A UNIFIED APPROACH TO GENERALIZED NETWORK SENSITIVITIES WITH APPLICATIONS TO POWER SYSTEM ANALYSIS AND PLANNING

By

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A UNIFIED APPROACH TO GENERALIZED POWER SYSTEM SENSITIVITIES

ABSTRACT

This thesis presents a new methodology for describing adjoint network approaches to sensitivity calculations performed in various power system analysis and planning studies. Difficulties observed by previous workers regarding the exact modelling of some power network elements are overcome by proper techniques employed with special complex notation. A generalized version of the Tellegen's theorem-based approach is developed which provides the required sensitivities based on the exact a.c. load flow model for any chosen set of real and/or complex variables of practical interest. A theoretical consistency study is performed to allow proper modelling of adjoint elements for direct treatment of general complex functions. A simplified version with many desirable features is described for real function sensitivities. It employs a simple adjoint network. General sensitivity expressions common to all relevant power system studies are derived and tabulated. A new method for solving the load flow problem using Tellegen's theorem is described with several advantages claimed. A special elimination technique is used to describe the Newton-Raphson method for load flow solution in a compact complex mode. A complex version of the Lagrange multiplier approach is developed and applied to allow a general number of complex dependent variables to be defined in a particular problem. A generalized version of the class of methods of sensitivity calculations which exploit the Jacobian matrix of the load flow analysis in formulating the adjoint equations is developed. Generalized sensitivity

expressions common to different modes of formulating power flow equations, e.g., cartesian and polar, are derived and tabulated for direct programming use. A unified comprehensive comparison between the Lagrange multiplier and Tellegen's theorem approaches to sensitivity calculations in electrical networks is presented.

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1

INTRODUCTION

In the context of steady-state computer-aided power system analysis and planning, functions of system variables are routinely defined in various studies to incorporate cost criteria, security assessment, reliability indices, etc. The system variables are related through a set of equality constraints representing, for example, power flow equations. Inequality constraints may also be defined to indicate, for example, physical limitations on practical variables.

The ratio between a small change Δf in a function f which may denote a dependent variable and a related small change Δu_j in an independent variable u_j indicates the sensitivity of f with respect to u_j. This ratio is generally a function of other system variables. It is very valuable in numerous power system analysis and planning problems. Using the Taylor series expansion, which relates Δf to increasing powers of Δu_j, the change in f may be calculated to any degree of accuracy.

First-order changes of functions of interest play a very important role in sensitivity calculations not only because they are relatively easy to calculate but also due to their direct contribution to gradient evaluations required by most optimization techniques used in different planning studies.

The use of second-order sensitivities, although requiring more elaborate calculation, also finds applications in investigations of the sensitivity of a function w.r.t. certain variables at an optimal solution represented by a stationary point of the function w.r.t. these variables.

This thesis employs a suitable notation and proper techniques to develop, unify, describe, improve and compare methods of evaluating first-order changes and gradients of functions of interest subject to equality constraints which may represent power flow equations. The term sensitivity calculations is used to indicate both first-order changes and gradient evaluation.

In Chapter 2 important approaches to sensitivity calculations in power system analysis and design problems are classified and generally described. Their contribution to solving some practical problems is outlined. The material presented in this chapter provides an adequate background for some of the studies presented in the subsequent chapters.

In Chapter 3 a generalized version of the Tellegen's theorem-based approach is developed. Using a special complex notation and proper techniques, the difficulties encountered in treating exact component models of power networks are overcome. This generalized version provides a methodology for handling complex functions. Sensitivity expressions which are common to all relevant power system studies are derived and tabulated.

A special class of the family of adjoint systems of Chapter 3 is considered in Chapter 4. A simplified version possessing many desirable features is described where a simpler adjoint network is obtained.

Sensitivity expressions for this simplified version are derived and tabulated for direct programming purposes for a wide variety of real functions.

The material presented in Chapter 5 provides a useful theoretical investigation for consistent definition of the adjoint system which provides complex function sensitivities, directly, as in typical electronic circuits. A unified study for consistent selection of arbitrary adjoining coefficients introduced in the generalized version of Chapter 3 is presented where the restrictions imposed by the number and type of elements of the network as well as the function considered are investigated.

In Chapter 6, some new concepts are introduced and utilized for presenting and studying several special topics in the context of complex analysis of power networks. A new Tellegen's theorem-based method for solving the load flow problem is proposed. This method employs a simple and mostly constant adjoint matrix of coefficients and enjoys the same rate of convergence as the Newton-Raphson method. Using a special complex notation, the Newton-Raphson method for solving the load flow problem has been interpreted formally in terms of first-order changes of problem variables and, hence, described in a compact complex form using a special elimination technique. The Lagrange multiplier approach described and applied in the real case is formulated in the complex mode to handle general complex functions and variables. Applications to power system sensitivity analysis are presented.

A unified study of the class of adjoint network approaches to power system sensitivity analysis which exploit the Jacobian matrix of the load flow solution is presented in Chapter 7. Generalized sensitivity expressions common to different modes of formulating the power flow equations, e.g., cartesian and polar are easily derived, compactly described and tabulated and effectively used for both real and complex forms of performance functions as well as control variables defined in a particular study.

A comprehensive comparison between the commonly used Lagrange multiplier and Tellegen's theorem approaches to sensitivity calculations in electrical networks is presented in Chapter 8 where the two approaches are described on a unified basis.

2

SENSITIVITY CALCULATIONS IN POWER NETWORKS

2.1 INTRODUCTION

Techniques for evaluating first-order changes and derivatives of performance functions subject to power network equations have been described in the context of applications in optimal load flow and planning problems (Sasson and Merrill 1974).

In this chapter, important methods of sensitivity calculations in power system analysis and design problems are classified and described in general. The notation used and the modes of formulation which contribute to developing a successful sensitivity approach are presented. Applications to some practical power system problems are also discussed.

2.2 EFFICIENT TECHNIQUES FOR SENSITIVITY CALCULATIONS

2.2.1 Requirements for a Successful Sensitivity Approach

Due to the inherently large size of power networks with various branch types, simplicity of derivation and formulation, flexibility in modelling different components of the power system and efficiency in computations represent basic requirements for a successful sensitivity approach.

2.2.2 General Survey

Some techniques (Fischl and Puntel 1972, Irisarri, Levner and Sasson 1979, Püttgen and Sullivan 1978, Wu and Sullivan 1976) address the previous requirements by approximating the a.c. load flow model describing the steady state behaviour of the power system. Other methods (Bandler and El-Kady 1979, Bandler and El-Kady 1980a, 1980b and 1980d, Dommel and Tinney 1968, Fischl and Wasley 1978) employ the exact a.c. load flow model. In some applications (Ejebe and Wollenberg 1979) both exact and approximated models have been used. The elements of the Jacobian matrix of the load flow solution are exploited in some approaches (Bandler and El-Kady 1980f, Dommel and Tinney 1968, Fischl and Wasley 1978) while the flexibility in modelling different power system elements provided by using suitable network theorems is gained in others (Bandler and El-Kady 1980a, Clements and Ringlee 1977, Püttgen and Sullivan 1978).

2.3 NOTATION

The different interpretation of the variables used to describe various power system components in equations poses a difficulty in choosing a suitable notation which facilitates the derivation and subsequent formulation of equations and expressions employed (Peschon, Piercy, Tinney and Tveit 1968).

2.3.1 State Variable Notation

The most successful notation used in describing the power flow equations and other physical constraints and interpreting the relationships between different variables is the state variable notation (Bandler and El-Kady 1980g, Dommel and Tinney 1968, Peschon et al. 1968) commonly used in control theory. Throughout the thesis this notation, which contributes significantly to an easier understanding of the equations, will be used.

The control or design variables are denoted by the column vector $\underset{\sim}{u}$ of n_u components. We also denote by the n_x - component vector $\underset{\sim}{x}$ the state variables or the dependent variables to be determined by solving the set of equality constraints, denoted by $\underset{\sim}{h}(\underset{\sim}{x},\underset{\sim}{u}) = \underset{\sim}{0}$, describing the steady-state behaviour of a particular power system.

2.3.2 Classification of Independent Variables

In the literature, the vector $\underset{\sim}{u}$ may be either classified further (Bandler and El-Kady 1980a) into subvectors associated with different bus and line branches in the power network or restricted (Dommel and Tinney 1968) to represent only the practically controlable variables, e.g., the real power P_g at a generator bus while some other variables, called fixed parameters, are assigned other symbols.

In general, we shall use $\underset{\sim}{u}$ to denote the independent variables to be specified in the equations describing a particular system. We may classify $\underset{\sim}{u}$ and $\underset{\sim}{x}$, whenever necessary, into appropriate subvectors associated with different power system steady-state component models (Sullivan 1977).

2.4 GENERAL FORMULATION

2.4.1 Power Flow Equations

Most of the literature in the area of power system analysis and design employs the real mode of formulation to describe the power flow equations and to derive, subsequently, the sensitivity expressions required in a particular study.

The power flow equations (Van Ness and Griffin 1961) are basically expressed in the complex form

$$V_m^* \sum_{i=1}^{n} (Y_{mi} V_m) = S_m^*, \; m = 1, \ldots, n, \tag{2.1}$$

where V_m is the mth bus voltage, Y_{mi} is an element of the bus admittance matrix (Stagg and El-Abiad 1968), $S_m = P_m + jQ_m$ is the mth bus power, P_m and Q_m denoting, respectively, the injected real and reactive powers, $j = \sqrt{-1}$, n denotes the number of buses and * denotes the complex conjugate.

The variables in (2.1) are, generally speaking, functions of the state $\underset{\sim}{x}$ and control $\underset{\sim}{u}$ variables of the system. Equations (2.1), whether written in the rectangular or in the polar form (Van Ness and Griffin 1961), are usually separated into real and imaginary parts in solving the load flow problem.

2.4.2 The Real Mode of Formulation

The real mode of formulation has been suggested upon the application (Van Ness and Griffin 1961, Tinney and Hart 1967) of the well-known Newton-Raphson method, which is superior in its quadratic convergence and ability to solve ill-conditioned problems, to the solution of the load flow problem. The reason (Stott 1974) is that the Newton-Raphson method is a derivative-based method and, mathematically speaking, the complex load flow equations are nonanalytic and cannot be differentiated in complex form. See Bandler and El-Kady (1980e).

The subsequent sensitivity calculations have been automatically performed in most of the literature in the same real mode.

2.4.3 First-Order Changes of Functions and Constraints

In general, we write the first-order change of a continuous function f in the form

$$\delta f = \sum_{i=1}^{n_x} \left(\frac{\partial f}{\partial x_i} \delta x_i \right) + \sum_{k=1}^{n_u} \left(\frac{\partial f}{\partial u_k} \delta u_k \right), \tag{2.2}$$

where δ denotes first - order change, x_i is the ith state variable and u_k is the kth control variable. We also write the first-order changes of the set of equality constraints $\underset{\sim}{h}(\underset{\sim}{x},\underset{\sim}{u}) = \underset{\sim}{0}$ in the form

$$\delta h_j = \sum_{i=1}^{n_x} \left(\frac{\partial h_j}{\partial x_i} \delta x_i \right) + \sum_{k=1}^{n_u} \left(\frac{\partial h_j}{\partial u_k} \delta u_k \right) = 0, \; j = 1, \ldots, n_x, \tag{2.3}$$

where h_j denotes the jth equality constraint.

The basic forms (2.2) and (2.3) are essential for the techniques employed to evaluate total derivatives of f w.r.t. $\underset{\sim}{u}$ by expressing δf solely in terms of the δu_k.

2.5 METHODS OF SENSITIVITY CALCULATIONS

Excluding the method based on approximate explicit expression of $\underset{\sim}{x}$ in terms of $\underset{\sim}{u}$ (Galiana and Banakar 1980) there are basically three methods for eliminating $\delta\underset{\sim}{x}$ from (2.2) and (2.3): the sensitivity matrix method, the adjoint or Lagrange multiplier method and the method based on Tellegen's theorem (Tellegen 1952, Director and Rohrer 1969a, 1969b).

2.5.1 The Sensitivity Matrix Method

In the sensitivity matrix method (Dommel and Tinney 1968, Peschon et al. 1968), the sensitivity matrix $\underset{\sim}{S}$ is defined by

$$\underset{\sim}{S} \triangleq - [(\frac{\partial h^T}{\partial \underset{\sim}{x}})^T]^{-1} (\frac{\partial h^T}{\partial \underset{\sim}{u}})^T, \tag{2.4}$$

where $(\partial \underset{\sim}{h}^T/\partial \underset{\sim}{x})^T$ and $(\partial \underset{\sim}{h}^T/\partial \underset{\sim}{u})^T$ are the Jacobian matrices of $\underset{\sim}{h}$ w.r.t. $\underset{\sim}{x}$ and $\underset{\sim}{u}$, respectively. Hence, from (2.3)

$$\delta\underset{\sim}{x} = \underset{\sim}{S}\ \delta\underset{\sim}{u}, \tag{2.5}$$

where $\delta\underset{\sim}{x}$ and $\delta\underset{\sim}{u}$ are column vectors of δx_i and δu_k, respectively, of (2.2). Substituting (2.5) into (2.2), we get

$$\delta f = [\frac{\partial f}{\partial \underset{\sim}{u}} + \underset{\sim}{S}^T \frac{\partial f}{\partial \underset{\sim}{x}}]^T \delta\underset{\sim}{u}, \tag{2.6}$$

from which

$$\frac{df}{d\underset{\sim}{u}} = \frac{\partial f}{\partial \underset{\sim}{u}} + \underset{\sim}{S}^T \frac{\partial f}{\partial \underset{\sim}{x}} . \tag{2.7}$$

The application of the sensitivity matrix method requires n_u repeat solutions of a system of linear equations formed from (2.4) for the elements of $\underset{\sim}{S}$. This task usually makes this method less preferable (Dommel and Tinney 1968, Fischl and Wasley 1978) unless the sensitivity matrix is needed for other purposes.

2.5.2 The Method of Lagrange Multipliers

The method of Lagrange multipliers (Dommel and Tinney 1968) is the most common one not only because it requires only one solution of a set of linear adjoint equations (as compared with the sensitivity matrix method) but also because it utilizes, in various applications, the elements of the Jacobian matrix available from the basic load flow solution.

The Lagrange multiplier method is commonly referred to for a general set of equality constraints (Director and Sullivan 1978). When the equality constraints represent power flow equations, the method may be interpreted as an adjoint network method (Bandler and El-Kady 1980f, Fischl and Wasley 1978).

The Lagrange multipliers are defined by

$$\underset{\sim}{\lambda} \overset{\Delta}{=} \left(\frac{\partial \underset{\sim}{h}^T}{\partial \underset{\sim}{x}} \right)^{-1} \frac{\partial f}{\partial \underset{\sim}{x}}, \tag{2.8}$$

hence, from (2.2) and (2.3)

$$\delta f = \left[\frac{\partial f}{\partial \underset{\sim}{u}} - \left(\frac{\partial \underset{\sim}{h}^T}{\partial \underset{\sim}{u}} \right) \underset{\sim}{\lambda} \right]^T \delta \underset{\sim}{u}, \tag{2.9}$$

from which

$$\frac{df}{d\underset{\sim}{u}} = \frac{\partial f}{\partial \underset{\sim}{u}} - \frac{\partial \underset{\sim}{h}^T}{\partial \underset{\sim}{u}} \underset{\sim}{\lambda}. \tag{2.10}$$

In practice, the set of linear equations formed by (2.8) is solved for the Lagrange multipliers $\underset{\sim}{\lambda}$ and the first-order change and total derivatives of f are then calculated from (2.9) and (2.10), respectively.

When the set of equality constraints $\underset{\sim}{h}(\underset{\sim}{x},\underset{\sim}{u}) = \underset{\sim}{0}$ represents the power flow equations (2.1), the 2n x 2n matrix of coefficients $(\partial \underset{\sim}{h}^T / \partial \underset{\sim}{x})$ of (2.8) may constitute the transpose of the Jacobian matrix of the load flow solution by the Newton-Raphson method. The exploitation of this fact necessitates expressing f in terms of $\underset{\sim}{x}$ which, in this case, represents 2n bus quantities (the unknown variables in power flow equations). Transformations are required to handle functions of other variables, e.g., line variables.

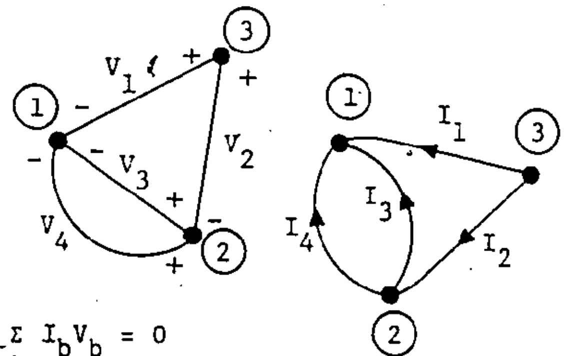
We remark that an extended vector $\underset{\sim}{x}$ which contains all variables of interest can be defined (Director and Sullivan 1978) so that general functions of line quantities may be directly handled. In this case, the size of the matrix of coefficients in (2.8) is determined by the total number of states considered.

2.5.3 Method Based on Tellegen's Theorem

The method based on Tellegen's theorem exploits the powerful features of the theorem to achieve both the compactness of the adjoint system of equations to be solved and the flexibility in handling line quantities.

Tellegen's theorem, see Fig. 2.1, which depends solely upon Kirchhoff's laws and the topology of the network, states that

$$\sum_b \hat{I}_b V_b = 0 \text{ and } \sum_b \hat{V}_b I_b = 0, \tag{2.11}$$



$$\sum_b I_b V_b = 0$$

$$\sum_b (I_b \delta V_b + V_b \delta I_b) = 0$$

Fig. 2.1 Illustration of Tellegen's theorem

where I_b and V_b are, respectively, the current and voltage of branch b of the network and ^ distinguishes the corresponding variables associated with the topologically similar adjoint network. The summations in (2.11) are taken over all branches. In addition to the current and voltage variables, the inclusion of the power variables S_b is required to accommodate the power flow model. Hence, we may use

$$S_b = V_b I_b^* . \tag{2.12}$$

Tellegen's theorem has been successfully applied to power system analysis and design problems since Fischl and Puntel (1972). In the beginning, the approximated d.c. load flow model was used. This found applications in transmission system planning problems (Fischl and Puntel 1972, Puntel, Reppen, Ringlee, Platts, Ryan and P.J. Sullivan 1973) in which the d.c. model may be considered of sufficient accuracy. The d.c.

load flow model is, however, characterized by the restrictive assumptions of neglecting transmission losses, excluding reactive power flows and considering flat voltage profiles which make it inadequate (Bandler and El-Kady 1979) for other studies requiring a more accurate model and more information.

Different versions of improved, approximate a.c. load flow models have been successively developed for application to different power system studies. The relatively difficult steady-state component models in power networks impose an observed difficulty in applying Tellegen's theorem to the exact a.c. load flow model. A proper methodology has been required to overcome this difficulty.

In general, a method of sensitivity calculations based on Tellegen's theorem incorporates the following steps. A perturbed Tellegen sum is formulated as

$$\hat{\eta}_x^T \delta x + \hat{\eta}_u^T \delta u = 0, \qquad (2.13)$$

where the state x and control u variables are defined in accordance with the power flow model considered and the vectors $\hat{\eta}_x$ and $\hat{\eta}_u$ are, in general, linear functions of the formulated adjoint network current and voltage variables. Hence, the $\hat{\eta}_x$ and $\hat{\eta}_u$ of (2.13) are related through Kirchhoff's current and voltage laws formulating a set of linear network equations to be solved for the unknown adjoint variables. The adjoint network is defined by setting

$$\hat{\eta}_x = \frac{\partial f}{\partial x} . \qquad (2.14)$$

hence, from (2.2) and (2.13), we get

$$\delta f = \left(\frac{\partial f}{\partial \underset{\sim}{u}} - \hat{\underset{\sim}{\eta}}_u \right)^T \delta \underset{\sim}{u}, \qquad (2.15)$$

from which

$$\frac{df}{d\underset{\sim}{u}} = \frac{\partial f}{\partial \underset{\sim}{u}} - \hat{\underset{\sim}{\eta}}_u. \qquad (2.16)$$

In practice, the adjoint network is defined for a given function by (2.14) and solved for the variables $\hat{\underset{\sim}{\eta}}_u$ which are then substituted into (2.15) and (2.16) to obtain first-order change and total derivatives of f w.r.t. control variables.

The matrix of coefficients of the adjoint system of equations has to be calculated at a base-case point. The LU factors of this matrix may be stored and different functions can be treated by repeat forward and backward substitutions.

2.5.4 Discussion

Based upon the foregoing description, we may conclude that the Tellegen theorem-based method has the advantage over the method of Lagrange multipliers regarding the flexibility of modelling the different elements of the network. It has, however, the disadvantage that the adjoint matrix of coefficients has to be calculated at a load flow solution.

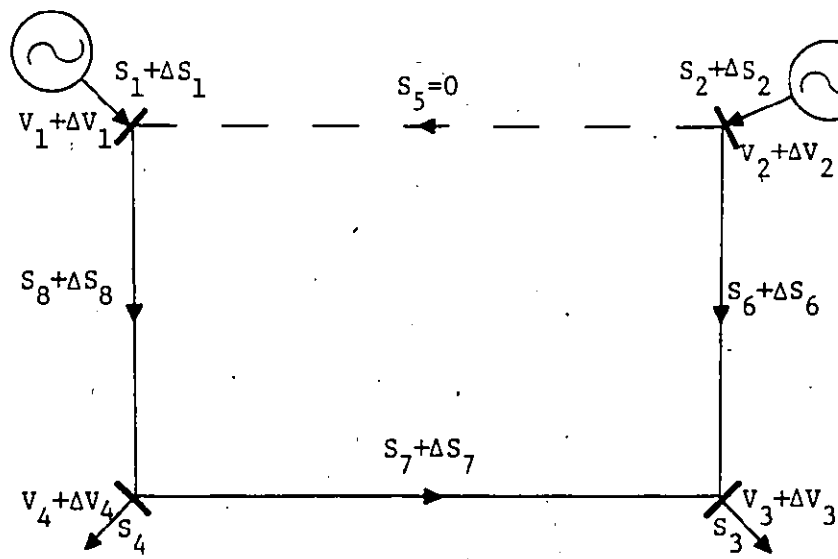
It is important to notice that when optimal solutions are required upon altering one or more system parameters from the base-case point, the adjoint matrix of coefficients in both methods has to be calculated at different iterations of the load flow solution included in each of the main optimization iterations towards the optimum.

