\ell-1}\sum_{i=1}^{\ell} a_i(\ell)\int_0^p (1-u)^{\gamma_i-\gamma_{\ell+1}-1}[1-F(\min[\xi_1, 2\xi_p - \xi_u])]^{\gamma_{\ell+1}}du.$$

Proof: We are interested in the region where $|X^{\mathcal{R}}_{\ell+1:r:n} - \xi_p| < |X^{\mathcal{R}}_{\ell:r:n} - \xi_p|$. Squaring both sides, expanding, and factoring we obtain the following:

$$\begin{aligned}
& |X^{\mathcal{R}}_{\ell+1:r:n} - \xi_p| < |X^{\mathcal{R}}_{\ell:r:n} - \xi_p| \\
\Leftrightarrow\ & (X^{\mathcal{R}}_{\ell+1:r:n})^2 - 2\xi_p X^{\mathcal{R}}_{\ell+1:r:n} + \xi_p^2 < (X^{\mathcal{R}}_{\ell:r:n})^2 - 2\xi_p X^{\mathcal{R}}_{\ell:r:n} + \xi_p^2 \\
\Leftrightarrow\ & (X^{\mathcal{R}}_{\ell+1:r:n})^2 - (X^{\mathcal{R}}_{\ell:r:n})^2 - 2\xi_p X^{\mathcal{R}}_{\ell+1:r:n} + 2\xi_p X^{\mathcal{R}}_{\ell:r:n} < 0 \\
\Leftrightarrow\ & (X^{\mathcal{R}}_{\ell+1:r:n} - X^{\mathcal{R}}_{\ell:r:n})(X^{\mathcal{R}}_{\ell+1:r:n} + X^{\mathcal{R}}_{\ell:r:n} - 2\xi_p) < 0.
\end{aligned}$$

Since the first term is always non-negative, the second term must be negative.

We also note that if $X^{\mathcal{R}}_{\ell:r:n} \geq \xi_p$ then $0 \leq 2X^{\mathcal{R}}_{\ell:r:n} - 2\xi_p \leq X^{\mathcal{R}}_{\ell+1:r:n} + X^{\mathcal{R}}_{\ell:r:n} - 2\xi_p$. Thus we require $X^{\mathcal{R}}_{\ell:r:n} < \xi_p$.

Finally, we can obtain the probability $\pi^{\mathcal{R}}_{\ell+1,\ell}(\xi_p)$ by integrating the joint density as in equation (6.1.1) over the regions $X^{\mathcal{R}}_{\ell+1:r:n} + X^{\mathcal{R}}_{\ell:r:n} - 2\xi_p < 0$ and $X^{\mathcal{R}}_{\ell:r:n} < \xi_p$.

$$\begin{aligned}
{\pi^{\mathcal{R}}}_{\ell+1,\ell}(\xi_p) &= \int_{\xi_0}^{\xi_p}\int_x^{\min(\xi_1, 2\xi_p - x)} f^{X^{\mathcal{R}}_{\ell:r:n}, X^{\mathcal{R}}_{\ell+1:r:n}}(x,y)dy\, dx \\
&= c_\ell \sum_{i=1}^{\ell} a_i(\ell)\int_{\xi_0}^{\xi_p}\{1-F(x)\}^{\gamma_i-\gamma_{\ell+1}-1}f(x)\left[\int_x^{\min(\xi_1, 2\xi_p - x)} f(y)\{1-F(y)\}^{\gamma_{\ell+1}-1}dy\right]dx
\end{aligned}$$

$$= c_\ell \sum_{i=1}^{\ell} a_i(\ell) \int_{\xi_0}^{\xi_p} \{1-F(x)\}^{\gamma_i-\gamma_{\ell+1}-1} f(x) \left[\frac{-1}{\gamma_{\ell+1}} \{1-F(y)\}^{\gamma_{\ell+1}} \Big|_{y=x}^{\min(\xi_1, 2\xi_p - x)} \right] dx$$

$$= c_{\ell-1} \sum_{i=1}^{\ell} a_i(\ell) \int_{\xi_0}^{\xi_p} \{1-F(x)\}^{\gamma_i-1} f(x)\, dx$$

$$- c_{\ell-1} \sum_{i=1}^{\ell} a_i(\ell) \int_{\xi_0}^{\xi_p} \{1-F(x)\}^{\gamma_i-\gamma_{\ell+1}-1} [1-F(\min[\xi_1, 2\xi_p - x])]^{\gamma_{\ell+1}} f(x) dx$$

It is clear from equation (1.2.3), that the first term in the last expression is $F^{X^{\mathcal{R}}_{\ell:r:n}}(\xi_p)$. Upon making the transformation $u = F(x)$ in the second expression, we then obtain the Lemma. ∎

It can be noted that the while the upper bound of the distribution ξ_1 appears in Lemma 6.1.3, the lower bound ξ_0 does not. Had we obtained $\pi^{\mathcal{R}}_{\ell,\ell+1}(\xi_p)$ in a similar manner, the reverse would be true.

Theorem 6.1.4 *Given a Progressively Type-II censored sample from an absolutely continuous distribution with PDF $f(x)$ and CDF $F(x)$, then for $\ell = 1, 2, \ldots, r$ and any fixed quantile ξ_p $(0 < p < 1)$, the probability that $X^{\mathcal{R}}_{\ell+1:r:n}$ is simultaneously Pitman closer to ξ_p than all other PCOS, is given by*

$$\pi^{\mathcal{R}}_\ell(\xi_p) = \begin{cases} 1 - \pi^{\mathcal{R}}_{2,1}(\xi_p) & \text{if } \ell = 1 \\ \pi^{\mathcal{R}}_{\ell,\ell-1}(\xi_p) - \pi^{\mathcal{R}}_{\ell+1,\ell}(\xi_p) & \text{if } \ell = 2, \ldots, r-1 \\ \pi^{\mathcal{R}}_{r,r-1}(\xi_p) & \text{if } \ell = r \end{cases} .$$

Proof: If $\ell = 1$ then $\pi^{\mathcal{R}}_1(\xi_p) = P(|X^{\mathcal{R}}_{1:r:n} - \xi_p| < \min_{j>1} |X^{\mathcal{R}}_{j:r:n} - \xi_p|) = P(|X^{\mathcal{R}}_{1:r:n} - \xi_p| < |X^{\mathcal{R}}_{2:r:n} - \xi_p|) = \pi^{\mathcal{R}}_{1,2}(\xi_p) = 1 - \pi^{\mathcal{R}}_{2,1}(\xi_p)$. Similarly for $\ell = r$, $\pi^{\mathcal{R}}_r(\xi_p) = P(|X^{\mathcal{R}}_{r:r:n} - \xi_p| <$

$\min_{j<r} |X^{\mathcal{R}}_{j:r:n} - \xi_p|) = P(|X^{\mathcal{R}}_{r:r:n} - \xi_p| < |X^{\mathcal{R}}_{r-1:r:n} - \xi_p|) = \pi^{\mathcal{R}}_{r,r-1}(\xi_p)$.

Consider now $\ell = 2, \ldots, r-1$. Then we have that

$$
\begin{aligned}
\pi^{\mathcal{R}}_{\ell}(\xi_p) &= P(|X^{\mathcal{R}}_{\ell:r:n} - \xi_p| < \min_{1\le j\le r, j\ne \ell} |X^{\mathcal{R}}_{j:r:n} - \xi_p|) \\
&= P(|X^{\mathcal{R}}_{\ell:r:n} - \xi_p| < \min_{j=\ell-1,\ell+1} |X^{\mathcal{R}}_{j:r:n} - \xi_p|) \\
&= P\left(\{|X^{\mathcal{R}}_{\ell:r:n} - \xi_p| < |X^{\mathcal{R}}_{\ell-1:r:n} - \xi_p|\} \bigcap \{|X^{\mathcal{R}}_{\ell:r:n} - \xi_p| < |X^{\mathcal{R}}_{\ell+1:r:n} - \xi_p|\}\right) \\
&= P\left(\{|X^{\mathcal{R}}_{\ell:r:n} - \xi_p| < |X^{\mathcal{R}}_{\ell-1:r:n} - \xi_p|\} \bigcap \{|X^{\mathcal{R}}_{\ell+1:r:n} - \xi_p| < |X^{\mathcal{R}}_{\ell:r:n} - \xi_p|\}^{C}\right) \\
&= P\left(|X^{\mathcal{R}}_{\ell:r:n} - \xi_p| < |X^{\mathcal{R}}_{\ell-1:r:n} - \xi_p|\right) \\
&\quad - P\left(\{|X^{\mathcal{R}}_{\ell:r:n} - \xi_p| < |X^{\mathcal{R}}_{\ell-1:r:n} - \xi_p|\} \bigcap \{|X^{\mathcal{R}}_{\ell+1:r:n} - \xi_p| < |X^{\mathcal{R}}_{\ell:r:n} - \xi_p|\}\right) \\
&= \pi^{\mathcal{R}}_{\ell,\ell-1}(\xi_p) \\
&\quad - P\left(X^{\mathcal{R}}_{\ell-1:r:n} < X^{\mathcal{R}}_{\ell:r:n} < \xi_p \le X^{\mathcal{R}}_{\ell+1:r:n}, X^{\mathcal{R}}_{\ell+1:r:n} \text{ is closer to } \xi_p \text{ than } X^{\mathcal{R}}_{\ell-1:r:n}\right) \\
&= \pi^{\mathcal{R}}_{\ell,\ell-1}(\xi_p) - \pi^{\mathcal{R}}_{\ell+1,\ell}(\xi_p)
\end{aligned}
$$

Where E^C denotes the complement of the event E. ∎

Corollary 6.1.5 *For any symmetric distribution F, for the population median $\xi_{0.5}$, the pairwise PC probabilities in Lemma 6.1.3 and SCPs in Theorem 6.1.4 are distribution free. The expression in the lemma reduces to*

$$
\pi^{\mathcal{R}}_{\ell+1,\ell}(\xi_{0.5}) = F^{X^{\mathcal{R}}_{\ell:r:n}}(\xi_{0.5}) - c_{\ell-1} \sum_{i=1}^{\ell} a_i(\ell) B(0.5; \gamma_{\ell+1} + 1, \gamma_i - \gamma_{\ell+1}),
$$

where $B(x;\alpha,\beta) = \int_0^x u^{\alpha-1}(1-u)^{\beta-1} du$ is the incomplete beta function.

Proof: Firstly we note that for all continuous distributions F, and all $0 < p < 1$, $F^{X^{\mathcal{R}}_{\ell:r:n}}(\xi_p) = G^{\mathcal{R}}_{\ell:r:n}(p)$ where $G^{\mathcal{R}}_{\ell:r:n}$ is the uniform CDF under the same censoring scheme. Thus this probability is always distribution free.

Since F is symmetric, then for all $0 < u < 1$, $2\xi_{0.5} - \xi_u = \xi_{1-u}$. Considering the integral we have

$$\begin{aligned}
&\int_0^{0.5} (1-u)^{\gamma_i - \gamma_{\ell+1} - 1}[1 - F(\min[\xi_1, 2\xi_{0.5} - \xi_u])]^{\gamma_{\ell+1}} du \\
= &\int_0^{0.5} (1-u)^{\gamma_i - \gamma_{\ell+1} - 1}[1 - F(\min[\xi_1, \xi_{1-u}])]^{\gamma_{\ell+1}} du = \int_0^{0.5} u^{\gamma_{\ell+1}}(1-u)^{\gamma_i - \gamma_{\ell+1} - 1},
\end{aligned}$$

as desired. ∎

If we consider extreme quantiles, namely as $p \to 0$, or $p \to 1$, we obtain the following result.

Corollary 6.1.6 *For all censoring schemes $\mathcal{R}$ we have*

$$\pi_\ell^{\mathcal{R}}(\xi_p) \xrightarrow{p\to 0} \begin{cases} 1 & \text{if } \ell = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \pi_\ell^{\mathcal{R}}(\xi_p) \xrightarrow{p\to 1} \begin{cases} 1 & \text{if } \ell = r \\ 0 & \text{otherwise} \end{cases}$$

Proof: For the first case, we consider $\lim_{p\to 0} \pi_{\ell+1,\ell}$. Clearly $F^{X^{\mathcal{R}}_{\ell:r:n}}(\xi_p) \to 0$ as $p \to 0$ for all $\ell = 1, \ldots, r$. The integrand in Lemma 6.1.3 is non-negative and bounded by 1, so as $p \to 0$, the integral must go to 0. So $\lim_{p\to 0} \pi_{\ell+1,\ell} \to 0$ as $p \to 0$.

For the second case we again consider $\lim_{p\to 1} \pi_{\ell+1,\ell}$. Again, $F^{X^{\mathcal{R}}_{\ell:r:n}}(\xi_p) \to 1$ as $p \to 1$ for all $\ell = 1, \ldots, r$. For the integrand in Lemma 6.1.3, we can see that for fixed

$u \in (0,1)$, $\lim_{p\to 1}(1-u)^{\gamma_i-\gamma_{\ell+1}-1}[1-F(\min[\xi_1, 2\xi_p-\xi_u])]^{\gamma_{\ell+1}} = 0$ as $F(\min[\xi_1, 2\xi_p - \xi_u]) \geq F(2\xi_p - \xi_u) \geq F(\xi_p) = p$ when $p \geq u$. So we have pointwise convergence of the integrand to 0 (though not necessarily uniformly) as $p \to 1$ for $0 < u < 1$. Since the integrand is uniformly bounded for all $u \in (0,1)$ we may exchange the outer limit and integral so that the term goes to 0. So $\lim_{p\to 1} \pi_{\ell+1,\ell} \to 1$ as $p \to 1$.

The corollary then follows from application of Theorem 6.1.4. ∎

An alternative proof of Corollary 6.1.6 could be given by observing the definition of $\pi_\ell^{\mathcal{R}}(\xi_p)$ directly. That is, $\pi_{\ell_1,\ell_2}^{\mathcal{R}}$ converges to $P(X_{\ell_1:r:n}^{\mathcal{R}} < X_{\ell_2:r:n}^{\mathcal{R}})[P(X_{\ell_1:r:n}^{\mathcal{R}} > X_{\ell_2:r:n}^{\mathcal{R}})]$ when $p \to 0[1]$.

Proposition 6.1.7 *For a location-scale family with location μ and scale σ, the SCP is free of μ and σ.*

Proof: Let $Z = \frac{X-\mu}{\sigma}$ be the standardized distribution. Consider the pairwise PC probabilities $\pi_{\ell_1,\ell_2}^{\mathcal{R}}$, we have for $\ell_1 \neq \ell_2$

$$\begin{aligned}
\pi_{X_{\ell_1:r:n}^{\mathcal{R}}, X_{\ell_2:r:n}^{\mathcal{R}}}(\xi_p) &= P\left(|X_{\ell_1:r:n}^{\mathcal{R}} - \xi_p| < |X_{\ell_2:r:n}^{\mathcal{R}} - \xi_p|\right) \\
&= P\left(\left|\frac{X_{\ell_1:r:n}^{\mathcal{R}} - \mu}{\sigma} - \frac{\xi_p - \mu}{\sigma}\right| < \left|\frac{X_{\ell_2:r:n}^{\mathcal{R}} - \mu}{\sigma} - \frac{\xi_p - \mu}{\sigma}\right|\right) \\
&= P\left(|Z_{\ell_1:r:n}^{\mathcal{R}} - \xi_p^Z| < |Z_{\ell_2:r:n}^{\mathcal{R}} - \xi_p^Z|\right) = \pi_{Z_{\ell_1:r:n}^{\mathcal{R}}, Z_{\ell_2:r:n}^{\mathcal{R}}}(\xi_p^Z),
\end{aligned}$$

where ξ_p^Z is the quantile function of the standard distribution Z which does not depend on μ or σ. Since the pairwise PC probabilities of Lemma 6.1.3 do not depend on μ or σ, neither do the SCPs of Theorem 6.1.4. ∎

Thus when we consider special cases of location-scale families, it suffices to consider the standard distribution.

6.2 Special Cases

In this section we obtain simple explicit results for the standard uniform$(0, 1)$ and standard exponential distributions. For brevity we present expressions for the pairwise PC probabilities $\pi^{\mathcal{R}}_{\ell+1,\ell}(\xi_p)$ rather than the SCPs $\pi^{\mathcal{R}}_{\ell}(\xi_p)$. One thing that should be noted, is that due to the alternate representation of PCOS, the representations here are much simpler than those in Balakrishnan et al. (2010c). That is, there is no recursion necessary in the calculations. However, the representations may not be as stable numerically, so use for ordinary order statistics of large sizes may be less than ideal.

6.2.1 Exponential Distribution

For the standard exponential distribution we have the quantile function $\xi_p = -\log(1-p)$ for $0 \leq p < 1$, CDF $F(x) = 1 - e^{-x}$ for $0 \leq x$, and lower/upper bounds as $\xi_0 = 0/\xi_1 = \infty$. It follows that $1 - F(2\xi_p - \xi_u) = (1-p)^2/(1-u)$ for $0 \leq u < p$.

We first evaluate the integral of Lemma 6.1.3, i.e.,

$$\int_0^p (1-u)^{\gamma_i - \gamma_{\ell+1} - 1} \left[\frac{(1-p)^2}{1-u}\right]^{\gamma_{\ell+1}} du = (1-p)^{2\gamma_{\ell+1}} \int_0^p (1-u)^{\gamma_i - 2\gamma_{\ell+1} - 1} du,$$

so that

$$\pi^{\mathcal{R}}_{\ell+1,\ell}(\xi_p) = F^{X^{\mathcal{R}}_{\ell:r:n}}(\xi_p) - c_{\ell-1}(1-2)^{2\gamma_{\ell+1}} \sum_{i=1}^{\ell} a_i(\ell) K(p;\gamma_i,\gamma_{\ell+1}), \tag{6.2.1}$$

where

$$K(p;\gamma_i,\gamma_{\ell+1}) = \begin{cases} \frac{1}{\gamma_i-2\gamma_{\ell+1}}[1-(1-p)^{\gamma_i-2\gamma_{\ell+1}}] & \text{if } \gamma_i - 2\gamma_{\ell+1} \neq 0 \\ -\ln(1-p) & \text{if } \gamma_i - 2\gamma_{\ell+1} = 0 \end{cases}.$$

6.2.2 Uniform Distribution

For the uniform distribution we have the quantile function $\xi_p = p$, CDF $F(x) = x$ for $0 \leq x \leq 1$, and lower/upper bounds as $\xi_0 = 0/\xi_1 = 1$. Thus for $0 \leq u \leq p$, $F(\min[1, 2\xi_p - \xi_u]) = F(\min[1, 2p - u])$. Again we need to evaluate the integral as in Lemma 6.1.3.

We consider the two cases $0 < p < 0.5$ and $0.5 \leq p < 1$ separately. For the former, $\min[1, 2p-u] = 2p-u$ so with the substitution $v = 1-(1-u)/((2(1-p))$ we get,

$$\int_0^p (1-u)^{\gamma_i-\gamma_{\ell+1}-1}\{1-(2p-u)\}^{\gamma_{\ell+1}}du = [2(1-p)]^{\gamma_i}\int_{1-\frac{1}{2(1-p)}}^{1/2}(1-v)^{\gamma_i-\gamma_{\ell+1}-1}v^{\gamma_{\ell+1}}dv$$
$$= [2(1-p)]^{\gamma_i}\left\{B\left(\frac{1}{2};\gamma_{\ell+1}+1,\gamma_i-\gamma_{\ell+1}\right) - B\left(1-\frac{1}{2(1-p)};\gamma_{\ell+1}+1,\gamma_i-\gamma_{\ell+1}\right)\right\}.$$

For the case $0.5 \leq p < 1$ we have $\min[1, 2p-u]$ is 1 if $u \leq 2p-1$ and $2p-u$ if

$u > 2p - 1$. So with the same substitution as above we get,

$$\int_{2p-1}^{p} (1-u)^{\gamma_i - \gamma_{\ell+1} - 1} \{1 - (2p - u)\}^{\gamma_{\ell+1}} du$$
$$= [2(1-p)]^{\gamma_i} \int_0^{1/2} (1-v)^{\gamma_i - \gamma_{\ell+1} - 1} v^{\gamma_{\ell+1}} dv = [2(1-p)]^{\gamma_i} B\left(\frac{1}{2}; \gamma_{\ell+1} + 1, \gamma_i - \gamma_{\ell+1}\right).$$

Combining the two cases yields the following expression for the pairwise PC probability as

$$\pi^{\mathcal{R}}_{\ell+1,\ell}(\xi_p) = F^{X^{\mathcal{R}}_{\ell:r:n}}(\xi_p) - c_{\ell-1}(1-2)^{2\gamma_{\ell+1}} \sum_{i=1}^{\ell} a_i(\ell)[2(1-p)]^{\gamma_i} \tag{6.2.2}$$
$$\times \left\{ B\left(\frac{1}{2}; \gamma_{\ell+1} + 1, \gamma_i - \gamma_{\ell+1}\right) - B\left(\max\left(0, 1 - \frac{1}{2(1-p)}\right); \gamma_{\ell+1} + 1, \gamma_i - \gamma_{\ell+1}\right)\right\}.$$

6.2.3 Other Distributions

For most distributions Lemma 6.1.3 will not yield elementary functions. We consider the standard cauchy, normal, and skew normal distributions, all of which do not yield simple explicit representations. As a result we obtain the PC probabilities in Section 6.3 for these distributions with numerical integration.

6.3 Numerical Illustration

We consider a numerical example to illustrate the methods presented in this section. The SCPs $\pi^{\mathcal{R}}_{\ell}(\xi_p)$ for the standard exponential, uniform, normal, cauchy, and skew normal distributions are presented in Tables 6.1-6.6 respectively for the following censoring schemes,

$$\mathcal{R}_1 = (20, 0, 0, 0, 0, 0, 0, 0, 0, 0), \qquad \mathcal{R}_2 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 20),$$

$$\mathcal{R}_3 = (2, 2, 2, 2, 2, 2, 2, 2, 2, 2), \qquad \mathcal{R}_4 = (5, 5, 0, 0, 0, 0, 0, 0, 5, 5),$$

$$\mathcal{R}_5 = (0, 0, 0, 0, 20, 0, 0, 0, 0, 0), \qquad \mathcal{R}_6 = (0, 0, 0, 0, 10, 10, 0, 0, 0, 0),$$

$$\mathcal{R}_7 = (4, 4, 4, 4, 4, 0, 0, 0, 0, 0), \qquad \mathcal{R}_8 = (0, 0, 0, 0, 0, 4, 4, 4, 4, 4).$$

In the above schemes we have $n = 30$ and $m = 10$. For the skew normal distribution we take the shape parameter α to be 1 and -1.

We can notice that across the distributions the probabilities for the central quantiles do not vary significantly, and in fact many vary by less than 0.001. Thus, when choosing an optimal order statistic for a given scheme $\mathcal{R}$, the different distributions will typically choose the same order statistic except for some small regions of p where the preference changes from the ℓ-th to $\ell + 1$-th PCOS. However, there are some notably large differences at the extremes, both for small/large p, and particularly for the largest and smallest order statistics. The most notable of these are $X^{\mathcal{R}_1}_{10:10:30}$ for $p = 0.95$ which varies from 0.7250 for the cauchy distribution to 0.8657 for the uniform distribution, and $X^{\mathcal{R}_1}_{1:10:30}$ for $p = 0.05$ which varies from 0.3554 for the cauchy distribution to 0.6711 for the exponential distribution.

We can notice that for schemes with right censoring $(\mathcal{R}_2, \mathcal{R}_4, \mathcal{R}_8)$, then $X^{\mathcal{R}}_{10:10:30}$

is closest to most upper quantiles. However, it may be closest by virtue of being the largest item much lower than the quantile. So to get a better idea of the spread of this item one could modify the scheme as follows: $\mathcal{R} = (R_1, \ldots, R_{r-1}, R_r) \to \mathcal{R}^* = (R_1, \ldots, R_{r-1}, 1, R_r - 1)$. Namely, we imagine that we had observed the next failure which would give a better idea of which quantiles $X^{\mathcal{R}}_{r:r:n}$ is closest to in the Pitman sense.

On a similar note however, schemes with right censoring appear to be more robust to the difference in distribution. In $\mathcal{R}_2$, $\mathcal{R}_4$, and $\mathcal{R}_8$ there is little variation between the distributions. This is most notable for right censoring. However, due to asymptotic nature of order statistics, this is an unsurprising fact.

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.6711	0.2794	0.0447	0.0045	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
0.10	0.3116	0.4636	0.1780	0.0402	0.0059	0.0006	0.0000	0.0000	0.0000	0.0000
0.15	0.1227	0.4328	0.2974	0.1142	0.0279	0.0046	0.0005	0.0000	0.0000	0.0000
0.20	0.0432	0.3181	0.3458	0.2002	0.0724	0.0173	0.0027	0.0003	0.0000	0.0000
0.25	0.0138	0.2038	0.3267	0.2670	0.1339	0.0439	0.0095	0.0013	0.0001	0.0000
0.30	0.0040	0.1186	0.2674	0.2963	0.1983	0.0860	0.0245	0.0044	0.0005	0.0000
0.35	0.0011	0.0638	0.1957	0.2860	0.2493	0.1396	0.0511	0.0119	0.0016	0.0001
0.40	0.0003	0.0319	0.1299	0.2459	0.2745	0.1956	0.0906	0.0266	0.0045	0.0003
0.45	0.0001	0.0148	0.0786	0.1903	0.2694	0.2421	0.1409	0.0518	0.0110	0.0010
0.50	0.0000	0.0063	0.0433	0.1331	0.2376	0.2682	0.1954	0.0896	0.0236	0.0027
0.55	0.0000	0.0025	0.0215	0.0838	0.1885	0.2671	0.2439	0.1399	0.0461	0.0067
0.60	0.0000	0.0009	0.0095	0.0471	0.1337	0.2389	0.2747	0.1982	0.0821	0.0149
0.65	0.0000	0.0003	0.0037	0.0231	0.0836	0.1902	0.2782	0.2554	0.1345	0.0311
0.70	0.0000	0.0001	0.0012	0.0096	0.0450	0.1323	0.2503	0.2975	0.2031	0.0610
0.75	0.0000	0.0000	0.0003	0.0032	0.0200	0.0778	0.1953	0.3088	0.2814	0.1131
0.80	0.0000	0.0000	0.0001	0.0008	0.0068	0.0365	0.1263	0.2772	0.3528	0.1995
0.85	0.0000	0.0000	0.0000	0.0001	0.0016	0.0121	0.0619	0.2021	0.3874	0.3347
0.90	0.0000	0.0000	0.0000	0.0000	0.0002	0.0022	0.0187	0.1033	0.3434	0.5323
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0018	0.0227	0.1867	0.7887

(a) Scheme $\mathcal{R}_1$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.3899	0.3317	0.1830	0.0698	0.0200	0.0046	0.0008	0.0001	0.0000	0.0000
0.10	0.0854	0.2009	0.2546	0.2163	0.1363	0.0673	0.0269	0.0090	0.0025	0.0008
0.15	0.0157	0.0664	0.1418	0.1995	0.2059	0.1654	0.1072	0.0575	0.0260	0.0146
0.20	0.0026	0.0164	0.0518	0.1065	0.1595	0.1853	0.1732	0.1336	0.0866	0.0846
0.25	0.0004	0.0033	0.0143	0.0401	0.0817	0.1287	0.1628	0.1696	0.1484	0.2507
0.30	0.0000	0.0005	0.0031	0.0115	0.0307	0.0630	0.1038	0.1407	0.1599	0.4867
0.35	0.0000	0.0001	0.0006	0.0026	0.0088	0.0231	0.0482	0.0830	0.1197	0.7139
0.40	0.0000	0.0000	0.0001	0.0005	0.0020	0.0065	0.0169	0.0363	0.0654	0.8723
0.45	0.0000	0.0000	0.0000	0.0001	0.0003	0.0014	0.0045	0.0120	0.0267	0.9550
0.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0009	0.0030	0.0081	0.9877
0.55	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0005	0.0018	0.9975
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0003	0.9996
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

(b) Scheme $\mathcal{R}_2$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.4046	0.3489	0.1768	0.0559	0.0118	0.0017	0.0002	0.0000	0.0000	0.0000
0.10	0.0912	0.2342	0.2940	0.2227	0.1108	0.0373	0.0085	0.0012	0.0001	0.0000
0.15	0.0170	0.0848	0.1944	0.2637	0.2323	0.1373	0.0544	0.0139	0.0021	0.0001
0.20	0.0028	0.0228	0.0841	0.1817	0.2530	0.2355	0.1464	0.0586	0.0137	0.0014
0.25	0.0004	0.0050	0.0276	0.0892	0.1854	0.2570	0.2376	0.1413	0.0490	0.0076
0.30	0.0001	0.0009	0.0073	0.0339	0.1019	0.2041	0.2729	0.2347	0.1179	0.0264
0.35	0.0000	0.0001	0.0016	0.0104	0.0442	0.1260	0.2403	0.2957	0.2132	0.0686
0.40	0.0000	0.0000	0.0003	0.0026	0.0155	0.0625	0.1693	0.2978	0.3085	0.1435
0.45	0.0000	0.0000	0.0000	0.0005	0.0044	0.0252	0.0976	0.2469	0.3715	0.2538
0.50	0.0000	0.0000	0.0000	0.0001	0.0010	0.0083	0.0463	0.1710	0.3809	0.3923
0.55	0.0000	0.0000	0.0000	0.0000	0.0002	0.0022	0.0180	0.0993	0.3367	0.5435
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0005	0.0057	0.0480	0.2576	0.6883
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0014	0.0189	0.1695	0.8101
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0059	0.0945	0.8994
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0013	0.0431	0.9556
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0151	0.9847
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0036	0.9964
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004	0.9996
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

(c) Scheme $\mathcal{R}_3$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.4303	0.3760	0.1494	0.0367	0.0066	0.0009	0.0001	0.0000	0.0000	0.0000
0.10	0.1024	0.3020	0.3158	0.1796	0.0721	0.0218	0.0052	0.0010	0.0002	0.0000
0.15	0.0197	0.1295	0.2675	0.2650	0.1793	0.0899	0.0350	0.0108	0.0029	0.0004
0.20	0.0033	0.0411	0.1505	0.2315	0.2348	0.1737	0.0987	0.0443	0.0181	0.0041
0.25	0.0005	0.0106	0.0655	0.1470	0.2100	0.2155	0.1681	0.1031	0.0594	0.0203
0.30	0.0001	0.0023	0.0235	0.0740	0.1432	0.1961	0.2025	0.1635	0.1288	0.0660
0.35	0.0000	0.0004	0.0072	0.0307	0.0785	0.1400	0.1868	0.1938	0.2048	0.1577
0.40	0.0000	0.0001	0.0019	0.0107	0.0355	0.0811	0.1375	0.1806	0.2533	0.2994
0.45	0.0000	0.0000	0.0004	0.0031	0.0133	0.0386	0.0824	0.1357	0.2516	0.4749
0.50	0.0000	0.0000	0.0001	0.0008	0.0041	0.0151	0.0404	0.0829	0.2039	0.6528
0.55	0.0000	0.0000	0.0000	0.0002	0.0010	0.0048	0.0160	0.0411	0.1349	0.8020
0.60	0.0000	0.0000	0.0000	0.0000	0.0002	0.0012	0.0051	0.0162	0.0720	0.9052
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0012	0.0050	0.0303	0.9633
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0011	0.0096	0.9891
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0021	0.9977
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.9997
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

(d) Scheme $\mathcal{R}_4$

Table 6.1: SCP for exponential distribution $[\pi_{\ell:r:n}(\xi_p)]$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.3899	0.3317	0.1830	0.0698	0.0241	0.0014	0.0000	0.0000	0.0000	0.0000
0.10	0.0854	0.2009	0.2546	0.2163	0.2056	0.0343	0.0027	0.0001	0.0000	0.0000
0.15	0.0157	0.0664	0.1418	0.1995	0.4090	0.1452	0.0206	0.0018	0.0001	0.0000
0.20	0.0026	0.0164	0.0518	0.1065	0.4435	0.3006	0.0689	0.0091	0.0007	0.0000
0.25	0.0004	0.0033	0.0143	0.0401	0.3427	0.4208	0.1472	0.0283	0.0029	0.0001
0.30	0.0000	0.0005	0.0031	0.0115	0.2129	0.4610	0.2374	0.0640	0.0090	0.0005
0.35	0.0000	0.0001	0.0006	0.0026	0.1134	0.4274	0.3160	0.1164	0.0219	0.0017
0.40	0.0000	0.0000	0.0001	0.0005	0.0537	0.3507	0.3651	0.1808	0.0447	0.0045
0.45	0.0000	0.0000	0.0000	0.0001	0.0230	0.2614	0.3774	0.2480	0.0798	0.0103
0.50	0.0000	0.0000	0.0000	0.0000	0.0089	0.1793	0.3549	0.3075	0.1282	0.0211
0.55	0.0000	0.0000	0.0000	0.0000	0.0031	0.1137	0.3063	0.3489	0.1882	0.0398
0.60	0.0000	0.0000	0.0000	0.0000	0.0010	0.0664	0.2427	0.3645	0.2558	0.0697
0.65	0.0000	0.0000	0.0000	0.0000	0.0003	0.0353	0.1756	0.3505	0.3233	0.1151
0.70	0.0000	0.0000	0.0000	0.0000	0.0001	0.0167	0.1144	0.3080	0.3803	0.1804
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0068	0.0654	0.2435	0.4140	0.2703
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0023	0.0313	0.1675	0.4107	0.3883
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0005	0.0114	0.0939	0.3585	0.5356
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0026	0.0369	0.2531	0.7074
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0062	0.1088	0.8848

(e) Scheme $\mathcal{R}_5$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.3899	0.3317	0.1830	0.0698	0.0218	0.0037	0.0001	0.0000	0.0000	0.0000
0.10	0.0854	0.2009	0.2546	0.2163	0.1609	0.0745	0.0070	0.0004	0.0000	0.0000
0.15	0.0157	0.0664	0.1418	0.1995	0.2634	0.2600	0.0487	0.0043	0.0002	0.0000
0.20	0.0026	0.0164	0.0518	0.1065	0.2207	0.4332	0.1467	0.0205	0.0015	0.0000
0.25	0.0004	0.0033	0.0143	0.0401	0.1219	0.4761	0.2788	0.0586	0.0064	0.0003
0.30	0.0000	0.0005	0.0031	0.0115	0.0492	0.3994	0.3959	0.1208	0.0183	0.0011
0.35	0.0000	0.0001	0.0006	0.0026	0.0152	0.2772	0.4599	0.1997	0.0414	0.0034
0.40	0.0000	0.0000	0.0001	0.0005	0.0036	0.1670	0.4612	0.2810	0.0782	0.0085
0.45	0.0000	0.0000	0.0000	0.0001	0.0007	0.0897	0.4126	0.3494	0.1294	0.0182
0.50	0.0000	0.0000	0.0000	0.0000	0.0001	0.0436	0.3360	0.3927	0.1925	0.0352
0.55	0.0000	0.0000	0.0000	0.0000	0.0000	0.0192	0.2515	0.4046	0.2623	0.0624
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0075	0.1736	0.3846	0.3313	0.1030
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0026	0.1098	0.3374	0.3899	0.1603
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0008	0.0629	0.2711	0.4278	0.2375
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0317	0.1962	0.4351	0.3368
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0134	0.1238	0.4037	0.4590
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0044	0.0637	0.3300	0.6020
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0009	0.0229	0.2180	0.7582
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0035	0.0874	0.9090

(f) Scheme $\mathcal{R}_6$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.4212	0.3668	0.1665	0.0402	0.0050	0.0002	0.0000	0.0000	0.0000	0.0000
0.10	0.0982	0.2765	0.3375	0.2117	0.0674	0.0081	0.0006	0.0000	0.0000	0.0000
0.15	0.0186	0.1114	0.2728	0.3366	0.2087	0.0458	0.0057	0.0004	0.0000	0.0000
0.20	0.0031	0.0333	0.1451	0.3174	0.3481	0.1261	0.0240	0.0028	0.0002	0.0000
0.25	0.0004	0.0081	0.0591	0.2183	0.4072	0.2317	0.0637	0.0104	0.0010	0.0000
0.30	0.0001	0.0016	0.0196	0.1195	0.3748	0.3270	0.1258	0.0281	0.0035	0.0002
0.35	0.0000	0.0003	0.0054	0.0542	0.2877	0.3809	0.2011	0.0600	0.0097	0.0007
0.40	0.0000	0.0000	0.0013	0.0208	0.1901	0.3820	0.2738	0.1076	0.0225	0.0020
0.45	0.0000	0.0000	0.0002	0.0067	0.1098	0.3379	0.3270	0.1680	0.0453	0.0051
0.50	0.0000	0.0000	0.0000	0.0018	0.0557	0.2672	0.3489	0.2338	0.0810	0.0116
0.55	0.0000	0.0000	0.0000	0.0004	0.0247	0.1901	0.3358	0.2941	0.1311	0.0239
0.60	0.0000	0.0000	0.0000	0.0001	0.0094	0.1215	0.2924	0.3367	0.1944	0.0455
0.65	0.0000	0.0000	0.0000	0.0000	0.0030	0.0692	0.2294	0.3513	0.2660	0.0810
0.70	0.0000	0.0000	0.0000	0.0000	0.0008	0.0345	0.1602	0.3322	0.3363	0.1361
0.75	0.0000	0.0000	0.0000	0.0000	0.0002	0.0146	0.0971	0.2802	0.3908	0.2170
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0050	0.0489	0.2045	0.4115	0.3301
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0012	0.0186	0.1210	0.3796	0.4796
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0044	0.0499	0.2821	0.6635
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0087	0.1276	0.8634

(g) Scheme $\mathcal{R}_7$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.3899	0.3317	0.1830	0.0698	0.0200	0.0047	0.0008	0.0001	0.0000	0.0000
0.10	0.0854	0.2009	0.2546	0.2163	0.1363	0.0709	0.0273	0.0070	0.0011	0.0001
0.15	0.0157	0.0664	0.1418	0.1995	0.2059	0.1782	0.1201	0.0551	0.0154	0.0020
0.20	0.0026	0.0164	0.0518	0.1065	0.1595	0.2038	0.2141	0.1580	0.0721	0.0152
0.25	0.0004	0.0033	0.0143	0.0401	0.0817	0.1442	0.2228	0.2515	0.1805	0.0612
0.30	0.0000	0.0005	0.0031	0.0115	0.0307	0.0717	0.1579	0.2665	0.2969	0.1610
0.35	0.0000	0.0001	0.0006	0.0026	0.0088	0.0266	0.0821	0.2060	0.3578	0.3154
0.40	0.0000	0.0000	0.0001	0.0005	0.0020	0.0076	0.0324	0.1218	0.3363	0.4994
0.45	0.0000	0.0000	0.0000	0.0001	0.0003	0.0016	0.0098	0.0563	0.2557	0.6761
0.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0023	0.0205	0.1604	0.8164
0.55	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004	0.0059	0.0838	0.9099
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0013	0.0363	0.9624
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0128	0.9870
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0036	0.9964
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0007	0.9993
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.9999
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

(h) Scheme $\mathcal{R}_8$

Table 6.1: SCP for exponential distribution $[\pi_{\ell:r:n}(\xi_p)]$

p	ℓ 1	2	3	4	5	6	7	8	9	10
0.05	0.6668	0.2832	0.0452	0.0045	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
0.10	0.2992	0.4722	0.1811	0.0409	0.0060	0.0006	0.0000	0.0000	0.0000	0.0000
0.15	0.1084	0.4384	0.3030	0.1165	0.0285	0.0046	0.0005	0.0000	0.0000	0.0000
0.20	0.0326	0.3171	0.3510	0.2044	0.0741	0.0177	0.0028	0.0003	0.0000	0.0000
0.25	0.0080	0.1983	0.3285	0.2718	0.1371	0.0451	0.0098	0.0013	0.0001	0.0000
0.30	0.0016	0.1122	0.2652	0.2996	0.2027	0.0884	0.0252	0.0046	0.0005	0.0000
0.35	0.0002	0.0587	0.1908	0.2864	0.2535	0.1434	0.0528	0.0123	0.0017	0.0001
0.40	0.0000	0.0287	0.1244	0.2430	0.2769	0.2003	0.0938	0.0277	0.0047	0.0004
0.45	0.0000	0.0132	0.0740	0.1853	0.2688	0.2463	0.1456	0.0542	0.0116	0.0011
0.50	0.0000	0.0056	0.0401	0.1275	0.2338	0.2701	0.2009	0.0939	0.0252	0.0030
0.55	0.0000	0.0022	0.0197	0.0789	0.1825	0.2654	0.2485	0.1462	0.0494	0.0073
0.60	0.0000	0.0007	0.0086	0.0435	0.1270	0.2333	0.2762	0.2057	0.0882	0.0167
0.65	0.0000	0.0002	0.0033	0.0210	0.0779	0.1819	0.2744	0.2616	0.1443	0.0354
0.70	0.0000	0.0001	0.0010	0.0086	0.0411	0.1235	0.2408	0.2982	0.2158	0.0710
0.75	0.0000	0.0000	0.0003	0.0029	0.0179	0.0706	0.1819	0.2998	0.2922	0.1345
0.80	0.0000	0.0000	0.0000	0.0007	0.0060	0.0321	0.1131	0.2569	0.3506	0.2407
0.85	0.0000	0.0000	0.0000	0.0001	0.0013	0.0103	0.0528	0.1755	0.3565	0.4035
0.90	0.0000	0.0000	0.0000	0.0000	0.0001	0.0018	0.0150	0.0817	0.2769	0.6244
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0013	0.0156	0.1172	0.8657

(a) Scheme $\mathcal{R}_1$

p	ℓ 1	2	3	4	5	6	7	8	9	10
0.05	0.3869	0.3333	0.1840	0.0701	0.0201	0.0046	0.0008	0.0001	0.0000	0.0000
0.10	0.0835	0.2003	0.2550	0.2172	0.1370	0.0676	0.0271	0.0090	0.0025	0.0008
0.15	0.0152	0.0656	0.1412	0.1994	0.2063	0.1660	0.1077	0.0579	0.0262	0.0147
0.20	0.0025	0.0161	0.0512	0.1058	0.1592	0.1854	0.1736	0.1341	0.0870	0.0851
0.25	0.0004	0.0032	0.0140	0.0397	0.0812	0.1283	0.1626	0.1698	0.1488	0.2520
0.30	0.0000	0.0005	0.0031	0.0113	0.0303	0.0625	0.1033	0.1404	0.1599	0.4885
0.35	0.0000	0.0001	0.0005	0.0026	0.0087	0.0228	0.0478	0.0826	0.1193	0.7157
0.40	0.0000	0.0000	0.0001	0.0005	0.0019	0.0064	0.0167	0.0360	0.0650	0.8735
0.45	0.0000	0.0000	0.0000	0.0001	0.0003	0.0014	0.0044	0.0118	0.0264	0.9556
0.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0009	0.0029	0.0080	0.9879
0.55	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0005	0.0018	0.9975
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0003	0.9996
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

(b) Scheme $\mathcal{R}_2$

p	ℓ 1	2	3	4	5	6	7	8	9	10
0.05	0.4013	0.3507	0.1779	0.0563	0.0119	0.0017	0.0002	0.0000	0.0000	0.0000
0.10	0.0889	0.2333	0.2947	0.2240	0.1116	0.0376	0.0085	0.0013	0.0001	0.0000
0.15	0.0163	0.0834	0.1932	0.2639	0.2334	0.1385	0.0550	0.0141	0.0021	0.0001
0.20	0.0027	0.0222	0.0828	0.1803	0.2528	0.2366	0.1478	0.0594	0.0139	0.0015
0.25	0.0004	0.0048	0.0269	0.0877	0.1837	0.2566	0.2390	0.1432	0.0500	0.0078
0.30	0.0000	0.0009	0.0070	0.0330	0.0999	0.2019	0.2725	0.2370	0.1205	0.0273
0.35	0.0000	0.0001	0.0015	0.0100	0.0429	0.1231	0.2374	0.2962	0.2173	0.0714
0.40	0.0000	0.0000	0.0003	0.0025	0.0148	0.0602	0.1649	0.2945	0.3123	0.1504
0.45	0.0000	0.0000	0.0000	0.0005	0.0042	0.0240	0.0934	0.2397	0.3711	0.2671
0.50	0.0000	0.0000	0.0000	0.0001	0.0010	0.0078	0.0434	0.1619	0.3719	0.4139
0.55	0.0000	0.0000	0.0000	0.0000	0.0002	0.0020	0.0165	0.0910	0.3171	0.5731
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004	0.0051	0.0423	0.2297	0.7225
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0012	0.0159	0.1398	0.8430
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0047	0.0700	0.9251
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0010	0.0278	0.9712
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0082	0.9917
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0016	0.9984
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.9999
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

(c) Scheme $\mathcal{R}_3$

p	ℓ 1	2	3	4	5	6	7	8	9	10
0.05	0.4267	0.3782	0.1506	0.0369	0.0066	0.0009	0.0001	0.0000	0.0000	0.0000
0.10	0.0993	0.3005	0.3180	0.1811	0.0727	0.0220	0.0052	0.0010	0.0002	0.0000
0.15	0.0187	0.1263	0.2675	0.2665	0.1807	0.0907	0.0353	0.0109	0.0030	0.0004
0.20	0.0031	0.0392	0.1489	0.2313	0.2358	0.1750	0.0996	0.0448	0.0183	0.0041
0.25	0.0004	0.0099	0.0641	0.1457	0.2097	0.2162	0.1692	0.1041	0.0599	0.0207
0.30	0.0001	0.0021	0.0228	0.0727	0.1421	0.1958	0.2031	0.1646	0.1294	0.0675
0.35	0.0000	0.0004	0.0069	0.0299	0.0774	0.1389	0.1864	0.1943	0.2043	0.1616
0.40	0.0000	0.0001	0.0018	0.0104	0.0347	0.0799	0.1364	0.1801	0.2500	0.3067
0.45	0.0000	0.0000	0.0004	0.0030	0.0130	0.0378	0.0812	0.1345	0.2446	0.4856
0.50	0.0000	0.0000	0.0001	0.0007	0.0040	0.0147	0.0395	0.0817	0.1940	0.6653
0.55	0.0000	0.0000	0.0000	0.0001	0.0010	0.0046	0.0156	0.0402	0.1247	0.8137
0.60	0.0000	0.0000	0.0000	0.0000	0.0002	0.0012	0.0049	0.0158	0.0641	0.9139
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0012	0.0048	0.0256	0.9682
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0011	0.0076	0.9911
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0015	0.9983
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.9998
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

(d) Scheme $\mathcal{R}_4$

Table 6.2: SCP for uniform distribution $[\pi_{\ell:r:n}(\xi_p)]$

p	ℓ 1	2	3	4	5	6	7	8	9	10
0.05	0.3869	0.3333	0.1840	0.0701	0.0242	0.0014	0.0000	0.0000	0.0000	0.0000
0.10	0.0835	0.2003	0.2550	0.2172	0.2063	0.0348	0.0027	0.0001	0.0000	0.0000
0.15	0.0152	0.0656	0.1412	0.1994	0.4072	0.1485	0.0211	0.0018	0.0001	0.0000
0.20	0.0025	0.0161	0.0512	0.1058	0.4341	0.3094	0.0709	0.0093	0.0007	0.0000
0.25	0.0004	0.0032	0.0140	0.0397	0.3240	0.4341	0.1522	0.0292	0.0030	0.0001
0.30	0.0000	0.0005	0.0031	0.0113	0.1883	0.4737	0.2464	0.0666	0.0094	0.0005
0.35	0.0000	0.0001	0.0005	0.0026	0.0889	0.4333	0.3278	0.1220	0.0231	0.0018
0.40	0.0000	0.0000	0.0001	0.0005	0.0342	0.3468	0.3763	0.1899	0.0475	0.0048
0.45	0.0000	0.0000	0.0000	0.0001	0.0104	0.2490	0.3837	0.2603	0.0855	0.0112
0.50	0.0000	0.0000	0.0000	0.0000	0.0023	0.1629	0.3529	0.3205	0.1380	0.0233
0.55	0.0000	0.0000	0.0000	0.0000	0.0004	0.0983	0.2952	0.3584	0.2031	0.0448
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0549	0.2251	0.3653	0.2745	0.0802
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0282	0.1561	0.3390	0.3413	0.1355
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0130	0.0973	0.2844	0.3885	0.2168
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0052	0.0532	0.2123	0.4006	0.3286
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0017	0.0244	0.1366	0.3669	0.4704
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004	0.0085	0.0708	0.2859	0.6343
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0018	0.0253	0.1712	0.8016
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0038	0.0563	0.9398

(e) Scheme $\mathcal{R}_5$

p	ℓ 1	2	3	4	5	6	7	8	9	10
0.05	0.3869	0.3333	0.1840	0.0701	0.0219	0.0037	0.0001	0.0000	0.0000	0.0000
0.10	0.0835	0.2003	0.2550	0.2172	0.1614	0.0751	0.0071	0.0004	0.0000	0.0000
0.15	0.0152	0.0656	0.1412	0.1994	0.2625	0.2617	0.0498	0.0044	0.0002	0.0000
0.20	0.0025	0.0161	0.0512	0.1058	0.2176	0.4329	0.1511	0.0211	0.0016	0.0001
0.25	0.0004	0.0032	0.0140	0.0397	0.1184	0.4679	0.2889	0.0607	0.0066	0.0003
0.30	0.0000	0.0005	0.0031	0.0113	0.0468	0.3802	0.4115	0.1261	0.0192	0.0012
0.35	0.0000	0.0001	0.0005	0.0026	0.0141	0.2489	0.4769	0.2097	0.0437	0.0036
0.40	0.0000	0.0000	0.0001	0.0005	0.0033	0.1352	0.4729	0.2957	0.0833	0.0091
0.45	0.0000	0.0000	0.0000	0.0001	0.0006	0.0610	0.4131	0.3664	0.1389	0.0199
0.50	0.0000	0.0000	0.0000	0.0000	0.0001	0.0223	0.3235	0.4072	0.2078	0.0391
0.55	0.0000	0.0000	0.0000	0.0000	0.0000	0.0063	0.2294	0.4103	0.2832	0.0707
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0013	0.1485	0.3762	0.3544	0.1195
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0883	0.3139	0.4069	0.1907
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0478	0.2374	0.4269	0.2879
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0231	0.1604	0.4054	0.4111
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0095	0.0940	0.3422	0.5544
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0030	0.0446	0.2463	0.7061
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0006	0.0147	0.1367	0.8481
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0020	0.0418	0.9562

(f) Scheme $\mathcal{R}_6$

p	ℓ 1	2	3	4	5	6	7	8	9	10
0.05	0.4177	0.3688	0.1676	0.0405	0.0051	0.0003	0.0000	0.0000	0.0000	0.0000
0.10	0.0954	0.2752	0.3388	0.2135	0.0682	0.0082	0.0006	0.0000	0.0000	0.0000
0.15	0.0178	0.1091	0.2708	0.3379	0.2114	0.0467	0.0058	0.0004	0.0000	0.0000
0.20	0.0029	0.0320	0.1418	0.3151	0.3512	0.1294	0.0246	0.0029	0.0002	0.0000
0.25	0.0004	0.0076	0.0566	0.2128	0.4064	0.2386	0.0657	0.0108	0.0010	0.0000
0.30	0.0001	0.0015	0.0184	0.1136	0.3667	0.3365	0.1303	0.0292	0.0036	0.0002
0.35	0.0000	0.0003	0.0050	0.0500	0.2726	0.3900	0.2086	0.0627	0.0102	0.0007
0.40	0.0000	0.0000	0.0011	0.0185	0.1718	0.3866	0.2833	0.1127	0.0238	0.0021
0.45	0.0000	0.0000	0.0002	0.0058	0.0928	0.3352	0.3359	0.1763	0.0483	0.0055
0.50	0.0000	0.0000	0.0000	0.0015	0.0431	0.2576	0.3536	0.2446	0.0869	0.0127
0.55	0.0000	0.0000	0.0000	0.0003	0.0171	0.1766	0.3331	0.3051	0.1412	0.0266
0.60	0.0000	0.0000	0.0000	0.0001	0.0057	0.1082	0.2816	0.3435	0.2092	0.0517
0.65	0.0000	0.0000	0.0000	0.0000	0.0016	0.0590	0.2128	0.3488	0.2835	0.0942
0.70	0.0000	0.0000	0.0000	0.0000	0.0004	0.0283	0.1423	0.3172	0.3501	0.1618
0.75	0.0000	0.0000	0.0000	0.0000	0.0001	0.0116	0.0823	0.2540	0.3895	0.2625
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0038	0.0394	0.1736	0.3824	0.4007
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0009	0.0143	0.0948	0.3173	0.5727
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0032	0.0355	0.2009	0.7604
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0055	0.0694	0.9249

(g) Scheme $\mathcal{R}_7$

p	ℓ 1	2	3	4	5	6	7	8	9	10
0.05	0.3869	0.3333	0.1840	0.0701	0.0201	0.0047	0.0008	0.0001	0.0000	0.0000
0.10	0.0835	0.2003	0.2550	0.2172	0.1370	0.0712	0.0275	0.0071	0.0011	0.0001
0.15	0.0152	0.0656	0.1412	0.1994	0.2063	0.1786	0.1207	0.0556	0.0155	0.0020
0.20	0.0025	0.0161	0.0512	0.1058	0.1592	0.2032	0.2144	0.1590	0.0730	0.0156
0.25	0.0004	0.0032	0.0140	0.0397	0.0812	0.1429	0.2217	0.2517	0.1825	0.0627
0.30	0.0000	0.0005	0.0031	0.0113	0.0303	0.0706	0.1557	0.2644	0.2983	0.1655
0.35	0.0000	0.0001	0.0005	0.0026	0.0087	0.0260	0.0801	0.2016	0.3554	0.3250
0.40	0.0000	0.0000	0.0001	0.0005	0.0019	0.0073	0.0312	0.1170	0.3273	0.5147
0.45	0.0000	0.0000	0.0000	0.0001	0.0003	0.0016	0.0094	0.0529	0.2408	0.6950
0.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0021	0.0187	0.1437	0.8351
0.55	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004	0.0052	0.0698	0.9246
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0011	0.0273	0.9715
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0085	0.9914
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0020	0.9980
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.9997
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