The choice of a suitable method for sensitivity calculations depends on various factors such as the kind of application considered, the types of elements defined in the power system and the available storage and facilities in computations.

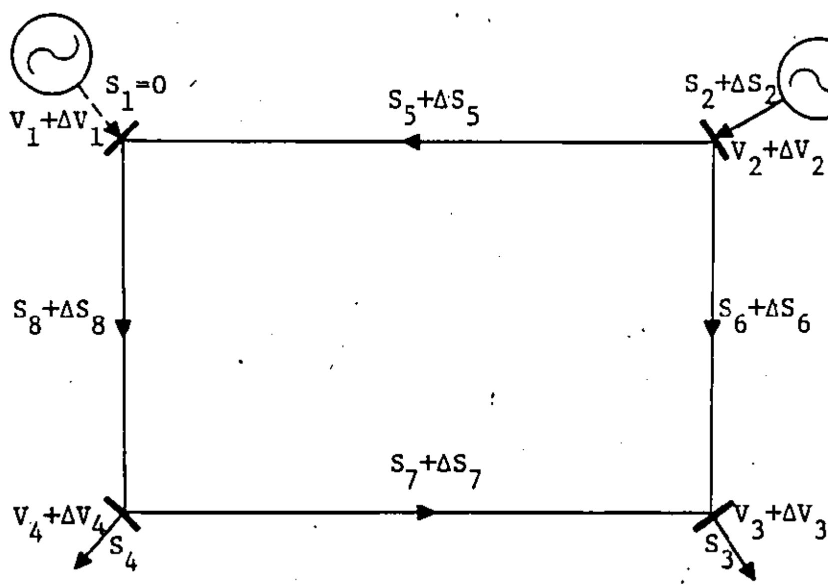
2.6 APPLICATIONS OF FIRST-ORDER CHANGE

Efficient sensitivity calculations may be performed to evaluate first-order changes of functions of interest corresponding to certain variations in the control variables defined in a particular study. These first-order changes are valuable in estimating the effects of transmission system contingencies and ranking them (Ejebe and Wollenberg 1979, Irisarri, Levner and Sasson 1979), generation outages, device malfunctions and other defects expected in power systems operation which may result in subsequent service deterioration.

In contingency analysis the changes in system performances, upon sustaining some of the above contingencies, are calculated using the d.c., the approximate a.c. or the exact a.c. load flow model. As illustrated before, the a.c. load flow models have the advantage of both accurate contingency evaluation and inclusion of the reactive power flows. Fig. 2.2 (Happ 1976) illustrates the contingency evaluation for line or generator loss.



(a) Line loss



(b) Generator loss

Fig. 2.2 Illustration of contingency evaluation

2.7 APPLICATIONS TO PRACTICAL DESIGN PROBLEMS

As stated before, sensitivity calculations are performed to evaluate gradients of functions of interest subject to equality constraints relating the state and control variables of the system. These gradients may be supplied to optimization routines employed in different power system design problems.

In practice, functional inequality constraints as well as upper and lower limits on the control variables must be considered in optimization to reflect the physical limitations on different system components.

2.7.1 Power System Design Problem

A typical power system design problem may be stated as the general nonlinear programming problem

$$\underset{\underset{\sim}{u}}{\text{Minimize}}\ f(\underset{\sim}{x},\ \underset{\sim}{u}) \tag{2.17}$$

subject to

$$\underset{\sim}{h}(\underset{\sim}{x},\ \underset{\sim}{u}) = \underset{\sim}{0} \tag{2.18}$$

and

$$\underset{\sim}{g}(\underset{\sim}{x},\ \underset{\sim}{u}) \geq \underset{\sim}{0}, \tag{2.19}$$

where the column vector $\underset{\sim}{g}(\underset{\sim}{x},\ \underset{\sim}{u})$ represents n_g inequality constraints.

Considering the general formulation of the problem (2.17)–(2.19) with continuous real variables and assuming proper convexity, the Kuhn-Tucker relations (Kuhn and Tucker 1951) provide a set of necessary conditions which the solution must satisfy at the minimum of f. Techniques of constrained optimization (Gill and Murray 1974, Wilde and

Beightler 1967) are employed.

2.7.2 Comments

A wide variety of problems in computerized operation and planning of power systems falls into the form (2.17)-(2.19). The type of the objective function f as well as the existence and the nature of both equality and inequality constraints depend on the study performed.

Several approaches have been described and successfully applied to handle functional inequality constraints in many power system problems. For example, some of the approaches (Carpentier 1973, Peschon, Bree and Hajdu 1971) utilize the generalized reduced gradient (GRG) method. Others (Sasson 1969a, 1969b, Fischl and Wasley 1978) employ penalty function methods. Features of both methods may be incorporated (Wu, Gross, Luini and Look 1979).

In these approaches, the total derivatives (called the reduced gradient) of a formulated objective function w.r.t. control variables may be evaluated by methods of sensitivity calculations described before.

2.7.3 The Optimal Load Flow Problem

In the optimal power flow problem (Dommel and Tinney 1968), a feasible power flow solution w.r.t. constraints on both control and state variables is found which minimizes some cost criterion.

In general, the adjustable control variables assigned include the real power P_g from generating plants available for adjustable dispatch, voltage magnitude $|V_g|$ at P, V-buses, tap transformer and phase shifter ratios and parameters of shunt control elements.

Some of the inequality constraints represent limits on the capability of adjustable control devices, e.g., real and imaginary transformer tap ratios, and other equipment capacities such as the generating capacity. The others represent the system security requirements which include line flow current and power constraints under normal and contingency conditions. The violation of inequality constraints may lead to inadequate service due to component outages.

A number of problems can be defined by a different choice of the objective function of (2.17) and constraints (2.18) and (2.19). The economic dispatch and minimum loss problems (Dommel and Tinney 1968, Happ 1977, Sasson 1969a), optimal load curtailment under emergency conditions (Peschon et al. 1971) and VAR flow control (Sullivan 1972) are examples.

2.7.4 Power System Planning Studies

Many power system planning problems can be formulated as nonlinear programming problems in the form (2.17)-(2.19). The objective function f, the design variables and the constraints are defined in a particular planning study to reflect economy, reliability, security and efficiency requirements.

The power flow model which simulates the steady-state power flows and voltages in the network under planning considerations is described either exactly or approximately according to accuracy requirements.

In automated power network design problems (Director and Sullivan 1978, Fischl and Puntel 1972, Puntel et al. 1973), for example, the objective function f may be formulated to represent line overloading. The control variables to be adjusted are line admittances representing the required additions to support the existing transmission capacity. The inclusion of inequality constraints imposed on the design variables by, for example, the right-of-ways may be included.

A contingency analysis may be required after designing a nominal network. In this kind of study, first-order changes of functions of interest simulating line overloading due to assigned parameter changes and line or generator outages are employed in the adequacy checks.

Many other applications of the methods of sensitivity evaluation described before can be identified in which either first-order changes or total derivatives of functions of interest are concerned.

3

APPROACH BASED ON TELLEGEN'S THEOREM: A GENERALIZED VERSION

3.1 INTRODUCTION

In trying to apply the powerful features of Tellegen's theorem to power system sensitivity calculations, previous workers approximated the a.c. load flow model to allow direct application of Tellegen's theorem. Moreover, the theoretical possibility of obtaining sensitivities of complex functions via one adjoint analysis as in typical electronic circuits has not been previously investigated.

In this chapter, a generalized version of the Tellegen's theorem-based approach is developed which, using a special complex notation, overcomes the difficulties observed (Püttgen 1976) in treating exact steady-state component models of power networks and provides a theoretical basis for treating complex functions.

3.2 PERTURBED STEADY-STATE COMPONENT MODELS

3.2.1 Component Models of Typical Electronic Circuits

A simple, exact application of Tellegen's theorem is possible in typical linear electronic circuits. This is essentially because of the relatively simpler component models used. The steady-state component models representing typical sources and passive elements are shown, respectively, in Fig. 3.1 and Fig. 3.2. Observe that the current I,

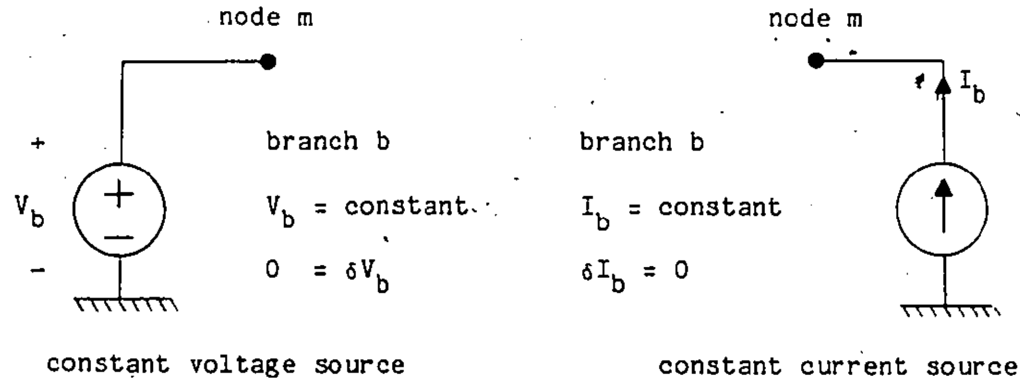
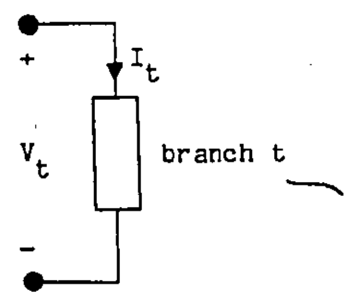


Fig. 3.1 Modelling of typical sources in electronic circuits



$$I_t = Y_t V_t$$

$$\delta I_t = Y_t \delta V_t + V_t \delta Y_t$$

Fig. 3.2 Modelling of typical passive element

voltage V and admittance Y variables appear in perturbed component models as complex quantities. The conjugate of a complex variable does not appear in the perturbed component model that might require separation into real and imaginary parts in subsequent analysis.

3.2.2 Typical Electronic Network Equations

In general, these simple models of network elements lead to a compact set of complex linear equations, e.g., the nodal equations, which is usually solved in the complex mode without the need for separating real and imaginary parts, which would require (Stewart 1973) about twice the computer memory.

Basically, we can arrive at a compact set of complex linear equations to analyze a network as long as the branch models have the perturbed form

$$\sigma_b \ \delta I_b = \tau_b \ \delta V_b + W_b^S, \tag{3.1}$$

where I_b and V_b are branch current and voltage variables, respectively, and σ_b, τ_b and W_b^S are coefficients associated with branch type.

3.2.3 Component Models of Power Networks

In power networks, the steady-state models of some components do not fall into the perturbed form (3.1). Examples of modelling load branch and generator branch connected to a voltage controlled bus are shown, respectively, in Fig. 3.3 and Fig. 3.4. Note that the complex conjugate of the current and voltage variables are used to express element models.

$S_\ell = V_\ell \, I_\ell^* = \text{constant}$

$\delta S_\ell = 0$

or

$V_\ell \, \delta I_\ell^* = -I_\ell^* \, \delta V_\ell$

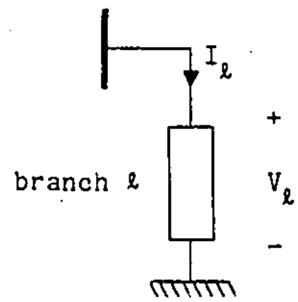


Fig. 3.3 Load branch modelling

$P_g, \; |V_g| = \text{constant}$

$\delta P_g = 0$

or

$V_g \, \delta I_g^* + V_g^* \, \delta I_g = - I_g^* \, \delta V_g - I_g \, \delta V_g^*$

$\delta |V_g| = 0$

or

$0 = V_g^* \, \delta V_g + V_g \, \delta V_g^*$

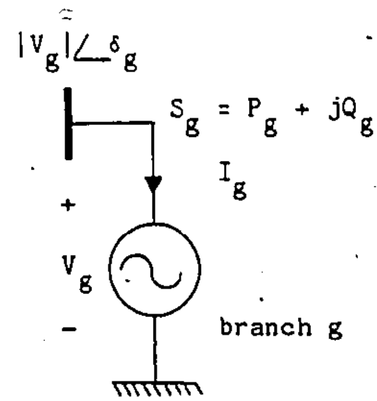
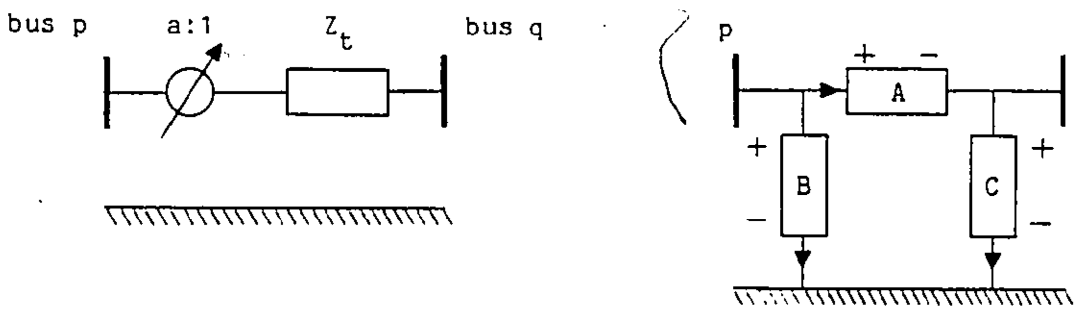


Fig. 3.4 Modelling of generator branch connected to P, V-type bus

3.2.4 Modelling of Power Transformers with Complex Turns Ratio

It is known (Sullivan 1977) that power transformers with complex turns ratio (phase shifting transformers) cannot be modelled by an equivalent π-network using the ordinary passive elements of Fig. 3.2. However, as shown in Fig. 3.5, a construction of an equivalent π-network is possible using more general branch models. The model of Fig. 3.5 represents a member of possible constructions derived in Appendix A.



$$I_A - a^* I_A^* = \frac{1}{Z_t a^*} V_A - \frac{a^*}{Z_t^* a} V_A^*$$

$$I_B - a^* I_B^* = \frac{1}{Z_t a^*} \left(\frac{1}{a} - 1 \right) V_B - \frac{a^*}{Z_t^* a} \left(\frac{1}{a^*} - 1 \right) V_B^*$$

$$I_C + a I_C^* = \frac{-1}{Z_t} \left(\frac{1}{a} - 1 \right) V_C - \frac{a}{Z_t^*} \left(\frac{1}{a^*} - 1 \right) V_C^*$$

Fig. 3.5 Modelling of transformers with complex turns ratio

3.2.5 General Perturbed Form of Component Models

In conclusion, we deal with power network component models of the more general perturbed form

$$\sigma_b \, \delta I_b + \bar{\sigma}_b \, \delta I_b^* = \tau_b \, \delta V_b + \bar{\tau}_b \, \delta V_b^* + W_b^S \qquad (3.2)$$

and in this chapter and throughout most of the thesis we manage to use both the theory and techniques which can handle these general element models without approximation.

3.3 THE CONJUGATE NOTATION

In order to perform the sensitivity calculations based on the exact element models of different power system components in a simple and straightforward way, we employ a special complex notation described in this section.

3.3.1 Formal Partial Derivatives

We denote by C and R, respectively, the field of complex numbers and the field of real numbers. The vector space over C, of n-tuples $(\zeta_1, \ldots, \zeta_n)$, $\zeta_i \in C$ is denoted by C^n. Similarly, R^n stands for the vector space over R, of n-tuples $(\zeta_{1m}, \ldots, \zeta_{nm})$, $m = 1, 2$ and $\zeta_{im} \in R$. Also, we write

$$\underset{\sim}{\zeta} = \underset{\sim}{\zeta}_1 + j\, \underset{\sim}{\zeta}_2, \tag{3.3}$$

where $\underset{\sim}{\zeta}$ is a column vector of components ζ_i given by

$$\zeta_i = \zeta_{i1} + j\, \zeta_{i2}, \tag{3.4}$$

$\underset{\sim}{\zeta}_1, \underset{\sim}{\zeta}_2 \in R^n$, $\zeta_{i1}, \zeta_{i2} \in R$, $i = 1, 2, .., n$.

For a continuously differentiable complex valued function f on an open set $\Omega \subset C^n$, we define the formal (Fuks 1963) or symbolic (Ahlfors 1966) partial derivatives

$$\frac{\partial f}{\partial \underset{\sim}{\zeta}} \triangleq \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}_1} - j \frac{\partial f}{\partial \underset{\sim}{\zeta}_2} \right)/2 \tag{3.5}$$

and

$$\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} \triangleq \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}_1} + j \frac{\partial f}{\partial \underset{\sim}{\zeta}_2} \right)/2, \tag{3.6}$$

where $\partial f/\partial \underset{\sim}{\zeta}$, $\partial f/\partial \underset{\sim}{\zeta}^*$, $\partial f/\partial \underset{\sim}{\zeta}_1$ and $\partial f/\partial \underset{\sim}{\zeta}_2$ are column vectors.

Note that in formal derivatives, the Cauchy-Riemann differential equations may be written (Fuks 1963) as

$$\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} = \underset{\sim}{0}. \tag{3.7}$$

3.3.2 First-Order Change in Terms of Formal Derivatives

We consider the nonsingular transformation

$$\begin{bmatrix} \underset{\sim}{\zeta}_1 \\ \underset{\sim}{\zeta}_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \underset{\sim}{1}^n & \underset{\sim}{1}^n \\ -\underset{\sim}{j}^n & \underset{\sim}{j}^n \end{bmatrix} \begin{bmatrix} \underset{\sim}{\zeta} \\ \underset{\sim}{\zeta}^* \end{bmatrix} , \tag{3.8}$$

where $\underset{\sim}{1}^n$ is the identity matrix of order n and

$$\underset{\sim}{j}^n \triangleq j\, \underset{\sim}{1}^n. \tag{3.9}$$

Equation (3.8) may be written in the perturbed form

$$\begin{bmatrix} \delta\underset{\sim}{\zeta}_1 \\ \delta\underset{\sim}{\zeta}_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \underset{\sim}{1}^n & \underset{\sim}{1}^n \\ -\underset{\sim}{j}^n & \underset{\sim}{j}^n \end{bmatrix} \begin{bmatrix} \delta\underset{\sim}{\zeta} \\ \delta\underset{\sim}{\zeta}^* \end{bmatrix} . \tag{3.10}$$

Note that

$$\delta\underset{\sim}{\zeta}^* = (\delta\underset{\sim}{\zeta})^*. \tag{3.11}$$

The first-order change of f is given by

$$\delta f = \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}_1} \right)^T \delta\underset{\sim}{\zeta}_1 + \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}_2} \right)^T \delta\underset{\sim}{\zeta}_2 \tag{3.12}$$

or, using (3.10),

$$\delta f = \frac{1}{2}\left[\left(\frac{\partial f}{\partial \underset{\sim}{\zeta}_1} \right)^T - \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}_2} \right)^T \underset{\sim}{j}^n \right] \delta\underset{\sim}{\zeta} + \frac{1}{2}\left[\left(\frac{\partial f}{\partial \underset{\sim}{\zeta}_1} \right)^T + \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}_2} \right)^T \underset{\sim}{j}^n \right] \delta\underset{\sim}{\zeta}^*. \tag{3.13}$$

Hence, from (3.5) and (3.6)

$$\delta f = \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}} \right)^T \delta \underset{\sim}{\zeta} + \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} \right)^T \delta \underset{\sim}{\zeta}^*. \qquad (3.14)$$

Equation (3.14) expresses δf in terms of the variations in $\underset{\sim}{\zeta}$ and $\underset{\sim}{\zeta}^*$ using the formal derivatives $\partial f/\partial \underset{\sim}{\zeta}$ and $\partial f/\partial \underset{\sim}{\zeta}^*$ of (3.5) and (3.6), respectively.

3.3.3 Pure Real and Pure Imaginary Functions

It can be shown that, for arbitrary $\underset{\sim}{\zeta}$, if

$$\underset{\sim}{\mu}^T \underset{\sim}{\zeta} + \underset{\sim}{\bar{\mu}}^T \underset{\sim}{\zeta}^* = \underset{\sim}{\mu}^{\bullet T} \underset{\sim}{\zeta} + \underset{\sim}{\bar{\mu}}^{\bullet T} \underset{\sim}{\zeta}^*, \qquad (3.15)$$

where $\underset{\sim}{\mu}$, $\underset{\sim}{\bar{\mu}}$, $\underset{\sim}{\mu}^\bullet$ and $\underset{\sim}{\bar{\mu}}^\bullet$ are appropriate vectors of complex scalars, then

$$\underset{\sim}{\mu} = \underset{\sim}{\mu}^\bullet \text{ and } \underset{\sim}{\bar{\mu}} = \underset{\sim}{\bar{\mu}}^\bullet. \qquad (3.16)$$

For a pure real function f, we write

$$\delta f = \delta f^* = (\delta f)^* \qquad (3.17)$$

or, using (3.14),

$$\left(\frac{\partial f}{\partial \underset{\sim}{\zeta}} \right)^T \delta \underset{\sim}{\zeta} + \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} \right)^T \delta \underset{\sim}{\zeta}^* = \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}} \right)^{*T} \delta \underset{\sim}{\zeta}^* + \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} \right)^{*T} \delta \underset{\sim}{\zeta}, \qquad (3.18)$$

hence, from (3.15) and (3.16)

$$\frac{\partial f}{\partial \underset{\sim}{\zeta}} = \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} \right)^*. \qquad (3.19)$$

Also, for a pure imaginary function f, we write

$$\delta f = - \delta f^* = - (\delta f)^* \qquad (3.20)$$

or

$$\left(\frac{\partial f}{\partial \underset{\sim}{\zeta}}\right)^T \delta\underset{\sim}{\zeta} + \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}^*}\right)^T \delta\underset{\sim}{\zeta}^* = -\left(\frac{\partial f}{\partial \underset{\sim}{\zeta}}\right)^{*T} \delta\underset{\sim}{\zeta}^* - \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}^*}\right)^{*T} \delta\underset{\sim}{\zeta}, \quad (3.21)$$

hence, from (3.15) and (3.16)

$$\frac{\partial f}{\partial \underset{\sim}{\zeta}} = -\left(\frac{\partial f}{\partial \underset{\sim}{\zeta}^*}\right)^*. \quad (3.22)$$

3.3.4 Remark

We remark (Fuks 1963) that the terminology of formal derivatives arises because of the possibility of obtaining them formally using the ordinary differentiation rules. The use of the conjugate notation facilitates the derivations and subsequent formulation of the equations to be solved.