(h) Scheme $\mathcal{R}_8$

Table 6.2: SCP for uniform distribution $[\pi_{\ell:r:n}(\xi_p)]$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.5224	0.4122	0.0594	0.0056	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000
0.10	0.1766	0.5555	0.2130	0.0473	0.0068	0.0007	0.0000	0.0000	0.0000	0.0000
0.15	0.0493	0.4549	0.3305	0.1282	0.0314	0.0051	0.0006	0.0000	0.0000	0.0000
0.20	0.0119	0.3065	0.3635	0.2166	0.0792	0.0190	0.0030	0.0003	0.0000	0.0000
0.25	0.0025	0.1859	0.3292	0.2800	0.1432	0.0474	0.0103	0.0014	0.0001	0.0000
0.30	0.0005	0.1048	0.2612	0.3027	0.2078	0.0915	0.0263	0.0048	0.0004	0.0001
0.35	0.0001	0.0556	0.1869	0.2859	0.2564	0.1463	0.0543	0.0127	0.0017	0.0001
0.40	0.0000	0.0277	0.1221	0.2415	0.2777	0.2022	0.0953	0.0283	0.0048	0.0004
0.45	0.0000	0.0129	0.0732	0.1843	0.2686	0.2470	0.1465	0.0547	0.0117	0.0011
0.50	0.0000	0.0056	0.0401	0.1275	0.2338	0.2701	0.2009	0.0939	0.0252	0.0030
0.55	0.0000	0.0022	0.0199	0.0796	0.1834	0.2658	0.2479	0.1452	0.0488	0.0072
0.60	0.0000	0.0008	0.0088	0.0444	0.1288	0.2350	0.2761	0.2037	0.0863	0.0161
0.65	0.0000	0.0002	0.0034	0.0217	0.0799	0.1851	0.2763	0.2596	0.1404	0.0334
0.70	0.0000	0.0001	0.0011	0.0090	0.0427	0.1274	0.2456	0.2988	0.2100	0.0654
0.75	0.0000	0.0000	0.0003	0.0030	0.0189	0.0742	0.1892	0.3059	0.2876	0.1208
0.80	0.0000	0.0000	0.0001	0.0008	0.0064	0.0345	0.1208	0.2703	0.3553	0.2119
0.85	0.0000	0.0000	0.0000	0.0001	0.0015	0.0114	0.0584	0.1933	0.3824	0.3530
0.90	0.0000	0.0000	0.0000	0.0000	0.0002	0.0021	0.0174	0.0963	0.3287	0.5554
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0016	0.0204	0.1687	0.8092

(a) Scheme $\mathcal{R}_1$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.3333	0.3591	0.2023	0.0772	0.0221	0.0050	0.0009	0.0001	0.0000	0.0000
0.10	0.0731	0.1945	0.2568	0.2226	0.1417	0.0703	0.0282	0.0094	0.0026	0.0008
0.15	0.0139	0.0626	0.1384	0.1986	0.2076	0.1682	0.1097	0.0591	0.0268	0.0151
0.20	0.0023	0.0154	0.0499	0.1043	0.1583	0.1854	0.1745	0.1353	0.0881	0.0865
0.25	0.0003	0.0031	0.0137	0.0390	0.0803	0.1275	0.1623	0.1701	0.1494	0.2541
0.30	0.0000	0.0005	0.0030	0.0112	0.0300	0.0620	0.1028	0.1400	0.1599	0.4905
0.35	0.0000	0.0001	0.0005	0.0025	0.0086	0.0226	0.0476	0.0822	0.1190	0.7168
0.40	0.0000	0.0000	0.0001	0.0005	0.0019	0.0063	0.0166	0.0359	0.0648	0.8739
0.45	0.0000	0.0000	0.0000	0.0001	0.0003	0.0014	0.0044	0.0118	0.0263	0.9557
0.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0009	0.0029	0.0080	0.9879
0.55	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0005	0.0018	0.9975
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0003	0.9996
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

(b) Scheme $\mathcal{R}_2$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.3416	0.3803	0.1992	0.0634	0.0134	0.0019	0.0002	0.0000	0.0000	0.0000
0.10	0.0761	0.2245	0.2978	0.2328	0.1180	0.0402	0.0092	0.0014	0.0001	0.0000
0.15	0.0145	0.0783	0.1879	0.2634	0.2377	0.1432	0.0576	0.0149	0.0023	0.0002
0.20	0.0024	0.0209	0.0796	0.1765	0.2517	0.2392	0.1516	0.0618	0.0147	0.0016
0.25	0.0004	0.0046	0.0259	0.0853	0.1807	0.2555	0.2411	0.1465	0.0520	0.0082
0.30	0.0000	0.0008	0.0068	0.0321	0.0978	0.1993	0.2718	0.2393	0.1235	0.0285
0.35	0.0000	0.0001	0.0015	0.0098	0.0420	0.1212	0.2352	0.2962	0.2202	0.0738
0.40	0.0000	0.0000	0.0003	0.0024	0.0146	0.0594	0.1631	0.2929	0.3136	0.1536
0.45	0.0000	0.0000	0.0000	0.0005	0.0041	0.0238	0.0927	0.2383	0.3707	0.2699
0.50	0.0000	0.0000	0.0000	0.0001	0.0010	0.0078	0.0434	0.1619	0.3719	0.4139
0.55	0.0000	0.0000	0.0000	0.0000	0.0002	0.0021	0.0167	0.0922	0.3205	0.5684
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004	0.0052	0.0436	0.2377	0.7130
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0013	0.0169	0.1508	0.8310
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0051	0.0803	0.9143
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0012	0.0346	0.9642
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0113	0.9886
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0024	0.9976
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.9997
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

(c) Scheme $\mathcal{R}_3$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.3563	0.4157	0.1768	0.0426	0.0075	0.0010	0.0001	0.0000	0.0000	0.0000
0.10	0.0815	0.2842	0.3333	0.1928	0.0778	0.0236	0.0056	0.0011	0.0002	0.0000
0.15	0.0159	0.1143	0.2660	0.2715	0.1864	0.0942	0.0368	0.0114	0.0031	0.0004
0.20	0.0027	0.0353	0.1447	0.2302	0.2380	0.1781	0.1019	0.0460	0.0187	0.0043
0.25	0.0004	0.0090	0.0619	0.1435	0.2090	0.2172	0.1711	0.1056	0.0607	0.0215
0.30	0.0001	0.0020	0.0220	0.0714	0.1408	0.1953	0.2037	0.1657	0.1298	0.0693
0.35	0.0000	0.0004	0.0067	0.0295	0.0766	0.1381	0.1861	0.1946	0.2036	0.1645
0.40	0.0000	0.0001	0.0017	0.0102	0.0344	0.0794	0.1359	0.1799	0.2484	0.3098
0.45	0.0000	0.0000	0.0004	0.0030	0.0129	0.0376	0.0810	0.1343	0.2433	0.4876
0.50	0.0000	0.0000	0.0001	0.0007	0.0040	0.0147	0.0395	0.0817	0.1940	0.6653
0.55	0.0000	0.0000	0.0000	0.0001	0.0010	0.0047	0.0157	0.0403	0.1261	0.8120
0.60	0.0000	0.0000	0.0000	0.0000	0.0002	0.0012	0.0049	0.0159	0.0661	0.9117
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0012	0.0048	0.0271	0.9666
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0011	0.0083	0.9903
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0018	0.9980
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.9998
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

(d) Scheme $\mathcal{R}_4$

Table 6.3: SCP for normal distribution $[\pi_{\ell:r:n}(\xi_p)]$

	ℓ									
p	1	2	3	4	5	6	7	8	9	10
0.05	0.3333	0.3591	0.2023	0.0772	0.0264	0.0017	0.0001	0.0000	0.0000	0.0000
0.10	0.0731	0.1945	0.2568	0.2226	0.2098	0.0400	0.0031	0.0001	0.0000	0.0000
0.15	0.0139	0.0626	0.1384	0.1986	0.3945	0.1666	0.0234	0.0020	0.0001	0.0000
0.20	0.0023	0.0154	0.0499	0.1043	0.4017	0.3381	0.0773	0.0101	0.0007	0.0000
0.25	0.0003	0.0031	0.0137	0.0390	0.2854	0.4609	0.1629	0.0313	0.0032	0.0001
0.30	0.0000	0.0005	0.0030	0.0112	0.1576	0.4885	0.2583	0.0703	0.0099	0.0006
0.35	0.0000	0.0001	0.0005	0.0025	0.0712	0.4359	0.3371	0.1267	0.0241	0.0019
0.40	0.0000	0.0000	0.0001	0.0005	0.0270	0.3434	0.3809	0.1943	0.0489	0.0049
0.45	0.0000	0.0000	0.0000	0.0001	0.0087	0.2460	0.3845	0.2627	0.0867	0.0114
0.50	0.0000	0.0000	0.0000	0.0000	0.0023	0.1629	0.3529	0.3205	0.1380	0.0233
0.55	0.0000	0.0000	0.0000	0.0000	0.0005	0.1004	0.2974	0.3573	0.2006	0.0438
0.60	0.0000	0.0000	0.0000	0.0000	0.0001	0.0575	0.2304	0.3663	0.2692	0.0765
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0302	0.1633	0.3452	0.3354	0.1258
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0143	0.1045	0.2971	0.3880	0.1961
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0059	0.0588	0.2297	0.4140	0.2916
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0020	0.0278	0.1544	0.4008	0.4151
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0005	0.0101	0.0845	0.3390	0.5660
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0023	0.0323	0.2290	0.7364
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0052	0.0913	0.9034

(e) Scheme $\mathcal{R}_5$

	ℓ									
p	1	2	3	4	5	6	7	8	9	10
0.05	0.3333	0.3591	0.2023	0.0772	0.0238	0.0041	0.0002	0.0000	0.0000	0.0000
0.10	0.0731	0.1945	0.2568	0.2226	0.1645	0.0800	0.0080	0.0004	0.0000	0.0000
0.15	0.0139	0.0626	0.1384	0.1986	0.2579	0.2681	0.0555	0.0048	0.0002	0.0000
0.20	0.0023	0.0154	0.0499	0.1043	0.2093	0.4284	0.1655	0.0230	0.0017	0.0001
0.25	0.0003	0.0031	0.0137	0.0390	0.1127	0.4482	0.3103	0.0653	0.0071	0.0003
0.30	0.0000	0.0005	0.0030	0.0112	0.0446	0.3532	0.4324	0.1335	0.0203	0.0013
0.35	0.0000	0.0001	0.0005	0.0025	0.0135	0.2254	0.4902	0.2182	0.0457	0.0038
0.40	0.0000	0.0000	0.0001	0.0005	0.0032	0.1212	0.4770	0.3027	0.0859	0.0094
0.45	0.0000	0.0000	0.0000	0.0001	0.0006	0.0560	0.4123	0.3697	0.1411	0.0203
0.50	0.0000	0.0000	0.0000	0.0000	0.0001	0.0223	0.3235	0.4072	0.2078	0.0391
0.55	0.0000	0.0000	0.0000	0.0000	0.0000	0.0076	0.2334	0.4102	0.2798	0.0691
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0022	0.1556	0.3805	0.3481	0.1137
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0005	0.0956	0.3252	0.4026	0.1761
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0535	0.2544	0.4329	0.2591
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0266	0.1793	0.4297	0.3644
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0112	0.1101	0.3873	0.4914
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0036	0.0552	0.3051	0.6361
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0007	0.0193	0.1917	0.7883
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0029	0.0707	0.9263

(f) Scheme $\mathcal{R}_6$

	ℓ									
p	1	2	3	4	5	6	7	8	9	10
0.05	0.3511	0.4033	0.1922	0.0471	0.0059	0.0003	0.0000	0.0000	0.0000	0.0000
0.10	0.0795	0.2618	0.3453	0.2280	0.0754	0.0093	0.0007	0.0000	0.0000	0.0000
0.15	0.0154	0.1001	0.2611	0.3412	0.2235	0.0518	0.0064	0.0005	0.0000	0.0000
0.20	0.0026	0.0294	0.1335	0.3068	0.3577	0.1401	0.0266	0.0031	0.0002	0.0000
0.25	0.0004	0.0071	0.0530	0.2029	0.4018	0.2523	0.0699	0.0115	0.0010	0.0000
0.30	0.0001	0.0015	0.0173	0.1074	0.3546	0.3484	0.1361	0.0307	0.0038	0.0002
0.35	0.0000	0.0003	0.0047	0.0474	0.2604	0.3964	0.2145	0.0649	0.0106	0.0007
0.40	0.0000	0.0000	0.0011	0.0178	0.1640	0.3878	0.2874	0.1152	0.0245	0.0022
0.45	0.0000	0.0000	0.0002	0.0057	0.0899	0.3343	0.3375	0.1780	0.0489	0.0056
0.50	0.0000	0.0000	0.0000	0.0015	0.0431	0.2576	0.3536	0.2446	0.0869	0.0127
0.55	0.0000	0.0000	0.0000	0.0003	0.0179	0.1787	0.3339	0.3035	0.1395	0.0261
0.60	0.0000	0.0000	0.0000	0.0001	0.0064	0.1116	0.2852	0.3423	0.2049	0.0496
0.65	0.0000	0.0000	0.0000	0.0000	0.0019	0.0624	0.2194	0.3512	0.2771	0.0880
0.70	0.0000	0.0000	0.0000	0.0000	0.0005	0.0307	0.1504	0.3259	0.3454	0.1472
0.75	0.0000	0.0000	0.0000	0.0000	0.0001	0.0129	0.0896	0.2694	0.3946	0.2334
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0044	0.0444	0.1922	0.4067	0.3524
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0011	0.0167	0.1109	0.3646	0.5067
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0039	0.0444	0.2601	0.6914
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0075	0.1094	0.8828

(g) Scheme $\mathcal{R}_7$

	ℓ									
p	1	2	3	4	5	6	7	8	9	10
0.05	0.3333	0.3591	0.2023	0.0772	0.0221	0.0051	0.0009	0.0001	0.0000	0.0000
0.10	0.0731	0.1945	0.2568	0.2226	0.1417	0.0737	0.0288	0.0075	0.0012	0.0001
0.15	0.0139	0.0626	0.1384	0.1986	0.2076	0.1799	0.1231	0.0575	0.0163	0.0021
0.20	0.0023	0.0154	0.0499	0.1043	0.1583	0.2017	0.2148	0.1614	0.0755	0.0164
0.25	0.0003	0.0031	0.0137	0.0390	0.0803	0.1409	0.2196	0.2518	0.1856	0.0657
0.30	0.0000	0.0005	0.0030	0.0112	0.0300	0.0695	0.1534	0.2618	0.2994	0.1712
0.35	0.0000	0.0001	0.0005	0.0025	0.0086	0.0257	0.0788	0.1985	0.3528	0.3325
0.40	0.0000	0.0000	0.0001	0.0005	0.0019	0.0073	0.0308	0.1152	0.3229	0.5214
0.45	0.0000	0.0000	0.0000	0.0001	0.0003	0.0016	0.0093	0.0523	0.2379	0.6985
0.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0021	0.0187	0.1437	0.8351
0.55	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004	0.0053	0.0717	0.9227
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0011	0.0294	0.9694
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0097	0.9901
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0025	0.9975
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0005	0.9995
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.9999
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

(h) Scheme $\mathcal{R}_8$

Table 6.3: SCP for normal distribution $[\pi_{\ell:r:n}(\xi_p)]$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.3554	0.5384	0.0960	0.0096	0.0006	0.0000	0.0000	0.0000	0.0000	0.0000
0.10	0.0994	0.5674	0.2600	0.0628	0.0094	0.0009	0.0001	0.0000	0.0000	0.0000
0.15	0.0243	0.4236	0.3567	0.1499	0.0384	0.0064	0.0007	0.0000	0.0000	0.0000
0.20	0.0053	0.2760	0.3678	0.2349	0.0899	0.0222	0.0036	0.0004	0.0000	0.0000
0.25	0.0010	0.1666	0.3221	0.2898	0.1543	0.0526	0.0117	0.0016	0.0001	0.0000
0.30	0.0002	0.0950	0.2518	0.3046	0.2162	0.0977	0.0286	0.0053	0.0006	0.0000
0.35	0.0000	0.0514	0.1798	0.2835	0.2606	0.1519	0.0573	0.0136	0.0019	0.0001
0.40	0.0000	0.0262	0.1183	0.2383	0.2784	0.2057	0.0981	0.0295	0.0051	0.0004
0.45	0.0000	0.0126	0.0718	0.1824	0.2679	0.2482	0.1482	0.0557	0.0120	0.0011
0.50	0.0000	0.0056	0.0401	0.1275	0.2338	0.2701	0.2009	0.0939	0.0252	0.0030
0.55	0.0000	0.0023	0.0204	0.0809	0.1851	0.2664	0.2467	0.1434	0.0478	0.0070
0.60	0.0000	0.0008	0.0093	0.0462	0.1322	0.2378	0.2753	0.1999	0.0833	0.0152
0.65	0.0000	0.0003	0.0037	0.0232	0.0839	0.1906	0.2783	0.2550	0.1340	0.0309
0.70	0.0000	0.0001	0.0012	0.0100	0.0463	0.1348	0.2523	0.2964	0.1997	0.0591
0.75	0.0000	0.0000	0.0003	0.0035	0.0214	0.0816	0.2008	0.3101	0.2751	0.1072
0.80	0.0000	0.0000	0.0001	0.0009	0.0077	0.0400	0.1347	0.2849	0.3464	0.1853
0.85	0.0000	0.0000	0.0000	0.0002	0.0019	0.0143	0.0703	0.2179	0.3891	0.3063
0.90	0.0000	0.0000	0.0000	0.0000	0.0002	0.0029	0.0238	0.1228	0.3665	0.4837
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0029	0.0346	0.2373	0.7250

(a) Scheme $\mathcal{R}_1$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.2919	0.3641	0.2224	0.0886	0.0259	0.0059	0.0011	0.0002	0.0000	0.0000
0.10	0.0647	0.1842	0.2549	0.2283	0.1486	0.0749	0.0304	0.0102	0.0029	0.0009
0.15	0.0125	0.0587	0.1336	0.1962	0.2089	0.1716	0.1131	0.0615	0.0280	0.0159
0.20	0.0021	0.0145	0.0479	0.1017	0.1563	0.1852	0.1759	0.1374	0.0900	0.0891
0.25	0.0003	0.0029	0.0132	0.0379	0.0787	0.1261	0.1616	0.1704	0.1505	0.2583
0.30	0.0000	0.0005	0.0029	0.0109	0.0294	0.0611	0.1018	0.1393	0.1597	0.4944
0.35	0.0000	0.0001	0.0005	0.0025	0.0084	0.0223	0.0470	0.0816	0.1184	0.7192
0.40	0.0000	0.0000	0.0001	0.0004	0.0019	0.0062	0.0165	0.0356	0.0644	0.8749
0.45	0.0000	0.0000	0.0000	0.0001	0.0003	0.0013	0.0044	0.0117	0.0262	0.9559
0.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0009	0.0029	0.0080	0.9879
0.55	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0005	0.0018	0.9975
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0003	0.9996
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

(b) Scheme $\mathcal{R}_2$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.2955	0.3847	0.2238	0.0764	0.0169	0.0025	0.0002	0.0000	0.0000	0.0000
0.10	0.0662	0.2096	0.2951	0.2427	0.1285	0.0454	0.0107	0.0016	0.0001	0.0000
0.15	0.0129	0.0720	0.1790	0.2597	0.2425	0.1511	0.0629	0.0169	0.0027	0.0002
0.20	0.0022	0.0193	0.0749	0.1699	0.2482	0.2423	0.1582	0.0667	0.0165	0.0018
0.25	0.0003	0.0043	0.0244	0.0813	0.1751	0.2524	0.2439	0.1526	0.0563	0.0094
0.30	0.0000	0.0008	0.0064	0.0306	0.0942	0.1943	0.2694	0.2428	0.1298	0.0316
0.35	0.0000	0.0001	0.0014	0.0094	0.0405	0.1177	0.2306	0.2953	0.2256	0.0794
0.40	0.0000	0.0000	0.0003	0.0024	0.0142	0.0579	0.1597	0.2894	0.3156	0.1607
0.45	0.0000	0.0000	0.0000	0.0005	0.0041	0.0234	0.0913	0.2356	0.3697	0.2754
0.50	0.0000	0.0000	0.0000	0.0001	0.0010	0.0078	0.0434	0.1619	0.3719	0.4139
0.55	0.0000	0.0000	0.0000	0.0000	0.0002	0.0021	0.0171	0.0944	0.3261	0.5601
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004	0.0055	0.0465	0.2511	0.6964
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0014	0.0191	0.1701	0.8094
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0063	0.1007	0.8927
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0016	0.0514	0.9469
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0220	0.9777
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0075	0.9925
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0018	0.9982
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.9998

(c) Scheme $\mathcal{R}_3$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.3016	0.4188	0.2140	0.0544	0.0097	0.0013	0.0001	0.0000	0.0000	0.0000
0.10	0.0689	0.2575	0.3429	0.2091	0.0870	0.0268	0.0064	0.0012	0.0002	0.0000
0.15	0.0136	0.1008	0.2583	0.2758	0.1949	0.1005	0.0398	0.0124	0.0034	0.0005
0.20	0.0024	0.0311	0.1374	0.2266	0.2402	0.1831	0.1062	0.0485	0.0197	0.0049
0.25	0.0004	0.0080	0.0584	0.1393	0.2070	0.2185	0.1742	0.1086	0.0621	0.0234
0.30	0.0000	0.0018	0.0209	0.0691	0.1382	0.1940	0.2044	0.1677	0.1304	0.0735
0.35	0.0000	0.0003	0.0064	0.0286	0.0750	0.1365	0.1853	0.1950	0.2020	0.1708
0.40	0.0000	0.0001	0.0017	0.0100	0.0338	0.0785	0.1350	0.1794	0.2451	0.3163
0.45	0.0000	0.0000	0.0004	0.0030	0.0128	0.0373	0.0806	0.1339	0.2406	0.4915
0.50	0.0000	0.0000	0.0001	0.0007	0.0040	0.0147	0.0395	0.0817	0.1940	0.6653
0.55	0.0000	0.0000	0.0000	0.0001	0.0010	0.0047	0.0158	0.0406	0.1290	0.8087
0.60	0.0000	0.0000	0.0000	0.0000	0.0002	0.0012	0.0050	0.0161	0.0701	0.9072
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0012	0.0050	0.0306	0.9629
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0011	0.0103	0.9883
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0026	0.9972
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004	0.9996
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

(d) Scheme $\mathcal{R}_4$

Table 6.4: SCP for cauchy distribution $[\pi_{\ell:r:n}(\xi_p)]$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.2919	0.3641	0.2224	0.0886	0.0299	0.0030	0.0001	0.0000	0.0000	0.0000
0.10	0.0647	0.1842	0.2549	0.2283	0.2047	0.0583	0.0045	0.0002	0.0000	0.0000
0.15	0.0125	0.0587	0.1336	0.1962	0.3502	0.2151	0.0309	0.0026	0.0001	0.0000
0.20	0.0021	0.0145	0.0479	0.1017	0.3287	0.3968	0.0947	0.0127	0.0009	0.0000
0.25	0.0003	0.0029	0.0132	0.0379	0.2166	0.5010	0.1871	0.0371	0.0039	0.0002
0.30	0.0000	0.0005	0.0029	0.0109	0.1120	0.5010	0.2814	0.0792	0.0114	0.0007
0.35	0.0000	0.0001	0.0005	0.0025	0.0487	0.4305	0.3524	0.1368	0.0266	0.0021
0.40	0.0000	0.0000	0.0001	0.0004	0.0187	0.3338	0.3871	0.2026	0.0519	0.0053
0.45	0.0000	0.0000	0.0000	0.0001	0.0067	0.2403	0.3850	0.2671	0.0891	0.0118
0.50	0.0000	0.0000	0.0000	0.0000	0.0023	0.1629	0.3529	0.3205	0.1380	0.0233
0.55	0.0000	0.0000	0.0000	0.0000	0.0008	0.1043	0.3009	0.3551	0.1965	0.0424
0.60	0.0000	0.0000	0.0000	0.0000	0.0003	0.0628	0.2389	0.3659	0.2603	0.0717
0.65	0.0000	0.0000	0.0000	0.0000	0.0001	0.0351	0.1760	0.3510	0.3231	0.1147
0.70	0.0000	0.0000	0.0000	0.0000	0.0001	0.0179	0.1187	0.3119	0.3766	0.1749
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0080	0.0718	0.2536	0.4110	0.2556
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0030	0.0373	0.1839	0.4155	0.3602
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0008	0.0154	0.1129	0.3799	0.4909
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0043	0.0522	0.2960	0.6474
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0005	0.0125	0.1627	0.8243

(e) Scheme $\mathcal{R}_5$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.2919	0.3641	0.2224	0.0886	0.0274	0.0053	0.0003	0.0000	0.0000	0.0000
0.10	0.0647	0.1842	0.2549	0.2283	0.1668	0.0885	0.0118	0.0006	0.0000	0.0000
0.15	0.0125	0.0587	0.1336	0.1962	0.2475	0.2706	0.0740	0.0065	0.0003	0.0000
0.20	0.0021	0.0145	0.0479	0.1017	0.1953	0.4033	0.2039	0.0290	0.0022	0.0001
0.25	0.0003	0.0029	0.0132	0.0379	0.1038	0.3984	0.3568	0.0777	0.0086	0.0004
0.30	0.0000	0.0005	0.0029	0.0109	0.0409	0.2998	0.4690	0.1509	0.0236	0.0015
0.35	0.0000	0.0001	0.0005	0.0025	0.0125	0.1859	0.5078	0.2358	0.0507	0.0043
0.40	0.0000	0.0000	0.0001	0.0004	0.0030	0.1001	0.4793	0.3153	0.0915	0.0102
0.45	0.0000	0.0000	0.0000	0.0001	0.0006	0.0488	0.4093	0.3752	0.1451	0.0211
0.50	0.0000	0.0000	0.0000	0.0000	0.0001	0.0223	0.3235	0.4072	0.2078	0.0391
0.55	0.0000	0.0000	0.0000	0.0000	0.0000	0.0099	0.2398	0.4095	0.2741	0.0667
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0044	0.1674	0.3846	0.3373	0.1062
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0020	0.1097	0.3382	0.3902	0.1600
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0009	0.0667	0.2771	0.4252	0.2302
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004	0.0368	0.2090	0.4354	0.3186
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0177	0.1415	0.4147	0.4260
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0069	0.0817	0.3588	0.5525
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0018	0.0359	0.2662	0.6961
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0083	0.1406	0.8509