3.4 AUGMENTED FORM OF TELLEGEN'S THEOREM

3.4.1 Tellegen Terms and Group Terms for A.C. Power Model

The expressions of (2.11) represent the basic form of Tellegen's theorem. Since the V_b and $\hat{V}_b$ of (2.11) satisfy Kirchhoff's voltage law (KVL), the V_b^* and $\hat{V}_b^*$ also satisfy KVL. Similarly, since the I_b and $\hat{I}_b$ of (2.11) satisfy Kirchhoff's current law (KCL), the I_b^* and $\hat{I}_b^*$ also satisfy KCL. Hence, in addition to (2.11) the following valid variations of Tellegen's theorem can be considered (Penfield, Spence and Duinker 1970)

$$\sum_b \hat{I}_b^* V_b^* = 0, \tag{3.23}$$

$$\sum_b \hat{V}_b^* I_b^* = 0, \tag{3.24}$$

$$\sum_b \hat{I}_b V_b^* = 0, \tag{3.25}$$

$$\sum_b \hat{V}_b I_b^* = 0, \tag{3.26}$$

$$\sum_b \hat{I}_b^* V_b = 0, \tag{3.27}$$

$$\sum_b \hat{V}_b^* I_b = 0. \tag{3.28}$$

Note that, in the case of identical original and adjoint networks, we set $\hat{V}_b = V_b$ and $\hat{I}_b = I_b$.

In addition to the Tellegen terms (2.11) and (3.23)-(3.28) we also consider valid expressions in terms of certain groups of elements in the form

$$\sum_{b \in B_k} C_b^k = 0 \tag{3.29}$$

and

$$\sum_{b \in B_k} C_b^{k*} = 0, \tag{3.30}$$

where C_b^k and C_b^{k*} are complex functions of the variables V_b and I_b and their complex conjugate V_b^* and I_b^*, and B_k is the set of branch elements forming the kth group. An example of the group terms is the KVL for a local loop of the network.

3.4.2 Extended Tellegen Sum and Adjoining Coefficients

The extended Tellegen sum is now written as

$$\sum_b [\alpha \hat{I}_b V_b + \bar{\alpha} \hat{I}_b^* V_b^* - \beta \hat{V}_b I_b - \bar{\beta} \hat{V}_b^* I_b^* + \xi \hat{I}_b V_b^* + \bar{\xi} \hat{I}_b^* V_b$$

$$- \nu \hat{V}_b I_b^* - \bar{\nu} \hat{V}_b^* I_b + \sum_k \Gamma_k \lambda_{bk} C_b^k + \sum_k \bar{\Gamma}_k \lambda_{bk} C_b^{k*}] = 0, \quad (3.31)$$

where the Tellegen terms and group terms have been adjoined in an appropriate sequence via the complex coefficients α, $\bar{\alpha}$, β, $\bar{\beta}$, ξ, $\bar{\xi}$, ν, $\bar{\nu}$, Γ_k and $\bar{\Gamma}_k$,

$$\lambda_{bk} = \begin{cases} 0 & \text{if } b \notin B_k \\ 1 & \text{if } b \in B_k \end{cases} . \quad (3.32)$$

Note that in cases where

$$\bar{\alpha} = \alpha^*, \quad (3.33)$$

$$\bar{\beta} = \beta^*, \quad (3.34)$$

$$\bar{\xi} = \xi^*, \quad (3.35)$$

$$\bar{\nu} = \nu^* \quad (3.36)$$

and

$$\bar{\Gamma}_k = \Gamma_k^* \text{ for all } k, \quad (3.37)$$

the extended Tellegen's sum (3.31) is a real quantity.

3.4.3 Perturbed Tellegen Sum

The sum (3.31) is written in terms of first-order changes in V and I as

$$\sum_b [\alpha\, \hat{I}_b\, \delta V_b + \bar{\alpha}\, \hat{I}_b^*\, \delta V_b^* - \beta\, \hat{V}_b\, \delta I_b - \bar{\beta}\, \hat{V}_b^*\, \delta I_b^* + \xi\, \hat{I}_b\, \delta V_b^* + \bar{\xi}\, \hat{I}_b^*\, \delta V_b$$
$$- \nu\, \hat{V}_b\, \delta I_b^* - \bar{\nu}\, \hat{V}_b^*\, \delta I_b + \sum_k \Gamma_k\, \lambda_{bk} (C_{bv}^k\, \delta V_b + \bar{C}_{bv}^k\, \delta V_b^* + C_{bi}^k\, \delta I_b$$
$$+ \bar{C}_{bi}^k\, \delta I_b^*) + \sum_k \bar{\Gamma}_k\, \lambda_{bk} (C_{bv}^{k*}\, \delta V_b + \bar{C}_{bv}^{k*}\, \delta V_b^* + C_{bi}^{k*}\, \delta I_b + \bar{C}_{bi}^{k*}\, \delta I_b^*)] = 0 \tag{3.38}$$

or

$$\sum_b [(\alpha\, \hat{I}_b + \bar{\xi}\, \hat{I}_b^* + \sum_k \Gamma_k\, \lambda_{bk}\, C_{bv}^k + \sum_k \bar{\Gamma}_k\, \lambda_{bk}\, C_{bv}^{k*})\, \delta V_b$$
$$+ (\bar{\alpha}\, \hat{I}_b^* + \xi\, \hat{I}_b + \sum_k \Gamma_k\, \lambda_{bk}\, \bar{C}_{bv}^k + \sum_k \bar{\Gamma}_k\, \lambda_{bk}\, \bar{C}_{bv}^{k*})\, \delta V_b^*$$
$$+ (-\beta\, \hat{V}_b - \bar{\nu}\, \hat{V}_b^* + \sum_k \Gamma_k\, \lambda_{bk}\, C_{bi}^k + \sum_k \bar{\Gamma}_k\, \lambda_{bk}\, C_{bi}^{k*})\, \delta I_b$$
$$+ (-\bar{\beta}\, \hat{V}_b^* - \nu\, \hat{V}_b + \sum_k \Gamma_k\, \lambda_{bk}\, \bar{C}_{bi}^k + \sum_k \bar{\Gamma}_k\, \lambda_{bk}\, \bar{C}_{bi}^{k*})\, \delta I_b^*] = 0, \tag{3.39}$$

where C_{bu}^k, C_{bu}^{k*}, $\bar{C}_{bu}^k$ and $\bar{C}_{bu}^{k*}$ stand for $\partial C_b^k/\partial U$, $\partial C_b^{k*}/\partial U$, $\partial C_b^k/\partial U^*$, $\partial C_b^{k*}/\partial U^*$, respectively, u denoting v or i and U denoting V or I.

3.5 ELEMENT VARIABLES VIA BASIC VARIABLES

3.5.1 Element Jacobian Matrices

The perturbed Tellegen sum (3.39) has been written in terms of first-order changes of V_b, V_b^*, I_b and I_b^*. We shall call these variables the basic variables and denote them by the vector

$$\underset{\sim}{w}_b = \begin{bmatrix} \underset{\sim}{w}_{bv} \\ \hline \underset{\sim}{w}_{bi} \end{bmatrix} \triangleq \begin{bmatrix} V_b \\ V_b^* \\ \hline I_b \\ I_b^* \end{bmatrix} . \tag{3.40}$$

Now, for each element, and according to its type, another set of variables called the element variables is of practical interest. The element variables will be denoted by the vector $\underset{\sim}{z}_b$ of four components describing the practical state $\underset{\sim}{x}_b$ and control $\underset{\sim}{u}_b$ variables associated with element b as

$$\underset{\sim}{z}_b = \begin{bmatrix} \underset{\sim}{x}_b \\ \underset{\sim}{u}_b \end{bmatrix} , \qquad (3.41)$$

where $\underset{\sim}{x}_b$ and $\underset{\sim}{u}_b$ are two component vectors. $\delta\underset{\sim}{z}_b$ can be expressed in terms of $\delta\underset{\sim}{w}_b$ in the form

$$\delta\underset{\sim}{z}_b = \begin{bmatrix} \delta\underset{\sim}{x}_b \\ \delta\underset{\sim}{u}_b \end{bmatrix} = \underset{\sim}{J}_b \delta\underset{\sim}{w}_b , \qquad (3.42)$$

where

$$\underset{\sim}{J}_b \triangleq \left(\frac{\partial \underset{\sim}{z}_b^T}{\partial \underset{\sim}{w}_b} \right)^T \qquad (3.43)$$

is a Jacobian matrix.

3.5.2 Transformed Form of Extended Tellegen Sum

From (3.42)

$$\delta\underset{\sim}{w}_b = \underset{\sim}{J}_b^{-1} \, \delta\underset{\sim}{z}_b . \qquad (3.44)$$

A term of (3.39) associated with the bth branch is written in the more convenient form

$$\hat{\underset{\sim}{f}}_b^T \, \delta\underset{\sim}{w}_b , \qquad (3.45)$$

where

$$\hat{f}_b = \begin{bmatrix} \hat{f}_{bi} \\ \hdashline \hat{f}_{bv} \end{bmatrix} \triangleq \begin{bmatrix} \alpha \hat{I}_b + \bar{\xi} \hat{I}_b^* + \sum_k \Gamma_k \lambda_{bk} C_{bv}^k + \sum_k \bar{\Gamma}_k \lambda_{bk} C_{bv}^{k*} \\ \bar{\alpha} \hat{I}_b^* + \xi \hat{I}_b + \sum_k \Gamma_k \lambda_{bk} \bar{C}_{bv}^k + \sum_k \bar{\Gamma}_k \lambda_{bk} \bar{C}_{bv}^{k*} \\ \hline -\beta \hat{V}_b - \bar{\nu} \hat{V}_b^* + \sum_k \Gamma_k \lambda_{bk} C_{bi}^k + \sum_k \bar{\Gamma}_k \lambda_{bk} C_{bi}^{k*} \\ -\bar{\beta} \hat{V}_b^* - \nu \hat{V}_b + \sum_k \Gamma_k \lambda_{bk} \bar{C}_{bi}^k + \sum_k \bar{\Gamma}_k \lambda_{bk} \bar{C}_{bi}^{k*} \end{bmatrix}, \quad (3.46)$$

hence (3.39) is written as

$$\sum_b \hat{f}_b^T \delta w_b = 0 \quad (3.47)$$

or, using (3.44), as

$$\sum_b \hat{f}_b^T J_b^{-1} \delta z_b = 0 \quad (3.48)$$

or

$$\sum_b ((J_b^{-1})^T \hat{f}_b)^T \delta z_b = 0. \quad (3.49)$$

3.6 TRANSFORMED ADJOINT VARIABLES AND NETWORK SENSITIVITIES

Let

$$\hat{\eta}_b = \begin{bmatrix} \hat{\eta}_{bx} \\ \hat{\eta}_{bu} \end{bmatrix}$$

$$\triangleq (J_b^{-1})^T \hat{f}_b \quad (3.50)$$

be transformed adjoint variables associated with the bth branch, where $\hat{\eta}_{bx}$ and $\hat{\eta}_{bu}$ are two component vectors. Then from (3.49)

$$\sum_b \hat{\eta}_b^T \, \delta z_b = 0 \tag{3.51}$$

or, from (3.41)

$$\sum_b (\hat{\eta}_{bx}^T \, \delta x_b + \hat{\eta}_{bu}^T \, \delta u_b) = 0. \tag{3.52}$$

Now, for a general complex function f of all state vectors x_b and all control vectors u_b we set

$$\hat{\eta}_{bx} = \frac{\partial f}{\partial x_b} , \tag{3.53}$$

hence

$$\delta f = \sum_b [(\frac{\partial f}{\partial x_b})^T \, \delta x_b + (\frac{\partial f}{\partial u_b})^T \, \delta u_b]$$

$$= \sum_b [\hat{\eta}_{bx}^T \, \delta x_b + (\frac{\partial f}{\partial u_b})^T \, \delta u_b]. \tag{3.54}$$

Then, from (3.52)

$$\delta f = \sum_b [(\frac{\partial f}{\partial u_b})^T - \hat{\eta}_{bu}^T]\delta u_b , \tag{3.55}$$

so that

$$\frac{df}{du_b} = \frac{\partial f}{\partial u_b} - \hat{\eta}_{bu} . \tag{3.56}$$

In the case when u_b is a function of some real design variables we write

$$\delta u_b = \sum_i \frac{\partial u_b}{\partial \zeta_{bi}} \Delta \zeta_{bi} , \tag{3.57}$$

where ζ_{bi} is the ith design variable associated with u_b and $\Delta\zeta_{bi}$ denotes the change in ζ_{bi}. In practice, ζ_{bi} represent, for example, the parameters of shunt control elements. From (3.55)

$$\frac{df}{d\zeta_{bi}} = [(\frac{\partial f}{\partial \underset{\sim}{u}_b})^T - \hat{\underset{\sim}{\eta}}_{bu}^T] \frac{\partial \underset{\sim}{u}_b}{\partial \zeta_{bi}} . \tag{3.58}$$

Note that (3.53) defines the adjoint elements while (3.56) or (3.58) provides the required gradients.

3.7 GENERAL ADJOINT FORMULATION

We define an adjoint vector analogous to $\underset{\sim}{w}_b$ of (3.40) as

$$\hat{\underset{\sim}{w}}_b = \begin{bmatrix} \hat{\underset{\sim}{w}}_{bv} \\ \hline \hat{\underset{\sim}{w}}_{bi} \end{bmatrix} \triangleq \begin{bmatrix} \hat{V}_b \\ \hat{V}_b^* \\ \hline \hat{I}_b \\ \hat{I}_b^* \end{bmatrix} \tag{3.59}$$

and write the matrix $(\underset{\sim}{J}_b^{-1})^T$ of (3.50) in a partitioned form

$$(\underset{\sim}{J}_b^{-1})^T = \begin{bmatrix} \underset{\sim}{M}_{11}^b & \underset{\sim}{M}_{12}^b \\ \underset{\sim}{M}_{21}^b & \underset{\sim}{M}_{22}^b \end{bmatrix} , \tag{3.60}$$

where $\underset{\sim}{M}_{11}^b$, $\underset{\sim}{M}_{12}^b$, $\underset{\sim}{M}_{21}^b$ and $\underset{\sim}{M}_{22}^b$ are 2x2 matrices.

Using (3.46) and (3.60) the vectors $\hat{\underset{\sim}{\eta}}_{bx}$ and $\hat{\underset{\sim}{\eta}}_{bu}$ of (3.50) are given by

$$\hat{\underset{\sim}{\eta}}_{bx} = \underset{\sim}{M}_{11}^b \hat{\underset{\sim}{f}}_{bi} + \underset{\sim}{M}_{12}^b \hat{\underset{\sim}{f}}_{bv} \tag{3.61}$$

and

$$\hat{\underset{\sim}{n}}_{bu} = \underset{\sim}{M}^b_{21}\ \hat{\underset{\sim}{f}}_{bi} + \underset{\sim}{M}^b_{22}\ \hat{\underset{\sim}{f}}_{bv}. \tag{3.62}$$

The vectors $\hat{\underset{\sim}{f}}_{bi}$ and $\hat{\underset{\sim}{f}}_{bv}$ are written in terms of $\underset{\sim}{w}_b$ and $\hat{\underset{\sim}{w}}_b$ as

$$\hat{\underset{\sim}{f}}_{bi} = \underset{\sim}{\Lambda}^b_i\ \underset{\sim}{w}_{bi} + \overline{\underset{\sim}{\Lambda}}_i\ \hat{\underset{\sim}{w}}_{bi} \tag{3.63}$$

and

$$\hat{\underset{\sim}{f}}_{bv} = \underset{\sim}{\Lambda}^b_v\ \underset{\sim}{w}_{bv} + \overline{\underset{\sim}{\Lambda}}_v\ \hat{\underset{\sim}{w}}_{bv}, \tag{3.64}$$

where $\underset{\sim}{\Lambda}^b_i$, $\overline{\underset{\sim}{\Lambda}}_i$, $\underset{\sim}{\Lambda}^b_v$ and $\overline{\underset{\sim}{\Lambda}}_v$ are 2x2 matrices. The elements of $\overline{\underset{\sim}{\Lambda}}_v$ and $\overline{\underset{\sim}{\Lambda}}_i$ consist of the adjoint coefficients α, $\overline{\alpha}$, ξ, $\overline{\xi}$, β, $\overline{\beta}$, ν and $\overline{\nu}$.

For the set of terms considered in Tellegen sum (3.31) the matrices $\underset{\sim}{\Lambda}^b_i$, $\overline{\underset{\sim}{\Lambda}}_i$, $\underset{\sim}{\Lambda}^b_v$ and $\overline{\underset{\sim}{\Lambda}}_v$ are given from (3.46) by

$$\underset{\sim}{\Lambda}^b_i = \begin{bmatrix} \sum_k \Gamma_k\ \lambda_{bk}\ C^k_{bv}/I_b & \sum_k \overline{\Gamma}_k\ \lambda_{bk}\ C^{k*}_{bv}/I^*_b \\ \sum_k \Gamma_k\ \lambda_{bk}\ \overline{C}^k_{bv}/I_b & \sum_k \overline{\Gamma}_k\ \lambda_{bk}\ \overline{C}^{k*}_{bv}/I^*_b \end{bmatrix}, \tag{3.65}$$

$$\overline{\underset{\sim}{\Lambda}}_i = \begin{bmatrix} \alpha & \overline{\xi} \\ \xi & \overline{\alpha} \end{bmatrix}, \tag{3.66}$$

$$\underset{\sim}{\Lambda}^b_v = \begin{bmatrix} \sum_k \Gamma_k\ \lambda_{bk}\ C^k_{bi}/V_b & \sum_k \overline{\Gamma}_k\ \lambda_{bk}\ C^{k*}_{bi}/V^*_b \\ \sum_k \Gamma_k\ \lambda_{bk}\ \overline{C}^k_{bi}/V_b & \sum_k \overline{\Gamma}_k\ \lambda_{bk}\ \overline{C}^{k*}_{bi}/V^*_b \end{bmatrix} \tag{3.67}$$

and

$$\bar{\Lambda}_v = - \begin{bmatrix} \beta & \bar{\nu} \\ \nu & \bar{\beta} \end{bmatrix} . \tag{3.68}$$

Note that if C_b^k of (3.29) has the form

$$C_b^k = \tilde{V}_b \tilde{I}_b, \tag{3.69}$$

where

$$\tilde{V}_b = \pm V_b \tag{3.70}$$

and

$$\tilde{I}_b = \pm I_b, \tag{3.71}$$

the elements of Λ_i^b and Λ_v^b consist solely of the adjoining coefficients Γ_k and $\bar{\Gamma}_k$. Note also that $\tilde{V}_b = \pm 1$ in (3.65) and $\tilde{I}_b = \pm 1$ in (3.67) lead to corresponding zero matrices.

For use later we now define

$$N_{ib}^k = \Gamma_k C_{bv}^k + \bar{\Gamma}_k C_{bv}^{k*}, \tag{3.72}$$

$$\bar{N}_{ib}^k = \Gamma_k \bar{C}_{bv}^k + \bar{\Gamma}_k \bar{C}_{bv}^{k*}, \tag{3.73}$$

$$N_{vb}^k = \Gamma_k C_{bi}^k + \bar{\Gamma}_k C_{bi}^{k*} \tag{3.74}$$

and

$$\bar{N}_{vb}^k = \Gamma_k \bar{C}_{bi}^k + \bar{\Gamma}_k \bar{C}_{bi}^{k*}. \tag{3.75}$$

Using (3.63) and (3.64), equation (3.61) is written as

$$\bar{\underset{\sim}{\theta}}_{bi}\ \hat{\underset{\sim}{w}}_{bi} = \bar{\underset{\sim}{\theta}}_{bv}\ \hat{\underset{\sim}{w}}_{bv} + \underset{\sim}{\theta}_{b}, \tag{3.76}$$

where the 2x2 matrices $\bar{\underset{\sim}{\theta}}_{bi}$ and $\bar{\underset{\sim}{\theta}}_{bv}$ are given by

$$\bar{\underset{\sim}{\theta}}_{bi} = \underset{\sim}{M}_{11}^{b}\ \bar{\underset{\sim}{\Lambda}}_{i} \tag{3.77}$$

and

$$\bar{\underset{\sim}{\theta}}_{bv} = - \underset{\sim}{M}_{12}^{b}\ \bar{\underset{\sim}{\Lambda}}_{v} \tag{3.78}$$

and the vector $\underset{\sim}{\theta}_b$ is given by

$$\underset{\sim}{\theta}_{b} = \hat{\underset{\sim}{n}}_{bx} - \underset{\sim}{M}_{11}^{b}\ \underset{\sim}{\Lambda}_{i}^{b}\ \underset{\sim}{w}_{bi} - \underset{\sim}{M}_{12}^{b}\ \underset{\sim}{\Lambda}_{v}^{b}\ \underset{\sim}{w}_{bv}. \tag{3.79}$$

Note that the choice of the coefficients α, $\bar{\alpha}$, etc., is subject to the consistency of (3.76).

3.8 POWER SYSTEM ELEMENT VARIABLES

3.8.1 Notation

We consider the total number of branches to be n_B consisting of n_L loads, n_G generators, one slack generator and $n_T = n_B - n_L - n_G - 1$ other branch elements.

The buses are ordered such that subscripts $\ell = 1, 2, \ldots, n_L$ identify load branches, $g = n_L+1, \ldots, n_L+n_G$ identify generator branches and $n = n_L + n_G + 1$ identifies the slack generator branch. Subscripts $t = n + 1, \ldots, n_B$ are used to identify other branches.