(f) Scheme $\mathcal{R}_6$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.2994	0.4070	0.2228	0.0616	0.0087	0.0005	0.0000	0.0000	0.0000	0.0000
0.10	0.0680	0.2400	0.3415	0.2458	0.0905	0.0133	0.0009	0.0000	0.0000	0.0000
0.15	0.0134	0.0897	0.2439	0.3373	0.2407	0.0661	0.0083	0.0006	0.0000	0.0000
0.20	0.0023	0.0263	0.1219	0.2894	0.3591	0.1648	0.0322	0.0038	0.0002	0.0000
0.25	0.0004	0.0064	0.0482	0.1868	0.3845	0.2791	0.0799	0.0134	0.0012	0.0000
0.30	0.0000	0.0013	0.0158	0.0981	0.3294	0.3676	0.1488	0.0343	0.0043	0.0002
0.35	0.0000	0.0002	0.0044	0.0435	0.2390	0.4045	0.2260	0.0699	0.0116	0.0008
0.40	0.0000	0.0000	0.0010	0.0166	0.1514	0.3878	0.2947	0.1202	0.0259	0.0023
0.45	0.0000	0.0000	0.0002	0.0054	0.0853	0.3320	0.3400	0.1811	0.0502	0.0058
0.50	0.0000	0.0000	0.0000	0.0015	0.0431	0.2576	0.3536	0.2446	0.0869	0.0127
0.55	0.0000	0.0000	0.0000	0.0004	0.0196	0.1825	0.3351	0.3006	0.1366	0.0253
0.60	0.0000	0.0000	0.0000	0.0001	0.0080	0.1181	0.2905	0.3389	0.1977	0.0467
0.65	0.0000	0.0000	0.0000	0.0000	0.0029	0.0694	0.2300	0.3515	0.2655	0.0806
0.70	0.0000	0.0000	0.0000	0.0000	0.0009	0.0365	0.1646	0.3342	0.3320	0.1318
0.75	0.0000	0.0000	0.0000	0.0000	0.0003	0.0167	0.1043	0.2880	0.3855	0.2053
0.80	0.0000	0.0000	0.0000	0.0000	0.0001	0.0063	0.0563	0.2194	0.4117	0.3062
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0018	0.0240	0.1405	0.3948	0.4390
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0067	0.0671	0.3204	0.6055
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0007	0.0164	0.1818	0.8011

(g) Scheme $\mathcal{R}_7$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.2919	0.3641	0.2224	0.0886	0.0259	0.0060	0.0010	0.0001	0.0000	0.0000
0.10	0.0647	0.1842	0.2549	0.2283	0.1486	0.0778	0.0314	0.0085	0.0014	0.0001
0.15	0.0125	0.0587	0.1336	0.1962	0.2089	0.1812	0.1270	0.0612	0.0182	0.0025
0.20	0.0021	0.0145	0.0479	0.1017	0.1563	0.1986	0.2145	0.1652	0.0804	0.0189
0.25	0.0003	0.0029	0.0132	0.0379	0.0787	0.1372	0.2153	0.2507	0.1909	0.0729
0.30	0.0000	0.0005	0.0029	0.0109	0.0294	0.0675	0.1490	0.2560	0.2995	0.1844
0.35	0.0000	0.0001	0.0005	0.0025	0.0084	0.0250	0.0764	0.1924	0.3462	0.3485
0.40	0.0000	0.0000	0.0001	0.0004	0.0019	0.0071	0.0300	0.1116	0.3141	0.5347
0.45	0.0000	0.0000	0.0000	0.0001	0.0003	0.0016	0.0091	0.0512	0.2325	0.7052
0.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0021	0.0187	0.1437	0.8351
0.55	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004	0.0054	0.0753	0.9188
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0012	0.0337	0.9650
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0129	0.9869
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0042	0.9958
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0011	0.9989
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.9997
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

(h) Scheme $\mathcal{R}_8$

Table 6.4: SCP for cauchy distribution $[\pi_{\ell:r:n}(\xi_p)]$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.5256	0.4095	0.0590	0.0055	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000
0.10	0.1790	0.5543	0.2121	0.0471	0.0068	0.0007	0.0000	0.0000	0.0000	0.0000
0.15	0.0504	0.4552	0.3297	0.1278	0.0312	0.0051	0.0006	0.0000	0.0000	0.0000
0.20	0.0123	0.3073	0.3632	0.2161	0.0789	0.0189	0.0030	0.0003	0.0000	0.0000
0.25	0.0026	0.1866	0.3293	0.2796	0.1428	0.0472	0.0103	0.0014	0.0001	0.0000
0.30	0.0005	0.1053	0.2616	0.3025	0.2074	0.0912	0.0262	0.0048	0.0005	0.0000
0.35	0.0001	0.0559	0.1873	0.2860	0.2561	0.1460	0.0541	0.0127	0.0017	0.0001
0.40	0.0000	0.0279	0.1225	0.2417	0.2775	0.2019	0.0950	0.0282	0.0048	0.0004
0.45	0.0000	0.0130	0.0735	0.1846	0.2687	0.2467	0.1462	0.0545	0.0117	0.0011
0.50	0.0000	0.0056	0.0402	0.1278	0.2341	0.2700	0.2006	0.0936	0.0251	0.0030
0.55	0.0000	0.0022	0.0200	0.0798	0.1837	0.2659	0.2477	0.1448	0.0486	0.0072
0.60	0.0000	0.0008	0.0088	0.0446	0.1291	0.2353	0.2760	0.2033	0.0860	0.0160
0.65	0.0000	0.0002	0.0034	0.0218	0.0802	0.1855	0.2765	0.2593	0.1399	0.0332
0.70	0.0000	0.0001	0.0011	0.0091	0.0429	0.1278	0.2460	0.2987	0.2094	0.0649
0.75	0.0000	0.0000	0.0003	0.0031	0.0190	0.0745	0.1898	0.3063	0.2871	0.1200
0.80	0.0000	0.0000	0.0001	0.0008	0.0065	0.0347	0.1213	0.2710	0.3552	0.2106
0.85	0.0000	0.0000	0.0000	0.0001	0.0015	0.0115	0.0587	0.1942	0.3831	0.3509
0.90	0.0000	0.0000	0.0000	0.0000	0.0002	0.0021	0.0175	0.0970	0.3306	0.5526
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0016	0.0207	0.1709	0.8067

(a) Scheme $\mathcal{R}_1$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.3340	0.3589	0.2020	0.0771	0.0220	0.0050	0.0009	0.0001	0.0000	0.0000
0.10	0.0733	0.1947	0.2568	0.2224	0.1416	0.0702	0.0282	0.0094	0.0026	0.0008
0.15	0.0139	0.0627	0.1385	0.1986	0.2076	0.1681	0.1097	0.0591	0.0268	0.0151
0.20	0.0023	0.0155	0.0500	0.1044	0.1583	0.1854	0.1744	0.1353	0.0880	0.0864
0.25	0.0003	0.0031	0.0137	0.0390	0.0804	0.1276	0.1624	0.1701	0.1494	0.2540
0.30	0.0000	0.0005	0.0030	0.0112	0.0300	0.0621	0.1028	0.1401	0.1599	0.4904
0.35	0.0000	0.0001	0.0005	0.0025	0.0086	0.0226	0.0476	0.0823	0.1191	0.7167
0.40	0.0000	0.0000	0.0001	0.0005	0.0019	0.0063	0.0166	0.0359	0.0648	0.8739
0.45	0.0000	0.0000	0.0000	0.0001	0.0003	0.0014	0.0044	0.0118	0.0264	0.9557
0.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0009	0.0029	0.0080	0.9879
0.55	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0005	0.0018	0.9975
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0003	0.9996
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

(b) Scheme $\mathcal{R}_2$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.3424	0.3800	0.1988	0.0633	0.0133	0.0019	0.0002	0.0000	0.0000	0.0000
0.10	0.0763	0.2247	0.2978	0.2326	0.1178	0.0401	0.0092	0.0014	0.0001	0.0000
0.15	0.0146	0.0785	0.1881	0.2634	0.2376	0.1431	0.0575	0.0149	0.0023	0.0002
0.20	0.0025	0.0210	0.0797	0.1767	0.2517	0.2391	0.1514	0.0617	0.0146	0.0015
0.25	0.0004	0.0046	0.0259	0.0854	0.1808	0.2556	0.2410	0.1463	0.0518	0.0082
0.30	0.0000	0.0008	0.0068	0.0322	0.0980	0.1995	0.2718	0.2391	0.1233	0.0284
0.35	0.0000	0.0001	0.0015	0.0098	0.0421	0.1214	0.2354	0.2962	0.2199	0.0735
0.40	0.0000	0.0000	0.0003	0.0024	0.0147	0.0596	0.1634	0.2932	0.3134	0.1531
0.45	0.0000	0.0000	0.0000	0.0005	0.0042	0.0239	0.0929	0.2388	0.3708	0.2689
0.50	0.0000	0.0000	0.0000	0.0001	0.0010	0.0078	0.0436	0.1625	0.3726	0.4125
0.55	0.0000	0.0000	0.0000	0.0000	0.0002	0.0021	0.0168	0.0926	0.3217	0.5667
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004	0.0052	0.0439	0.2392	0.7112
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0013	0.0170	0.1522	0.8294
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0052	0.0814	0.9132
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0012	0.0353	0.9635
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0116	0.9883
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0025	0.9975
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.9997
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

(c) Scheme $\mathcal{R}_3$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.3573	0.4154	0.1763	0.0425	0.0075	0.0010	0.0001	0.0000	0.0000	0.0000
0.10	0.0818	0.2847	0.3331	0.1925	0.0776	0.0235	0.0056	0.0011	0.0002	0.0000
0.15	0.0159	0.1147	0.2661	0.2714	0.1862	0.0941	0.0367	0.0114	0.0031	0.0004
0.20	0.0027	0.0355	0.1449	0.2303	0.2379	0.1780	0.1018	0.0459	0.0187	0.0043
0.25	0.0004	0.0091	0.0620	0.1436	0.2091	0.2172	0.1710	0.1055	0.0607	0.0215
0.30	0.0001	0.0020	0.0221	0.0715	0.1409	0.1953	0.2036	0.1656	0.1298	0.0692
0.35	0.0000	0.0004	0.0067	0.0295	0.0766	0.1382	0.1861	0.1946	0.2037	0.1641
0.40	0.0000	0.0001	0.0017	0.0103	0.0345	0.0795	0.1360	0.1800	0.2487	0.3093
0.45	0.0000	0.0000	0.0004	0.0030	0.0129	0.0377	0.0811	0.1344	0.2438	0.4868
0.50	0.0000	0.0000	0.0001	0.0007	0.0040	0.0147	0.0396	0.0818	0.1946	0.6645
0.55	0.0000	0.0000	0.0000	0.0001	0.0010	0.0047	0.0157	0.0404	0.1267	0.8114
0.60	0.0000	0.0000	0.0000	0.0000	0.0002	0.0012	0.0049	0.0159	0.0665	0.9113
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0012	0.0048	0.0274	0.9663
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0011	0.0084	0.9903
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0018	0.9980
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.9998
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

(d) Scheme $\mathcal{R}_4$

Table 6.5: SCP for skew normal distribution ($\alpha = 1$) [$\pi_{\ell:r:n}(\xi_p)$]

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.3340	0.3589	0.2020	0.0771	0.0263	0.0017	0.0001	0.0000	0.0000	0.0000
0.10	0.0733	0.1947	0.2568	0.2224	0.2097	0.0398	0.0031	0.0001	0.0000	0.0000
0.15	0.0139	0.0627	0.1385	0.1986	0.3951	0.1659	0.0233	0.0020	0.0001	0.0000
0.20	0.0023	0.0155	0.0500	0.1044	0.4033	0.3367	0.0770	0.0101	0.0007	0.0000
0.25	0.0003	0.0031	0.0137	0.0390	0.2877	0.4593	0.1622	0.0312	0.0032	0.0001
0.30	0.0000	0.0005	0.0030	0.0112	0.1599	0.4875	0.2574	0.0700	0.0099	0.0006
0.35	0.0000	0.0001	0.0005	0.0025	0.0730	0.4358	0.3361	0.1262	0.0240	0.0019
0.40	0.0000	0.0000	0.0001	0.0005	0.0281	0.3441	0.3802	0.1935	0.0486	0.0049
0.45	0.0000	0.0000	0.0000	0.0001	0.0092	0.2471	0.3842	0.2619	0.0863	0.0113
0.50	0.0000	0.0000	0.0000	0.0000	0.0026	0.1640	0.3532	0.3197	0.1373	0.0232
0.55	0.0000	0.0000	0.0000	0.0000	0.0006	0.1013	0.2981	0.3569	0.1997	0.0435
0.60	0.0000	0.0000	0.0000	0.0000	0.0001	0.0580	0.2314	0.3663	0.2682	0.0760
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0306	0.1643	0.3458	0.3345	0.1248
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0145	0.1053	0.2982	0.3875	0.1945
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0059	0.0594	0.2310	0.4143	0.2894
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0020	0.0281	0.1556	0.4021	0.4122
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0005	0.0102	0.0854	0.3414	0.5625
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0023	0.0327	0.2318	0.7331
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0053	0.0932	0.9013

(e) Scheme $\mathcal{R}_5$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.3340	0.3589	0.2020	0.0771	0.0238	0.0041	0.0002	0.0000	0.0000	0.0000
0.10	0.0733	0.1947	0.2568	0.2224	0.1645	0.0799	0.0080	0.0004	0.0000	0.0000
0.15	0.0139	0.0627	0.1385	0.1986	0.2580	0.2680	0.0553	0.0048	0.0002	0.0000
0.20	0.0023	0.0155	0.0500	0.1044	0.2097	0.4287	0.1648	0.0229	0.0017	0.0001
0.25	0.0003	0.0031	0.0137	0.0390	0.1130	0.4495	0.3089	0.0650	0.0070	0.0003
0.30	0.0000	0.0005	0.0030	0.0112	0.0447	0.3554	0.4308	0.1329	0.0202	0.0012
0.35	0.0000	0.0001	0.0005	0.0025	0.0136	0.2280	0.4889	0.2173	0.0455	0.0038
0.40	0.0000	0.0000	0.0001	0.0005	0.0032	0.1235	0.4765	0.3015	0.0855	0.0093
0.45	0.0000	0.0000	0.0000	0.0001	0.0006	0.0576	0.4127	0.3686	0.1403	0.0201
0.50	0.0000	0.0000	0.0000	0.0000	0.0001	0.0233	0.3246	0.4065	0.2067	0.0388
0.55	0.0000	0.0000	0.0000	0.0000	0.0000	0.0081	0.2348	0.4100	0.2785	0.0686
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0024	0.1569	0.3811	0.3469	0.1127
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0006	0.0966	0.3264	0.4018	0.1746
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0542	0.2560	0.4327	0.2570
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0270	0.1808	0.4306	0.3616
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0114	0.1114	0.3894	0.4879
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0037	0.0559	0.3081	0.6323
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0008	0.0196	0.1947	0.7849
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0029	0.0725	0.9245

(f) Scheme $\mathcal{R}_6$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.3520	0.4030	0.1918	0.0470	0.0059	0.0003	0.0000	0.0000	0.0000	0.0000
0.10	0.0798	0.2622	0.3452	0.2276	0.0752	0.0093	0.0007	0.0000	0.0000	0.0000
0.15	0.0154	0.1004	0.2615	0.3412	0.2231	0.0516	0.0064	0.0005	0.0000	0.0000
0.20	0.0026	0.0295	0.1338	0.3073	0.3575	0.1395	0.0265	0.0031	0.0002	0.0000
0.25	0.0004	0.0071	0.0532	0.2035	0.4021	0.2515	0.0696	0.0114	0.0010	0.0000
0.30	0.0001	0.0015	0.0174	0.1079	0.3556	0.3475	0.1357	0.0305	0.0038	0.0002
0.35	0.0000	0.0003	0.0048	0.0477	0.2617	0.3958	0.2139	0.0646	0.0105	0.0007
0.40	0.0000	0.0000	0.0011	0.0179	0.1653	0.3876	0.2867	0.1148	0.0244	0.0022
0.45	0.0000	0.0000	0.0002	0.0057	0.0909	0.3346	0.3369	0.1774	0.0487	0.0055
0.50	0.0000	0.0000	0.0000	0.0015	0.0437	0.2583	0.3534	0.2440	0.0865	0.0126
0.55	0.0000	0.0000	0.0000	0.0003	0.0183	0.1795	0.3342	0.3029	0.1388	0.0259
0.60	0.0000	0.0000	0.0000	0.0001	0.0066	0.1123	0.2858	0.3420	0.2041	0.0492
0.65	0.0000	0.0000	0.0000	0.0000	0.0020	0.0629	0.2203	0.3513	0.2762	0.0874
0.70	0.0000	0.0000	0.0000	0.0000	0.0005	0.0309	0.1512	0.3266	0.3447	0.1461
0.75	0.0000	0.0000	0.0000	0.0000	0.0001	0.0130	0.0902	0.2705	0.3944	0.2317
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0044	0.0448	0.1934	0.4075	0.3499
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0011	0.0169	0.1119	0.3666	0.5036
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0039	0.0450	0.2628	0.6882
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0076	0.1115	0.8806

(g) Scheme $\mathcal{R}_7$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.3340	0.3589	0.2020	0.0771	0.0220	0.0051	0.0008	0.0001	0.0000	0.0000
0.10	0.0733	0.1947	0.2568	0.2224	0.1416	0.0737	0.0288	0.0075	0.0012	0.0001
0.15	0.0139	0.0627	0.1385	0.1986	0.2076	0.1798	0.1230	0.0575	0.0163	0.0021
0.20	0.0023	0.0155	0.0500	0.1044	0.1583	0.2018	0.2147	0.1613	0.0754	0.0164
0.25	0.0003	0.0031	0.0137	0.0390	0.0804	0.1410	0.2197	0.2518	0.1855	0.0655
0.30	0.0000	0.0005	0.0030	0.0112	0.0300	0.0696	0.1536	0.2620	0.2993	0.1708
0.35	0.0000	0.0001	0.0005	0.0025	0.0086	0.0257	0.0789	0.1989	0.3531	0.3317
0.40	0.0000	0.0000	0.0001	0.0005	0.0019	0.0073	0.0309	0.1155	0.3237	0.5202
0.45	0.0000	0.0000	0.0000	0.0001	0.0003	0.0016	0.0093	0.0525	0.2389	0.6973
0.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0022	0.0188	0.1447	0.8340
0.55	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004	0.0053	0.0724	0.9219
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0011	0.0298	0.9690
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0099	0.9899
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0026	0.9974
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0005	0.9995
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.9999
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

(h) Scheme $\mathcal{R}_8$

Table 6.5: SCP for skew normal distribution ($\alpha = 1$) [$\pi_{\ell:r:n}(\xi_p)$]

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.5166	0.4173	0.0601	0.0056	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000
0.10	0.1726	0.5576	0.2145	0.0476	0.0069	0.0007	0.0000	0.0000	0.0000	0.0000
0.15	0.0476	0.4544	0.3318	0.1290	0.0315	0.0051	0.0006	0.0000	0.0000	0.0000
0.20	0.0114	0.3053	0.3641	0.2174	0.0796	0.0190	0.0030	0.0003	0.0000	0.0000
0.25	0.0024	0.1848	0.3291	0.2806	0.1437	0.0476	0.0103	0.0014	0.0001	0.0000
0.30	0.0004	0.1042	0.2607	0.3029	0.2083	0.0918	0.0264	0.0048	0.0005	0.0000
0.35	0.0001	0.0552	0.1864	0.2858	0.2567	0.1467	0.0545	0.0128	0.0017	0.0001
0.40	0.0000	0.0276	0.1217	0.2412	0.2778	0.2026	0.0956	0.0284	0.0049	0.0004
0.45	0.0000	0.0129	0.0729	0.1839	0.2684	0.2472	0.1469	0.0549	0.0118	0.0011
0.50	0.0000	0.0055	0.0399	0.1271	0.2336	0.2702	0.2012	0.0942	0.0253	0.0030
0.55	0.0000	0.0022	0.0198	0.0793	0.1831	0.2657	0.2481	0.1456	0.0490	0.0072
0.60	0.0000	0.0008	0.0088	0.0442	0.1285	0.2347	0.2761	0.2041	0.0866	0.0162
0.65	0.0000	0.0002	0.0034	0.0216	0.0797	0.1847	0.2761	0.2599	0.1408	0.0336
0.70	0.0000	0.0001	0.0011	0.0090	0.0426	0.1271	0.2452	0.2988	0.2105	0.0657
0.75	0.0000	0.0000	0.0003	0.0030	0.0188	0.0740	0.1888	0.3057	0.2880	0.1214
0.80	0.0000	0.0000	0.0001	0.0008	0.0064	0.0344	0.1204	0.2697	0.3554	0.2129
0.85	0.0000	0.0000	0.0000	0.0001	0.0015	0.0113	0.0582	0.1927	0.3819	0.3544
0.90	0.0000	0.0000	0.0000	0.0000	0.0002	0.0020	0.0173	0.0959	0.3276	0.5570
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0016	0.0203	0.1675	0.8105

(a) Scheme $\mathcal{R}_1$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.3320	0.3595	0.2028	0.0774	0.0221	0.0050	0.0009	0.0001	0.0000	0.0000
0.10	0.0728	0.1942	0.2568	0.2228	0.1419	0.0704	0.0283	0.0094	0.0026	0.0008
0.15	0.0138	0.0625	0.1382	0.1985	0.2077	0.1683	0.1098	0.0592	0.0268	0.0151
0.20	0.0023	0.0154	0.0498	0.1042	0.1582	0.1854	0.1745	0.1354	0.0882	0.0865
0.25	0.0003	0.0031	0.0137	0.0389	0.0802	0.1275	0.1623	0.1701	0.1495	0.2543
0.30	0.0000	0.0005	0.0030	0.0112	0.0300	0.0620	0.1027	0.1400	0.1598	0.4907
0.35	0.0000	0.0001	0.0005	0.0025	0.0086	0.0226	0.0475	0.0822	0.1190	0.7170
0.40	0.0000	0.0000	0.0001	0.0005	0.0019	0.0063	0.0166	0.0358	0.0648	0.8740
0.45	0.0000	0.0000	0.0000	0.0001	0.0003	0.0014	0.0044	0.0118	0.0263	0.9557
0.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0009	0.0029	0.0080	0.9880
0.55	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0005	0.0018	0.9975
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0003	0.9996
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

(b) Scheme $\mathcal{R}_2$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.3401	0.3807	0.1999	0.0637	0.0134	0.0019	0.0002	0.0000	0.0000	0.0000
0.10	0.0757	0.2240	0.2978	0.2332	0.1183	0.0403	0.0092	0.0014	0.0001	0.0000
0.15	0.0145	0.0781	0.1876	0.2633	0.2379	0.1435	0.0578	0.0150	0.0023	0.0002
0.20	0.0024	0.0209	0.0794	0.1763	0.2516	0.2394	0.1519	0.0619	0.0147	0.0016
0.25	0.0004	0.0046	0.0258	0.0851	0.1804	0.2554	0.2413	0.1467	0.0521	0.0082
0.30	0.0000	0.0008	0.0068	0.0320	0.0976	0.1990	0.2717	0.2395	0.1238	0.0286
0.35	0.0000	0.0001	0.0015	0.0097	0.0419	0.1210	0.2349	0.2962	0.2206	0.0742
0.40	0.0000	0.0000	0.0003	0.0024	0.0146	0.0593	0.1628	0.2926	0.3138	0.1543
0.45	0.0000	0.0000	0.0000	0.0005	0.0041	0.0237	0.0924	0.2378	0.3705	0.2710
0.50	0.0000	0.0000	0.0000	0.0001	0.0009	0.0077	0.0433	0.1614	0.3712	0.4154
0.55	0.0000	0.0000	0.0000	0.0000	0.0002	0.0020	0.0167	0.0918	0.3193	0.5701
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004	0.0052	0.0434	0.2364	0.7146
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0013	0.0168	0.1496	0.8323
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0051	0.0794	0.9152
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0012	0.0341	0.9647
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0111	0.9887
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0024	0.9976
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.9998
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

(c) Scheme $\mathcal{R}_3$

p	ℓ = 1	2	3	4	5	6	7	8	9	10
0.05	0.3545	0.4162	0.1778	0.0428	0.0075	0.0010	0.0001	0.0000	0.0000	0.0000
0.10	0.0810	0.2834	0.3338	0.1933	0.0780	0.0236	0.0056	0.0011	0.0002	0.0000
0.15	0.0158	0.1138	0.2658	0.2718	0.1867	0.0944	0.0369	0.0114	0.0031	0.0004
0.20	0.0027	0.0351	0.1444	0.2301	0.2381	0.1783	0.1021	0.0461	0.0188	0.0043
0.25	0.0004	0.0090	0.0617	0.1433	0.2090	0.2173	0.1712	0.1058	0.0608	0.0216
0.30	0.0001	0.0019	0.0220	0.0713	0.1407	0.1952	0.2037	0.1658	0.1299	0.0695
0.35	0.0000	0.0004	0.0067	0.0294	0.0764	0.1380	0.1860	0.1946	0.2036	0.1649
0.40	0.0000	0.0001	0.0017	0.0102	0.0344	0.0793	0.1358	0.1799	0.2481	0.3105
0.45	0.0000	0.0000	0.0004	0.0030	0.0129	0.0376	0.0809	0.1342	0.2427	0.4883
0.50	0.0000	0.0000	0.0001	0.0007	0.0040	0.0147	0.0395	0.0816	0.1934	0.6661
0.55	0.0000	0.0000	0.0000	0.0001	0.0010	0.0047	0.0156	0.0403	0.1256	0.8126
0.60	0.0000	0.0000	0.0000	0.0000	0.0002	0.0012	0.0049	0.0159	0.0657	0.9121
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0012	0.0048	0.0270	0.9668
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0011	0.0083	0.9904
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0017	0.9981
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.9998
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