3.8.2 Element Variables of System Components

The element variables for a load are usually defined as

$$\bar{\underset{\sim}{z}}_\ell = \begin{bmatrix} \underset{\sim}{x}_\ell \\ \hline \underset{\sim}{u}_\ell \end{bmatrix} \triangleq \begin{bmatrix} |V_\ell| \\ \delta_\ell \\ \hline P_\ell \\ Q_\ell \end{bmatrix} = \begin{bmatrix} (V_\ell V_\ell^*)^{1/2} \\ \tan^{-1}[j(V_\ell^* - V_\ell)/(V_\ell + V_\ell^*)] \\ \hline (V_\ell I_\ell^* + V_\ell^* I_\ell)/2 \\ j(V_\ell^* I_\ell - V_\ell I_\ell^*)/2 \end{bmatrix} \tag{3.80}$$

or, for example, as

$$\tilde{\underset{\sim}{z}}_\ell \triangleq \begin{bmatrix} V_\ell \\ V_\ell^* \\ S_\ell \\ S_\ell^* \end{bmatrix} = \begin{bmatrix} V_\ell \\ V_\ell^* \\ V_\ell I_\ell^* \\ V_\ell^* I_\ell \end{bmatrix} . \tag{3.81}$$

The element variables for a generator are usually defined as

$$\bar{\underset{\sim}{z}}_g = \begin{bmatrix} \underset{\sim}{x}_g \\ \hline \underset{\sim}{u}_g \end{bmatrix} \triangleq \begin{bmatrix} \delta_g \\ Q_g \\ \hline |V_g| \\ P_g \end{bmatrix} = \begin{bmatrix} \tan^{-1}[j(V_g^* - V_g)/(V_g + V_g^*)] \\ j(V_g^* I_g - V_g I_g^*)/2 \\ \hline (V_g V_g^*)^{1/2} \\ (V_g I_g^* + V_g^* I_g)/2 \end{bmatrix} \tag{3.82}$$

or, for example, as

$$\tilde{z}_g \triangleq \begin{bmatrix} V_g \\ I_g \\ |V_g|^2 \\ S_g+S_g^* \end{bmatrix} = \begin{bmatrix} V_g \\ I_g \\ V_g V_g^* \\ V_g I_g^* + V_g^* I_g \end{bmatrix} . \tag{3.83}$$

The element variables for the slack generator are usually defined as

$$\bar{z}_n = \begin{bmatrix} x_n \\ -- \\ u_n \end{bmatrix} \triangleq \begin{bmatrix} P_n \\ Q_n \\ \hline |V_n| \\ \delta_n \end{bmatrix} = \begin{bmatrix} (V_n I_n^* + V_n^* I_n)/2 \\ j(V_n^* I_n - V_n I_n^*)/2 \\ \hline (V_n V_n^*)^{1/2} \\ \tan^{-1}[j(V_n^* - V_n)/(V_n + V_n^*)] \end{bmatrix} \tag{3.84}$$

or, for example, as

$$\tilde{z}_n \triangleq \begin{bmatrix} I_n \\ I_n^* \\ V_n \\ V_n^* \end{bmatrix} . \tag{3.85}$$

For other branches the element variables are defined according to the element type. The element variables for a transmission element, for example, may be defined as

$$\bar{\underset{\sim}{z}}_t = \begin{bmatrix} \underset{\sim}{x}_t \\ \hline \underset{\sim}{u}_t \end{bmatrix} \triangleq \begin{bmatrix} \mathrm{Re}\{I_t\} \\ \mathrm{Im}\{I_t\} \\ \hline G_t \\ B_t \end{bmatrix} = \begin{bmatrix} (I_t+I_t^*)/2 \\ j(I_t^*-I_t)/2 \\ \hline (I_t/V_t+I_t^*/V_t^*)/2 \\ j(I_t^*/V_t^*-I_t/V_t)/2 \end{bmatrix} \tag{3.86}$$

or, for example, as

$$\tilde{\underset{\sim}{z}}_t \triangleq \begin{bmatrix} I_t \\ I_t^* \\ Y_t \\ Y_t^* \end{bmatrix} = \begin{bmatrix} I_t \\ I_t^* \\ I_t/V_t \\ I_t^*/V_t^* \end{bmatrix} . \tag{3.87}$$

Real and/or complex element variables of any branch type can be defined and classified in the same way. We shall only consider, without loss of generality, the above most important branch types. Other branch types can be treated in a similar straightforward manner.

We shall use $\bar{\underset{\sim}{z}}_b$ to denote $\bar{\underset{\sim}{z}}_\ell$, $\bar{\underset{\sim}{z}}_g$, $\bar{\underset{\sim}{z}}_n$ and $\bar{\underset{\sim}{z}}_t$ of (3.80), (3.82), (3.84) and (3.86), respectively. Also, we use $\tilde{\underset{\sim}{z}}_b$ to denote $\tilde{\underset{\sim}{z}}_\ell$, $\tilde{\underset{\sim}{z}}_g$, $\tilde{\underset{\sim}{z}}_n$ and $\tilde{\underset{\sim}{z}}_t$ of (3.81), (3.83), (3.85) and (3.87), respectively.

3.8.3 Standard Tabulated Expressions

Using the results of Appendix B the corresponding matrices $\bar{\underset{\sim}{\theta}}_{bi}$ and $\bar{\underset{\sim}{\theta}}_{bv}$ and vector $\underset{\sim}{\theta}_b$ for different power system elements are shown in Table 3.1 for the set of element variables $\bar{\underset{\sim}{z}}_b$ and in Table 3.2 for the set of element variables $\tilde{\underset{\sim}{z}}_b$.

TABLE 3.1a

ELEMENTS OF $\bar{\theta}_{bi}$, $\bar{\theta}_{bv}$ AND θ_b USING ELEMENT VARIABLES $\bar{z}_b$

	Load Elements	Generator Elements
$\bar{\theta}_{bi}$	$\begin{bmatrix} (\alpha V_\ell+\xi V_\ell^*)/\lvert V_\ell\rvert & (\bar{\xi}V_\ell+\bar{\alpha}V_\ell^*)/\lvert V_\ell\rvert \\ j(\alpha V_\ell-\xi V_\ell^*) & j(\bar{\xi}V_\ell-\bar{\alpha}V_\ell^*) \end{bmatrix}$	$\begin{bmatrix} j(\alpha V_g-\xi V_g^*) & j(\bar{\xi}V_g-\bar{\alpha}V_g^*) \\ 0 & 0 \end{bmatrix}$
$\bar{\theta}_{bv}$	$\begin{bmatrix} -(\beta S_\ell^*/V_\ell^*+\nu S_\ell/V_\ell)/\lvert V_\ell\rvert & -(\bar{\nu}S_\ell^*/V_\ell^*+\bar{\beta}S_\ell/V_\ell)/\lvert V_\ell\rvert \\ j(\beta S_\ell^*/V_\ell^*-\nu S_\ell/V_\ell) & j(\bar{\nu}S_\ell^*/V_\ell^*-\bar{\beta}S_\ell/V_\ell) \end{bmatrix}$	$\begin{bmatrix} j(\beta S_g^*/V_g^*-\nu S_g/V_g) & j(\bar{\nu}S_g^*/V_g^*-\bar{\beta}S_g/V_g) \\ j(-\beta/V_g^*+\nu/V_g) & j(-\bar{\nu}/V_g^*+\bar{\beta}/V_g) \end{bmatrix}$
θ_b	$\begin{bmatrix} \frac{\partial f}{\partial\lvert V_\ell\rvert} - \sum_k \lambda_{\ell k}[V_\ell N_{i\ell}^k+V_\ell^*\bar{N}_{i\ell}^k-S_\ell^*N_{v\ell}^k/V_\ell^*-S_\ell\bar{N}_{v\ell}^k/V_\ell]/\lvert V_\ell\rvert \\ \frac{\partial f}{\partial\delta_\ell} - \sum_k j\lambda_{\ell k}[V_\ell N_{i\ell}^k-V_\ell^*\bar{N}_{i\ell}^k+S_\ell^*N_{v\ell}^k/V_\ell^*-S_\ell\bar{N}_{v\ell}^k/V_\ell] \end{bmatrix}$	$\begin{bmatrix} \frac{\partial f}{\partial\delta_g} - \sum_k j\lambda_{gk}[V_g N_{ig}^k-V_g^*\bar{N}_{ig}^k+S_g^*N_{vg}^k/V_g^*-S_g\bar{N}_{vg}^k/V_g] \\ \frac{\partial f}{\partial Q_g} - \sum_k j\lambda_{gk}[-N_{vg}^k/V_g^*+\bar{N}_{vg}^k/V_g] \end{bmatrix}$

TABLE 3.1b

ELEMENTS OF $\bar{\theta}_{bi}$, $\bar{\theta}_{bv}$ AND θ_b USING ELEMENT VARIABLES $\bar{z}_b$

	Slack Generator	Transmission Elements
$\bar{\theta}_{bi}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} (\alpha/Y_t+\xi/Y_t^*) & (\bar{\xi}/Y_t+\bar{\alpha}/Y_t^*) \\ j(\alpha/Y_t-\xi/Y_t^*) & j(\bar{\xi}/Y_t-\bar{\alpha}/Y_t^*) \end{bmatrix}$
$\bar{\theta}_{bv}$	$\begin{bmatrix} (\beta/V_n^*+\nu/V_n) & (\bar{\nu}/V_n^*+\bar{\beta}/V_n) \\ j(-\beta/V_n^*+\nu/V_n) & j(-\bar{\nu}/V_n^*+\bar{\beta}/V_n) \end{bmatrix}$	$\begin{bmatrix} (\beta+\nu) & (\bar{\nu}+\bar{\beta}) \\ j(\beta-\nu) & j(\bar{\nu}-\bar{\beta}) \end{bmatrix}$
$\bar{\theta}_b$	$\begin{bmatrix} \frac{\partial f}{\partial P_n} - \sum_k \lambda_{nk}[N_{vn}^k/V_n^*+\bar{N}_{vn}^k/V_n] \\ \frac{\partial f}{\partial Q_n} - \sum_k j\lambda_{nk}[-N_{vn}^k/V_n^*+\bar{N}_{vn}^k/V_n] \end{bmatrix}$	$\begin{bmatrix} \frac{\partial f}{\partial Re(I_t)} - \sum_k \lambda_{tk}[N_{it}^k/Y_t+\bar{N}_{it}^k/Y_t^*+N_{vt}^k+\bar{N}_{vt}^k] \\ \frac{\partial f}{\partial Im(I_t)} - \sum_k j\lambda_{tk}[N_{it}^k/Y_t-\bar{N}_{it}^k/Y_t^*+N_{vt}^k-\bar{N}_{vt}^k] \end{bmatrix}$

TABLE 3.2a

ELEMENTS OF $\bar{\theta}_{bi}$, $\bar{\theta}_{bv}$ AND θ_b USING ELEMENT VARIABLES $\tilde{z}_b$

	Load Elements	Generator Elements
$\bar{\theta}_{bi}$	$\begin{bmatrix} \alpha & \bar{\xi} \\ \xi & \bar{\alpha} \end{bmatrix}$	$\begin{bmatrix} (\alpha-\xi V_g^*/V_g) & (\bar{\xi}-\bar{\alpha}V_g^*/V_g) \\ 0 & 0 \end{bmatrix}$
$\bar{\theta}_{bv}$	$\begin{bmatrix} -\nu S_\ell/V_\ell^2 & -\bar{\beta}S_\ell/V_\ell^2 \\ -\beta S_\ell^*/V_\ell^{*2} & -\bar{\nu}S_\ell^*/V_\ell^{*2} \end{bmatrix}$	$\begin{bmatrix} -j2\nu Q_g/V_g^2 & -j2\bar{\beta}Q_g/V_g^2 \\ \beta-\nu V_g^*/V_g & \bar{\nu}-\bar{\beta}V_g^*/V_g \end{bmatrix}$
θ_b	$\begin{bmatrix} \frac{\partial f}{\partial V_\ell} - \sum_k \lambda_{\ell k}[N_{i\ell}^k - S_\ell \bar{N}_{v\ell}^k/V_\ell^2] \\ \frac{\partial f}{\partial V_\ell^*} - \sum_k \lambda_{\ell k}[\bar{N}_{i\ell}^k - S_\ell^* N_{v\ell}^k/V_\ell^{*2}] \end{bmatrix}$	$\begin{bmatrix} \frac{\partial f}{\partial V_g} - \sum_k \lambda_{gk}[N_{ig}^k - V_g^*\bar{N}_{ig}^k/V_g - j2Q_g\bar{N}_{vg}^k/V_g^2] \\ \frac{\partial f}{\partial I_g} - \sum_k \lambda_{gk}[N_{vg}^k - \bar{N}_{vg}^k V_g^*/V_g] \end{bmatrix}$

TABLE 3.2b

ELEMENTS OF $\bar{\theta}_{bi}$, $\bar{\theta}_{bv}$ AND θ_b USING ELEMENT VARIABLES $\tilde{z}_b$

	Slack Generator	Transmission Elements
$\bar{\theta}_{bi}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} \alpha/Y_t & \bar{\xi}/Y_t \\ \xi/Y_t^* & \bar{\alpha}/Y_t^* \end{bmatrix}$
$\bar{\theta}_{bv}$	$\begin{bmatrix} \beta & \bar{\nu} \\ \nu & \bar{\beta} \end{bmatrix}$	$\begin{bmatrix} \beta & \bar{\nu} \\ \nu & \bar{\beta} \end{bmatrix}$
θ_b	$\begin{bmatrix} \frac{\partial f}{\partial I_n} - \sum_k \lambda_{nk} N_{vn}^k \\ \frac{\partial f}{\partial I_n^*} - \sum_k \lambda_{nk} \bar{N}_{vn}^k \end{bmatrix}$	$\begin{bmatrix} \frac{\partial f}{\partial I_t} - \sum_k \lambda_{tk}[N_{it}^k/Y_t + N_{vt}^k] \\ \frac{\partial f}{\partial I_t^*} - \sum_k \lambda_{tk}[\bar{N}_{it}^k/Y_t^* + \bar{N}_{vt}^k] \end{bmatrix}$

It is important to notice that $\bar{\theta}_{bi}$, $\bar{\theta}_{bv}$ and θ_b of Tables 3.1 and 3.2 are common to all relevant power system studies as long as the element variables considered are $\bar{z}_b$ and $\tilde{z}_b$, respectively.

3.9 THE ADJOINT EQUATIONS

3.9.1 General Derivation

We write the matrices $\bar{\underline{\theta}}_{bi}$ and $\bar{\underline{\theta}}_{bv}$ and vector $\underline{\theta}_b$ of (3.76) in the form

$$\bar{\underline{\theta}}_{bi} = \begin{bmatrix} \tilde{\phi}_b^1 & \bar{\phi}_b^1 \\ \tilde{\phi}_b^2 & \bar{\phi}_b^2 \end{bmatrix} , \tag{3.88}$$

$$\bar{\underline{\theta}}_{bv} = \begin{bmatrix} \tilde{\psi}_b^1 & \bar{\psi}_b^1 \\ \tilde{\psi}_b^2 & \bar{\psi}_b^2 \end{bmatrix} \tag{3.89}$$

and

$$\underline{\theta}_b = \begin{bmatrix} \hat{W}_b^{S1} \\ \hat{W}_b^{S2} \end{bmatrix} , \tag{3.90}$$

hence, the adjoint current-voltage relationship for element b has, from (3.76), the form

$$\tilde{\phi}_b^k \hat{I}_b + \bar{\phi}_b^k \hat{I}_b^* = \tilde{\psi}_b^k \hat{V}_b + \bar{\psi}_b^k \hat{V}_b^* + \hat{W}_b^{Sk}, \tag{3.91}$$

where k = 1, 2 denotes the first and second complex equation of (3.76), respectively, or, when separated into real and imaginary parts,

$$(\tilde{\phi}_{b1}^i + \bar{\phi}_{b1}^i)\hat{I}_{b1} + (\bar{\phi}_{b2}^i - \tilde{\phi}_{b2}^i)\hat{I}_{b2} = (\tilde{\psi}_{b1}^i + \bar{\psi}_{b1}^i)\hat{V}_{b1} + (\bar{\psi}_{b2}^i - \tilde{\psi}_{b2}^i)\hat{V}_{b2} + \hat{W}_{b1}^{Si}, \quad i = 1, 2 \tag{3.92}$$

and

$$(\tilde{\phi}^j_{b2}+\bar{\phi}^j_{b2})\hat{I}_{b1} + (\tilde{\phi}^j_{b1}-\bar{\phi}^j_{b1})\hat{I}_{b2} = (\tilde{\psi}^j_{b2}+\bar{\psi}^j_{b2})\hat{V}_{b1} + (\tilde{\psi}^j_{b1}-\bar{\psi}^j_{b1})\hat{V}_{b2} + \hat{W}^{Sj}_{b2}, \quad j = 1, 2, \quad (3.93)$$

where

$$\tilde{\phi}^k_b = \tilde{\phi}^k_{b1} + j\tilde{\phi}^k_{b2}, \quad (3.94)$$

$$\bar{\phi}^k_b = \bar{\phi}^k_{b1} + j\bar{\phi}^k_{b2}, \quad (3.95)$$

$$\tilde{\psi}^k_b = \tilde{\psi}^k_{b1} + j\tilde{\psi}^k_{b2}, \quad (3.96)$$

$$\bar{\psi}^k_b = \bar{\psi}^k_{b1} + j\bar{\psi}^k_{b2}, \quad (3.97)$$

$$\hat{V}_b = \hat{V}_{b1} + j\hat{V}_{b2}, \quad (3.98)$$

$$\hat{I}_b = \hat{I}_{b1} + j\hat{I}_{b2} \quad (3.99)$$

and

$$\hat{W}^{Sk}_b = \hat{W}^{Sk}_{b1} + j\hat{W}^{Sk}_{b2}, \quad k = 1, 2. \quad (3.100)$$

3.9.2 General Adjoint Element Modelling

In order to uniquely define the adjoint currents $\hat{I}_b$ in terms of the adjoint voltages $\hat{V}_b$ the system of four linear equations (3.92) and (3.93) has rank 2. Two of the four equations are used to describe the adjoint element. We write these two equations in the form

$$\phi^b_{11}\ \hat{I}_{b1} + \phi^b_{12}\ \hat{I}_{b2} = \psi^b_{11}\ \hat{V}_{b1} + \psi^b_{12}\ \hat{V}_{b2} + \hat{W}^S_{b1} \quad (3.101)$$

and

$$\phi^b_{21}\ \hat{I}_{b1} + \phi^b_{22}\ \hat{I}_{b2} = \psi^b_{21}\ \hat{V}_{b1} + \psi^b_{22}\ \hat{V}_{b2} + \hat{W}^S_{b2}, \quad (3.102)$$

where

$$\phi^b_{11} = \tilde{\phi}^i_{b1} + \bar{\phi}^i_{b1}, \quad (3.103)$$

$$\phi^b_{12} = \bar{\phi}^i_{b2} - \tilde{\phi}^i_{b2}, \quad (3.104)$$

$$\psi^b_{11} = \tilde{\psi}^i_{b1} + \bar{\psi}^i_{b1}, \quad (3.105)$$

$$\psi_{12}^{b} = \bar{\psi}_{b2}^{i} - \tilde{\psi}_{b2}^{i}, \tag{3.106}$$

$$\hat{W}_{b1}^{S} = \hat{W}_{b1}^{Si}, \qquad i = 1 \text{ or } 2 \tag{3.107}$$

and

$$\phi_{21}^{b} = \tilde{\phi}_{b2}^{j} + \bar{\phi}_{b2}^{j}, \tag{3.108}$$

$$\phi_{22}^{b} = \tilde{\phi}_{b1}^{j} - \bar{\phi}_{b1}^{j}, \tag{3.109}$$

$$\psi_{21}^{b} = \tilde{\psi}_{b2}^{j} + \bar{\psi}_{b2}^{j}, \tag{3.110}$$

$$\psi_{22}^{b} = \tilde{\psi}_{b1}^{j} - \bar{\psi}_{b1}^{j}, \tag{3.111}$$

$$\hat{W}_{b2}^{S} = \hat{W}_{b2}^{Sj}, \qquad j = 1 \text{ or } 2. \tag{3.112}$$

3.9.3 Adjoint Modelling of Transmission Elements

Equations (3.101) and (3.102) are written for transmission elements in the form

$$\hat{I}_{t1} = \bar{Y}_{t1} \hat{V}_{t1} - \bar{Y}_{t2} \hat{V}_{t2} + \hat{I}_{t1}^{S} \tag{3.113}$$

and

$$\hat{I}_{t2} = \tilde{Y}_{t2} \hat{V}_{t1} + \tilde{Y}_{t1} \hat{V}_{t2} + \hat{I}_{t2}^{S}, \tag{3.114}$$

where

$$\begin{bmatrix} \bar{Y}_{t1} & -\bar{Y}_{t2} \\ \tilde{Y}_{t2} & \tilde{Y}_{t1} \end{bmatrix} = \frac{1}{\Delta_t} \begin{bmatrix} \phi_{22}^{t} & -\phi_{12}^{t} \\ -\phi_{21}^{t} & \phi_{11}^{t} \end{bmatrix} \begin{bmatrix} \psi_{11}^{t} & \psi_{12}^{t} \\ \psi_{21}^{t} & \psi_{22}^{t} \end{bmatrix} \tag{3.115}$$

and

$$\begin{bmatrix} \hat{I}_{t1}^{S} \\ \hat{I}_{t2}^{S} \end{bmatrix} = \frac{1}{\Delta_t} \begin{bmatrix} \phi_{22}^{t} & -\phi_{12}^{t} \\ -\phi_{21}^{t} & \phi_{11}^{t} \end{bmatrix} \begin{bmatrix} \hat{W}_{t1}^{S} \\ \hat{W}_{t2}^{S} \end{bmatrix} \tag{3.116}$$

and where

$$\Delta_t \triangleq \phi_{11}^t \phi_{22}^t - \phi_{12}^t \phi_{21}^t \neq 0. \quad (3.117)$$

We define the complex quantities

$$\overline{Y}_t \triangleq \overline{Y}_{t1} + j\overline{Y}_{t2} \quad (3.118)$$

and

$$\tilde{Y}_t \triangleq \tilde{Y}_{t1} + j\tilde{Y}_{t2}. \quad (3.119)$$

Equations (3.113) and (3.114) are written in the matrix form

$$\begin{bmatrix} \overline{\underline{Y}}^p_{T1} & -\overline{\underline{Y}}^p_{T2} \\ \tilde{\underline{Y}}^p_{T2} & \tilde{\underline{Y}}^p_{T1} \end{bmatrix} \begin{bmatrix} \hat{\underline{V}}_{T1} \\ \hat{\underline{V}}_{T2} \end{bmatrix} = \begin{bmatrix} \hat{\underline{I}}_{T1} - \hat{\underline{I}}^S_{T1} \\ \hat{\underline{I}}_{T2} - \hat{\underline{I}}^S_{T2} \end{bmatrix}, \quad (3.120)$$

where $\overline{\underline{Y}}^p_{T1}$, $\overline{\underline{Y}}^p_{T2}$, $\tilde{\underline{Y}}^p_{T1}$ and $\tilde{\underline{Y}}^p_{T2}$ are diagonal matrices consisting of the Y_{t1}, Y_{t2}, $\tilde{Y}_{t1}$ and $\tilde{Y}_{t2}$, respectively, and $\hat{\underline{V}}_{T1}$, $\hat{\underline{V}}_{T2}$, $\hat{\underline{I}}_{T1}$, $\hat{\underline{I}}_{T2}$, $\hat{\underline{I}}^S_{T1}$ and $\hat{\underline{I}}^S_{T2}$ are vectors of components $\hat{V}_{t1}$, $\hat{V}_{t2}$, $\hat{I}_{t1}$, $\hat{I}_{t2}$, $\hat{I}^S_{t1}$ and $\hat{I}^S_{t2}$, respectively. For later use, let

$$\overline{\underline{Y}}^p_T \triangleq \overline{\underline{Y}}^p_{T1} + j\overline{\underline{Y}}^p_{T2}, \quad (3.121)$$

$$\tilde{\underline{Y}}^p \triangleq \tilde{\underline{Y}}^p_{T1} + j\tilde{\underline{Y}}^p_{T2}, \quad (3.122)$$

$$\hat{\underline{V}}_T \triangleq \hat{\underline{V}}_{T1} + j\hat{\underline{V}}_{T2}, \quad (3.123)$$

$$\hat{\underline{I}}_T \triangleq \hat{\underline{I}}_{T1} + j\hat{\underline{I}}_{T2} \quad (3.124)$$

and

$$\hat{\underline{I}}^S_T \triangleq \hat{\underline{I}}^S_{T1} + j\hat{\underline{I}}^S_{T2}. \quad (3.125)$$

3.9.4 Adjoint Modelling of Bus Elements

We define the 2nx2n matrices

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} \\ \boldsymbol{\Phi}_{21} & \boldsymbol{\Phi}_{22} \end{bmatrix} \qquad (3.126)$$

and

$$\boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{\Psi}_{11} & \boldsymbol{\Psi}_{12} \\ \boldsymbol{\Psi}_{21} & \boldsymbol{\Psi}_{22} \end{bmatrix}, \qquad (3.127)$$

where

$$\boldsymbol{\Phi}_{11} \triangleq \text{diag } \{\phi^m_{11}\}, \qquad (3.128)$$

$$\boldsymbol{\Phi}_{12} \triangleq \text{diag } \{\phi^m_{12}\}, \qquad (3.129)$$

$$\boldsymbol{\Phi}_{21} \triangleq \text{diag } \{\phi^m_{21}\}, \qquad (3.130)$$

$$\boldsymbol{\Phi}_{22} \triangleq \text{diag } \{\phi^m_{22}\}, \qquad (3.131)$$

$$\boldsymbol{\Psi}_{11} \triangleq \text{diag } \{\psi^m_{11}\}, \qquad (3.132)$$

$$\boldsymbol{\Psi}_{12} \triangleq \text{diag } \{\psi^m_{12}\}, \qquad (3.133)$$

$$\boldsymbol{\Psi}_{21} \triangleq \text{diag } \{\psi^m_{21}\} \qquad (3.134)$$

and

$$\boldsymbol{\Psi}_{22} \triangleq \text{diag } \{\psi^m_{22}\} \qquad (3.135)$$

are nxn diagonal matrices, m can be ℓ, g or n.

Equations (3.101) and (3.102) are written for the bus elements using (3.128)-(3.135) in the matrix form

$$\begin{bmatrix} \boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} \\ \boldsymbol{\Phi}_{21} & \boldsymbol{\Phi}_{22} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{I}}^B_{M1} \\ \hat{\mathbf{I}}^B_{M2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Psi}_{11} & \boldsymbol{\Psi}_{12} \\ \boldsymbol{\Psi}_{21} & \boldsymbol{\Psi}_{22} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{V}}^B_{M1} \\ \hat{\mathbf{V}}^B_{M2} \end{bmatrix} + \begin{bmatrix} \hat{\mathbf{W}}^{SB}_{M1} \\ \hat{\mathbf{W}}^{SB}_{M2} \end{bmatrix}. \qquad (3.136)$$

where $\hat{\mathbf{I}}^B_{M1}$, $\hat{\mathbf{I}}^B_{M2}$, $\hat{\mathbf{V}}^B_{M1}$, $\hat{\mathbf{V}}^B_{M2}$, $\hat{\mathbf{W}}^{SB}_{M1}$ and $\hat{\mathbf{W}}^{SB}_{M2}$ are vectors of components $\hat{I}_{m1}$, $\hat{I}_{m2}$, $\hat{V}_{m1}$, $\hat{V}_{m2}$, $\hat{W}^S_{m1}$ and $\hat{W}^S_{m2}$, respectively. We let

$$\hat{\underset{\sim}{I}}{}^{B}_{M} \triangleq \hat{\underset{\sim}{I}}{}^{B}_{M1} + j\hat{\underset{\sim}{I}}{}^{B}_{M2}, \tag{3.137}$$

$$\hat{\underset{\sim}{V}}{}^{B}_{M} \triangleq \hat{\underset{\sim}{V}}{}^{B}_{M1} + j\hat{\underset{\sim}{V}}{}^{B}_{M2} \tag{3.138}$$

and

$$\hat{\underset{\sim}{W}}{}^{SB}_{M} \triangleq \hat{\underset{\sim}{W}}{}^{SB}_{M1} + j\hat{\underset{\sim}{W}}{}^{SB}_{M2}. \tag{3.139}$$

KCL is written as

$$\begin{bmatrix} \underset{\sim}{A}_M & | & \underset{\sim}{A}_T \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{I}}{}^{B}_{M} \\ \hat{\underset{\sim}{I}}_{T} \end{bmatrix} = \underset{\sim}{0}, \tag{3.140}$$

where

$$\underset{\sim}{A} \triangleq \begin{bmatrix} \underset{\sim}{A}_M & | & \underset{\sim}{A}_T \end{bmatrix} \tag{3.141}$$

is the reduced incidence matrix of dimension $n x n_B$ (n buses, n_B branches) whose elements a_{ij} are given by

$a_{ij} = 1$ if branch j is incident at bus i and oriented away from it,

$a_{ij} = -1$ if branch j is incident at bus i and oriented toward it and

$a_{ij} = 0$ if branch j is not incident at bus i.

Now we define

$$\overline{\underset{\sim}{Y}}_T = \overline{\underset{\sim}{Y}}_{T1} + j\overline{\underset{\sim}{Y}}_{T2} \triangleq \underset{\sim}{A}_T\, \overline{\underset{\sim}{Y}}{}^{P}_{T}\, \underset{\sim}{A}{}^{T}_{T}, \tag{3.142}$$

$$\tilde{\underset{\sim}{Y}}_T = \tilde{\underset{\sim}{Y}}_{T1} + j\tilde{\underset{\sim}{Y}}_{T2} \triangleq \underset{\sim}{A}_T\, \tilde{\underset{\sim}{Y}}{}^{P}_{T}\, \underset{\sim}{A}{}^{T}_{T}, \tag{3.143}$$

$$\hat{J}_M = \hat{J}_{M1} + j\hat{J}_{M2} \triangleq A_T \hat{I}_T^S \qquad (3.144)$$

and

$$\hat{W}_M^S = \hat{W}_{M1}^S + j\hat{W}_{M2}^S \triangleq A_M \hat{W}_M^{SB}. \qquad (3.145)$$

Also the bus voltages

$$\hat{V}_M = \hat{V}_{M1} + j\hat{V}_{M2} \triangleq A_M \hat{V}_M^B \qquad (3.146)$$

are related to $\hat{V}_T$ through the relationship

$$\hat{V}_T = A_T^T \hat{V}_M. \qquad (3.147)$$

3.9.5 Formulation of Adjoint Equations

Eliminating $\hat{I}_T$ and $\hat{I}_M^B$ from (3.120), (3.136) and (3.140) and using (3.142)-(3.147) we arrive at the final set of adjoint equations to be solved in the form

$$\begin{bmatrix} (\Phi_{11}\bar{Y}_{T1}+\Phi_{12}\tilde{Y}_{T2}+\Psi_{11}) & (-\Phi_{11}\bar{Y}_{T2}+\Phi_{12}\tilde{Y}_{T1}+\Psi_{12}) \\ (\Phi_{21}\bar{Y}_{T1}+\Phi_{22}\tilde{Y}_{T2}+\Psi_{21}) & (-\Phi_{21}\bar{Y}_{T2}+\Phi_{22}\tilde{Y}_{T1}+\Psi_{22}) \end{bmatrix} \begin{bmatrix} \hat{V}_{M1} \\ \hat{V}_{M2} \end{bmatrix} = -\begin{bmatrix} \Phi_{11}\hat{J}_{M1}+\Phi_{12}\hat{J}_{M2}+\hat{W}_{M1}^S \\ \Phi_{21}\hat{J}_{M1}+\Phi_{22}\hat{J}_{M2}+\hat{W}_{M2}^S \end{bmatrix}. \qquad (3.148)$$

Note that multiplying (3.120) from left by the matrix

$$\bar{A}_T = \begin{bmatrix} A_T & 0 \\ 0 & A_T \end{bmatrix}, \qquad (3.149)$$

substituting $\hat{V}_T$ from (3.147) and using (3.140), (3.142), (3.143) and (3.144) we get

$$\begin{bmatrix} \underset{\sim}{\bar{Y}}_{T1} & -\underset{\sim}{\bar{Y}}_{T2} \\ \underset{\sim}{\tilde{Y}}_{T2} & \underset{\sim}{\tilde{Y}}_{T1} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\hat{V}}_{M1} \\ \underset{\sim}{\hat{V}}_{M2} \end{bmatrix} = - \begin{bmatrix} \underset{\sim}{\hat{I}}_{M1} + \underset{\sim}{\hat{J}}_{M1} \\ \underset{\sim}{\hat{I}}_{M2} + \underset{\sim}{\hat{J}}_{M2} \end{bmatrix}, \tag{3.150}$$

where

$$\underset{\sim}{\hat{I}}_M = \underset{\sim}{\hat{I}}_{M1} + j\underset{\sim}{\hat{I}}_{M2} = \underset{\sim}{A}_M \underset{\sim}{\hat{I}}_M^B. \tag{3.151}$$

The form of (3.150) is that of the conventional nodal equations. It can be used for solution purposes if the RHS is voltage independent, e.g., as for typical linear electronic circuit cases.

3.10 GRADIENT EVALUATION

The solution of the adjoint system (3.148) provides the adjoint variables $\hat{\underset{\sim}{w}}_b$ of (3.59). The required gradients are then calculated using (3.56) or (3.58). The vector $\hat{\underset{\sim}{\eta}}_{bu}$ is obtained from (3.62) where $\hat{\underset{\sim}{f}}_{bi}$ and $\hat{\underset{\sim}{f}}_{bv}$ are calculated from (3.63), (3.64) and (3.65)-(3.68). Using the results of Appendix B matrices $\underset{\sim}{M}_{21}^b$ and $\underset{\sim}{M}_{22}^b$ of (3.62) for different power system elements are shown in Table 3.3 for the set of element variables $\bar{\underset{\sim}{z}}_b$ and in Table 3.4 for the set of element variables $\tilde{\underset{\sim}{z}}_b$.

3.11 CONCLUSIONS

In this chapter, the foundation of an exact adjoint network approach to general power system sensitivity analysis and planning problems has been laid. A family of adjoint systems of equations has been derived so that a wide variety of special problems can be handled.

TABLE 3.3a

MATRICES M_{21}^{b} AND M_{22}^{b} USING ELEMENT VARIABLES $\bar{z}_b$

	Load Elements	Generator Elements
M_{21}^{b}	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} V_g/\lvert V_g\rvert & V_g^*/\lvert V_g\rvert \\ 0 & 0 \end{bmatrix}$
M_{22}^{b}	$\begin{bmatrix} 1/V_\ell^* & 1/V_\ell \\ -j/V_\ell^* & j/V_\ell \end{bmatrix}$	$\begin{bmatrix} -S_g^*/(V_g^*\lvert V_g\rvert) & -S_g/(V_g\lvert V_g\rvert) \\ 1/V_g^* & 1/V_g \end{bmatrix}$

Instead of approximating the a.c. load flow model to cope with the form of Tellegen's theorem and technique usually employed in typical electronic circuit analysis, we use an augmented form of Tellegen's theorem and employ a proper technique to deal with the relatively difficult element models which exist in power networks.

The approach has been described via a generalized version in which the concepts of arbitrary complex adjoining coefficients and group terms are introduced. The exploitation of these concepts provide, as shown later, the theoretical methodology of handling the complex functions directly, as in typical electronic circuits. We have derived and tabulated standard sensitivity expressions common to all relevant power system studies.

The concepts stated in this chapter are general. While they have been applied with power systems in mind, they are applicable to other systems as well.

TABLE 3.3b

MATRICES $\underset{\sim}{M}_{21}^{b}$ AND $\underset{\sim}{M}_{22}^{b}$ USING ELEMENT VARIABLES $\underset{\sim}{\bar{z}}_b$

	Slack Generator	Transmission Elements
$\underset{\sim}{M}_{21}^{b}$	$\begin{bmatrix} V_n/\lvert V_n\rvert & V_n^*/\lvert V_n\rvert \\ jV_n & -jV_n^* \end{bmatrix}$	$\begin{bmatrix} -V_t/Y_t & -V_t^*/Y_t^* \\ -jV_t/Y_t & jV_t^*/Y_t^* \end{bmatrix}$
$\underset{\sim}{M}_{22}^{b}$	$\begin{bmatrix} -S_n^*/(V_n^*\lvert V_n\rvert) & -S_n/(V_n\lvert V_n\rvert) \\ jS_n^*/V_n^* & -jS_n/V_n \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

TABLE 3.4

MATRICES $\underset{\sim}{M}_{21}^{b}$ AND $\underset{\sim}{M}_{22}^{b}$ USING ELEMENT VARIABLES $\underset{\sim}{\tilde{z}}_b$

	Load Elements	Generator Elements	Slack Generator	Transmission Elements
$\underset{\sim}{M}_{21}^{b}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1/V_g \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -V_t/Y_t & 0 \\ 0 & -V_t^*/Y_t^* \end{bmatrix}$
$\underset{\sim}{M}_{22}^{b}$	$\begin{bmatrix} 0 & 1/V_\ell \\ 1/V_\ell^* & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -S_g^*/(V_g^* V_g^2) \\ 0 & 1/V_g \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

4

SPECIAL CLASS OF ADJOINT SYSTEMS

4.1 INTRODUCTION

A generalized version of a Tellegen's theorem-based approach to power network sensitivity calculations has been developed in Chapter 3, where a family of adjoint systems based on the exact a.c. power flow model was described. Theoretically speaking, this generalized version provides the possibility of handling complex functions via one adjoint analysis. For practical purposes, however, a simplified version may be required, although restricted to real functions, where a relatively simple adjoint system and sensitivity expressions are employed.

In this chapter, we consider a class of the family of adjoint systems in which the extended Tellegen sum is a real quantity. We discuss in detail an important practical case in which the adjoining complex coefficients are set to particular values which result in an adjoint system of a simplified structure allowing a wide variety of real functions to be handled. Numerical results are also presented for a 6-bus sample power system.

4.2 REAL EXTENDED TELLEGEN SUM

4.2.1 General Form of Real Tellegen Sum

Equation (3.31) describes the general form of the extended Tellegen sum. To obtain a real extended Tellegen sum, it is sufficient for the adjoining complex coefficients to satisfy the conditions

$$\overline{\alpha} = \alpha^*, \ \overline{\beta} = \beta^*, \ \overline{\xi} = \xi^*, \ \overline{\nu} = \nu^* \tag{4.1}$$

and

$$\overline{\Gamma}_k = \Gamma_k^* \text{ for all } k. \tag{4.2}$$

Hence, the extended Tellegen sum is written as

$$\sum_b [\alpha \hat{I}_b V_b + \alpha^* \hat{I}_b^* V_b^* - \beta \hat{V}_b I_b - \beta^* \hat{V}_b^* I_b^* + \xi \hat{I}_b V_b^* + \xi^* \hat{I}_b^* V_b$$

$$- \nu \hat{V}_b I_b^* - \nu^* \hat{V}_b^* I_b + \sum_k \Gamma_k \lambda_{bk} C_b^k + \sum_k \Gamma_k^* \lambda_{bk} C_b^{k*}] = 0. \tag{4.3}$$

Expressing (4.3) in terms of first-order changes in V and I, the resulting perturbed sum is also a real quantity and equations (3.76) are consistent for all values of α, β, ξ, ν and Γ_k.

Equation (4.3) describes a class of adjoint networks corresponding to different values of the arbitrary complex coefficients α, β, ξ, ν and Γ_k.

4.2.2 A Member of the Class

One member of this class (Bandler and El-Kady 1980c) may be defined by setting

$$\alpha = \beta = \Gamma_1 = 1 \tag{4.4}$$

and

$$\xi = \nu = \Gamma_k = 0, \ k \neq 1, \tag{4.5}$$

where the group term of (3.29) associated with Γ_1 is given by

$$\sum_{b \in B_1} C_b^1 = \sum_b S_b = 0 \tag{4.6}$$

representing power conservation in the network.

4.3 AN IMPORTANT SPECIAL CASE

We consider the special case (Püttgen 1976)

$$\alpha = \beta = 1 \tag{4.7}$$

$$\xi = \nu = 0 \tag{4.8}$$

and

$$\Gamma_k = 0 \text{ for all } k \tag{4.9}$$

which, as we shall see, provides a special structure of the adjoint network.

The matrices $\underset{\sim}{\Lambda}_i^b$, $\overline{\underset{\sim}{\Lambda}}_i$, $\underset{\sim}{\Lambda}_v^b$ and $\overline{\underset{\sim}{\Lambda}}_v$ of (3.65)-(3.68) are given by

$$\underset{\sim}{\Lambda}_i^b = \underset{\sim}{\Lambda}_i = \underset{\sim}{0}, \tag{4.10}$$

$$\overline{\underset{\sim}{\Lambda}}_i = \underset{\sim}{1}, \tag{4.11}$$

$$\underset{\sim}{\Lambda}_v^b = \underset{\sim}{\Lambda}_v = \underset{\sim}{0} \tag{4.12}$$

and

$$\overline{\underset{\sim}{\Lambda}}_v = -\underset{\sim}{1}, \tag{4.13}$$

where $\underset{\sim}{1}$ is a unity matrix of order 2, hence, the matrices $\overline{\underset{\sim}{\theta}}_{bi}$, $\overline{\underset{\sim}{\theta}}_{bv}$ and vector $\underset{\sim}{\theta}_b$ of (3.76) are simply

$$\overline{\underset{\sim}{\theta}}_{bi} = \underset{\sim}{M}_{11}^b, \tag{4.14}$$

$$\overline{\underset{\sim}{\theta}}_{bv} = \underset{\sim}{M}_{12}^b \tag{4.15}$$

and

$$\underset{\sim}{\theta}_b = \hat{\underset{\sim}{\eta}}_{bx}. \tag{4.16}$$

Tables 4.1 and 4.2 show the corresponding matrices $\overline{\underset{\sim}{\theta}}_{bi}$ and $\overline{\underset{\sim}{\theta}}_{bv}$ and vector $\underset{\sim}{\theta}_b$ for different power system elements considering the sets of element variables $\overline{\underset{\sim}{z}}_b$ and $\tilde{\underset{\sim}{z}}_b$, respectively.