(d) Scheme $\mathcal{R}_4$

Table 6.6: SCP for skew normal distribution ($\alpha = -1$) [$\pi_{\ell:r:n}(\xi_p)$]

p	ℓ 1	2	3	4	5	6	7	8	9	10
0.05	0.3320	0.3595	0.2028	0.0774	0.0265	0.0017	0.0001	0.0000	0.0000	0.0000
0.10	0.0728	0.1942	0.2568	0.2228	0.2099	0.0403	0.0031	0.0002	0.0000	0.0000
0.15	0.0138	0.0625	0.1382	0.1985	0.3935	0.1679	0.0235	0.0020	0.0001	0.0000
0.20	0.0023	0.0154	0.0498	0.1042	0.3992	0.3403	0.0778	0.0102	0.0007	0.0000
0.25	0.0003	0.0031	0.0137	0.0389	0.2821	0.4631	0.1638	0.0315	0.0032	0.0001
0.30	0.0000	0.0005	0.0030	0.0112	0.1546	0.4897	0.2596	0.0707	0.0100	0.0006
0.35	0.0000	0.0001	0.0005	0.0025	0.0692	0.4359	0.3383	0.1274	0.0242	0.0019
0.40	0.0000	0.0000	0.0001	0.0005	0.0259	0.3425	0.3817	0.1951	0.0492	0.0050
0.45	0.0000	0.0000	0.0000	0.0001	0.0082	0.2448	0.3847	0.2637	0.0872	0.0114
0.50	0.0000	0.0000	0.0000	0.0000	0.0022	0.1619	0.3525	0.3213	0.1387	0.0235
0.55	0.0000	0.0000	0.0000	0.0000	0.0005	0.0997	0.2966	0.3577	0.2014	0.0441
0.60	0.0000	0.0000	0.0000	0.0000	0.0001	0.0570	0.2295	0.3662	0.2701	0.0771
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0300	0.1625	0.3447	0.3361	0.1266
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0142	0.1039	0.2963	0.3883	0.1973
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0058	0.0585	0.2287	0.4138	0.2932
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0019	0.0276	0.1535	0.3998	0.4171
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0005	0.0100	0.0839	0.3375	0.5681
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0022	0.0320	0.2273	0.7384
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0052	0.0902	0.9044

(e) Scheme $\mathcal{R}_5$

p	ℓ 1	2	3	4	5	6	7	8	9	10
0.05	0.3320	0.3595	0.2028	0.0774	0.0239	0.0041	0.0002	0.0000	0.0000	0.0000
0.10	0.0728	0.1942	0.2568	0.2228	0.1646	0.0803	0.0081	0.0004	0.0000	0.0000
0.15	0.0138	0.0625	0.1382	0.1985	0.2576	0.2684	0.0559	0.0049	0.0002	0.0000
0.20	0.0023	0.0154	0.0498	0.1042	0.2088	0.4279	0.1666	0.0232	0.0017	0.0001
0.25	0.0003	0.0031	0.0137	0.0389	0.1123	0.4463	0.3122	0.0657	0.0071	0.0003
0.30	0.0000	0.0005	0.0030	0.0112	0.0443	0.3503	0.4346	0.1343	0.0205	0.0013
0.35	0.0000	0.0001	0.0005	0.0025	0.0134	0.2224	0.4918	0.2194	0.0460	0.0038
0.40	0.0000	0.0000	0.0001	0.0005	0.0032	0.1187	0.4776	0.3041	0.0865	0.0095
0.45	0.0000	0.0000	0.0000	0.0001	0.0006	0.0542	0.4119	0.3710	0.1419	0.0204
0.50	0.0000	0.0000	0.0000	0.0000	0.0001	0.0213	0.3224	0.4080	0.2089	0.0394
0.55	0.0000	0.0000	0.0000	0.0000	0.0000	0.0072	0.2320	0.4102	0.2809	0.0696
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0020	0.1544	0.3799	0.3492	0.1145
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0005	0.0948	0.3242	0.4033	0.1773
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0530	0.2532	0.4329	0.2608
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0263	0.1782	0.4290	0.3665
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0111	0.1093	0.3858	0.4938
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0036	0.0547	0.3032	0.6385
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0007	0.0191	0.1899	0.7902
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0028	0.0698	0.9273

(f) Scheme $\mathcal{R}_6$

p	ℓ 1	2	3	4	5	6	7	8	9	10
0.05	0.3494	0.4039	0.1931	0.0474	0.0060	0.0003	0.0000	0.0000	0.0000	0.0000
0.10	0.0791	0.2611	0.3453	0.2286	0.0758	0.0094	0.0007	0.0000	0.0000	0.0000
0.15	0.0153	0.0997	0.2606	0.3413	0.2241	0.0521	0.0064	0.0005	0.0000	0.0000
0.20	0.0026	0.0292	0.1330	0.3062	0.3580	0.1409	0.0268	0.0031	0.0002	0.0000
0.25	0.0004	0.0071	0.0528	0.2021	0.4012	0.2535	0.0702	0.0115	0.0011	0.0000
0.30	0.0000	0.0014	0.0172	0.1069	0.3532	0.3495	0.1368	0.0308	0.0038	0.0002
0.35	0.0000	0.0002	0.0047	0.0471	0.2588	0.3972	0.2154	0.0652	0.0106	0.0007
0.40	0.0000	0.0000	0.0011	0.0176	0.1626	0.3879	0.2882	0.1157	0.0246	0.0022
0.45	0.0000	0.0000	0.0002	0.0056	0.0889	0.3338	0.3380	0.1786	0.0492	0.0056
0.50	0.0000	0.0000	0.0000	0.0015	0.0425	0.2569	0.3537	0.2453	0.0873	0.0127
0.55	0.0000	0.0000	0.0000	0.0003	0.0176	0.1780	0.3337	0.3041	0.1400	0.0263
0.60	0.0000	0.0000	0.0000	0.0001	0.0063	0.1111	0.2846	0.3425	0.2056	0.0499
0.65	0.0000	0.0000	0.0000	0.0000	0.0019	0.0620	0.2188	0.3510	0.2778	0.0885
0.70	0.0000	0.0000	0.0000	0.0000	0.0004	0.0305	0.1497	0.3254	0.3459	0.1480
0.75	0.0000	0.0000	0.0000	0.0000	0.0001	0.0128	0.0891	0.2686	0.3947	0.2347
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0043	0.0441	0.1914	0.4061	0.3540
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0011	0.0166	0.1103	0.3634	0.5087
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0038	0.0441	0.2585	0.6934
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0075	0.1083	0.8840

(g) Scheme $\mathcal{R}_7$

p	ℓ 1	2	3	4	5	6	7	8	9	10
0.05	0.3320	0.3595	0.2028	0.0774	0.0221	0.0051	0.0009	0.0001	0.0000	0.0000
0.10	0.0728	0.1942	0.2568	0.2228	0.1419	0.0738	0.0289	0.0075	0.0012	0.0001
0.15	0.0138	0.0625	0.1382	0.1985	0.2077	0.1799	0.1232	0.0577	0.0164	0.0021
0.20	0.0023	0.0154	0.0498	0.1042	0.1582	0.2016	0.2148	0.1615	0.0756	0.0165
0.25	0.0003	0.0031	0.0137	0.0389	0.0802	0.1407	0.2194	0.2518	0.1859	0.0659
0.30	0.0000	0.0005	0.0030	0.0112	0.0300	0.0694	0.1531	0.2615	0.2994	0.1719
0.35	0.0000	0.0001	0.0005	0.0025	0.0086	0.0256	0.0786	0.1981	0.3524	0.3336
0.40	0.0000	0.0000	0.0001	0.0005	0.0019	0.0072	0.0307	0.1148	0.3221	0.5227
0.45	0.0000	0.0000	0.0000	0.0001	0.0003	0.0016	0.0093	0.0521	0.2368	0.6999
0.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0021	0.0186	0.1427	0.8362
0.55	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004	0.0052	0.0710	0.9234
0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0011	0.0290	0.9698
0.65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0096	0.9903
0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0024	0.9975
0.75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0005	0.9995
0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.9999
0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

(h) Scheme $\mathcal{R}_8$

Table 6.6: SCP for skew normal distribution ($\alpha = -1$) [$\pi_{\ell:r:n}(\xi_p)$]

Chapter 7

Pitman Closeness as a Criterion for Optimal Censoring Schemes

When designing a life-testing experiment, the experimenter would fix the sample size and allowable number of failures, and would like to obtain the highest amount of "information" possible. This typically means minimizing the mean square error or variance for some plausible class of distributions. The exponential distribution is one such distribution assumed in life testing.

For a single parameter exponential distribution with scale parameter θ, the Best Linear Unbiased Estimator (BLUE) of θ based on a progressively censored sample with censoring scheme $\mathcal{R}$ is $\theta^* = \dfrac{1}{r}\displaystyle\sum_{i=1}^{r}(R_i+1)X_{i:r:n}^{\mathcal{R}}$ However, it is a well known property of the exponential distribution that $\dfrac{r\theta^*}{m} \sim \chi^2_{2r}$ (See Balakrishnan and Aggarwala, 2000; Viveros and Balakrishnan, 1994). The marginal distribution of the BLUE is free of the censoring scheme, so no determination of optimality can be determined that uses only the marginal distributions of the BLUE for various censoring schemes.

In this chapter we provide a method for comparing the BLUE of the exponential distribution, across possible censoring schemes. We will make use of the Pitman Criterion for comparing the BLUE, in particular, we will make use of the joint distribution of the BLUE for two censoring schemes applied to the same hypothetical sample. An experimenter will typically run the experiment once, so it stands to reason that they should choose the scheme that will be closest that one particular time. Pitman's measure of closeness is well suited to answer the question when viewed in this light.

We further conjecture that right censoring is optimal generally. However, we only demonstrate this in specific cases, and leave a general proof as an open problem.

7.1 Comparison of Censoring Schemes

Consider two progressive censoring schemes $\mathcal{R}_1$ and $\mathcal{R}_2$. An experimenter will place n items on failure. If the items are left until all n have failed, we will obtain $X_1, \ldots, X_n$ which will all be finite with probability 1. The progressive censoring schemes $\mathcal{R}_1$ and $\mathcal{R}_2$ are applied to the hypothetical data and the BLUEs $\theta^*_{\mathcal{R}_1}$ and $\theta^*_{\mathcal{R}_2}$ are obtained respectively. Since the BLUEs are generated from the same underlying data, they will be dependent.

Definition 7.1.1 *For two given progressive censoring schemes $\mathcal{R}_1$ and $\mathcal{R}_2$ we define* *the PC probability between the two schemes as*

$$\pi\left(\theta^*_{\mathcal{R}_1}, \theta^*_{\mathcal{R}_2}\right) = P(|\theta^*_{\mathcal{R}_1} - \theta| < |\theta^*_{\mathcal{R}_2} - \theta|),$$

where the dependence structure is generated as above.

If $\theta^*_{\mathcal{R}_1}$ is uniformly closer to θ than $\theta^*_{\mathcal{R}_2}$, we say that $\mathcal{R}_1$ is better than $\mathcal{R}_2$ in the sense of Pitman closeness.

Lemma 7.1.2 *If* $\pi\left(\theta^*_{\mathcal{R}_1}, \theta^*_{\mathcal{R}_2} \middle| \theta = 1\right) = q$, *then* $\pi\left(\theta^*_{\mathcal{R}_1}, \theta^*_{\mathcal{R}_2} \middle| \theta\right) = \pi\left(\theta^*_{\mathcal{R}_1}, \theta^*_{\mathcal{R}_2}\right) = q, \forall\, \theta > 0.$

Proof: For any $\theta > 0$, we have

$$\begin{aligned}\pi\left(\theta^*_{\mathcal{R}_1}, \theta^*_{\mathcal{R}_2} \middle| \theta\right) &= P\left(|\theta^*_{\mathcal{R}_1} - \theta| < |\theta^*_{\mathcal{R}_2} - \theta|\right) = P\left(|\theta^*_{\mathcal{R}_1}/\theta - 1| < |\theta^*_{\mathcal{R}_2}/\theta - 1|\right) \\ &= \pi\left(\theta^*_{\mathcal{R}_1}, \theta^*_{\mathcal{R}_2} \middle| \theta = 1\right)\end{aligned}$$

The last equality follows from the fact that $X^{\mathcal{R}}_{1:r:n}/\theta, \ldots, X^{\mathcal{R}}_{r:r:n}/\theta$ is equal in distribution to a progressively Type-II censored sample from the standard exponential distribution. ∎

Consequently we can write the PC probability as $\pi\left(\theta^*_{\mathcal{R}_1}, \theta^*_{\mathcal{R}_2} \middle| \theta\right) = \pi\left(\theta^*_{\mathcal{R}_1}, \theta^*_{\mathcal{R}_2}\right)$, since it free of θ. Furthermore, we can restrict ourselves to the standard exponential for simplicity, as we will now assume unless otherwise stated.

Following along the lines of the proof of Lemma 6.1.3 we can find the region that $\theta^*_{\mathcal{R}_1}$ is closer to θ than $\theta^*_{\mathcal{R}_2}$ as $\left(\theta^*_{\mathcal{R}_1} - \theta^*_{\mathcal{R}_2}\right)\left(\theta^*_{\mathcal{R}_1} + \theta^*_{\mathcal{R}_2} - 2\right) = ab < 0$. Here $a = \left(\theta^*_{\mathcal{R}_1} - \theta^*_{\mathcal{R}_2}\right)$ and $b = \left(\theta^*_{\mathcal{R}_1} + \theta^*_{\mathcal{R}_2} - 2\right)$.

The BLUE $\theta^*_{\mathcal{R}_1}$ is Pitman closer to θ than $\theta^*_{\mathcal{R}_2}$ if either $(a > 0, b < 0)$ or $(a < 0, b > 0)$.

Obtaining the BLUE from the n complete observations is equivalent to expressing the joint distribution of $X^{\mathcal{R}}_{1:r:n}, \ldots, X^{\mathcal{R}}_{r:r:n}$ as a mixture of the usual order statistics from the complete sample. Thus $(X^{\mathcal{R}}_{1:r:n}, \ldots, X^{\mathcal{R}}_{r:r:n}) = (X_{i_1:n}, \ldots, X_{i_r:n})$ for some $1 \le i_1 < \cdots < i_r \le n$ depending on which items are removed in the sampling and where $(X_{i_1:n}, \ldots, X_{i_r:n})$ is a component in the mixture distribution.

To simplify the matter, we can use the independent spacing properties of the exponential distribution (See Balakrishnan and Aggarwala, 2000). Namely, the order statistics from a complete sample can be written as $X_{i:n} = \sum_{k=1}^{i} \frac{1}{n-k+1} Z_k$, where $Z_1, \ldots, Z_n$ are i.i.d standard exponential.

Finally we can obtain the PC probability conditional on the two component mixtures by integration over $\Re^{n+}$ subject to the above linear constraints set in terms of $Z_1, \ldots, Z_n$. We describe this in detail in the next section.

7.1.1 A General Algorithm

Here we describe in detail general algorithm to obtain the PC probabilities to compare any two progressively censored schemes as discussed in Section 7.1.

Given two potential censoring schemes $\mathcal{R}_1$ and $\mathcal{R}_2$, we can obtain the PC probability $\pi(\theta^*_{\mathcal{R}_1}, \theta^*_{\mathcal{R}_2})$ for any r and n as follows.

1. Express the Type-II progressively censored samples as a mixture of the usual order statistics (see Guilbaud, 2004), given by

$$f^{\mathcal{R}}_{\tilde{X}}(x_1, \ldots, x_r) = \sum_{1 \le i^{\mathcal{R}}_1, \ldots, i^{\mathcal{R}}_r \le n} w^{\mathcal{R}}_{i^{\mathcal{R}}_1, \ldots, i^{\mathcal{R}}_r} f_{X_{i_1:n}, \ldots, X_{i_r:n}}(x_1, \ldots, x_r); \tag{7.1.1}$$

2. Based on each mixture component vector $(X_{i_1:n}, \ldots, X_{i_r:n})$, obtain the estimator $\theta^*_{\mathcal{R}}(i^{\mathcal{R}}_1, \ldots, i^{\mathcal{R}}_r)$, as $\theta^*_{\mathcal{R}}$, conditioned on $(X^{\mathcal{R}}_{1:r:n}, \ldots, X^{\mathcal{R}}_{r:r:n}) = (X_{i_1:n}, \ldots, X_{i_r:n})$, given by

$$\theta^*_{\mathcal{R}}(i^{\mathcal{R}}_1, \ldots, i^{\mathcal{R}}_r) = \frac{1}{r} \sum_{j=1}^{r} (R_i + 1) X_{i_j:n}; \tag{7.1.2}$$

which yields the final mixture form for the estimator $\theta^*_{\mathcal{R}}$ as

$$\theta^*_{\mathcal{R}} = \sum_{1 \le i^{\mathcal{R}}_1, \ldots, i^{\mathcal{R}}_r \le n} w^{\mathcal{R}}_{i^{\mathcal{R}}_1, \ldots, i^{\mathcal{R}}_r} \theta^*_{\mathcal{R}}(i^{\mathcal{R}}_1, \ldots, i^{\mathcal{R}}_r). \tag{7.1.3}$$

3. Using the independent spacing property of the order statistics from the exponential distribution, express the usual order statistics as $X_{j:n} = \sum_{k=1}^{j} \frac{1}{n-k+1} Z_k$, where $Z_1, \ldots, Z_n$ are i.i.d. exponential with mean 1;

4. For each pairwise combination between the component distributions of $\theta^*_{\mathcal{R}_1}$ and $\theta^*_{\mathcal{R}_2}$, obtain the two linear constraints as described previously (a and b);

5. Integrate the joint density of the independent exponential random variables,

$Z_1, \ldots, Z_n$ over $\mathbb{R}^{n+}$ subject to the above two linear constraints. This gives the conditional PC probability $\pi\left(\theta^*_{\mathcal{R}_1}(i_1^{\mathcal{R}_1}, \ldots, i_r^{\mathcal{R}_1}), \theta^*_{\mathcal{R}_2}(i_1^{\mathcal{R}_2}, \ldots, i_r^{\mathcal{R}_2})\right)$. It should be mentioned that the two linear constraints can be turned into affine constraints with the simple algorithm as considered in Schechter (1998);

6. The PC probability $\pi(\theta^*_{\mathcal{R}_1}, \theta^*_{\mathcal{R}_2})$ can be finally computed as the weighted sum of the above conditional probabilities. Namely,

$$\pi(\theta^*_{\mathcal{R}_1}, \theta^*_{\mathcal{R}_2}) = \sum_{\substack{1 \le i_1^{\mathcal{R}_1}, \ldots, i_r^{\mathcal{R}_1} \le n \\ 1 \le i_1^{\mathcal{R}_2}, \ldots, i_r^{\mathcal{R}_2} \le n}} w^{\mathcal{R}_1}_{i_1^{\mathcal{R}_1}, \ldots, i_r^{\mathcal{R}_1}} w^{\mathcal{R}_2}_{i_1^{\mathcal{R}_2}, \ldots, i_r^{\mathcal{R}_2}} \pi\left(\theta^*_{\mathcal{R}_1}(i_1^{\mathcal{R}_1}, \ldots, i_r^{\mathcal{R}_1}), \theta^*_{\mathcal{R}_2}(i_1^{\mathcal{R}_2}, \ldots, i_r^{\mathcal{R}_2})\right).$$

7.1.2 Some Special Cases

We demonstrate parts of the algorithm for the cases $n = 3$, $m = 2$, and $n = 4$,$m = 3$. In particular we compare right censoring to all other schemes.

For the case $n = 3$, $m = 2$, there are only two possible censoring schemes, $\mathcal{S} = (0, 1)$ and $\mathcal{R}_1 = (1, 0)$. For these cases we can write the BLUEs as a mixture of the usual order statistics as follows:

$$\theta^*_1 = \frac{1}{2} X_{1:3} + X_{2:3} \tag{7.1.4}$$

$$\theta^*_2 = \begin{cases} X_{1:3} + \frac{1}{2} X_{2:3} & \text{with probability } 1/2 \\ X_{1:3} + \frac{1}{2} X_{3:3} & \text{with probability } 1/2 \end{cases}. \tag{7.1.5}$$

Expressing the usual order statistics by means of the exponential spacings, we obtain

$$\theta_1^* = \frac{1}{2}z_1 + \frac{1}{2}z_2, \tag{7.1.6}$$

$$\theta_2^* = \begin{cases} \frac{1}{2}z_1 + \frac{1}{4}z_2 & \text{with probability } 1/2 \\ \frac{1}{2}z_1 + \frac{1}{4}z_2 + \frac{1}{2}z_3 & \text{with probability } 1/2 \end{cases}. \tag{7.1.7}$$

Table 7.1 gives the PC probability between $\theta_{\mathcal{S}}^*$ and $\theta_{\mathcal{R}_1}^*$. For each component of the mixture distribution in (7.1.6) and (7.1.7), the unconditional conditional PC probability subject to the constraints $a > 0 (< 0)$, and $b < 0 (> 0)$ are shown explicitly in Column 3, with a decimal equivalent in Column 4. Combining these results, we readily obtain $\pi\left(\theta_{\mathcal{S}}^*, \theta_{\mathcal{R}_1}^*\right) = 0.5657$, and thus right censoring is optimal in the case of $n = 3$, $m = 2$.

Constraints	Conditions	Cond. Prob.	Value
$a = \frac{1}{4}z_2$	$a > 0, b < 0$	$1 - 4e^{-2} + 3e^{-\frac{8}{3}}$	0.3336
$b = z_1 + \frac{3}{4}z_2 - 2$	$a < 0, b > 0$	0	0
$a = \frac{1}{4}z_2 - \frac{1}{2}z_3$	$a > 0, b < 0$	$\frac{1}{3} - \frac{16}{3}e^{-3} + 9e^{-\frac{8}{3}} - 4e^{-2}$	0.0759
$b = z_1 + \frac{3}{4}z_2 + \frac{1}{2}z_3 - 2$	$a < 0, b > 0$	$4e^{-2} - \frac{16}{3}e^{-3} + 2e^{-4}$	0.1562
			0.5657

Table 7.1: PC probability $\pi\left(\theta_{\mathcal{S}}^*, \theta_{\mathcal{R}_1}^*\right)$ for n=3, r=2

When $n = 4$ and $m = 3$ there are 3 possible censoring schemes $\mathcal{S} = (0, 0, 1)$, $\mathcal{R}_1 = (0, 1, 0)$ and $\mathcal{R}_2 = (1, 0, 0)$. Again we can write the BLUEs as a mixture of the usual order statistics as follows.

$$\theta^*_{\mathcal{S}} = \frac{1}{3}X_{1:4} + \frac{1}{3}X_{2:4} + \frac{2}{3}X_{3:4}, \tag{7.1.8}$$

$$\theta^*_{\mathcal{R}_1} = \begin{cases} \frac{1}{3}X_{1:4} + \frac{2}{3}X_{2:4} + \frac{1}{3}X_{3:4} & \text{with probability } 1/2 \\ \frac{1}{3}X_{1:4} + \frac{2}{3}X_{2:4} + \frac{1}{3}X_{4:4} & \text{with probability } 1/2 \end{cases}, \tag{7.1.9}$$

$$\theta^*_{\mathcal{R}_2} = \begin{cases} \frac{2}{3}X_{1:4} + \frac{1}{3}X_{2:4} + \frac{1}{3}X_{3:4} & \text{with probability } 1/3 \\ \frac{2}{3}X_{1:4} + \frac{1}{3}X_{2:4} + \frac{1}{3}X_{4:4} & \text{with probability } 1/3 \\ \frac{2}{3}X_{1:4} + \frac{1}{3}X_{3:4} + \frac{1}{3}X_{4:4} & \text{with probability } 1/3 \end{cases}. \tag{7.1.10}$$

Once again, expressing now the usual order statistics by means of the exponential spacings, we obtain

$$\theta^*_{\mathcal{S}} = \frac{1}{3}z_1 + \frac{1}{3}z_2 + \frac{1}{3}z_3, \tag{7.1.11}$$

$$\theta^*_{\mathcal{R}_1} = \begin{cases} \frac{1}{3}z_1 + \frac{1}{3}z_2 + \frac{1}{6}z_3 & \text{with probability } 1/2 \\ \frac{1}{3}z_1 + \frac{1}{3}z_2 + \frac{1}{6}z_3 + \frac{1}{3}z_4 & \text{with probability } 1/2 \end{cases}, \tag{7.1.12}$$

$$\theta^*_{\mathcal{R}_2} = \begin{cases} \frac{1}{3}z_1 + \frac{2}{9}z_2 + \frac{1}{6}z_3 & \text{with probability } 1/3 \\ \frac{1}{3}z_1 + \frac{2}{9}z_2 + \frac{1}{6}z_3 + \frac{1}{3}z_4 & \text{with probability } 1/3 \\ \frac{1}{3}z_1 + \frac{2}{9}z_2 + \frac{1}{3}z_3 + \frac{1}{3}z_4 & \text{with probability } 1/3 \end{cases}. \tag{7.1.13}$$

For each component of the mixture forms in (7.1.12) and (7.1.13), the conditional PC probability subject to the constraints $a > 0$ (< 0) and $b < 0$ (> 0) are shown explicitly in Column 3 while the unconditional decimal equivalents are in Column 4 of Table 7.2. Combining these results suitably, we find in Table 7.2 the values of $\pi\left(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}_1}\right)$ and $\pi\left(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}_2}\right)$ to be 0.5526 and 0.5363, respectively. These readily imply once again that the right censoring case is the optimal one in the sense of Pitman closeness.

We can notice that in each case, the first mixture component seems to contribute the largest probability that $\theta^*_{\mathcal{R}_S}$ is closer than the alternative. This corresponds to the highest observations being censored. As a result, the BLUE under right censoring is surely no less than the other BLUE, and as a result the other censoring schemes tend to underestimate θ.

We have obtained the PC probabilities in this section for the cases $n = 3$ and $n = 4$, but as the sample size increases, the number of component distributions for the mixture representations grows rapidly. Thus the algorithm presented in here may not be feasible for computation. In these cases, Monte Carlo simulations may be preferable.

7.2 Simulation Study

In Tables 7.3-7.8, we present the PC probabilities comparing right censoring to other progressive censoring schemes for various sample sizes and proportion of censoring. Where possible we present all possible comparisons ($n = 5, 6, 7$), otherwise we present a broad selection of comparisons. All simulated probabilities were obtained with

Constraints	Conditions	Cond. Prob.	Value
$a = \frac{1}{6}z_3$	$a > 0, b < 0$	$1 - 9e^{-4} - 4e^{-3}$	0.3180
$b = \frac{2}{3}z_1 + \frac{2}{3}z_2 + \frac{1}{2}z_3 - 2$	$a < 0, b > 0$	0	0
$a = \frac{1}{6}z_3 - \frac{1}{3}z_4$	$a > 0, b < 0$	$\frac{1}{3} + 4e^{-3} + \frac{32}{3}e^{-\frac{9}{2}} - 27e^{-4}$	0.0783
$b = \frac{2}{3}z_1 + \frac{2}{3}z_2 + \frac{1}{2}z_3 + \frac{1}{3}z_4 - 2$	$a < 0, b > 0$	$4e^{-3} + \frac{32}{3}e^{-\frac{9}{2}} - 3e^{-6}$	0.1563
			0.5526

(a) $\pi\left(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}_1}\right)$

Constraints	Conditions	Cond. Prob.	Value
$a = \frac{1}{9}z_2 - \frac{1}{6}z_3$	$a > 0, b < 0$	$1 - 24e^{-3} + 50e^{-\frac{18}{5}} - 27e^{-4}$	0.2256
$b = \frac{2}{3}z_1 + \frac{5}{9}z_2 + \frac{1}{2}z_3 - 2$	$a < 0, b > 0$	0	0
$a = \frac{1}{9}z_2 + \frac{1}{6}z_3 - \frac{1}{3}z_4$	$a > 0, b < 0$	$\frac{1}{2} - 36e^{-3} + 125e^{-\frac{18}{5}} - \frac{243}{2}e^{-4} + 32e^{-\frac{9}{2}}$	0.0844
$b = \frac{2}{3}z_1 + \frac{5}{9}z_2 + \frac{1}{2}z_3 + \frac{1}{3}z_4 - 2$	$a < 0, b > 0$	$12e^{-3} - \frac{81}{2}e^{-4} + 32e^{-\frac{9}{2}} - 3e^{-6}$	0.0679
$a = \frac{1}{9}z_2 - \frac{1}{3}z_4$	$a > 0, b < 0$	$\frac{1}{4} + 24e^{-3} - \frac{125}{2}e^{-\frac{18}{5}} + \frac{81}{4}e^{-4}$	0.0360
$b = \frac{2}{3}z_1 + \frac{5}{9}z_2 + \frac{2}{3}z_3 + \frac{1}{3}z_4 - 2$	$a < 0, b > 0$	$\frac{81}{4}e^{-4} - \frac{3}{2}e^{-6}$	0.1224
			0.5363

(b) $\pi\left(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}_2}\right)$

Table 7.2: PC probabilities $\pi\left(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}_1}\right)$ and $\pi\left(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}_2}\right)$ for $n = 4$, $r = 3$

1,000,000 Monte Carlo simulations.