4.4 MODELLING OF SYSTEM ADJOINT ELEMENTS

Using the results outlined in Table 4.1, the equations defining the adjoint elements for the set of element variables $\bar{z}_b$ are, for a load

$$\hat{I}_\ell = -(S_\ell/V_\ell^2)\hat{V}_\ell^* + [|V_\ell| \frac{\partial f}{\partial |V_\ell|} - j\frac{\partial f}{\partial \delta_\ell}]/(2V_\ell), \tag{4.17}$$

TABLE 4.1a

ELEMENTS OF $\bar{\theta}_{bi}$, $\bar{\theta}_{bv}$ AND θ_b FOR SIMPLIFIED VERSION USING ELEMENT VARIABLES $\bar{z}_b$

	Load Elements	Generator Elements
$\bar{\theta}_{bi}$	$\begin{bmatrix} V_\ell/\|V_\ell\| & V_\ell^*/\|V_\ell\| \\ jV_\ell & -jV_\ell^* \end{bmatrix}$	$\begin{bmatrix} jV_g & -jV_g^* \\ 0 & 0 \end{bmatrix}$
$\bar{\theta}_{bv}$	$\begin{bmatrix} -S_\ell^*/(V_\ell^*\|V_\ell\|) & -S_\ell/(V_\ell\|V_\ell\|) \\ jS_\ell^*/V_\ell^* & -jS_\ell/V_\ell \end{bmatrix}$	$\begin{bmatrix} jS_g^*/V_g^* & -jS_g/V_g \\ -j/V_g^* & j/V_g \end{bmatrix}$
θ_b	$\begin{bmatrix} \frac{\partial f}{\partial \|V_\ell\|} \\ \frac{\partial f}{\partial \delta_\ell} \end{bmatrix}$	$\begin{bmatrix} \frac{\partial f}{\partial \delta_g} \\ \frac{\partial f}{\partial Q_g} \end{bmatrix}$

TABLE 4.1b

ELEMENTS OF $\bar{\underset{\sim}{\theta}}_{bi}$, $\bar{\underset{\sim}{\theta}}_{bv}$ AND $\underset{\sim}{\theta}_b$ FOR SIMPLIFIED VERSION USING ELEMENT VARIABLES $\bar{\underset{\sim}{z}}_b$

	Slack Generator	Transmission Elements
$\bar{\underset{\sim}{\theta}}_{bi}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1/Y_t & 1/Y_t^* \\ j/Y_t & -j/Y_t^* \end{bmatrix}$
$\bar{\underset{\sim}{\theta}}_{bv}$	$\begin{bmatrix} 1/V_n^* & 1/V_n \\ -j/V_n^* & j/V_n \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}$
$\underset{\sim}{\theta}_b$	$\begin{bmatrix} \frac{\partial f}{\partial P_n} \\ \frac{\partial f}{\partial Q_n} \end{bmatrix}$	$\begin{bmatrix} \frac{\partial f}{\partial Re\{I_t\}} \\ \frac{\partial f}{\partial Im\{I_t\}} \end{bmatrix}$

for a generator

$$V_g \hat{I}_g - V_g^* \hat{I}_g^* = (S_g^*/V_g^*) \hat{V}_g - (S_g/V_g) \hat{V}_g^* - j \frac{\partial f}{\partial \delta_g} \tag{4.18}$$

and

$$V_g \hat{V}_g - V_g^* \hat{V}_g^* = - j V_g V_g^* \frac{\partial f}{\partial Q_g} , \tag{4.19}$$

where each of (4.18) and (4.19) represents only one condition, for the slack generator

$$\hat{V}_n = -V_n^* \left(\frac{\partial f}{\partial P_n} + j \frac{\partial f}{\partial Q_n} \right)/2 \tag{4.20}$$

and for a transmission element

$$\hat{I}_t = Y_t \hat{V}_t + Y_t \left[\frac{\partial f}{\partial \mathrm{Re}\{I_t\}} - j \frac{\partial f}{\partial \mathrm{Im}\{I_t\}} \right]/2. \tag{4.21}$$

Similarly, using the results outlined in Table 4.2, the equations defining the adjoint elements using the set of element variables $\tilde{z}_b$ are, for a load

$$\hat{I}_\ell = -(S_\ell/V_\ell^2)\, \hat{V}_\ell^* + \frac{\partial f}{\partial V_\ell}, \tag{4.22}$$

for a generator

$$V_g \hat{I}_g - V_g^* \hat{I}_g^* = -j\, 2\, Q_g \hat{V}_g^*/V_g + V_g \frac{\partial f}{\partial V_g} \tag{4.23}$$

and

$$V_g \hat{V}_g - V_g^* \hat{V}_g^* = -V_g \frac{\partial f}{\partial I_g}, \tag{4.24}$$

where each of (4.23) and (4.24) represents only one condition, for the slack generator

$$\hat{V}_n = -\frac{\partial f}{\partial I_n} \tag{4.25}$$

and for a transmission element

$$\hat{I}_t = Y_t \hat{V}_t + Y_t \frac{\partial f}{\partial I_t}. \tag{4.26}$$

Note incidentally that, for transmission elements, equations (4.21) and (4.26) have the form

$$\hat{I}_t = Y_t \hat{V}_t + \hat{I}_t^S. \tag{4.27}$$

TABLE 4.2

ELEMENTS OF $\bar{\theta}_{bi}$, $\bar{\theta}_{bv}$ and θ_b FOR SIMPLIFED VERSION USING ELEMENT VARIABLES $\tilde{z}_b$

	Load Elements	Generator Elements	Slack Generator	Transmission Elements
$\bar{\theta}_{bi}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & -V_g^*/V_g \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1/Y_t & 0 \\ 0 & 1/Y_t^* \end{bmatrix}$
$\bar{\theta}_{bv}$	$\begin{bmatrix} 0 & -S_\ell/V_\ell^2 \\ -S_\ell^*/V_\ell^{*2} & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -j2Q_g/V_g^2 \\ 1 & -V_g^*/V_g \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
θ_b	$\begin{bmatrix} \frac{\partial f}{\partial V_\ell} \\ \frac{\partial f}{\partial V_\ell^*} \end{bmatrix}$	$\begin{bmatrix} \frac{\partial f}{\partial V_g} \\ \frac{\partial f}{\partial I_g} \end{bmatrix}$	$\begin{bmatrix} \frac{\partial f}{\partial I_n} \\ \frac{\partial f}{\partial I_n^*} \end{bmatrix}$	$\begin{bmatrix} \frac{\partial f}{\partial I_t} \\ \frac{\partial f}{\partial I_t^*} \end{bmatrix}$

Comparing (3.113) and (3.114) with (4.27), we get

$$\bar{Y}_{t1} = \tilde{Y}_{t1} = Y_{t1} \tag{4.28}$$

and

$$\bar{Y}_{t2} = \tilde{Y}_{t2} = Y_{t2}, \tag{4.29}$$

where

$$Y_t = Y_{t1} + jY_{t2}, \tag{4.30}$$

hence, the line admittances of both original and adjoint systems are the same, and (3.150) has the form

$$\underset{\sim}{Y}_T \, \hat{\underset{\sim}{V}}_M = -(\hat{\underset{\sim}{I}}_M + \hat{\underset{\sim}{J}}_M), \tag{4.31}$$

where $\underset{\sim}{Y}_T$ is the symmetrical bus admittance matrix of the original system. Moreover, since the matrices $\underset{\sim}{\Phi}_{ij}$, $\underset{\sim}{\Psi}_{ij}$, i, j = 1, 2 of (3.126) and (3.127) are diagonal matrices, the adjoint matrix of coefficients is of the same sparsity as the Jacobian matrix of the original network.

4.5. SIMPLIFIED ADJOINT EQUATIONS

4.5.1 Case of General Element Variables

We write equation (4.31) in the form

$$\begin{bmatrix} \underset{\sim}{Y}_{LL} & \underset{\sim}{Y}_{LG} & \underset{\sim}{Y}_{LN} \\ \underset{\sim}{Y}_{GL} & \underset{\sim}{Y}_{GG} & \underset{\sim}{Y}_{GN} \\ \underset{\sim}{Y}_{NL} & \underset{\sim}{Y}_{NG} & Y_{nn} \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{V}}_L \\ \hat{\underset{\sim}{V}}_G \\ \hat{V}_n \end{bmatrix} = - \begin{bmatrix} \hat{\underset{\sim}{I}}_L + \hat{\underset{\sim}{J}}_L \\ \hat{\underset{\sim}{I}}_G + \hat{\underset{\sim}{J}}_G \\ \hat{I}_n + \hat{J}_n \end{bmatrix}, \tag{4.32}$$

where the nxn bus admittance matrix has been partitioned into blocks associated with the sets of load, generator and slack buses of appropriate dimension, and $\hat{\underset{\sim}{V}}_M$, $\hat{\underset{\sim}{I}}_M$ and $\hat{\underset{\sim}{J}}_M$ have been partitioned correspondingly.

We also write the diagonal matrices $\underset{\sim}{\Phi}_{ij}$ and $\underset{\sim}{\Psi}_{ij}$, i, j = 1, 2 and the vector $\hat{\underset{\sim}{W}}_M^S$ of (3.148) in the corresponding partitioned forms

$$\underset{\sim}{\Phi}_{ij} = \begin{bmatrix} \underset{\sim}{\Phi}_{ij}^L & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{\Phi}_{ij}^G & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \Phi_{ij}^n \end{bmatrix}, \tag{4.33}$$

$$\underset{\sim}{\Psi}_{ij} = \begin{bmatrix} \underset{\sim}{\Psi}_{ij}^L & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{\Psi}_{ij}^G & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \Psi_{ij}^n \end{bmatrix}, \tag{4.34}$$

where $\underset{\sim}{\Phi}_{ij}^L$, $\underset{\sim}{\Phi}_{ij}^G$, $\underset{\sim}{\Psi}_{ij}^L$ and $\underset{\sim}{\Psi}_{ij}^G$ are diagonal submatrices, and

$$\hat{\mathbf{W}}^S_M = \begin{bmatrix} \hat{\mathbf{W}}^S_L \\ \hat{\mathbf{W}}^S_G \\ \hat{\mathbf{W}}^S_n \end{bmatrix} . \qquad (4.35)$$

Using the forms (4.32)-(4.35) it is straightforward to show that the adjoint system of equations (3.148) can be written in the form

$$\left[\begin{array}{c|c|c|c}
\boldsymbol{\Phi}^L_{11}\mathbf{G}_{LL}+\boldsymbol{\Phi}^L_{12}\mathbf{B}_{LL}+\boldsymbol{\Psi}^L_{11} & \boldsymbol{\Phi}^L_{11}\mathbf{G}_{LG}+\boldsymbol{\Phi}^L_{12}\mathbf{B}_{LG} & -\boldsymbol{\Phi}^L_{11}\mathbf{B}_{LL}+\boldsymbol{\Phi}^L_{12}\mathbf{G}_{LL}+\boldsymbol{\Psi}^L_{12} & -\boldsymbol{\Phi}^L_{11}\mathbf{B}_{LG}+\boldsymbol{\Phi}^L_{12}\mathbf{G}_{LG} \\ \hline
\boldsymbol{\Phi}^G_{11}\mathbf{G}_{GL}+\boldsymbol{\Phi}^G_{12}\mathbf{B}_{GL} & \boldsymbol{\Phi}^G_{11}\mathbf{G}_{GG}+\boldsymbol{\Phi}^G_{12}\mathbf{B}_{GG}+\boldsymbol{\Psi}^G_{11} & -\boldsymbol{\Phi}^G_{11}\mathbf{B}_{GL}+\boldsymbol{\Phi}^G_{12}\mathbf{G}_{GL} & -\boldsymbol{\Phi}^G_{11}\mathbf{B}_{GG}+\boldsymbol{\Phi}^G_{12}\mathbf{G}_{GG}+\boldsymbol{\Psi}^G_{12} \\ \hline
\boldsymbol{\Phi}^L_{21}\mathbf{G}_{LL}+\boldsymbol{\Phi}^L_{22}\mathbf{B}_{LL}+\boldsymbol{\Psi}^L_{21} & \boldsymbol{\Phi}^L_{21}\mathbf{G}_{LG}+\boldsymbol{\Phi}^L_{22}\mathbf{B}_{LG} & -\boldsymbol{\Phi}^L_{21}\mathbf{B}_{LL}+\boldsymbol{\Phi}^L_{22}\mathbf{G}_{LL}+\boldsymbol{\Psi}^L_{22} & -\boldsymbol{\Phi}^L_{21}\mathbf{B}_{LG}+\boldsymbol{\Phi}^L_{22}\mathbf{G}_{LG} \\ \hline
\boldsymbol{\Phi}^G_{21}\mathbf{G}_{GL}\ \boldsymbol{\Phi}^G_{22}\mathbf{B}_{GL} & \boldsymbol{\Phi}^G_{21}\mathbf{G}_{GG}+\boldsymbol{\Phi}^G_{22}\mathbf{B}_{GG}+\boldsymbol{\Psi}^G_{21} & -\boldsymbol{\Phi}^G_{21}\mathbf{B}_{GL}+\boldsymbol{\Phi}^G_{22}\mathbf{G}_{GL} & -\boldsymbol{\Phi}^G_{21}\mathbf{B}_{GG}+\boldsymbol{\Phi}^G_{22}\mathbf{G}_{GG}+\boldsymbol{\Psi}^G_{22}
\end{array}\right]$$

$$\left[\begin{array}{c} \hat{\mathbf{V}}_{L1} \\ \hline \hat{\mathbf{V}}_{G1} \\ \hline \hat{\mathbf{V}}_{L2} \\ \hline \hat{\mathbf{V}}_{G2} \end{array}\right] = - \left[\begin{array}{c}
\boldsymbol{\Phi}^L_{11}\hat{\mathbf{J}}_{L1}+\boldsymbol{\Phi}^L_{12}\hat{\mathbf{J}}_{L2}+\hat{\mathbf{W}}^S_{L1} \\
(\boldsymbol{\Phi}^L_{11}\mathbf{G}_{LN}+\boldsymbol{\Phi}^L_{12}\mathbf{B}_{LN})\hat{\mathbf{V}}_{n1}-(\boldsymbol{\Phi}^L_{11}\mathbf{B}_{LN}-\boldsymbol{\Phi}^L_{12}\mathbf{G}_{LN})\hat{\mathbf{V}}_{n2} \\ \hline
\boldsymbol{\Phi}^G_{11}\hat{\mathbf{J}}_{G1}+\boldsymbol{\Phi}^G_{12}\hat{\mathbf{J}}_{G2}+\hat{\mathbf{W}}^S_{G1} \\
(\boldsymbol{\Phi}^G_{11}\mathbf{G}_{GN}+\boldsymbol{\Phi}^G_{12}\mathbf{B}_{GN})\hat{\mathbf{V}}_{n1}-(\boldsymbol{\Phi}^G_{11}\mathbf{B}_{GN}-\boldsymbol{\Phi}^G_{12}\mathbf{G}_{GN})\hat{\mathbf{V}}_{n2} \\ \hline
\boldsymbol{\Phi}^L_{21}\hat{\mathbf{J}}_{L1}+\boldsymbol{\Phi}^L_{22}\hat{\mathbf{J}}_{L2}+\hat{\mathbf{W}}^S_{L2} \\
(\boldsymbol{\Phi}^L_{21}\mathbf{G}_{LN}+\boldsymbol{\Phi}^L_{22}\mathbf{B}_{LN})\hat{\mathbf{V}}_{n1}-(\boldsymbol{\Phi}^L_{21}\mathbf{B}_{LN}-\boldsymbol{\Phi}^L_{22}\mathbf{G}_{LN})\hat{\mathbf{V}}_{n2} \\ \hline
\boldsymbol{\Phi}^G_{21}\hat{\mathbf{J}}_{G1}+\boldsymbol{\Phi}^G_{22}\hat{\mathbf{J}}_{G2}+\hat{\mathbf{W}}^S_{G2} \\
(\boldsymbol{\Phi}^G_{21}\mathbf{G}_{GN}+\boldsymbol{\Phi}^G_{22}\mathbf{B}_{GN})\hat{\mathbf{V}}_{n1}-(\boldsymbol{\Phi}^G_{21}\mathbf{B}_{GN}-\boldsymbol{\Phi}^G_{22}\mathbf{G}_{GN})\hat{\mathbf{V}}_{n2}
\end{array}\right] \qquad (4.36)$$

where

$$\underset{\sim}{Y}_{IJ} = \underset{\sim}{G}_{IJ} + j\,\underset{\sim}{B}_{IJ}, \tag{4.37}$$

$$\hat{\underset{\sim}{V}}_{K} = \hat{\underset{\sim}{V}}_{K1} + j\,\hat{\underset{\sim}{V}}_{K2}, \tag{4.38}$$

$$\hat{\underset{\sim}{J}}_{K} = \hat{\underset{\sim}{J}}_{K1} + j\,\hat{\underset{\sim}{J}}_{K2} \tag{4.39}$$

and

$$\hat{\underset{\sim}{W}}{}^{S}_{K} = \hat{\underset{\sim}{W}}{}^{S}_{K1} + j\,\hat{\underset{\sim}{W}}{}^{S}_{K2}, \tag{4.40}$$

I, J and K can be G, L, N or n.

Note that the form (4.36) is general for any set of element variables. The adjoint matrix of coefficients has dimension 2(n−1) x 2(n−1), where the slack bus equations have been substituted.

4.5.2 Case of Practical Element Variables

Tables 4.3 and 4.4 show the parameters of the adjoint system (4.36) for the sets of element variables $\bar{\underset{\sim}{z}}_b$ and $\tilde{\underset{\sim}{z}}_b$, respectively. For simplicity, only general elements of the diagonal matrices and of the vectors are shown.

The structure of the adjoint system (4.36) for any of the two cases is simplified to

$$\left[\begin{array}{c|c|c|c}
\underset{\sim}{G}_{LL} + \underset{\sim}{\Psi}^{L}_{11} & \underset{\sim}{G}_{LG} & -\underset{\sim}{B}_{LL} + \underset{\sim}{\Psi}^{L}_{12} & -\underset{\sim}{B}_{LG} \\
\hline
\underset{\sim}{\Phi}^{G}_{11}\underset{\sim}{G}_{GL} + \underset{\sim}{\Phi}^{G}_{12}\underset{\sim}{B}_{GL} & \underset{\sim}{\Phi}^{G}_{11}\underset{\sim}{G}_{GG} + \underset{\sim}{\Phi}^{G}_{12}\underset{\sim}{B}_{GG} + \underset{\sim}{\Psi}^{G}_{11} & -\underset{\sim}{\Phi}^{G}_{11}\underset{\sim}{B}_{GL} + \underset{\sim}{\Phi}^{G}_{12}\underset{\sim}{G}_{GL} & -\underset{\sim}{\Phi}^{G}_{11}\underset{\sim}{B}_{GG} + \underset{\sim}{\Phi}^{G}_{12}\underset{\sim}{G}_{GG} + \underset{\sim}{\Psi}^{G}_{12} \\
\hline
\underset{\sim}{B}_{LL} + \underset{\sim}{\Psi}^{L}_{21} & \underset{\sim}{B}_{LG} & \underset{\sim}{G}_{LL} + \underset{\sim}{\Psi}^{L}_{22} & \underset{\sim}{G}_{LG} \\
\hline
\underset{\sim}{0} & \underset{\sim}{\Psi}^{G}_{21} & \underset{\sim}{0} & \underset{\sim}{\Psi}^{G}_{22}
\end{array}\right]$$

$$\begin{bmatrix} \hat{V}_{L1} \\ \hline \hat{V}_{G1} \\ \hline \hat{V}_{L2} \\ \hline \hat{V}_{G2} \end{bmatrix} = - \begin{bmatrix} \hat{J}_{L1} + \hat{W}^S_{L1} + G_{LN} \hat{V}_{n1} - B_{LN} \hat{V}_{n2} \\ \hline \Phi^G_{11}[\hat{J}_{G1} + G_{GN}\hat{V}_{n1} - B_{GN}\hat{V}_{n2}] + \Phi^G_{12}[\hat{J}_{G2} + B_{GN}\hat{V}_{n1} + G_{GN}\hat{V}_{n2}] + \hat{W}^S_{G1} \\ \hline \hat{J}_{L2} + \hat{W}^S_{L2} + B_{LN} \hat{V}_{n1} + G_{LN} \hat{V}_{n2} \\ \hline \hat{W}^S_{G2} \end{bmatrix} \qquad (4.41)$$

Observe that the majority of elements of the adjoint matrix of coefficients of (4.41) are constant and are not required to be updated at different base-case points. Moreover, they simply constitute elements of the bus admittance matrix representing basic data of the problem available and already stored in computer memory.

4.6 CALCULATION OF TOTAL DERIVATIVES

The derivatives of the function f w.r.t. the control variables are calculated from (3.56) or (3.58). The vector $\hat{n}_{bu}$ is obtained from (3.62) where the vector $\hat{f}_b$ is given, for the considered case, by

$$\hat{f}_b = \begin{bmatrix} \hat{f}_{bi} \\ \hline \hat{f}_{bv} \end{bmatrix} = \begin{bmatrix} \hat{I}_b \\ \hat{I}^*_b \\ \hline -\hat{V}_b \\ -\hat{V}^*_b \end{bmatrix} \qquad (4.42)$$

TABLE 4.3

PARAMETERS OF ADJOINT SYSTEM USING ELEMENT VARIABLES $\bar{\underset{\sim}{z}}_b$

Load Elements

$$\phi^{\ell}_{11} = 1 \qquad \psi^{\ell}_{11} = \mathrm{Re}\{-S_\ell/V^2_\ell\}$$

$$\phi^{\ell}_{12} = 0 \qquad \psi^{\ell}_{12} = \mathrm{Im}\{-S_\ell/V^2_\ell\}$$

$$\hat{W}^S_{\ell 1} = \mathrm{Re}\{(|V_\ell| \frac{\partial f}{\partial |V_\ell|} - j \frac{\partial f}{\partial \delta_\ell})/(2V_\ell)\}$$

$$\phi^{\ell}_{21} = 0 \qquad \psi^{\ell}_{21} = \mathrm{Im}\{-S_\ell/V^2_\ell\}$$

$$\phi^{\ell}_{22} = 1 \qquad \psi^{\ell}_{22} = \mathrm{Re}\{S_\ell/V^2_\ell\}$$

$$\hat{W}^S_{\ell 2} = \mathrm{Im}\{(|V_\ell| \frac{\partial f}{\partial |V_\ell|} - j \frac{\partial f}{\partial \delta_\ell})/(2V_\ell)\}$$

Generator Elements

$$\phi^{g}_{11} = 2\mathrm{Im}\{V_g\} \qquad \psi^{g}_{11} = -2\mathrm{Im}\{S_g/V_g\}$$

$$\phi^{g}_{12} = 2\mathrm{Re}\{V_g\} \qquad \psi^{g}_{12} = 2\mathrm{Re}\{S_g/V_g\}$$

$$\hat{W}^S_{g1} = -\frac{\partial f}{\partial \delta_g}$$

$$\phi^{g}_{21} = 0 \qquad \psi^{g}_{21} = 2\mathrm{Im}\{V_g\}$$

$$\phi^{g}_{22} = 0 \qquad \psi^{g}_{22} = 2\mathrm{Re}\{V_g\}$$

$$\hat{W}^S_{g2} = |V_g|^2 \frac{\partial f}{\partial Q_g}$$

Slack Generator

$$\hat{V}_n = - V^*_n \left(\frac{\partial f}{\partial P_n} + j \frac{\partial f}{\partial Q_n} \right)/2$$

Transmission Elements

$$\hat{I}^S_t = Y_t\left(\frac{\partial f}{\partial \mathrm{Re}\{I_t\}} - j \frac{\partial f}{\partial \mathrm{Im}\{I_t\}} \right)/2, \text{ hence } \hat{J}_{m1} = \sum_t a_{mt} \mathrm{Re}\{\hat{I}^S_t\} \text{ and}$$

$$\hat{J}_{m2} = \sum_t a_{mt} \mathrm{Im}\{\hat{I}^S_t\}, \; m = \ell \text{ or } g.$$

The a_{mt} are elements of the reduced incidence matrix of (3.141).

TABLE 4.4

PARAMETERS OF ADJOINT SYSTEM USING ELEMENT VARIABLES $\tilde{\underset{\sim}{z}}_b$

Load Elements

$\phi_{11}^{\ell} = 1$ $\quad \psi_{11}^{\ell} = \mathrm{Re}\{-S_\ell/V_\ell^2\}$

$\phi_{12}^{\ell} = 0$ $\quad \psi_{12}^{\ell} = \mathrm{Im}\{-S_\ell/V_\ell^2\}$ $\quad \hat{W}_{\ell 1}^{S} = \mathrm{Re}\{ \frac{\partial f}{\partial V_\ell} \}$

$\phi_{21}^{\ell} = 0$ $\quad \psi_{21}^{\ell} = \mathrm{Im}\{-S_\ell/V_\ell^2\}$

$\phi_{22}^{\ell} = 1$ $\quad \psi_{22}^{\ell} = \mathrm{Re}\{S_\ell/V_\ell^2\}$ $\quad \hat{W}_{\ell 2}^{S} = \mathrm{Im}\{ \frac{\partial f}{\partial V_\ell} \}$

Generator Elements

$\phi_{11}^{g} = 2\mathrm{Im}\{V_g\}$ $\quad \psi_{11}^{g} = \mathrm{Im}\{-j2Q_g/V_g\}$

$\phi_{12}^{g} = 2\mathrm{Re}\{V_g\}$ $\quad \psi_{12}^{g} = \mathrm{Re}\{j2Q_g/V_g\}$ $\quad \hat{W}_{g1}^{S} = \mathrm{Im}\{V_g \frac{\partial f}{\partial V_g} \}$

$\phi_{21}^{g} = 0$ $\quad \psi_{21}^{g} = 2\mathrm{Im}\{V_g\}$

$\phi_{22}^{g} = 0$ $\quad \psi_{22}^{g} = 2\mathrm{Re}\{V_g\}$ $\quad \hat{W}_{g2}^{S} = \mathrm{Im}\{V_g \frac{\partial f}{\partial I_g} \}$

Slack Generator

$$\hat{V}_n = -\frac{\partial f}{\partial I_n}$$

Transmission Elements

$\hat{I}_t^S = Y_t \frac{\partial f}{\partial I_t}$, hence $\hat{J}_{m1} = \sum_t a_{mt} \mathrm{Re}\{\hat{I}_t^S\}$ and $\hat{J}_{m2} = \sum_t a_{mt} \mathrm{Im}\{\hat{I}_t^S\}$, $m = \ell$ or g.

The a_{mt} are elements of the reduced incidence matrix of (3.141).

and the matrices M_{21}^{b} and M_{22}^{b} are given in Tables 3.3 and 3.4.

Tables 4.5 and 4.6 show the vector $\hat{n}_{bu}$ for different power system elements in the considered special case using the sets of element variables $\bar{z}_b$ and $\bar{\bar{z}}_b$, respectively.

4.7 NUMERICAL EXAMPLES

4.7.1 A 6-Bus Sample Power System

In this section, we present some numerical results to illustrate the use of the derived formulas. A 6-bus sample power system (Bandler and El-Kady 1979, Garver 1970) shown in Fig. 4.1 is considered.

Required data for the problem is shown in Tables 4.7 and 4.8. Powers injected into buses are shown. The corresponding a.c. load flow solution is shown in Table 4.9. Tables 4.10 and 4.11 show the coefficient matrices of the adjoint systems corresponding to element variables $\bar{z}_b$ and $\bar{\bar{z}}_b$, respectively. These matrices are common to all the sensitivity calculations.

So as not to be restricted to any particular application, we consider the following examples where we consider, without loss of generality, the sensitivities of some system states and a function representing the total transmission losses in the system. The control variables associated with the transmission elements are taken as the line conductances G_t and susceptances B_t.

The results presented have been checked by small perturbations about the base point.

TABLE 4.5a

THE VECTOR $\hat{\eta}_{bu}$ USING ELEMENT VARIABLES $\bar{z}_b$

	Load Elements	Generator Elements
$\hat{\eta}_{bu}$	$\begin{bmatrix} -\hat{V}_\ell/V_\ell^* - \hat{V}_\ell^*/V_\ell \\ j\hat{V}_\ell/V_\ell^* - j\hat{V}_\ell^*/V_\ell \end{bmatrix}$	$\begin{bmatrix} [V_g\hat{I}_g+V_g^*\hat{I}_g^*+S_g^*\hat{V}_g/V_g^*+S_g\hat{V}_g^*/V_g]/\lvert V_g\rvert \\ -\hat{V}_g/V_g^*-\hat{V}_g^*/V_g \end{bmatrix}$

TABLE 4.5b

THE VECTOR $\hat{\eta}_{bu}$ USING ELEMENT VARIABLES $\bar{z}_b$

	Slack Generator	Transmission Elements
$\hat{\eta}_{bu}$	$\begin{bmatrix} [V_n\hat{I}_n+V_n^*\hat{I}_n^*+S_n^*\hat{V}_n/V_n^*+S_n\hat{V}_n^*/V_n]/\lvert V_n\rvert \\ j[V_n\hat{I}_n-V_n^*\hat{I}_n^*-S_n^*\hat{V}_n/V_n^*+S_n\hat{V}_n^*/V_n] \end{bmatrix}$	$\begin{bmatrix} -V_t\hat{I}_t/Y_t-V_t^*\hat{I}_t^*/Y_t^* \\ j[-V_t\hat{I}_t/Y_t+V_t^*\hat{I}_t^*/Y_t^*] \end{bmatrix}$

TABLE 4.6

THE VECTOR $\hat{\eta}_{bu}$ USING ELEMENT VARIABLES $\tilde{z}_b$

	Load Elements	Generator Elements	Slack Generator	Transmission Elements
$\hat{\eta}_{bu}$	$\begin{bmatrix} -\hat{V}_\ell^*/V_\ell \\ -\hat{V}_\ell/V_\ell^* \end{bmatrix}$	$\begin{bmatrix} \hat{I}_g^*/V_g + S_g^*\hat{V}_g^*/(V_g^* V_g^2) \\ -\hat{V}_g^*/V_g \end{bmatrix}$	$\begin{bmatrix} \hat{I}_n \\ \hat{I}_n^* \end{bmatrix}$	$\begin{bmatrix} V_t\ \hat{I}_t/Y_t \\ V_t^*\ \hat{I}_t^*/Y_t^* \end{bmatrix}$

TABLE 4.7

BUS DATA FOR 6-BUS POWER SYSTEM

Bus Index, i	Bus Type	P_i (pu)	Q_i (pu)	$\|V_i\| \angle\delta_i$ (pu)
1	load	-2.40	0	– ∠–
2	load	-2.40	0	– ∠–
3	load	-1.60	-0.40	– ∠–
4	generator	-0.30	–	1.02 ∠–
5	generator	1.25	–	1.04 ∠–
6	slack	–	–	1.04 ∠0

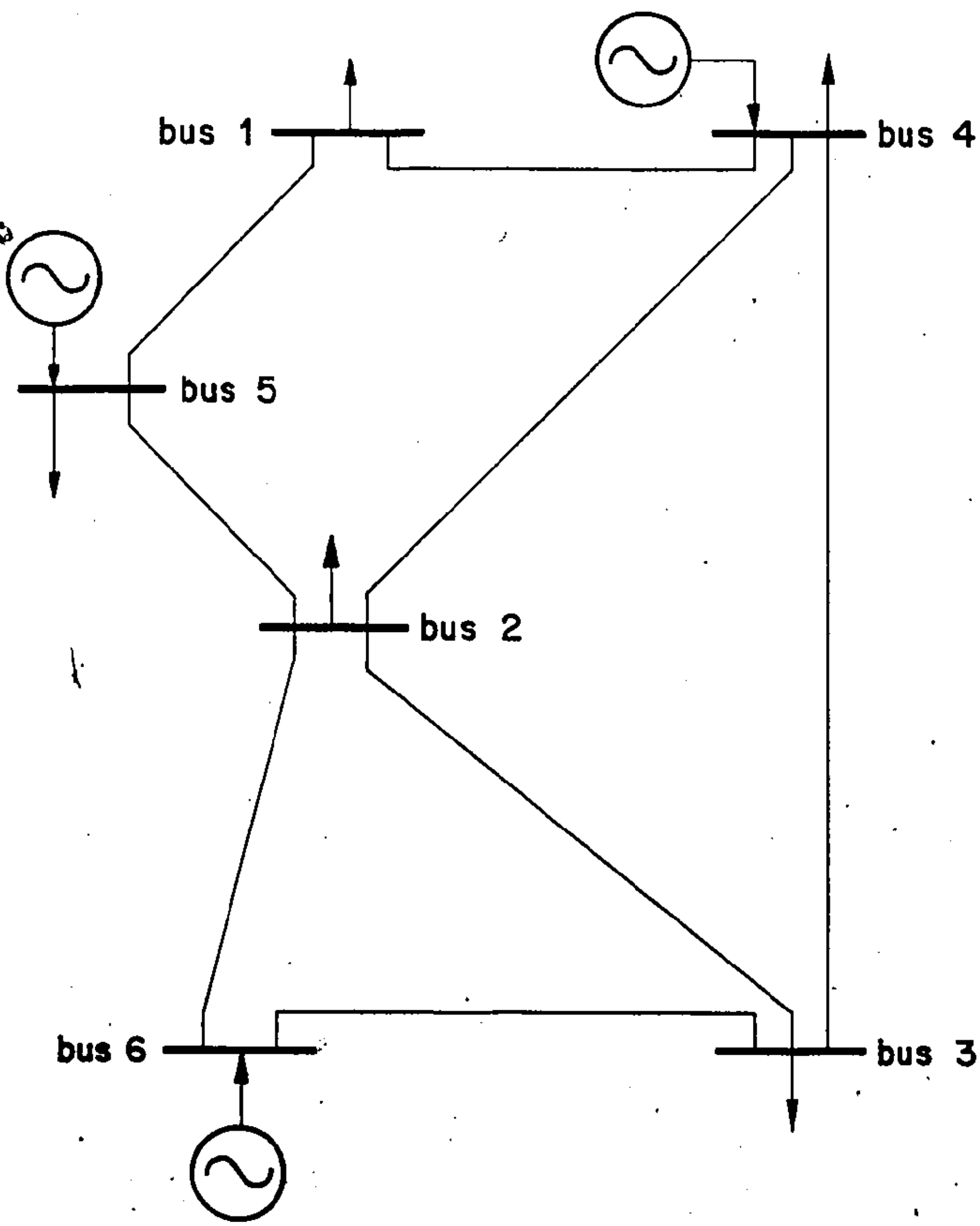


Fig. 4.1 6-bus sample power system

TABLE 4.8

LINE DATA FOR 6-BUS POWER SYSTEM

Branch Index, t	Terminal Buses	Resistance R_t (pu)	Reactance X_t (pu)	Number of Lines
7	1,4	0.05	0.20	1
8	1,5	0.025	0.10	2
9	2,3	0.10	0.40	1
10	2,4	0.10	0.40	1
11	2,5	0.05	0.20	1
12	2,6	0.01875	0.075	4
13	3,4	0.15	0.60	1
14	3,6	0.0375	0.15	2

TABLE 4.9

LOAD FLOW SOLUTION OF 6-BUS POWER SYSTEM

Load Buses

$V_1 = 0.9787 \angle -0.6602$

$V_2 = 0.9633 \angle -0.2978$

$V_3 = 0.9032 \angle -0.3036$

Generator Buses

$Q_4 = 0.7866,\quad \delta_4 = -0.5566$

$Q_5 = 0.9780,\quad \delta_5 = -0.4740$

Slack Bus

$P_6 = 6.1298,\quad Q_6 = 1.3546$

TABLE 4.10

ADJOINT MATRIX OF COEFFICIENTS USING ELEMENT VARIABLES $\bar{z}_b$

2.9085	0	0	- 1.1765	- 2.3529	11.6900	0	0	-4.7059	-9.4118
0	3.3490	- 0.5882	- 0.5882	- 1.1765	0	20.5097	-2.3529	-2.3529	-4.7059
0	- 0.5882	1.2179	- 0.3922	0	0	- 2.3529	8.6744	-1.5686	0
9.4189	4.7095	3.1396	-16.2693	0	3.0338	1.5169	1.0113	-4.2477	0
19.6518	9.8259	0	0	-26.7071	4.5821	2.2910	0	0	-8.1534
-16.5453	0	0	4.7059	9.4118	4.1503	0	0	-1.1765	-2.3529
0	-23.4119	2.3529	2.3529	4.7059	0	7.6314	-0.5882	-0.5882	-1.1765
0	2.3529	-11.7178	1.5686	0	0	- 0.5882	3.8802	-0.3922	0
0	0	0	- 1.0777	0	0	0	0	1.7321	0
0	0	0	0	- 0.9495	0	0	0	0	1.8506

TABLE 4.11

ADJOINT MATRIX OF COEFFICIENTS USING ELEMENT VARIABLES $\tilde{z}_b$

2.9085	0	0	- 1.1765	- 2.3529	11.6900	0	0	-4.7059	-9.4118
0	3.3490	- 0.5882	- 0.5882	- 1.1765	0	20.5097	-2.3529	-2.3529	-4.7059
0	- 0.5882	1.2179	- 0.3922	0	0	- 2.3529	8.6744	-1.5686	0
9.4189	4.7095	3.1396	-15.9585	0	3.0338	1.5169	1.0113	-4.7471	0
19.6519	9.8259	0	0	-27.8045	4.5821	2.2910	0	0	-6.0146
-16.5453	0	0	4.7059	9.4118	4.1503	0	0	-1.1765	-2.3529
0	-23.4119	2.3529	2.3529	4.7059	0	7.6314	-0.5882	-0.5882	-1.1765
0	2.3529	-11.7178	1.5686	0	0	- 0.5882	3.8802	-0.3922	0
0	0	0	- 1.0777	0	0	0	0	1.7321	0
0	0	0	0	- 0.9495	0	0	0	0	1.8506

4.7.2 Example 4.1

In this example, we consider the states associated with the load bus number 3. Element variables $\bar{z}_b$ are used. Table 4.12 shows the RHS vector of the adjoint equations for both states and the adjoint voltages resulting from the solution of (4.41). Table 4.13 shows the derivatives calculated by our approach.

4.7.3 Example 4.2

Now, we consider the states associated with the generator bus number 5. Element variables $\bar{z}_b$ are used. The RHS vector of the adjoint equations for both states and the adjoint voltages resulting from the solution of (4.41) are shown in Table 4.14. Table 4.15 shows the derivatives calculated by our approach.

Using element variables $\bar{z}_b$, the derivatives w.r.t. S_ℓ, for example, are shown in Table 4.16. Note that from (3.19)

$$\frac{\partial f}{\partial S_\ell^*} = \left(\frac{\partial f}{\partial S_\ell} \right)^*.$$

4.7.4 Example 4.3

In this example, we consider the function

$$f = \sum_t |I_t|^2 R_t$$

which represents the total transmission losses in the power network. Table 4.17 shows the RHS vector of the adjoint equations for this function and the adjoint voltages resulting from the solution of (4.41). Table 4.18 shows the derivatives calculated by our approach. Element variables $\bar{z}_b$ are used.

TABLE 4.12

RHS AND SOLUTON VECTORS OF THE ADJOINT NETWORKS FOR THE STATES OF BUS 3

Element No.	$f = \|V_3\|$		$f = \delta_3$	
	RHS Vector	Solution Vector	RHS Vector	Solution Vector
1	0	-0.0102	0	-0.0223
2	0	-0.0053	0	-0.0142
3	-0.4771	-0.0087	-0.1655	-0.0570
4	0	-0.0130	0	-0.0287
5	0	-0.0100	0	-0.0220
6	0	-0.0082	0	-0.0180
7	0	-0.0092	0	-0.0082
8	-0.1495	-0.0587	0.5283	-0.0188
9	0	-0.0081	0	-0.0178
10	0	-0.0051	0	-0.0113

TABLE 4.13

RESULTS OF EXAMPLE 4.1

Line Quantities

Line	Derivatives w.r.t. G_t		Derivatives w.r.t. B_t	
	$f = \|V_3\|$	$f = \delta_3$	$f = \|V_3\|$	$f = \delta_3$
1,4	-0.000544	-0.001205	0.000329	0.000743
1,5	-0.000729	-0.001595	-0.000962	-0.002133
2,3	0.001664	0.005312	-0.005748	-0.000166
2,4	0.001407	-0.003359	-0.003853	-0.008465
2,5	0.001507	-0.001260	-0.001870	-0.002965
2,6	-0.003937	-0.001242	-0.005161	-0.009986
3,4	0.027165	0.000744	-0.002716	0.015220
3,6	-0.028570	0.010158	-0.025622	-0.037461

Load Bus Quantities

Bus	Derivatives w.r.t. P_ℓ		Derivatives w.r.t. Q_ℓ	
	$f = \|V_3\|$	$f = \delta_3$	$f = \|V_3\|$	$f = \delta_3$
1	0.026681	0.058622	0.000512	0.001132
2	0.016034	0.033200	0.015022	0.007596
3	0.057311	0.132854	0.118208	0.001969

General Bus Quantities

Bus	Derivatives w.r.t. $\|V_g\|$		Derivatives w.r.t. P_g	
	$f = \|V_3\|$	$f = \delta_3$	$f = \|V_3\|$	$f = \delta_3$
4	0.194810	-0.008082	0.030046	0.066205
5	0.079778	0.056708	0.021688	0.047554

TABLE 4.14

RHS AND SOLUTION VECTORS OF THE ADJOINT NETWORKS

FOR THE STATES OF BUS 5

Element No.	$f = \delta_5$		$f = Q_5$	
	RHS Vector	Solution Vector	RHS Vector	Solution Vector
1	0	-0.0947	0	0.0600
2	0	-0.0365	0	0.0275
3	0	-0.0205	0	0.0328
4	0	-0.0745	0	0.1354
5	1.0	-0.1171	0	-0.0240
6	0	-0.0746	0	0.4884
7	0	-0.0247	0	0.1466
8	0	-0.0180	0	0.0584
9	0	-0.0464	0	0.0843
10	0	-0.0601	1.0816	0.5721

TABLE 4.15

RESULTS OF EXAMPLE 4.2

Line Quantities

Line	Derivatives w.r.t. G_t		Derivatives w.r.t. B_t	
	$f = \delta_5$	$f = Q_5$	$f = \delta_5$	$f = Q_5$
1,4	0.000289	0.063610	-0.007712	0.065953
1,5	-0.010462	-0.046596	0.001216	-0.004421
2,3	-0.002054	0.001612	0.000375	-0.010535
2,4	-0.006958	-0.043764	-0.021878	0.048458
2,5	-0.014328	0.163046	-0.030654	-0.023595
2,6	-0.005245	0.076305	-0.026501	0.050501
3,4	-0.015685	0.014771	-0.028828	0.054970
3,6	-0.002444	0.019837	-0.017490	0.038517

Load Bus Quantities

Bus	Derivatives w.r.t. P_ℓ		Derivatives w.r.t. Q_ℓ	
	$f = \delta_5$	$f = Q_5$	$f = \delta_5$	$f = Q_5$
1	0.246249	-0.709070	0.001696	-0.713165
2	0.087446	-0.143975	0.026717	-0.274202
3	0.055239	-0.107990	0.024564	-0.101658

Generator Bus Quantities

Bus	Derivatives w.r.t. $\|V_g\|$		Derivatives w.r.t. P_g	
	$f = \delta_5$	$f = Q_5$	$f = \delta_5$	$f = Q_5$
4	0.173629	-4.51867	0.172132	-0.312777
5	-0.088893	7.58088	0.253086	-0.461239

TABLE 4.16

DERIVATIVES W.R.T. COMPLEX LOAD POWERS OF EXAMPLE 4.2

Bus	Derivatives w.r.t. S_ℓ	
	$f = \delta_5$	$f = Q_5$
1	0.123125 - j0.000848	-0.354535 + j0.356583
2	0.043723 - j0.013359	-0.071988 + j0.137101
3	0.027620 - j0.012282	-0.053995 + j0.050829

TABLE 4.17

RHS AND SOLUTION VECTORS OF THE ADJOINT

NETWORK OF EXAMPLE 4.3

Element No.	RHS Vector	Solution Vector
1	0.4678	0.1692
2	0.3121	0.0852
3	0.3157	0.0828
4	-0.2337	0.1627
5	0.4732	0.1447
6	-0.3673	0.1440
7	-0.5174	0.0534
8	-0.3106	0.0707
9	0	0.1013
10	0	0.0743

TABLE 4.18

RESULTS OF EXAMPLE 4.3

Line Quantities

Line	Derivatives w.r.t. G_t	Derivatives w.r.t. B_t
1,4	0.016462	0.008741
1,5	0.048977	0.027370
2,3	0.003490	0.002102
2,4	0.084665	0.044962
2,5	0.045468	0.022680
2,6	0.103966	0.060904
3,4	0.089397	0.042758
3,6	0.113314	0.069869

Load Bus Quantities

Bus	Derivatives w.r.t. P_ℓ	Derivatives w.r.t. Q_ℓ
1	-0.453538	-0.020390
2	-0.201703	-0.054098
3	-0.221666	-0.094646

Generator Bus Quantities

Bus	Derivatives w.r.t. $\|V_g\|$	Derivatives w.r.t. P_g
4	-0.373561	-0.375812
5	-0.184047	-0.312838

4.7.5 Example 4.4

In this example, we investigate line removals by considering functions of the form

$$f = |I_t|^2.$$

The control variables associated with generator and load buses are maintained at their base-case values. Table 4.19 shows some results of different contingencies.

4.8 CONCLUSIONS

We have considered a class of adjoint systems in which the extended Tellegen sum is a real quantity. A detailed discussion of an important case in which the selection of the adjoining complex coefficients leads to an adjoint system of a special structure has been presented. The transmission line admittances of both the original and the adjoint systems turn out to be identical. The adjoint matrix of coefficients is of the same size and sparsity as the Jacobian matrix of the load flow analysis of the original power network. Moreover, most of its elements are constant and represent basic data of the problem available and already stored for computerized analysis.

The required sensitivity expressions for this special case have been derived and tabulated for direct use in sensitivity analysis and gradient evaluation. These sensitivity expressions are common to all relevant power system studies employing real functions. A number of

TABLE 4.19

CONTINGENCY RESULTS OF EXAMPLE 4.4

Function Line Index	Removed Line Index	Calculated Function Change	Exact Function Change
1,4	2,4	-0.200	-0.224
2,3	1,5*	0.002	0.005
2,3	2,3	-0.029	-0.021
2,4	2,4	-0.470	-0.404

* Only one line of branch 1,5 is removed.

relevant problems have been numerically explored for a 6-bus sample power system. The application of the formulas, which have been derived for two important sets of variables, has been illustrated.

5

CONSISTENT SELECTION OF ADJOINING COEFFICIENTS

5.1 INTRODUCTION

The approach described in Chapter 3 utilizes a generalized adjoint network concept with complex adjoining coefficients set to proper values which permit the required sensitivity evaluation for a general complex function.

The theoretical investigation of the consistent definition of the adjoint system is of particular importance because it provides the possibility of obtaining complex function sensitivities directly as in typical electronic circuits.

In this chapter, we present a unified study for consistent selection of the adjoining coefficients where the restrictions imposed by the number and type of elements of the network as well as the function considered are investigated. The study, hence, justifies the use of the approach described in Chapter 3 as a general network approach.