We denote $\mathcal{S} = (0, 0, \ldots, 0, n-r)$ as the usual Type-II right censoring scheme, and $\mathcal{R}$ as the alternative. The results for all tables seem to confirm the conjecture that right censoring is optimal in the Pitman closeness sense.

It is also of interest to note, that the near extremal scheme $\mathcal{T} = (0, \ldots, 0, 1, n-r)$, for fixed n and r, has the highest PC probability amongst all of the comparisons considered.

We can also note that when the sample size n is fixed, the PC probabilities tend to increase as the censoring proportion increases. For a fixed number of censored items and an increasing number of failures, the PC probabilities tend to decrease but not rapidly. So for even moderate sample sizes, there are schemes that $\mathcal{S}$ is moderately better than, and seemingly none that it is worse than.

i	1	2	3	4	$\pi(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}})$
$\mathcal{R}_1$	1	0	0	0	0.5225
$\mathcal{R}_2$	0	1	0	0	0.5303
$\mathcal{R}_3$	0	0	1	0	0.5442

(a) $r = 4$

i	1	2	3	$\pi(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}})$
$\mathcal{R}_1$	2	0	0	0.5203
$\mathcal{R}_2$	1	1	0	0.5261
$\mathcal{R}_3$	1	0	1	0.5562
$\mathcal{R}_4$	0	2	0	0.5343
$\mathcal{R}_5$	0	1	1	0.5685

(b) $r = 3$

i	1	2	$\pi(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}})$
$\mathcal{R}_1$	3	0	0.5321
$\mathcal{R}_2$	2	1	0.5644
$\mathcal{R}_3$	1	2	0.5931

(c) $r = 2$

Table 7.3: PC probabilities for different choices of r when $n = 5$

i	1	2	3	4	5	$\pi(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}})$
$\mathcal{R}_1$	1	0	0	0	0	0.5143
$\mathcal{R}_2$	0	1	0	0	0	0.5202
$\mathcal{R}_3$	0	0	1	0	0	0.5280
$\mathcal{R}_4$	0	0	0	1	0	0.5394

(a) $r = 5$

i	1	2	3	4	$\pi(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}})$
$\mathcal{R}_1$	2	0	0	0	0.5145
$\mathcal{R}_2$	1	1	0	0	0.5155
$\mathcal{R}_3$	1	0	1	0	0.5168
$\mathcal{R}_3$	1	0	0	1	0.5384
$\mathcal{R}_5$	0	2	0	0	0.5167
$\mathcal{R}_6$	0	1	1	0	0.5226
$\mathcal{R}_7$	0	1	0	1	0.5478
$\mathcal{R}_8$	0	0	2	0	0.5593
$\mathcal{R}_9$	0	0	1	1	0.5594

(b) $r = 4$

i	1	2	3	$\pi(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}})$
$\mathcal{R}_1$	3	0	0	0.5157
$\mathcal{R}_2$	2	1	0	0.5153
$\mathcal{R}_3$	2	0	1	0.5341
$\mathcal{R}_3$	1	2	0	0.5201
$\mathcal{R}_5$	1	1	1	0.5437
$\mathcal{R}_6$	1	0	2	0.5667
$\mathcal{R}_7$	0	3	0	0.5262
$\mathcal{R}_8$	0	2	1	0.5524
$\mathcal{R}_9$	0	1	2	0.5754

(c) $r = 3$

i	1	2	$\pi(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}})$
$\mathcal{R}_1$	4	0	0.5248
$\mathcal{R}_2$	3	1	0.5519
$\mathcal{R}_3$	2	2	0.5787
$\mathcal{R}_4$	1	3	0.5969

(d) $r = 2$

Table 7.4: PC probabilities for different choices of r when $n = 6$

i	1	2	3	4	5	6	$\pi(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}})$
$\mathcal{R}_1$	1	0	0	0	0	0	0.5112
$\mathcal{R}_2$	0	1	0	0	0	0	0.5140
$\mathcal{R}_3$	0	0	1	0	0	0	0.5182
$\mathcal{R}_4$	0	0	0	1	0	0	0.5249
$\mathcal{R}_5$	0	0	0	0	1	0	0.5363

(a) $r = 6$

i	1	2	$\pi(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}})$
$\mathcal{R}_1$	5	0	0.5206
$\mathcal{R}_2$	4	1	0.5430
$\mathcal{R}_3$	3	2	0.5657
$\mathcal{R}_4$	2	3	0.5856
$\mathcal{R}_5$	1	4	0.5982

(b) $r = 2$

i	1	2	3	4	5	$\pi(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}})$
$\mathcal{R}_1$	2	0	0	0	0	0.5134
$\mathcal{R}_2$	1	1	0	0	0	0.5129
$\mathcal{R}_3$	1	0	1	0	0	0.5121
$\mathcal{R}_4$	1	0	0	1	0	0.5127
$\mathcal{R}_5$	1	0	0	0	1	0.5284
$\mathcal{R}_6$	0	2	0	0	0	0.5135
$\mathcal{R}_7$	0	1	1	0	0	0.5137
$\mathcal{R}_8$	0	1	0	1	0	0.5157
$\mathcal{R}_9$	0	1	0	0	1	0.5346
$\mathcal{R}_{10}$	0	0	2	0	0	0.5152
$\mathcal{R}_{11}$	0	0	1	1	0	0.5202
$\mathcal{R}_{12}$	0	0	1	0	1	0.5424
$\mathcal{R}_{13}$	0	0	0	2	0	0.5265
$\mathcal{R}_{14}$	0	0	0	1	1	0.5528

(c) $r = 5$

i	1	2	3	4	$\pi(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}})$
$\mathcal{R}_1$	3	0	0	0	0.5131
$\mathcal{R}_2$	2	1	0	0	0.5134
$\mathcal{R}_3$	2	0	1	0	0.5122
$\mathcal{R}_4$	2	0	0	1	0.5213
$\mathcal{R}_5$	1	2	0	0	0.5133
$\mathcal{R}_6$	1	1	1	0	0.5120
$\mathcal{R}_7$	1	1	0	1	0.5243
$\mathcal{R}_8$	1	0	2	0	0.5147
$\mathcal{R}_9$	1	0	1	1	0.5315
$\mathcal{R}_{10}$	1	0	0	2	0.5495
$\mathcal{R}_{11}$	0	3	0	0	0.5141
$\mathcal{R}_{12}$	0	2	1	0	0.5135
$\mathcal{R}_{13}$	0	2	0	1	0.5282
$\mathcal{R}_{14}$	0	1	2	0	0.5176
$\mathcal{R}_{15}$	0	1	1	1	0.5377
$\mathcal{R}_{16}$	0	1	0	2	0.5573
$\mathcal{R}_{17}$	0	0	3	0	0.5218
$\mathcal{R}_{18}$	0	0	2	1	0.5451
$\mathcal{R}_{19}$	0	0	1	2	0.5648

(d) $r = 4$

i	1	2	3	$\pi(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}})$
$\mathcal{R}_1$	4	0	0	0.5145
$\mathcal{R}_2$	3	1	0	0.5143
$\mathcal{R}_3$	3	0	1	0.5253
$\mathcal{R}_4$	2	2	0	0.5126
$\mathcal{R}_5$	2	1	1	0.5279
$\mathcal{R}_6$	2	0	2	0.5438
$\mathcal{R}_7$	1	3	0	0.5168
$\mathcal{R}_8$	1	2	1	0.5360
$\mathcal{R}_9$	1	1	2	0.5555
$\mathcal{R}_{10}$	1	0	3	0.5725
$\mathcal{R}_{11}$	0	4	0	0.5203
$\mathcal{R}_{12}$	0	3	1	0.5417
$\mathcal{R}_{13}$	0	2	2	0.5627
$\mathcal{R}_{14}$	0	1	3	0.5781

(e) $r = 3$

Table 7.5: PC probabilities for different choices of r when $n = 7$

i	1	2	3	4	5	6	7	8	9	10	$\pi(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}})$
$\mathcal{R}_1$	5	0	0	0	0	0	0	0	0	0	0.5049
$\mathcal{R}_2$	4	1	0	0	0	0	0	0	0	0	0.5047
$\mathcal{R}_3$	4	0	0	0	0	0	0	0	0	1	0.5065
$\mathcal{R}_4$	3	2	0	0	0	0	0	0	0	0	0.5051
$\mathcal{R}_5$	3	0	0	0	0	0	0	0	0	2	0.5081
$\mathcal{R}_6$	2	0	0	0	2	0	0	0	0	1	0.5071
$\mathcal{R}_7$	2	1	0	0	0	0	0	0	1	1	0.5084
$\mathcal{R}_8$	1	1	1	1	1	0	0	0	0	0	0.5052
$\mathcal{R}_9$	1	0	1	0	1	0	1	0	1	0	0.5074
$\mathcal{R}_{10}$	0	0	0	1	2	1	1	0	0	0	0.5058
$\mathcal{R}_{11}$	0	0	0	1	1	1	1	1	0	0	0.5053
$\mathcal{R}_{12}$	0	0	0	0	2	3	0	0	0	0	0.5057
$\mathcal{R}_{13}$	0	1	0	1	0	1	0	1	0	1	0.5080
$\mathcal{R}_{14}$	0	0	0	0	0	1	1	1	1	1	0.5098
$\mathcal{R}_{15}$	1	1	0	0	0	0	0	0	1	2	0.5111
$\mathcal{R}_{16}$	1	0	0	0	0	2	0	0	0	2	0.5100
$\mathcal{R}_{17}$	2	0	0	0	0	0	0	0	0	3	0.5126
$\mathcal{R}_{18}$	0	0	0	0	0	0	0	0	2	3	0.5372
$\mathcal{R}_{19}$	1	0	0	0	0	0	0	0	0	4	0.5219
$\mathcal{R}_{20}$	0	0	0	0	0	0	0	0	1	4	0.5431

(a) $r = 10$

i	1	2	3	4	5	$\pi(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}})$
$\mathcal{R}_1$	10	0	0	0	0	0.5063
$\mathcal{R}_2$	9	1	0	0	0	0.5059
$\mathcal{R}_3$	8	2	0	0	0	0.5055
$\mathcal{R}_4$	8	0	2	0	0	0.5067
$\mathcal{R}_5$	8	0	0	0	2	0.5099
$\mathcal{R}_6$	6	2	2	0	0	0.5063
$\mathcal{R}_7$	6	0	2	0	2	0.5095
$\mathcal{R}_8$	4	0	2	0	4	0.5163
$\mathcal{R}_9$	2	2	2	2	2	0.5130
$\mathcal{R}_{10}$	3	2	0	2	3	0.5151
$\mathcal{R}_{11}$	1	2	4	2	1	0.5109
$\mathcal{R}_{12}$	0	3	4	3	0	0.5089
$\mathcal{R}_{13}$	0	0	10	0	0	0.5058
$\mathcal{R}_{14}$	0	5	0	5	0	0.5088
$\mathcal{R}_{15}$	2	0	2	0	6	0.5279
$\mathcal{R}_{16}$	0	0	2	2	6	0.5439
$\mathcal{R}_{17}$	2	0	0	0	8	0.5430
$\mathcal{R}_{18}$	0	0	2	0	8	0.5551
$\mathcal{R}_{19}$	0	0	0	2	8	0.5607
$\mathcal{R}_{20}$	0	0	0	1	9	0.5623

(b) $r = 5$

i	1	2	3	$\pi(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}})$
$\mathcal{R}_1$	12	0	0	0.5065
$\mathcal{R}_2$	11	1	0	0.5089
$\mathcal{R}_3$	10	2	0	0.5107
$\mathcal{R}_4$	8	2	2	0.5170
$\mathcal{R}_5$	7	3	2	0.5170
$\mathcal{R}_6$	6	3	3	0.5216
$\mathcal{R}_7$	5	4	3	0.5219
$\mathcal{R}_8$	7	0	5	0.5299
$\mathcal{R}_9$	5	7	0	0.5059
$\mathcal{R}_{10}$	6	0	6	0.5374
$\mathcal{R}_{11}$	4	4	4	0.5287
$\mathcal{R}_{12}$	5	2	5	0.5333
$\mathcal{R}_{13}$	0	7	5	0.5483
$\mathcal{R}_{14}$	5	0	7	0.5456
$\mathcal{R}_{15}$	3	4	5	0.5383
$\mathcal{R}_{16}$	3	3	6	0.5455
$\mathcal{R}_{17}$	2	3	7	0.5573
$\mathcal{R}_{18}$	2	2	8	0.5642
$\mathcal{R}_{19}$	0	2	10	0.5798
$\mathcal{R}_{20}$	0	1	11	0.5802

(c) $r = 3$

Table 7.6: PC probabilities for different choices of r when $n = 15$

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	$\pi(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}})$
$\mathcal{R}_1$	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.5040
$\mathcal{R}_2$	4	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0.5041
$\mathcal{R}_3$	2	0	0	0	0	0	0	1	0	0	0	0	0	0	2	0.5069
$\mathcal{R}_4$	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	0.5052
$\mathcal{R}_5$	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0.5064
$\mathcal{R}_6$	0	0	0	0	0	0	0	5	0	0	0	0	0	0	0	0.5042
$\mathcal{R}_7$	0	0	0	0	0	0	1	3	1	0	0	0	0	0	0	0.5044
$\mathcal{R}_8$	0	0	0	0	0	0	2	1	2	0	0	0	0	0	0	0.5043
$\mathcal{R}_9$	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0.5037
$\mathcal{R}_{10}$	0	0	0	0	0	0	0	0	0	0	0	0	0	1	4	0.5352

(a) $r = 15$

i	1	2	3	4	5	$\pi(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}})$
$\mathcal{R}_1$	15	0	0	0	0	0.5046
$\mathcal{R}_2$	14	1	0	0	0	0.5047
$\mathcal{R}_3$	7	0	1	0	7	0.5166
$\mathcal{R}_4$	5	0	5	0	5	0.5137
$\mathcal{R}_5$	3	3	3	3	3	0.5126
$\mathcal{R}_6$	0	5	5	5	0	0.5084
$\mathcal{R}_7$	0	2	11	2	0	0.5069
$\mathcal{R}_8$	0	7	8	0	0	0.5035
$\mathcal{R}_9$	0	0	8	7	0	0.5058
$\mathcal{R}_{10}$	0	0	0	1	14	0.5620

(b) $r = 5$

i	1	2	3	4	5	6	7	8	9	10	$\pi(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}})$
$\mathcal{R}_1$	10	0	0	0	0	0	0	0	0	0	0.5047
$\mathcal{R}_2$	9	1	0	0	0	0	0	0	0	0	0.5040
$\mathcal{R}_3$	3	0	0	0	2	2	0	0	0	3	0.5075
$\mathcal{R}_4$	2	0	2	0	2	0	2	0	2	0	0.5056
$\mathcal{R}_5$	1	1	1	1	1	1	1	1	1	1	0.5059
$\mathcal{R}_6$	0	0	0	0	5	5	0	0	0	0	0.5036
$\mathcal{R}_7$	0	0	0	3	4	3	0	0	0	0	0.5041
$\mathcal{R}_8$	0	2	0	2	0	2	0	2	0	2	0.5059
$\mathcal{R}_9$	0	0	1	1	3	3	1	1	0	0	0.5042
$\mathcal{R}_{10}$	0	0	0	0	0	0	0	0	1	9	0.5441

(c) $r = 10$

i	1	2	3	4	5	6	7	$\pi(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}})$
$\mathcal{R}_1$	13	0	0	0	0	0	0	0.5035
$\mathcal{R}_2$	12	1	0	0	0	0	0	0.5041
$\mathcal{R}_3$	5	0	0	3	0	0	5	0.5096
$\mathcal{R}_4$	2	2	2	1	2	2	2	0.5086
$\mathcal{R}_5$	1	1	1	7	1	1	1	0.5072
$\mathcal{R}_6$	0	0	0	13	0	0	0	0.5041
$\mathcal{R}_7$	0	0	2	8	3	0	0	0.5037
$\mathcal{R}_8$	0	0	3	8	2	0	0	0.5039
$\mathcal{R}_9$	0	0	4	5	4	0	0	0.5050
$\mathcal{R}_{10}$	0	0	0	0	0	1	12	0.5520

(d) $r = 7$

Table 7.7: PC probabilities for different choices of r when $n = 20$

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	$\pi(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}})$
$\mathcal{R}_1$	10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.5024
$\mathcal{R}_2$	9	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.5016
$\mathcal{R}_3$	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	0.5050
$\mathcal{R}_4$	3	0	0	0	0	0	0	0	0	2	2	0	0	0	0	0	0	0	0	3	0.5036
$\mathcal{R}_5$	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0.5035
$\mathcal{R}_6$	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0.5034
$\mathcal{R}_7$	0	0	0	0	0	0	0	0	0	5	5	0	0	0	0	0	0	0	0	0	0.5028
$\mathcal{R}_8$	0	0	0	0	0	0	0	0	2	2	4	2	0	0	0	0	0	0	0	0	0.5027
$\mathcal{R}_9$	0	0	0	0	0	0	0	0	2	4	2	2	0	0	0	0	0	0	0	0	0.5021
$\mathcal{R}_{10}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	9	0.5306

(a) $r = 20$

i	1	2	3	4	5	$\pi(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}})$
$\mathcal{R}_1$	25	0	0	0	0	0.5033
$\mathcal{R}_2$	24	1	0	0	0	0.5034
$\mathcal{R}_3$	12	0	1	0	12	0.5163
$\mathcal{R}_4$	10	0	5	0	10	0.5138
$\mathcal{R}_5$	5	5	5	5	5	0.5119
$\mathcal{R}_6$	0	12	1	12	0	0.5019
$\mathcal{R}_7$	0	5	15	5	0	0.5070
$\mathcal{R}_8$	1	5	13	5	1	0.5080
$\mathcal{R}_9$	0	0	25	0	0	0.5028
$\mathcal{R}_{10}$	0	0	0	1	24	0.5616

(b) $r = 5$

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	$\pi(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}})$
$\mathcal{R}_1$	15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.5018
$\mathcal{R}_2$	14	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0.5031
$\mathcal{R}_3$	5	0	0	0	0	0	0	5	0	0	0	0	0	0	5	0.5050
$\mathcal{R}_4$	3	0	0	3	0	0	3	0	0	3	0	0	3	0	0	0.5030
$\mathcal{R}_5$	0	0	3	0	0	3	0	0	3	0	0	3	0	0	3	0.5040
$\mathcal{R}_6$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0.5040
$\mathcal{R}_7$	0	0	0	0	0	3	3	3	3	3	0	0	0	0	0	0.5023
$\mathcal{R}_8$	0	0	0	0	0	0	5	5	5	0	0	0	0	0	0	0.5021
$\mathcal{R}_9$	0	0	0	0	0	0	0	15	0	0	0	0	0	0	0	0.5027
$\mathcal{R}_{10}$	0	0	0	0	0	0	0	0	0	0	0	0	0	1	14	0.5359

(c) $r = 15$

i	1	2	3	4	5	6	7	8	9	10	$\pi(\theta^*_{\mathcal{S}}, \theta^*_{\mathcal{R}})$
$\mathcal{R}_1$	20	0	0	0	0	0	0	0	0	0	0.5034
$\mathcal{R}_2$	19	1	0	0	0	0	0	0	0	0	0.5024
$\mathcal{R}_3$	5	0	0	0	5	5	0	0	0	5	0.5061
$\mathcal{R}_4$	4	0	4	0	4	0	4	0	4	0	0.5045
$\mathcal{R}_3$	0	4	0	4	0	4	0	4	0	4	0.5064
$\mathcal{R}_6$	2	2	2	2	2	2	2	2	2	2	0.5052
$\mathcal{R}_7$	0	0	0	0	10	10	0	0	0	0	0.5018
$\mathcal{R}_8$	0	0	0	6	8	6	0	0	0	0	0.5016
$\mathcal{R}_9$	0	0	0	0	6	8	6	0	0	0	0.5025
$\mathcal{R}_{10}$	0	0	0	0	0	0	0	0	1	19	0.5439

(d) $r = 10$

Table 7.8: PC probabilities for different choices of r when $n = 30$

Chapter 8

Conclusions and Further Work

In this thesis we have discussed inference under Type-II (right and doubly) and progressive Type-II censoring.

In Chapters 3 to 5 we have demonstrated how to obtain mixture representations for pooled order statistics based on multiple independent samples. We have shown how the marginal and bivariate representations can be used for nonparametric inference in the way of confidence intervals for quantiles, tolerance intervals, and prediction intervals. However, these mixture representations can be used efficiently wherever the results for regular order statistics is previously studied. In the case of Type-II censoring we have shown that there are significant gains in terms of coverage probability over the one-sample case. This gain is not clearly evident in progressive censoring.

However, the mixture distributions presented can be taxing in terms of computational power. As a result we suggested simulating the mixture weights. In the example considered in Chapter 4 we have shown that even with a small number of simulations, the absolute and relative errors in the final estimated probabilities are very small. Since one is typically interested in large probabilities, then the absolute and relative errors are negligible. Thus, we suggest that this is an ideal method of obtaining the mixture weights for the representations in terms of regular order statistics when the number of samples, and/or the number of censored items is large. However, simulation is not always easy. The mixture representation in Theorem 5.1.1 is a non-convex mixture of progressively censored samples. In some cases these correspond to an outcome in terms of the pooling of the samples. In others however, there is no clear interpretation of the component distributions in terms of the ordering of the complete or censored data.

We have briefly considered asymptotic results in Section 3.2.4. In particular we have shown that when the sample size is bounded uniformly, $F_{Z_{(\dot{r})}} \to 1$ as $B \to \infty$ regardless of the amount of censoring in each sample. This also applies to doubly Type-II censored samples in regards to $Z_{(1)}$ and $Z_{(\dot{r})}$. This result immediately applies to the progressively Type-II censoring case. However, in the progressive censoring scenario, there may not be any gain over the one sample case depending on the censoring schemes considered.

In Chapter 6 we have given some results concerning quantile estimation with PCOS, using Pitman's measure of closeness as an optimality criterion. We gave a symmetry result for the median, and for some special distributions gave explicit results. We considered numerical results with many censoring schemes and distributions and noted that the Pitman's criterion is quite robust, with regards to the distribution, for central order quantiles. This is most notable for symmetric distributions.

Progressive censoring schemes are the focus of Chapter 7. In this chapter we discuss how one can use Pitman's measure of closeness to determine an optimal censoring scheme for determining the scale parameter of a single parameter exponential distribution. Since the marginal distribution of the BLUE in this case is free of the censoring scheme, one must consider other methods. We demonstrate how to obtain the PC probability comparing two schemes and do exact calculations, and simulate values for larger sample sizes. The results support the conjecture that right censoring is the scheme that produces a BLUE that is Pitman Closer than the BLUE generated by any other censoring scheme.

8.1 Further Work

There are many problems which have presented themselves for future consideration. While the nonparametric methods in Chapters 3 to 5 cover a variety of situations, one may ask how to include more information. For example, if we know that the underlying distribution is symmetric, are there simple ways of including this into the nonparametric intervals? How can one include information of covariates, perhaps concomitants?

In Section 3.2.4 we showed that the sample maximum from the pooled samples may have a non-degenerate distribution on the support of the underlying distribution. In such a case some subsequence either converges in probability to the lower endpoint of the distribution, or has a non-degenerate distribution on the entire support. It is natural to ask what are necessary and/or sufficient conditions for the maximum to converge to the upper endpoint. Furthermore, can we normalize the maximum or central order statistics to obtain non-degenerate limiting distributions as with regular order statistics?

It also seems natural to consider an empirical type distribution based on the pooled data. A possible way to do this would be as $\widehat{F(x)} = \frac{n+1}{n} E_{Z_{(i)}}$ for $x \in [z_{(i)}, z_{(i+1)})$, $i = 1, 2, \ldots \dot{r} - 1$ and $\widehat{F(x)} = 1$ for $x \geq z_{(\dot{r})}$. In the case of a complete sample this would be the standard empirical distribution function. Perhaps this can be used to obtain Kolmogorov-Smirnov type tests.

In Chapter 5 we consider a non-convex mixture distribution. There seems to be no clear way to sample from such a distribution. One would need to simulate as to provide an estimate which is a valid probability distribution function.

We leave one clear open problem in Chapter 7. That is, we have conjectured that right censoring produces a Pitman best scheme (and left censoring Pitman worst), can this be formalized?

We leave these questions open and look forward to the forthcoming answers, and questions yet to come.

Appendix A

Code for Chapter 3

function:	qpxkn, qpziu, qpzi2
	input
q	The probability for the q-th quantile, ξ_q
k/l	Lower/Upper bounds in terms of OS indices
n(N)	The overall sample size
ii	An integer vector of OS indices
weights	A double vector of mixture probabilities (corresponding to ii) ii1/ii2, and weights1/weights2 correspond to $Z_{(i_1)}$ and $Z_{(i_2)}$ respectively
	output
Returns a double of $P(X_{k:n} < \xi_q < X_{l:n})/P(\xi_q < Z_{(i)})/P(Z_{(i_1)} < \xi_q < Z_{(i_2)})$	

```
qpxkn<-function(q,k,l,n){sum(dbinom(k:(l-1),n,q))}
```

```
qpziu<-function(ii,weights,N,q){sum(weights*sapply(ii,qpxkn,q=q,k=0,n=N)
    )}
```

```
qpzi2<-function(ii1,ii2,weights1,weights2,N,q){qpziu(ii2,weights2,N,q)-
    qpziu(ii1,weights1,N,q)}
```

function:	samp2
	input
m/n	Sample sizes
s/r	Number of observations (corresponding to n/m)
i	Index for i-th pooled OS $Z_{(i)}$
	output

Returns a matrix with 2 columns. The first column stores the index of an order statistic $X_{j:m+n}$. The second column stores the mixture weight $w_{ij} = P(Z_{(i)} = X_{j:m+n})$

```
samp2<-function(m,n,s,r,i){
wikl<-function(i,k,l,n){choose(i-1,k-1)*choose(sum(n)-i,l-k)/choose(sum(n),n[1])}
if(r<s){q<-c(m,n,s,r);m<-q[2];n<-q[1];s<-q[4];r<-q[3]}
if(1<=i&i<=min(r,s)){return(matrix(c(i,1),ncol=2))}
if(i<=max(r,s)){out<-cbind(i:(i+m-s),rep(0,m-s+1))
          for(l in (s+1):m){out[l-s+1,2]<-wikl(i-s+l,l+1,m+1,c(m,n))}
          out[1,2]<-1-sum(out[,2]);return(matrix(out,ncol=2))
     }
if(i<=r+s){
top<-max(m-s,n-r)
out<-cbind(i:(i+top),rep(0,top+1))
for(l in (s+1):m){out[l-s+1,2]<-wikl(i-s+l,l+1,m+1,c(m,n))}
for(l in (r+1):n){out[l-r+1,2]<-out[l-r+1,2]+wikl(i-r+l,l+1,n+1,c(m,n))}
out[1,2]<-1-sum(out[,2])
return(matrix(out,ncol=2))
}
stop("i not valid")
}
```

function:	wiklj
	input
i	The index such that $Z_{(j)} = X_{i:\dot{r}}$
k/l	Vector containing the number of observed and unobserved items from each sample that fall below/above $Z_{(j)}$
j	Vector of sample sizes
N	Overall sample size
	output
Returns a double with the weights $W_{\{i\},\{k\},\{l\},\{j\}}$ as in Chapter 3	

```
wiklj<-function(i,k,l,j,N){
if(i<sum(k)|N-i<sum(l)){return(0)}
exp(lgamma(i)+lgamma(N-i+1)-sum(lgamma(k+1))-sum(lgamma(l+1))-lgamma(i-
    sum(k))-lgamma(N-i-sum(l)+1)-lgamma(N+1)+sum(lgamma(j+1))+lgamma(N-
    sum(j)+1))
}
```

function:	sampkall
	input
n	Vector of sample sizes
r	Vector of number of observations in each sample
i	Index for i-th pooled OS $Z_{(i)}$
	output
Returns a matrix with 2 columns. The first column stores the index of an order statistic $X_{j:\dot{r}}$. The second column stores the mixture weight $w_{ij} = P(Z_{(i)} = X_{j:\dot{r}})$. This program is ONLY used for the case where $n_b = n$, $r_b = r$, $b = 1, 2, \ldots, B$.	

```
sampkall<-function(n,r,i){
B<-length(n);N<-sum(n);R<-sum(r);nmr<-n-r
# Calculate combinations for sigmabprime
tempsbp<-vector("list",B-1);for(yy in 1:(B-1)){tempsbp[[yy]]<-combn(1:(B
    -1),yy)}
# Calculate combinations for b' and b''alpha
tempnmr<-vector("list",B);for(yy in 1:B){tempnmr[[yy]]<-1:(n[yy]-r[yy])}
tempr<-vector("list",B);for(yy in 1:B){tempr[[yy]]<-0:r[yy]}
# Main Loop
out<-cbind(i:N,0);b<-1
for(k in 1:r[1]){
      bdim<-1
      while(bdim<B&&sum(r[-b][1:bdim])<=i-k){
            j<-1
            bweight<-B*choose(B-1,bdim)
            imkmRjp<-i-k-sum(r[-b][tempsbp[[bdim]][,j]])
```

```
          if(imkmRjp>=0&&i-k<=R-r[b]){ # Permutation b' satisfies condition
                tempind<-which(r[-b][-tempsbp[[bdim]][,j]]<imkmRjp)
                if(length(tempind)==0){# Case with b''alpha empty
                    if(bdim==B-1&&imkmRjp==0){ # b''alpha empty b''beta empty
                        xx<-as.matrix(expand.grid(tempnmr[-b]),ncol=bdim)
                        for(yy in 1:(dim(xx)[1])){out[1+sum(xx[yy,]),2]<-out[1+sum(xx[yy,]),2]+bweight*wiklj(i+sum(xx[yy,]),c(k-1,r[-b]+xx[yy,]),c(n[b]-k,nmr[-b]-xx[yy,]),n,N)}
                    }
                    if(bdim<B-1){ # b''alpha empty b''beta non-empty
                        xx<-as.matrix(expand.grid(tempnmr[-b][tempsbp[[bdim]][,j]]))
                        for(yy in 1:(dim(xx)[1])){out[1+sum(xx[yy,]),2]<-out[1+sum(xx[yy,]),2]+bweight*wiklj(i+sum(xx[yy,]),c(k-1,r[-b][tempsbp[[bdim]][,j]]+xx[yy,]),c(n[b]-k,nmr[-b][tempsbp[[bdim]][,j]]-xx[yy,]),c(n[b],n[-b][tempsbp[[bdim]][,j]]),N)}
                    }
                } else{ # b''alpha non-empty and b''beta empty
                    xx<-as.matrix(expand.grid(c(tempnmr[-b][tempsbp[[bdim]][,j]],tempr[-b][-tempsbp[[bdim]][,j]][tempind])))
                    if(length(tempind)==1){xx<-matrix(xx[which(xx[,-(1:bdim)]==imkmRjp),],ncol=bdim+1)}
                    if(length(tempind)>1){xx<-matrix(xx[which(apply(xx[,-(1:bdim)],1,sum)==imkmRjp),],ncol=bdim+length(tempind))}
                    for(yy in 1:(dim(xx)[1])){out[1+sum(xx[yy,1:bdim]),2]<-out[1+sum(xx[yy,1:bdim]),2]+bweight*wiklj(i+sum(xx[yy,1:bdim]),c(k-1,r[-b][tempsbp[[bdim]][,j]]+xx[yy,1:bdim],xx[yy,-(1:bdim)]),c(n[b]-k,nmr[-b][tempsbp[[bdim]][,j]]-xx[yy,1:bdim],n[-b][-tempsbp[[bdim]][,j]][tempind]-xx[yy,-(1:bdim)]),c(n[b],n[-b][tempsbp[[bdim]][,j]],n[-b][-tempsbp[[bdim]][,j]][tempind]),N)}
                }
          }
      bdim<-bdim+1
      }
}
out[1,2]<-1-sum(out[,2])
if(any(out[,2]<0)){print("Error has occurred some mixing probability is negative")}
return(matrix(out[which(out[,2]>0),],ncol=2,))
}
```

function:	sampk
	input
n	Vector of sample sizes
r	Vector of number of observations in each sample
i	Index for i-th pooled OS $Z_{(i)}$
	output

Returns a matrix with 2 columns. The first column stores the index of an order statistic $X_{j:m+n}$. The second column stores the mixture weight $w_{ij} = P(Z_{(i)} = X_{j:\dot{r}})$. This program will call sampkall or samp2 where appropriate

```
sampk<-function(n,r,i){

if(i-floor(i)!=0){cat("i not an integer, i set to", floor(i), "\n");i<-
    floor(i)}
if(any(n-floor(n)!=0)){cat("some n is not an integer, n set to", floor(n
    ), "\n");n<-floor(n)}
if(any(r-floor(r)!=0)){cat("some r is not an integer, r set to", floor(r
    ), "\n");r<-floor(r)}

B<-length(n);N<-sum(n);R<-sum(r)

# Terminating Conditions
if(B!=length(r)){stop("n and r not of same length")}
if(any(n<r)){stop("Some r value is less than its corresponding n")}
if(any(r<1)){stop("Some r value is less than 1")}
if(i<1){stop("invalid i, must be integer from 1 to sum of r")}
if(i>R){stop("i too large must be less than sum of r")}

# Sort n and r, first by r then within each r by n
temp1<-cbind(n,r)
temp1<-temp1[order(temp1[,2],temp1[,1]),]
n<-temp1[,1]; r<-temp1[,2]

# Merge complete samples into 1 sample and append as the last sample
mflag<-0
if(any(n==r)){
     temp3<-which(n==r)
     r<-c(r[-temp3],sum(r[temp3]))
     n<-c(n[-temp3],sum(n[temp3]))
     Borig<-B;B<-length(r);mflag<-1
}
nmr<-n-r

# Special Cases
if(B==1){return(matrix(c(i,1),ncol=2))}
if(B==2){return(samp2(n[1],n[2],r[1],r[2],i))}
if(length(unique(r))==1&length(unique(n))==1){return(sampkall(n,r,i))}

```

```
# Condition 1 - trivial
if(1<=i&i<=r[1]){return(matrix(c(i,1),ncol=2))}

# Calculate weights for Repetitions of (n_j,r_j)
nr<-rbind(n,r);weights<-rep(1,B)
for(yy in 1:(B-1)){
      if(weights[yy]!=0){
            for(zz in (yy+1):B){if(all(nr[,zz]==nr[,yy])){weights[c(yy,zz)]<-c(weights[yy]+1,0)}}
      }
}

# Calculate combinations for sigmabprime
tempsbp<-vector("list",B-1-mflag);for(yy in 1:(B-1-mflag)){tempsbp[[yy]]<-combn(1:(B-1-mflag),yy)}
if(mflag==1){tempsbp1<-vector("list",B-1)
for(yy in 1:(B-1)){tempsbp1[[yy]]<-combn(1:(B-1),yy)}}

# Calculate combinations for b' and b''alpha
tempnmr<-vector("list",B);for(yy in 1:B){tempnmr[[yy]]<-1:(n[yy]-r[yy])}
tempr<-vector("list",B);for(yy in 1:B){tempr[[yy]]<-0:r[yy]}

# Main Loop
out<-cbind(i:N,0)

if(mflag==1){ # When 1 (or more) complete sample(s) (i.e., mflag = 1)
for(b in 1:(B-1)){# Going over b where the bth sample is not complete
 if(weights[b]!=0){
  for(k in 1:r[b]){
   bdim<-1
         while(bdim<(B-1)&&sum(r[-b][1:bdim])<=i-k){
           for(j in 1:choose(B-2,bdim)){
     imkmRjp<-i-k-sum(r[-b][tempsbp[[bdim]][,j]])
       if(imkmRjp>=0&&i-k<=R-r[b]){     # Permutation b' satisfies condition
       tempind<-which(r[-c(b,B)][-tempsbp[[bdim]][,j]]<imkmRjp)
       if(length(tempind)==0){       # Case with b''alpha empty
              ##### b''alpha empty b''beta non-empty
         xx<-as.matrix(expand.grid(tempnmr[-b][tempsbp[[bdim]][,j]]))
         for(yy in 1:(dim(xx)[1])){
           out[1+sum(xx[yy,]),2]<-out[1+sum(xx[yy,]),2]+weights[b]*wiklj(i+sum(xx[yy,]),c(k-1,r[-b][tempsbp[[bdim]][,j]]+xx[yy,]),c(n[b]-k,nmr[-b][tempsbp[[bdim]][,j]]-xx[yy,]),c(n[b],n[-b][tempsbp[[bdim]][,j]]),N)
         }
       }

       else{          # Case with b''alpha non-empty

        if(bdim+1+length(tempind)<B){    ##### b''alpha non-empty and b''beta non-empty
```

```
            xx<-as.matrix(expand.grid(c(tempnmr[-b][tempsbp[[bdim]][,j]],tempr[-b][-tempsbp[[bdim]][,j]][tempind])))
           if(length(tempind)==1){xx<-matrix(xx[which(xx[,-(1:bdim)]<=imkmRjp),],ncol=bdim+1)}
            if(length(tempind)>1){xx<-matrix(xx[which(apply(xx[,-(1:bdim)],1,sum)<=imkmRjp),],ncol=bdim+length(tempind))}
           for(yy in 1:(dim(xx)[1])){
            out[1+sum(xx[yy,1:bdim]),2]<-out[1+sum(xx[yy,1:bdim]),2]+weights[b]*wiklj(i+sum(xx[yy,1:bdim]),c(k-1,r[-b][tempsbp[[bdim]][,j]]+xx[yy,1:bdim],xx[yy,-(1:bdim)]),c(n[b]-k,nmr[-b][tempsbp[[bdim]][,j]]-xx[yy,1:bdim],n[-b][-tempsbp[[bdim]][,j]][tempind]-xx[yy,-(1:bdim)]),c(n[b],n[-b][tempsbp[[bdim]][,j]],n[-b][-tempsbp[[bdim]][,j]][tempind]),N)
           }
          }

          else{       ##### b''alpha non-empty and b''beta empty
            xx<-as.matrix(expand.grid(c(tempnmr[-b][tempsbp[[bdim]][,j]],tempr[-b][-tempsbp[[bdim]][,j]][tempind])))
           if(length(tempind)==1){xx<-matrix(xx[which(xx[,-(1:bdim)]==imkmRjp),],ncol=bdim+1)}
            if(length(tempind)>1){xx<-matrix(xx[which(apply(xx[,-(1:bdim)],1,sum)==imkmRjp),],ncol=bdim+length(tempind))}
           for(yy in 1:(dim(xx)[1])){
            out[1+sum(xx[yy,1:bdim]),2]<-out[1+sum(xx[yy,1:bdim]),2]+weights[b]*wiklj(i+sum(xx[yy,1:bdim]),c(k-1,r[-b][tempsbp[[bdim]][,j]]+xx[yy,1:bdim],xx[yy,-(1:bdim)]),c(n[b]-k,nmr[-b][tempsbp[[bdim]][,j]]-xx[yy,1:bdim],n[-b][-tempsbp[[bdim]][,j]][tempind]-xx[yy,-(1:bdim)]),c(n[b],n[-b][tempsbp[[bdim]][,j]],n[-b][-tempsbp[[bdim]][,j]][tempind]),N)
           }
          }
         }
        }
       }
     bdim<-bdim+1
     }
   }
  }
}

b<-B # Going over b = B where the sample is complete
for(k in 1:r[b]){
 bdim<-1
 while(bdim<B&&sum(r[-b][1:bdim])<=i-k){
  for(j in 1:choose(B-1,bdim)){
  imkmRjp<-i-k-sum(r[-b][tempsbp1[[bdim]][,j]])
    if(imkmRjp>=0&&i-k<=R-r[b]){     # Permutation b' satisfies condition
    tempind<-which(r[-b][-tempsbp1[[bdim]][,j]]<imkmRjp)
    if(length(tempind)==0){      # Case with b''alpha empty
```

```
    if(bdim==B-1&&imkmRjp==0){    ##### b''alpha empty b''beta empty

      xx<-as.matrix(expand.grid(tempnmr[-b]),ncol=bdim)
      for(yy in 1:(dim(xx)[1])){
        out[1+sum(xx[yy,]),2]<-out[1+sum(xx[yy,]),2]+weights[b]*wiklj(i+sum(xx[yy,]),c(k-1,r[-b]+xx[yy,]),c(n[b]-k,nmr[-b]-xx[yy,]),n,N)
      }
    }

    if(bdim<B-1){       ##### b''alpha empty b''beta non-empty

      xx<-as.matrix(expand.grid(tempnmr[-b][tempsbp1[[bdim]][,j]]))
      for(yy in 1:(dim(xx)[1])){
        out[1+sum(xx[yy,]),2]<-out[1+sum(xx[yy,]),2]+weights[b]*wiklj(i+sum(xx[yy,]),c(k-1,r[-b][tempsbp1[[bdim]][,j]]+xx[yy,]),c(n[b]-k,nmr[-b][tempsbp1[[bdim]][,j]]-xx[yy,]),c(n[b],n[-b][tempsbp1[[bdim]][,j]]),N)
      }
    }
  }
  else{         # Case with b''alpha non-empty

    if(bdim+1+length(tempind)<B){    ##### b''alpha non-empty and b''beta non-empty

      xx<-as.matrix(expand.grid(c(tempnmr[-b][tempsbp1[[bdim]][,j]],tempr[-b][-tempsbp1[[bdim]][,j]][tempind])))
      if(length(tempind)==1){xx<-matrix(xx[which(xx[,-(1:bdim)]<=imkmRjp),],ncol=bdim+1)}
      if(length(tempind)>1){xx<-matrix(xx[which(apply(xx[,-(1:bdim)],1,sum)<=imkmRjp),],ncol=bdim+length(tempind))}
      for(yy in 1:(dim(xx)[1])){
        out[1+sum(xx[yy,1:bdim]),2]<-out[1+sum(xx[yy,1:bdim]),2]+weights[b]*wiklj(i+sum(xx[yy,1:bdim]),c(k-1,r[-b][tempsbp1[[bdim]][,j]]+xx[yy,1:bdim],xx[yy,-(1:bdim)]),c(n[b]-k,nmr[-b][tempsbp1[[bdim]][,j]]-xx[yy,1:bdim],n[-b][-tempsbp1[[bdim]][,j]][tempind]-xx[yy,-(1:bdim)]),c(n[b],n[-b][tempsbp1[[bdim]][,j]],n[-b][-tempsbp1[[bdim]][,j]][tempind]),N)
      }
    }

    else{         ##### b''alpha non-empty and b''beta empty

      xx<-as.matrix(expand.grid(c(tempnmr[-b][tempsbp1[[bdim]][,j]],tempr[-b][-tempsbp1[[bdim]][,j]][tempind])))
      if(length(tempind)==1){xx<-matrix(xx[which(xx[,-(1:bdim)]==imkmRjp),],ncol=bdim+1)}
      if(length(tempind)>1){xx<-matrix(xx[which(apply(xx[,-(1:bdim)],1,sum)==imkmRjp),],ncol=bdim+length(tempind))}
```

```
        for(yy in 1:(dim(xx)[1])){
          out[1+sum(xx[yy,1:bdim]),2]<-out[1+sum(xx[yy,1:bdim]),2]+weights[b]*wiklj(i+sum(xx[yy,1:bdim]),c(k-1,r[-b][tempsbp1[[bdim]][,j]]+xx[yy,1:bdim],xx[yy,-(1:bdim)]),c(n[b]-k,nmr[-b][tempsbp1[[bdim]][,j]]-xx[yy,1:bdim],n[-b][-tempsbp1[[bdim]][,j]][tempind]-xx[yy,-(1:bdim)]),c(n[b],n[-b][tempsbp1[[bdim]][,j]],n[-b][-tempsbp1[[bdim]][,j]][tempind]),N)
        }
       }
      }
     }
   }
  bdim<-bdim+1
  }
}
}

if(mflag!=1){ # When all samples have censoring)
for(b in 1:B){
  if(weights[b]!=0){
   for(k in 1:r[b]){
    bdim<-1
          while(bdim<B&&sum(r[-b][1:bdim])<=i-k){
             for(j in 1:choose(B-1,bdim)){
      imkmRjp<-i-k-sum(r[-b][tempsbp[[bdim]][,j]])
        if(imkmRjp>=0&&i-k<=R-r[b]){     # Permutation b' satisfies condition
        tempind<-which(r[-b][-tempsbp[[bdim]][,j]]<imkmRjp)
        if(length(tempind)==0){       # Case with b''alpha empty

          if(bdim==B-1&&imkmRjp==0){    ##### b''alpha empty b''beta empty
           xx<-as.matrix(expand.grid(tempnmr[-b]),ncol=bdim)
           for(yy in 1:(dim(xx)[1])){
             out[1+sum(xx[yy,]),2]<-out[1+sum(xx[yy,]),2]+weights[b]*wiklj(i+sum(xx[yy,]),c(k-1,r[-b]+xx[yy,]),c(n[b]-k,nmr[-b]-xx[yy,]),n,N)
           }
          }

          if(bdim<B-1){       ##### b''alpha empty b''beta non-empty
           xx<-as.matrix(expand.grid(tempnmr[-b][tempsbp[[bdim]][,j]]))
           for(yy in 1:(dim(xx)[1])){
             out[1+sum(xx[yy,]),2]<-out[1+sum(xx[yy,]),2]+weights[b]*wiklj(i+sum(xx[yy,]),c(k-1,r[-b][tempsbp[[bdim]][,j]]+xx[yy,]),c(n[b]-k,nmr[-b][tempsbp[[bdim]][,j]]-xx[yy,]),c(n[b],n[-b][tempsbp[[bdim]][,j]]),N)
           }
          }
        }
        else{         # Case with b''alpha non-empty

```

```
          if(bdim+1+length(tempind)<B){     ##### b''alpha non-empty and b''beta non-empty
            xx<-as.matrix(expand.grid(c(tempnmr[-b][tempsbp[[bdim]][,j]],tempr[-b][-tempsbp[[bdim]][,j]][tempind])))
           if(length(tempind)==1){xx<-matrix(xx[which(xx[,-(1:bdim)]<=imkmRjp),],ncol=bdim+1)}
            if(length(tempind)>1){xx<-matrix(xx[which(apply(xx[,-(1:bdim)],1,sum)<=imkmRjp),],ncol=bdim+length(tempind))}
           for(yy in 1:(dim(xx)[1])){
            out[1+sum(xx[yy,1:bdim]),2]<-out[1+sum(xx[yy,1:bdim]),2]+weights[b]*wiklj(i+sum(xx[yy,1:bdim]),c(k-1,r[-b][tempsbp[[bdim]][,j]]+xx[yy,1:bdim],xx[yy,-(1:bdim)]),c(n[b]-k,nmr[-b][tempsbp[[bdim]][,j]]-xx[yy,1:bdim],n[-b][-tempsbp[[bdim]][,j]][tempind]-xx[yy,-(1:bdim)]),c(n[b],n[-b][tempsbp[[bdim]][,j]],n[-b][-tempsbp[[bdim]][,j]][tempind]),N)
           }
          }

          else{        ##### b''alpha non-empty and b''beta empty
            xx<-as.matrix(expand.grid(c(tempnmr[-b][tempsbp[[bdim]][,j]],tempr[-b][-tempsbp[[bdim]][,j]][tempind])))
           if(length(tempind)==1){xx<-matrix(xx[which(xx[,-(1:bdim)]==imkmRjp),],ncol=bdim+1)}
            if(length(tempind)>1){xx<-matrix(xx[which(apply(xx[,-(1:bdim)],1,sum)==imkmRjp),],ncol=bdim+length(tempind))}
           for(yy in 1:(dim(xx)[1])){
            out[1+sum(xx[yy,1:bdim]),2]<-out[1+sum(xx[yy,1:bdim]),2]+weights[b]*wiklj(i+sum(xx[yy,1:bdim]),c(k-1,r[-b][tempsbp[[bdim]][,j]]+xx[yy,1:bdim],xx[yy,-(1:bdim)]),c(n[b]-k,nmr[-b][tempsbp[[bdim]][,j]]-xx[yy,1:bdim],n[-b][-tempsbp[[bdim]][,j]][tempind]-xx[yy,-(1:bdim)]),c(n[b],n[-b][tempsbp[[bdim]][,j]],n[-b][-tempsbp[[bdim]][,j]][tempind]),N)
           }
          }
        }
        } else{break()}
       }
      bdim<-bdim+1
      }
    }
  }
}
}
out[1,2]<-1-sum(out[,2]);out<-matrix(out[which(out[,2]>0),],ncol=2,)
if(any(out[,2]<0)){print("Error has occurred some mixing probability is negative")}
return(out)
}
```

Appendix B

Code for Chapter 4

function:	simdtiiw
	input
iter	The number of iterations to estimate the mixture weights
n	A vector of sample sizes
rL/rU	A vector of the number of Lower/Upper censored items
i	Vector of indices for the i-th pooled OS S$Z_{(i)}$
p	A vector of quantiles for the uniform distribution
alpha	A value for confidence bands of a $100(1-\alpha)\%$ confidence interval
	output

A list of length equal to the length of i. In each list is a vector containing (i, w_{ij}). The names of the vector are ("i","j"), where j is the index of the order statistic $X_{j:\dot{r}}$ and $w_{ij} = P(Z_{(i)} = X_{j:\dot{r}})$

```
simdtiiw<-function(iter,n,rL,rU,i){

wdsim1<-function(){  # Simulation function for when length(i) = 1
temp<- c(NULL,NULL)
for(j in 1:B){temp<-rbind(temp,cbind(sort(runif(n[j])),c(if(rL[j]>0){rep
    (0,rL[j])},rep(1,n[j]-rL[j]-rU[j]),if(rU[j]>0){rep(0,rU[j])})))}
temp<-temp[order(temp[,1]),]
which(cumsum(temp[,2])==i)[1]
}

wdsim1p<-function(){  # Simulation function for when length(i) > 1
temp<- c(NULL,NULL)
for(j in 1:B){temp<-rbind(temp,cbind(sort(runif(n[j])),c(if(rL[j]>0){rep
    (0,rL[j])},rep(1,n[j]-rL[j]-rU[j]),if(rU[j]>0){rep(0,rU[j])})))}
temp<-temp[order(temp[,1]),]
out<-NULL
for(j in 1:length(i)){out<-c(out,which(cumsum(temp[,2])==j)[1])}
return(out)
```

```
}

# Various

B<-length(n); i<-sort(unique(i))

# Sanitize input

if(any(i-floor(i)!=0)){cat("some i is not an integer, i set to", sort(unique(floor(i))), "\n"); i<-floor(i)}
if(any(n-floor(n)!=0)){cat("some n is not an integer, n set to", floor(n), "\n"); n<-floor(n)}
if(any(rL-floor(rL)!=0)){cat("some rL is not an integer, rL set to", floor(rL), "\n"); rL<-floor(rL)}
if(any(rU-floor(rU)!=0)){cat("some rU is not an integer, rU set to", floor(rU), "\n"); rU<-floor(rU)}

# Terminating conditions (i.e., invalid input)

if(B!=length(rL)){stop("n and rL not of same length")}
if(B!=length(rU)){stop("n and rU not of same length")}
if(any(n<=rL+rU)){stop("Some sample has no observed values (n<=rL+rU)")}
if(any(rL<0)){stop("Some rL value is less than 0")}
if(any(rU<0)){stop("Some rU value is less than 0")}
if(any(i<1)){stop("Some i is invalid, must be integer from 1 to number of observed values")}
if(any(i>sum(n-rL-rU))){stop("Some i is too large, must be no more than number of observed values")}

# output

if(length(i)==1){
    y<-i; names(y)<-"i"; out<-vector("list",1)
    out[[1]]<-c(y,table(replicate(iter,wdsim1()))/iter)
    return(out)
}
if(length(i)>1){
    dat<-replicate(iter,wdsim1p()); out<-vector("list",length(i))
    for(j in 1:length(i)){y<-i[j]; names(y)<-"i"; out[[j]]<-c(y,table(dat[j,])/iter)}
    return(out)
    }
}
```

function:	simdtiip
	input
outeriter	The number of outer iterations repeating the estimation
iter	The number of iterations to estimate the mixture weights
n	A vector of sample sizes
rL/rU	A vector of the number of Lower/Upper censored items
i	Index for i-th pooled OS $Z_{(i)}$
p	A vector of quantiles for the uniform distribution
alpha	A value in (0,0.5) for confidence bands of a $100(1-\alpha)\%$ confidence interval
	output
A matrix with dimension length(p) x 3. The first/third columns give the lower/upper confidence bands and the second column gives the mean estimated probability of $\widehat{F_{Z_{(i)}}(\xi_p)}$	

```r
simdtiip<-function(outeriter,iter,n,rL,rU,i,p=seq(0.01,0.99,by=0.01),
    alpha=0.002){

wdsim1<-function(){        # Simulation function for when length(i) = 1
temp<- c(NULL,NULL)
for(j in 1:B){temp<-rbind(temp,cbind(sort(runif(n[j])),c(if(rL[j]>0){rep
    (0,rL[j])},rep(1,n[j]-rL[j]-rU[j]),if(rU[j]>0){rep(0,rU[j])})))}
temp<-temp[order(temp[,1]),];return(which(cumsum(temp[,2])==i)[1])
}

wdsim1p<-function(){       # Simulation function for when length(i) > 1
temp<- c(NULL,NULL)
for(j in 1:B){temp<-rbind(temp,cbind(sort(runif(n[j])),c(if(rL[j]>0){rep
    (0,rL[j])},rep(1,n[j]-rL[j]-rU[j]),if(rU[j]>0){rep(0,rU[j])})))}
temp<-temp[order(temp[,1]),];out<-NULL;for(j in 1:length(i)){out<-c(out,
    which(cumsum(temp[,2])==j)[1])}
return(out)}

simdtiiw1<-function(iter,n,rL,rU,i){ # Inner simulation function
if(length(i)==1){y<-i;names(y)<-"i";return(c(y,table(replicate(iter,
    wdsim1()))/iter))}
if(length(i)>1){
dat<-replicate(iter,wdsim1p())
out<-list()
for(j in 1:length(i)){y<-i[j];names(y)<-"i";out[[j]]<-c(y,table(dat[j,])
    /iter)}
return(out)}
}


getp<-function(){        # Simulation subfunction - returns estimated
    quantiles P*_1,...P*_P
```

```
datw<-simdtiiw1(iter,n,rL,rU,i)[-1]
dati<-as.numeric(names(datw))
out<-NULL
for(j in 1:P){out<-c(out,qpziu(dati,datw,N,p[j]))}
return(out)
}

# Sanitize input

if(i-floor(i)!=0){cat("i is not an integer, i set to", floor(i), "\n");i<-floor(i)}
if(any(n-floor(n)!=0)){cat("Some n is not an integer, n set to", floor(n), "\n");n<-floor(n)}
if(any(rL-floor(rL)!=0)){cat("Some rL is not an integer, rL set to", floor(rL), "\n");rL<-floor(rL)}
if(any(rU-floor(rU)!=0)){cat("Some rU is not an integer, rU set to", floor(rU), "\n");rU<-floor(rU)}
if(any(p<0)|any(p>1)){cat("Some p < 0 or p > 1, these p were removed", "\n");p<-p[which(0<p&p<1)]}

# Various

B<-length(n);N<-sum(n)

# Terminating conditions (i.e., invalid input)

if(length(p)==0){stop("No valid p, include at least one quantile between 0 and 1")} else{p<-sort(unique(p));P<-length(p)}
if(B!=length(rL)){stop("n and rL not of same length")}
if(B!=length(rU)){stop("n and rU not of same length")}
if(any(n<=rL+rU)){stop("Some sample has no observed values (n<=rL+rU)")}
if(any(rL<0)){stop("Some rL value is less than 0")}
if(any(rU<0)){stop("Some rU value is less than 0")}
if(i<1){stop("i is invalid, must be integer from 1 to number of observed values")}
if(i>sum(n-rL-rU)){stop("i is too large, must be no more than number of observed values")}
if(alpha<=0|alpha>=0.5){stop("alpha must be a number between 0 and 0.5")}

# Output

tempout<-replicate(outeriter,getp())
out<-matrix(0,ncol=3,nrow=P)
out[,1]<-apply(tempout,1,quantile,probs=alpha/2,type=4)
out[,2]<-apply(tempout,1,mean)
out[,3]<-apply(tempout,1,quantile,probs=1-alpha/2,type=4)
out
}
```

Appendix C

Code for Chapter 5

The following function is also useful for Chapter 7.

function:	prosch
	input
R	The number of censored items
r	The number of observed failures
	output
A matrix where the rows are all possible censoring schemes given the number of observed and censored items. The number of columns is r.	

```
prosch<-function(R,r){
if(R==0){return(matrix(0,ncol=r,nrow=1))}
if(r==1){return(matrix(R,ncol=1,nrow=1))}
out<-NULL;for(i in R:0){out<-rbind(out,cbind(i,prosch(R-i,r-1)))}
return(out)
}
```

function:	ksprogsimos
	input
i	Vector of indices for i-th pooled OS $Z_{(i)}$
R	List of censoring schemes
iter	The number of iterations to estimate the mixture weights
	output

A list of length equal to the length of i. In each list is a vector containing (i, w_{ij}). The names of the vector are (”i”,”j”), where j is the index of the order statistic $X_{j:\dot{r}}$ and $w_{ij} = P(Z_{(i)} = X_{j:\dot{r}})$

```
ksprogsimos<-function(i,R,iter=1000000){

wdsim<-function(){
temp<-c(NULL,NULL)
for(j in 1:B){
 tdat<-sort(runif(n[j]));qdat<-tdat;datx<-double(r[j])

 if(r[j]>1){
  for(jj in 1:(r[j]-1)){
    datx[jj]<-tdat[1];tdat<-tdat[-1];if(R[[j]][jj]>0){tdat<-tdat[-sample(1:length(tdat),R[[j]][jj])]}
  }
 }
 datx[r[j]]<-tdat[1]

 tindex<-match(datx,qdat)
 zo<-rep(0,n[j]);zo[tindex]<-1
 temp<-rbind(temp,cbind(qdat,zo))
 }
temp<-temp[order(temp[,1]),]
return(match(i,cumsum(temp[,2])))
}

# Get number of schemes, sanitize input
B<-length(R)
for(j in 1:B){if(any(R[[j]]!=floor(R[[j]]))){cat("Some censoring amount is not an integer, R set to floor(R)" , "\n");R[[j]]<-floor(R[[j]])}}

# Extract information from schemes
r<-sapply(R,length);nmr<-sapply(R,sum);n<-r+nmr

# Merge complete samples
temp<-which(nmr==0)
if(length(temp)>1){
     R<-append(R[-temp],list(rep(0,sum(r[temp]))))
 r<-sapply(R,length);nmr<-sapply(R,sum);n<-r+nmr;B<-length(R)
}
```

```

# Make out
out<-matrix(0,ncol=2,nrow=sum(n));out[,1]<-1:sum(n);colnames(out)<-c("j:n","w_j")

# Calls
if(length(i)==1){out<-list(table(replicate(iter,wdsim())))}
if(length(i)>1){out<-apply(replicate(iter,wdsim()),1,table)}
for(j in 1:length(i)){out[[j]]<-out[[j]]/iter;out[[j]]<-c(i[j],out[[j]]);names(out[[j]])[1]<-"i"}
return(out)
}
```

function:	elenexp, elenunif, elenlog
	input
ini1/ini2	The output of ksprogsimos for i_1 and i_2
n	The overall sample size
	output

A double of the expected length for the interval $(Z_{(i_1)}, Z_{(i_2)})$ for the standard exponential/Uniform(0,1)/standard logistic distribution

```
elenexp<-function(ini1,ini2,n=60){
exj<-cumsum(1/(n-1:n+1))
return(sum(ini2[-1]*exj[as.numeric(as.numeric(names(ini2)[-1]))])-sum(ini1[-1]*exj[as.numeric(as.numeric(names(ini1)[-1]))]))
}
```

```
elenunif<-function(ini1,ini2,n=60){
uxj<-1:n/(n+1)
return(sum(ini2[-1]*uxj[as.numeric(as.numeric(names(ini2)[-1]))])-sum(ini1[-1]*uxj[as.numeric(as.numeric(names(ini1)[-1]))]))
}
```

```
elenlog<-function(ini1,ini2,n=60){
lxj<-digamma(1:n)-digamma(n-1:n+1)
return(sum(ini2[-1]*lxj[as.numeric(as.numeric(names(ini2)[-1]))])-sum(ini1[-1]*lxj[as.numeric(as.numeric(names(ini1)[-1]))]))
}
```

function:	ksprogsimpc
	input
i	Index for i-th pooled OS $Z_{(i)}$
R	List of censoring schemes
out	All possible permutations of progressive censoring schemes given by R Of length (1+i). First column is 0 (for weights) Generated as a subset from prosch
iter	The number of iterations to estimate the mixture weights
	output
The same matrix "out" (that is input) with the estimated mixture weights in column 1	

```
ksprogsimpc<-function(i,R,out,iter=1000000){

wdsim<-function(){
alldat<-NULL
for(j in 1:B){
 tdat<-sort(runif(n[j]));qdat<-tdat;datx<-double(r[j])

 if(r[j]>1){
  for(jj in 1:(r[j]-1)){
   datx[jj]<-tdat[1];tdat<-tdat[-1];if(R[[j]][jj]>0){tdat<-tdat[-sample
       (1:length(tdat),R[[j]][jj])]}
  }
 }
 datx[r[j]]<-tdat[1]
 alldat<-cbind(alldat,rbind(datx,R[[j]]))
 }
alldat<-alldat[,order(alldat[1,])]
alldat<-alldat[2,1:i]
for(j in 1:Lout){if(all(alldat==out[j,2:(i+1)])){return(j)}}
print(alldat)
}

# Out only up to i
out<-unique(out[,1:(i+1)])

# Get number of schemes, sanitize input
B<-length(R)
for(j in 1:B){if(any(R[[j]]!=floor(R[[j]]))){cat("Some censoring amount
    is not an integer, R set to floor(R)" , "\n");R[[j]]<-floor(R[[j]])
    }}

# Extract information from schemes
r<-sapply(R,length);nmr<-sapply(R,sum);n<-r+nmr;Lout<-dim(out)[[1]]
temp<-factor(replicate(iter,wdsim()),levels=1:(dim(out)[[1]]))
```

```
out[,1]<-table(temp)/iter
return(out)
}
```

Appendix D

Code for Chapter 6

The following code is written for Generalized Order Statistics (GOS) when $\gamma_j \neq -1$, for which progressive Type-II censoring is a special case. See Kamps and Cramer (2001) for a general overview of GOS, or Volterman et al. (2011) for a specific application in this case.

function:	pitgosexp, pitgosunif, pitgosnorm, pitgoscauchy, pitgosskewn
	input
i	The index of the PCOS ($X^{\mathcal{R}}_{i:r:n}$)
m	The censoring scheme portion ($R_1, ..., R_{r-1}$)
k	The final number of item removals plus 1, $R_r + 1$
p	The probability for the p-th quantile, ξ_p
tol	Tolerance for integration (cauchy and skew-norm only)
alpha	Skewness parameter (skew-norm only)
	output
Returns a double of the SCP probability $\pi_{i:r:n}$	

```
pitgosexp<-function(i,m,k,p){

Fxmki<-function(i){1-cjm1[i]*sum(aji[i,1:i]/gam[1:i]*(1-p)^gam[1:i])}

Aip1<-function(i){temp<-Fxmki(i)
 for(j in 1:i){
   if(gam[j]==2*gam[i]){temp<-temp-cjm1[i]*(1-p)^(2*gam[i])*aji[i,j]*log(1-p)
   } else{temp<-temp+cjm1[i]*(1-p)^(2*gam[i])*aji[i,j]/(gam[j]-2*gam[i])*(1-(1-p)^(gam[j]-2*gam[i]))}}
 return(temp)}

n<-length(m)+1
gam<-c(k+n-1:(n-1)+rev(cumsum(rev(m))),k)
cjm1<-cumprod(gam)
```

```
aji<-matrix(1,ncol=n,nrow=n)
for(j in 1:n){for(jj in 1:n){aji[jj,j]<-if(jj<j){1/prod(gam[1:jj]-gam[j])} else{1/prod(gam[1:jj][-j]-gam[j])}}}

if(i==1){return(1-Aip1(2))} else{if(i==n){return(Aip1(n))} else{return(Aip1(i)-Aip1(i+1))}}
}
```

```
pitgosunif<-function(i,m,k,p){

Fxmki<-function(i){1-cjm1[i]*sum(aji[i,1:i]/gam[1:i]*(1-p)^gam[1:i])}

Aip1<-function(i){
if(p<0.5){ return(Fxmki(i-1)-cjm1[i-1]*sum((2*(1-p))^(gam[1:(i-1)])*aji[i-1,1:(i-1)]*beta(gam[1:(i-1)]-gam[i],gam[i]+1)*(pbeta(0.5,gam[i]+1,gam[1:(i-1)]-gam[i])-pbeta(1-0.5/(1-p),gam[i]+1,gam[1:(i-1)]-gam[i])))))}
if(p>=0.5){return(Fxmki(i-1)-cjm1[i-1]*sum((2*(1-p))^(gam[1:(i-1)])*aji[i-1,1:(i-1)]*beta(gam[1:(i-1)]-gam[i],gam[i]+1)*(pbeta(0.5,gam[i]+1,gam[1:(i-1)]-gam[i])))))}
}

n<-length(m)+1
gam<-c(k+n-1:(n-1)+rev(cumsum(rev(m))),k)
cjm1<-cumprod(gam)
aji<-matrix(1,ncol=n,nrow=n)
for(j in 1:n){for(jj in 1:n){aji[jj,j]<-if(jj<j){1/prod(gam[1:jj]-gam[j])} else{1/prod(gam[1:jj][-j]-gam[j])}}}

if(i==1){return(1-Aip1(2))} else{ if(i==n){return(Aip1(n))} else{ return(Aip1(i)-Aip1(i+1)) }}
}
```

```
pitgosnorm<-function(i,m,k,p){

chip<-qnorm(p)

Fxmki<-function(i){1-cjm1[i]*sum(aji[i,1:i]/gam[1:i]*(1-p)^gam[1:i])}
fpint<-function(u,jl){j<-jl[1];l<-jl[2];(1-u)^(gam[j]-gam[l]-1)*(1-pnorm(2*chip-qnorm(u)))^(gam[l])}

Aip1<-function(i){
pint<-NULL;for(j in 1:(i-1)){pint<-c(pint,integrate(fpint,0,p,subdivisions=10000,jl=c(j,i))$value)}
return(Fxmki(i-1)-cjm1[i-1]*sum(aji[i-1,1:(i-1)]*pint[1:(i-1)]))}

n<-length(m)+1
```

```
gam<-c(k+n-1:(n-1)+rev(cumsum(rev(m))),k)
cjm1<-cumprod(gam)
aji<-matrix(1,ncol=n,nrow=n)
for(j in 1:n){for(jj in 1:n){aji[jj,j]<-if(jj<j){1/prod(gam[1:jj]-gam[j])} else{1/prod(gam[1:jj][-j]-gam[j])}}}

if(i==1){return(1-Aip1(2))} else{ if(i==n){return(Aip1(n))} else{ return(Aip1(i)-Aip1(i+1)) }}
}
```

```
pitgoscauchy<-function(i,m,k,p,tol=1e-10){

chip<-qcauchy(p)

Fxmki<-function(i){1-cjm1[i]*sum(aji[i,1:i]/gam[1:i]*(1-p)^gam[1:i])}
fpint<-function(u,jl){j<-jl[1]; l<-jl[2];(1-u)^(gam[j]-gam[l]-1)*(1-pcauchy(2*chip-qcauchy(u)))^(gam[l])}

Aip1<-function(i){
pint<-NULL; for(j in 1:(i-1)){pint<-c(pint,integrate(fpint,0,p,subdivisions=1000,rel.tol=tol,jl=c(j,i))$value)}
return(Fxmki(i-1)-cjm1[i-1]*sum(aji[i-1,1:(i-1)]*pint[1:(i-1)]))}

n<-length(m)+1
gam<-c(k+n-1:(n-1)+rev(cumsum(rev(m))),k)
cjm1<-cumprod(gam)
aji<-matrix(1,ncol=n,nrow=n)
for(j in 1:n){for(jj in 1:n){aji[jj,j]<-if(jj<j){1/prod(gam[1:jj]-gam[j])} else{1/prod(gam[1:jj][-j]-gam[j])}}}

if(i==1){return(1-Aip1(2))} else{ if(i==n){return(Aip1(n))} else{ return(Aip1(i)-Aip1(i+1)) }}
}
```

```
pitgosskewn<-function(i,m,k,p,alpha,tol=1e-10){

require(sn,quietly=TRUE)
chip<-qsn(p,shape=alpha,tol=tol)

Fxmki<-function(i){1-cjm1[i]*sum(aji[i,1:i]/gam[1:i]*(1-p)^gam[1:i])}
fpint<-function(u,jl){j<-jl[1]; l<-jl[2];(1-u)^(gam[j]-gam[l]-1)*(1-psn(2*chip-qsn(u,shape=alpha,tol=tol),shape=alpha))^(gam[l])}

Aip1<-function(i){
pint<-NULL; for(j in 1:(i-1)){pint<-c(pint,integrate(fpint,0,p,subdivisions=1000,rel.tol=tol,jl=c(j,i))$value)}
return(Fxmki(i-1)-cjm1[i-1]*sum(aji[i-1,1:(i-1)]*pint[1:(i-1)]))}
```

```

n<-length(m)+1
gam<-c(k+n-1:(n-1)+rev(cumsum(rev(m))),k)
cjm1<-cumprod(gam)
aji<-matrix(1,ncol=n,nrow=n)
for(j in 1:n){for(jj in 1:n){aji[jj,j]<-if(jj<j){1/prod(gam[1:jj]-gam[j])} else{1/prod(gam[1:jj][-j]-gam[j])}}}

if(i==1){return(1-Aip1(2))} else{ if(i==n){return(Aip1(n))} else{ return(Aip1(i)-Aip1(i+1)) }}
}
```

Appendix E

Code for Chapter 7

function:	psen
	input
N	Number of iterations for simulation
R1,R2	The two censoring schemes to be compared
	output
Double vector of length 2 containing an estimate of $\pi(\theta^*_{\mathcal{R}1}, \theta^*_{R2})$ and $\pi(\theta^*_{\mathcal{R}2}, \theta^*_{R1})$	

```
psen<-function(N,R1,R2){
if(length(R1)!=length(R2)|sum(R1)!=sum(R2)){stop("R's must be of same length and sum")}
m<-length(R1);n<-sum(R1)+m
count<-0
iter<-0
while(iter<N){
 tdat<-sort(rexp(n));tdat1<-tdat;tdat2<-tdat
 dat1<-rep(0,m);dat2<-rep(0,m);dat1[1]<-tdat[1];dat2[1]<-tdat[1]
 tdat1<-tdat1[-1];if(R1[1]>0){tdat1<-tdat1[-sample(1:(n-1),R1[1])]}
 tdat2<-tdat2[-1];if(R2[1]>0){tdat2<-tdat2[-sample(1:(n-1),R2[1])]}
     if(m>2){for(j in 2:(m-1)){
          dat1[j]<-tdat1[1];tdat1<-tdat1[-1];if(R1[j]>0){tdat1<-tdat1[-sample(1:length(tdat1),R1[j])]}
          dat2[j]<-tdat2[1];tdat2<-tdat2[-1];if(R2[j]>0){tdat2<-tdat2[-sample(1:length(tdat2),R2[j])]}
     }}
     dat1[m]<-tdat1[1];dat2[m]<-tdat2[1]
 xt1<-sum(dat1*(R1+1))/m;xt2<-sum(dat2*(R2+1))/m
 if(abs(xt1-1)<abs(xt2-1)){count<-count+1}
 iter<-iter+1
}
return(c(count,N-count)/N)
}
```

Appendix F

Glossary Chapters 3–5

ξ_p	the p-th quantile
B	number of independent samples
r_b, $\dot{r}$	the number of observed failures in the b-th sample/all B samples
n_b, n	sample size of the b-th sample ($n_b = r_b^L + r_b + r_b^U$)/all B samples
$\mathcal{R}^{(b)}$	The censoring scheme for the b-th sample
$X_{b,k:n_b}^{\mathcal{R}^{(b)}}$	the k-th PCOS from the b-th sample.
$Z_{(i)}$	the i-th order statistic from the pooled sample ($1 \leq i \leq \dot{r} \leq n$)
$\gamma_\ell^{(b)}, a_i^{(b)}(\ell), c_{\ell-1}^{(b)}$	as defined in Section 1.2.3 for the b-th sample
$\mathcal{P}(S)$, $\lvert S\rvert$	The powerset/cardinality of a set S
α	a set of indices such that sample $j \in \alpha$ if for some $k_j = 0, ..., r_j - 1$ then $X_{k_j:r_j:n_j}^{\mathcal{R}^{(j)}} < Z_{(i)} < X_{k_b+1:r_b:n_b}^{\mathcal{R}^{(j)}}$
β	a set of indices such that sample $j \in \beta$ if $X_{r_j:r_j:n_j}^{\mathcal{R}^{(j)}} < Z_{(i)}$

Table F.1: Notation for Chapter 5

ξ_p	the p-th quantile
B	number of independent samples
r_b, $\dot{r}$	the number of observed failures in the b-th sample/all B samples
r_b^L, $\dot{r}^L$	the number of items left censored in the b-th sample/all B samples
r_b^U, $\dot{r}^U$	the number of items right censored in the b-th sample/all B samples
n_b, n	sample size of the b-th sample ($n_b = r_b^L + r_b + r_b^U$)/all B samples
$X_{b,k:n_b}$	the k-th order statistic from the b-th sample. This is the $k - r_b^L$-th observed item in the b-th sample
$Z_{(i)}$	the i-th order statistic from the pooled sample ($1 \leq i \leq \dot{r} \leq n$)
$\mathcal{A}$	the set of indices excluding those corresponding to samples conditioned to be a specific pooled order statistic (when $Z_{(i)} = X_{b,k:n_b}$ then $\mathcal{A} = \{1, 2, ..., B\} \setminus \{b\}$)
$\{b'_L\}$	a subset of $\mathcal{A}$, such that some left censored items fall above $Z_{(i)}$ ($Z_{(i_1)}$)
$\{b'_{L1}\}$,$\{b'_{L2}\}$	a subset of $\{b'_L\}$, such that the first observed failure is above/below $Z_{(i_2)}$
$\{b'_U\}$	a subset of $\mathcal{A}$, such that some right censored items fall below $Z_{(i)}$ ($Z_{(i_2)}$)
$\{b'_{U1}\}$,$\{b'_{U2}\}$	a subset of $\{b'_U\}$, such that the last observed failure is below/above $Z_{(i_1)}$
$\{b'_{UL}\}$	a subset of $\mathcal{A}$, such that both left/right censored items fall above/below $Z_{(i_1)}/Z_{(i_2)}$ simultaneously.
$\{b''\}$	the complement of all $\{b'_U\}$, $\{b'_{U1}\}$, $\{b'_{UL}\}$, $\{b'_L\}$, and $\{b'_{L1}\}$ in $\mathcal{A}$. All of the left/right censored items are below/above $Z_{(i)}$ ($Z_{(i_1)}/Z_{(i_2)}$).
$\sigma_{\{b'\}}$	All possible valid combinations of $\{b'_U\}$ and $\{b'_L\}$ ($\{b'_U\}$, $\{b'_{U1}\}$, $\{b'_{UL}\}$, $\{b'_L\}$, and $\{b'_{L1}\}$). A combination $\sigma_{\{b'\}}$ is valid if $P(Z_{(i)} = X_{b,k:n_b} \mid \sigma_{\{b'\}}) > 0$
c_j, c_j^L, c_j^U	the number of observed/left censored/right censored failures below $Z_{(i)}$
$\dot{r}_S, \dot{c}_S$	the sum of r's or c's restricted to the set of samples S ($\dot{r}_S = \dot{c}_S = 0$)

Table F.2: Notation for Chapter 4

B	number of independent samples
n_b	sample size of the b-th sample
r_b	the number of observed failures for the b-th Type-II right censored sample ($1 \leq r_b \leq n_b$)
n	the total sample size of all B samples
$\dot{r}$	the total number of observed failures from all B samples
$X_{b,k:n_b}$	the k-th order statistic from the b-th sample
$Z_{(i)}$	the i-th order statistic from the pooled sample ($1 \leq i \leq \dot{r} \leq n$)
$\mathcal{A}$	the set of indices excluding those corresponding to samples conditioned to be a specific pooled order statistic (when $Z_{(i)} = X_{b,k:n_b}$ then $\mathcal{A} = \{1, 2, ..., B\} \setminus \{b\}$)
$\{b'\}$	a subset of $\mathcal{A}$ where at least one censored value from these samples fall below $Z_{(i)}$ (or $Z_{(i_2)}$ when two pooled order statistics are specified)
$\{b'_1\}, \{b'_2\}$	$\{b'_1\}$ is a subset of $\{b'\}$ such that the final observed value of the samples is below $Z_{(i_1)}$, and $\{b'_2\}$ is the compliment of $\{b'_1\} \in \{b'\}$, where the final observed value falls between $Z_{(i_1)}$ and $Z_{(i_2)}$
$\{b''\}$	the complement of $\{b'\}$ in $\mathcal{A}$ where none of the censored values from these samples fall below $Z_{(i)}$
$\{b''_\alpha\}, \{b''_\beta\}$	$\{b''_\beta\}$ is the subset of $\{b''\}$ such that the samples within are either complete ($r_j = n_j$) or $r_j \geq i - k - \dot{r}_{\{b'\}}$, and $\{b''_\alpha\}$ is the complement of $\{b''_\beta\} \in \{b''\}$
$\sigma_{\{b'\}}$	the collection of all valid sets $\{b'\}$. a set $\{b'\}$ is valid if $P(Z_{(i)} = X_{b,k:n_b} \mid X_{j,r_j:n_j} < X_{b,k:n_b},\ j \in \{b'\}) > 0$
$\sigma_{\{b'_1\}}$	the collection of all valid subsets $\{b'_1\} \in \{b'\}$ given a valid set $\{b'\}$, a set $\{b'_1\} \subset \{b'\}$ is valid if $P(Z_{(i_1)} = X_{b_o,k_1:n_{b_o}}, Z_{(i_2)} = X_{b,k_2,n_b} \mid \ \{b'\} \text{ is valid},\ X_{j_1,r_{j_1}:n_{j_1}} < X_{b_o,k_1:n_{b_o}} < X_{j_2,r_{j_2}:n_{j_2}} < X_{b,k_2:n_b},\ j_1 \in \{b'_1\}, j_2 \in \{b'_2\}) > 0$ (where $1 \leq k_1 < k_2 \leq n_b$ if $b_o = b$)
c_j	the number of censored values ($j \in \{b'\}$) or observed failures ($j \in \{b''\}$) below $Z_{(i)}$ from sample j
c_{j_1}, c_{j_2}	the number of censored values ($j \in \{b'\}$) or observed failures ($j \in \{b''\}$) below $Z_{(i_1)}$ and between $Z_{(i_1)}$ and $Z_{(i_2)}$, respectively, from sample j
$\dot{r}_S, \dot{c}_S$	the sum of r_b's and c_j's restricted over the set of indices S (where $\dot{r}_\emptyset = \dot{c}_\emptyset = 0$)

Table F.3: Notation for Chapter 3

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