5.2 REMARKS ON THE CONJUGATE NOTATION

We have described and utilized the conjugate notation in Chapter 3. In classical complex algebra, the variables of a system of complex linear equations are defined independently (Stewart 1973), e.g., x_1, x_2, etc., and this is the case in real algebra. Since the use of conjugate notation implies in some cases a set of complex variables and their complex conjugate to appear in the same linear equations, a special analysis is required to reveal the properties of such systems of linear equations regarding, for example, rank, consistency conditions, etc.

Throughout Chapters 3 and 4, the application of conjugate notation has been performed in a straightforward manner since the assumption of consistency of (3.76) was made when defining the adjoint elements. In this chapter, the consistency of (3.76) is discussed for a suitable selection of the adjoining coefficients. In order to facilitate the consistency study performed in the following sections, we state here the following theorems.

5.2.1 Theorem 5.1

Let $\underset{\sim}{\theta}, \underset{\sim}{\bar{\theta}} \in C^{m\times n}$, where

$$\underset{\sim}{\theta} = \underset{\sim}{\theta}_1 + j\, \underset{\sim}{\theta}_2 \tag{5.1}$$

and

$$\underset{\sim}{\bar{\theta}} = \underset{\sim}{\bar{\theta}}_1 + j\, \underset{\sim}{\bar{\theta}}_2, \tag{5.2}$$

$\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \underset{\sim}{\bar{\theta}}_1, \underset{\sim}{\bar{\theta}}_2 \in R^{m\times n}$. Then the two matrices $\underset{\sim}{\theta}^c \in C^{2m\times 2n}$ and $\underset{\sim}{\theta}^r \in R^{2m\times 2n}$ defined as

$$\underset{\sim}{\theta}^c \triangleq \begin{bmatrix} \underset{\sim}{\theta} & \underset{\sim}{\bar{\theta}} \\ \underset{\sim}{\bar{\theta}}^* & \underset{\sim}{\theta}^* \end{bmatrix} \tag{5.3}$$

and

$$\underset{\sim}{\theta}^r \triangleq \begin{bmatrix} (\underset{\sim}{\theta}_1 + \underset{\sim}{\bar{\theta}}_1) & (\underset{\sim}{\bar{\theta}}_2 - \underset{\sim}{\theta}_2) \\ (\underset{\sim}{\theta}_2 + \underset{\sim}{\bar{\theta}}_2) & (\underset{\sim}{\theta}_1 - \underset{\sim}{\bar{\theta}}_1) \end{bmatrix} \tag{5.4}$$

have the same rank.

<u>Proof</u>

Let $\mathbf{1}^{\ell}$ be the identity matrix of order ℓ and

$$\mathbf{j}^{\ell} \triangleq j\,\mathbf{1}^{\ell}, \tag{5.5}$$

and define the two unitary matrices

$$\mathbf{U}_L \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{1}^m & \mathbf{j}^m \\ \mathbf{1}^m & -\mathbf{j}^m \end{bmatrix} \tag{5.6}$$

and

$$\mathbf{U}_R \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{1}^n & \mathbf{1}^n \\ -\mathbf{j}^n & \mathbf{j}^n \end{bmatrix} \tag{5.7}$$

Since $\mathbf{U}_L$ and $\mathbf{U}_R$ are nonsingular (Stewart 1973),

$$\text{rank}[\mathbf{U}_L\, \boldsymbol{\theta}^r\, \mathbf{U}_R] = \text{rank}[\boldsymbol{\theta}^r].$$

But

$$\mathbf{U}_L\, \boldsymbol{\theta}^r\, \mathbf{U}_R = \boldsymbol{\theta}^c,$$

hence

$$\text{rank}[\boldsymbol{\theta}^r] = \text{rank}[\boldsymbol{\theta}^c] \quad \blacksquare$$

5.2.2 Theorem 5.2

Let $\boldsymbol{\theta}, \bar{\boldsymbol{\theta}} \in C^{m\times n}$ given by (5.1) and (5.2), and let $\boldsymbol{\theta} \in C^m$ and $\mathbf{w} \in C^n$, where

$$\boldsymbol{\theta} = \boldsymbol{\theta}_1 + j\,\boldsymbol{\theta}_2 \tag{5.8}$$

and

$$\mathbf{w} = \mathbf{w}_1 + j\,\mathbf{w}_2, \tag{5.9}$$

$\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in R^m$ and $\mathbf{w}_1, \mathbf{w}_2 \in R^n$. Then the system of complex linear equations

$$\underset{\sim}{\theta}\ \underset{\sim}{w} + \bar{\underset{\sim}{\theta}}\ \underset{\sim}{w}^* = \underset{\sim}{\theta} \tag{5.10}$$

has a solution $\underset{\sim}{w}$ if and only if

$$\text{rank}[(\underset{\sim}{\theta}^c,\ \underset{\sim}{\theta}^c)] = \text{rank}[\underset{\sim}{\theta}^c],$$

where

$\underset{\sim}{\theta}^c \in C^{2m}$ is defined as

$$\underset{\sim}{\theta}^c \triangleq \begin{bmatrix} \underset{\sim}{\theta} \\ \underset{\sim}{\theta}^* \end{bmatrix} \tag{5.11}$$

and $\underset{\sim}{\theta}^c$ is given by (5.3).

<u>Proof</u>

Separating (5.10) into real and imaginary parts using (5.1), (5.2), (5.8) and (5.9), we get

$$(\underset{\sim}{\theta}_1 + \bar{\underset{\sim}{\theta}}_1)\ \underset{\sim}{w}_1 + (\bar{\underset{\sim}{\theta}}_2 - \underset{\sim}{\theta}_2)\ \underset{\sim}{w}_2 = \underset{\sim}{\theta}_1 \tag{5.12}$$

and

$$(\underset{\sim}{\theta}_2 + \bar{\underset{\sim}{\theta}}_2)\ \underset{\sim}{w}_1 + (\underset{\sim}{\theta}_1 - \bar{\underset{\sim}{\theta}}_1)\ \underset{\sim}{w}_2 = \underset{\sim}{\theta}_2 \tag{5.13}$$

or, using (5.4),

$$\underset{\sim}{\theta}^r\ \underset{\sim}{w}^r = \underset{\sim}{\theta}^r, \tag{5.14}$$

where

$$\underset{\sim}{w}^r \triangleq \begin{bmatrix} \underset{\sim}{w}_1 \\ \underset{\sim}{w}_2 \end{bmatrix} \tag{5.15}$$

and

$$\underset{\sim}{\theta}^r \triangleq \begin{bmatrix} \underset{\sim}{\theta}_1 \\ \underset{\sim}{\theta}_2 \end{bmatrix} \tag{5.16}$$

We define the nonsingular matrix

$$\overline{\mathbf{U}}_R \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{1}^n & \mathbf{1}^n & \mathbf{0} \\ -\mathbf{j}^n & \mathbf{j}^n & \mathbf{0} \\ 0 & 0 & 2 \end{bmatrix}, \tag{5.17}$$

hence

$$\text{rank}[\mathbf{U}_L(\boldsymbol{\theta}^r,\ \boldsymbol{\theta}^r)\ \overline{\mathbf{U}}_R] = \text{rank}[(\boldsymbol{\theta}^r,\ \boldsymbol{\theta}^r)],$$

where $\mathbf{U}_L$ is given by (5.6). But

$$\mathbf{U}_L(\boldsymbol{\theta}^r,\ \boldsymbol{\theta}^r)\ \overline{\mathbf{U}}_R = (\boldsymbol{\theta}^c,\ \boldsymbol{\theta}^c),$$

hence

$$\text{rank}[(\boldsymbol{\theta}^r,\ \boldsymbol{\theta}^r)] = \text{rank}[(\boldsymbol{\theta}^c,\ \boldsymbol{\theta}^c)]. \tag{5.18}$$

Now, the system of equations (5.14) has a solution if and only if

$$\text{rank}\ [(\boldsymbol{\theta}^r,\ \boldsymbol{\theta}^r)] = \text{rank}[\boldsymbol{\theta}^r],$$

hence the theorem is proved using (5.18) and Theorem 5.1 ∎

5.3 CRITERIA FOR SELECTING ADJOINING COEFFICIENTS

In this section, we derive the required conditions which (3.76) must satisfy for proper definition of the adjoint system. First, equation (3.76) must be consistent. The results of the previous section allow us to state the following corollary.

5.3.1 Corollary 5.1

Equation (3.76) is consistent if and only if

$$\text{rank}[(\overline{\boldsymbol{\theta}}_b,\ \boldsymbol{\theta}_b^c)] = \text{rank}[(\overline{\boldsymbol{\theta}}_b)], \tag{5.19}$$

where

$$\bar{\theta}_b \triangleq \begin{bmatrix} -\bar{\theta}_{bv} & \bar{\theta}_{bi} \\ -\bar{\theta}^*_{bv}\bar{1} & \bar{\theta}^*_{bi}\bar{1} \end{bmatrix} . \tag{5.20}$$

$$\theta^c_b \triangleq \begin{bmatrix} \theta_b \\ \theta^*_b \end{bmatrix} \tag{5.21}$$

and

$$\bar{1} \triangleq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \tag{5.22}$$

<u>Proof</u>

The conjugate of (3.76) is written as

$$\bar{\theta}^*_{bi}\,\hat{W}^*_{bi} = \bar{\theta}^*_{bv}\,\hat{W}^*_{bv} + \theta^*_b. \tag{5.23}$$

Since

$$\hat{W}^*_{bi} = \bar{1}\,\hat{W}_{bi} \tag{5.24}$$

and

$$\hat{W}^*_{bv} = \bar{1}\,\hat{W}_{bv}, \tag{5.25}$$

hence

$$\bar{\theta}^*_{bi}\,\bar{1}\,\hat{W}_{bi} = \bar{\theta}^*_{bv}\,\bar{1}\,\hat{W}_{bv} + \theta^*_b. \tag{5.26}$$

Equations (3.76) and (5.26) are written together as

$$\bar{\theta}_b\,\hat{W}_b = \theta^c_b. \tag{5.27}$$

When the variables $\hat{W}_b$ and the corresponding columns of $\bar{\theta}_b$ are rearranged such that (3.76) has the same form as (5.10), the rank of $\bar{\theta}_b$ is preserved. Hence from Theorem 5.2, the corollary is proved ■

5.3.2 General Consistency Criterion

In order to uniquely define the adjoint elements with proper relations between adjoint variables, we also require, in addition to

(5.19), that the system of four real equations (3.76) has rank 2. Hence, from Theorem 5.1 and the previous corollary, the matrix $\overline{\theta}_b$ of (5.20) must be of rank 2. In summary, the conditions which (3.76) must satisfy are

$$\mathrm{rank}[(\overline{\theta}_b, \theta_b^c)] = \mathrm{rank}[\ \overline{\theta}_b\ ] = 2. \tag{5.28}$$

5.3.3 Remarks on the Adjoining Coefficients

Note that the elements of the matrices M_{11}^b and M_{12}^b of (3.61) and the complex adjoining coefficients form the matrix $\overline{\theta}_b$. The matrices M_{11}^b, M_{12}^b, M_{21}^b and M_{22}^b of (3.60) depend solely upon the element-type modelling. Moreover, the vector θ_b^c contains the derivatives of the function f w.r.t. the states associated with element b. Thus we require a proper selection of the adjoining coefficients which satisfy (5.28) for a particular element-type modelling and for a given function f.

Since the set of adjoining coefficients α, $\overline{\alpha}$, β, $\overline{\beta}$, ξ, $\overline{\xi}$, ν and $\overline{\nu}$ is common to all element types described in a particular system, we expect that the more element types in a system, the more restrictions, hence, the more difficulty there will be in selecting these adjoining coefficients to satisfy (5.28). On the other hand, the adjoining coefficients Γ_k and $\overline{\Gamma}_k$ are only common to those elements within certain group terms which may include a few element types. Consequently, we expect more flexibility in adjusting these coefficients to satisfy (5.28) for certain elements.

Examples of element types of particular systems are shown in Tables 5.1, 5.2 and 5.3. Tables 5.1 and 5.2 represent typical linear electronic circuits. Two element types are described for each system, namely, the node elements (e.g., source elements) and the line elements. One representation of the power system considered in Chapter 3 is shown in Table 5.3 in which four element types describing the load, the generator, the slack generator and the transmission branches are considered.

5.3.4 Difficulties due to Source Elements

A comparison between the electronic system of Tables 5.1 or 5.2 and the power system of Table 5.3 is of particular interest. The state and control variables associated with the source elements of an electronic system are simply the basic variables $\underset{\sim}{w}_b$ of (3.40) which classify them as either current sources (Table 5.1) or voltage sources (Table 5.2). In a power system, the situation is different. The state and control variables associated with the source elements are nonlinear functions of the basic variables $\underset{\sim}{w}_b$ which results in nonlinear load flow equations, and also difficulty with respect to the consistent selection of the adjoining coefficients as we shall see later on.

5.3.5 Alternative Consistency Conditions

At the end of this section we state some important forms of equation (3.76) which satisfy condition (5.28). As shown in Chapter 3, equation (3.76) has the form

TABLE 5.1

A TYPICAL LINEAR ELECTRONIC CIRCUIT WITH CURRENT SOURCES

Element Type	Symbol	$\underset{\sim}{x}_b$	$\underset{\sim}{u}_b$	$\underset{\sim}{M}^b_{11}$	$\underset{\sim}{M}^b_{12}$
Node Elements	j	$\begin{bmatrix} V_j \\ V^*_j \end{bmatrix}$	$\begin{bmatrix} I_j \\ I^*_j \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
Line Elements	t	$\begin{bmatrix} I_t \\ I^*_t \end{bmatrix}$	$\begin{bmatrix} Y_t \\ Y^*_t \end{bmatrix}$	$\begin{bmatrix} 1/Y_t & 0 \\ 0 & 1/Y^*_t \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$Y_t = I_t/V_t$ is the admittance of line t.

TABLE 5.2

A TYPICAL LINEAR ELECTRONIC CIRCUIT WITH VOLTAGE SOURCES

Element Type	Symbol	$\underset{\sim}{x}_b$	$\underset{\sim}{u}_b$	$\underset{\sim}{M}^b_{11}$	$\underset{\sim}{M}^b_{12}$
Node Elements	j	$\begin{bmatrix} I_j \\ I^*_j \end{bmatrix}$	$\begin{bmatrix} V_j \\ V^*_j \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Line Elements	t	$\begin{bmatrix} I_t \\ I^*_t \end{bmatrix}$	$\begin{bmatrix} Y_t \\ Y^*_t \end{bmatrix}$	$\begin{bmatrix} 1/Y_t & 0 \\ 0 & 1/Y^*_t \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

TABLE 5.3

A REPRESENTATION OF A POWER SYSTEM

Element Type	Symbol	$\underset{\sim}{x}_b$	$\underset{\sim}{u}_b$	$\underset{\sim}{M}^b_{11}$	$\underset{\sim}{M}^b_{12}$
Load Elements	ℓ	$\begin{bmatrix} V_\ell \\ V_\ell^* \end{bmatrix}$	$\begin{bmatrix} S_\ell \\ S_\ell^* \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & -I_\ell^*/V_\ell \\ -I_\ell/V_\ell^* & 0 \end{bmatrix}$
Elements Generator	g	$\begin{bmatrix} V_g \\ I_g \end{bmatrix}$	$\begin{bmatrix} \|V_g\|^2 \\ 2P_g \end{bmatrix}$	$\begin{bmatrix} 1 & -V_g^*/V_g \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -j2Q_g/V_g^2 \\ 1 & -V_g^*/V_g \end{bmatrix}$
Slack Generator	n	$\begin{bmatrix} I_n \\ I_n^* \end{bmatrix}$	$\begin{bmatrix} V_n \\ V_n^* \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Transmission Elements	t	$\begin{bmatrix} I_t \\ I_t^* \end{bmatrix}$	$\begin{bmatrix} Y_t \\ Y_t^* \end{bmatrix}$	$\begin{bmatrix} 1/Y_t & 0 \\ 0 & 1/Y_t^* \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$S_m = P_m + jQ_m$ is the power of element m, m can be ℓ, g or n.

$$\tilde{\phi}_b^k \hat{I}_b + \bar{\phi}_b^k \hat{I}_b^* = \tilde{\psi}_b^k \hat{V}_b + \bar{\psi}_b^k \hat{V}_b^* + \hat{W}_b^{Sk}, \qquad (5.29)$$

where k = 1, 2 denotes the first and second complex equation of (3.76), respectively. It can be shown that each of the following conditions is equivalent to (5.28).

Alternative Condition 1

$$\tilde{\phi}_b^1 = \tilde{\phi}_b^2,\ \bar{\phi}_b^1 = \bar{\phi}_b^2,\ \tilde{\psi}_b^1 = \tilde{\psi}_b^2,\ \bar{\psi}_b^1 = \bar{\psi}_b^2 \qquad (5.30)$$

and

$$\hat{W}_b^{S1} = \hat{W}_b^{S2}, \qquad (5.31)$$

in which the two complex equations of (5.29) are identical.

Alternative Condition 2

$$\tilde{\phi}_b^1 = \bar{\phi}_b^{2*},\ \bar{\phi}_b^1 = \tilde{\phi}_b^{2*},\ \tilde{\psi}_b^1 = \bar{\psi}_b^{2*},\ \bar{\psi}_b^1 = \tilde{\psi}_b^{2*} \qquad (5.32)$$

and

$$\hat{W}_b^{S1} = \hat{W}_b^{S2*}, \qquad (5.33)$$

in which the first complex equation of (5.29) is the conjugate of the second one.

Alternative Condition 3

$$\tilde{\phi}_b^1 = \bar{\phi}_b^{1*},\ \tilde{\psi}_b^1 = \bar{\psi}_b^{1*},\ \hat{W}_b^{S1} \text{ is real} \qquad (5.34)$$

or

$$\tilde{\phi}_b^1 = -\bar{\phi}_b^{1*},\ \tilde{\psi}_b^1 = -\bar{\psi}_b^{1*},\ \hat{W}_b^{S1} \text{ is imaginary} \qquad (5.35)$$

and

$$\tilde{\phi}_b^2 = \bar{\phi}_b^{2*},\ \tilde{\psi}_b^2 = \bar{\psi}_b^{2*},\ \hat{W}_b^{S2} \text{ is real} \qquad (5.36)$$

or

$$\tilde{\phi}_b^2 = -\bar{\phi}_b^{2*},\ \tilde{\psi}_b^2 = -\bar{\psi}_b^{2*} \text{ and } \hat{W}_b^{S2} \text{ is imaginary,} \tag{5.37}$$

in which each of the two equations of (5.29) represents one real equation.

Observe, for example, that for a real function f and under conditions (3.33)-(3.37), Table 3.1 shows a proper adjoint system by conditions (5.34) and (5.36) for all elements while Table 3.2 shows a proper adjoint system by conditions (5.32) and (5.33) for the load, slack generator, and transmission elements.

5.4 A SPECIAL CONSISTENCY CRITERION

In the previous section, we have derived the required conditions for proper definition of the adjoint system to be solved. Since we are searching for proper adjoining coefficients which satisfy condition (5.28) rather than checking the condition itself, the form (5.28) may not be adequate for direct use in selecting the various adjoining coefficients.

In this section, we state a special technique for selecting the adjoining coefficients. The technique presented is based on a few assumptions regarding the coefficients and hence it satisfies a somewhat more restricted criterion than (5.28). The technique, however, allows fast and easy selection of proper adjoining coefficients for different systems of different element types.

5.4.1 Description of Criterion

We write (3.46) in the form

$$\hat{f}_b = \begin{bmatrix} \hat{f}_{bi} \\ \hline \hat{f}_{bv} \end{bmatrix} = \begin{bmatrix} \hat{I}_b \\ \hat{\bar{I}}_b \\ \hline -\hat{U}_b \\ -\hat{\bar{U}}_b \end{bmatrix}, \tag{5.38}$$

where we have defined

$$\hat{I}_b \triangleq \alpha \hat{I}_b + \bar{\xi} \hat{I}_b^* + D_{ib}, \tag{5.39}$$

$$\hat{\bar{I}}_b \triangleq \bar{\alpha} \hat{I}_b^* + \xi \hat{I}_b + \bar{D}_{ib}, \tag{5.40}$$

$$\hat{U}_b \triangleq \beta \hat{V}_b + \bar{\nu} \hat{V}_b^* + D_{vb}, \tag{5.41}$$

$$\hat{\bar{U}}_b \triangleq \bar{\beta} \hat{V}_b^* + \nu \hat{V}_b + \bar{D}_{vb} \tag{5.42}$$

and, using (3.72)-(3.75),

$$D_{ib} \triangleq \sum_k \lambda_{bk} N_{ib}^k, \tag{5.43}$$

$$\bar{D}_{ib} \triangleq \sum_k \lambda_{bk} \bar{N}_{ib}^k, \tag{5.44}$$

$$D_{vb} \triangleq -\sum_k \lambda_{bk} N_{vb}^k \tag{5.45}$$

and

$$\bar{D}_{vb} \triangleq -\sum_k \lambda_{bk} \bar{N}_{vb}^k, \tag{5.46}$$

where λ_{bk} is given by (3.32).

Under the assumptions

$$\bar{\xi}\, \bar{\xi}^* - \alpha\, \alpha^* \neq 0 \tag{5.47}$$

and

$$\bar{\nu}\,\bar{\nu}^{*} - \beta\,\beta^{*} \neq 0, \tag{5.48}$$

one may write

$$\hat{\bar{I}}_b = A_i\,\hat{I}_b + \bar{A}_i\,\hat{I}^{*}_b + \tilde{A}_{ib} \tag{5.49}$$

and

$$\hat{\bar{U}}_b = A_v\,\hat{U}_b + \bar{A}_v\,\hat{U}^{*}_b + \tilde{A}_{vb}, \tag{5.50}$$

where

$$A_i \triangleq \frac{\bar{\xi}^{*}\,\bar{\alpha} - \xi\,\alpha^{*}}{\bar{\xi}\,\bar{\xi}^{*} - \alpha\,\alpha^{*}}, \tag{5.51}$$

$$\bar{A}_i \triangleq \frac{\xi\,\bar{\xi} - \alpha\,\bar{\alpha}}{\bar{\xi}\,\bar{\xi}^{*} - \alpha\,\alpha^{*}}, \tag{5.52}$$

$$\tilde{A}_{ib} \triangleq \bar{D}_{ib} - A_i\,D_{ib} - \bar{A}_i\,D^{*}_{ib}, \tag{5.53}$$

$$A_v \triangleq \frac{\bar{\nu}^{*}\,\bar{\beta} - \nu\,\beta^{*}}{\bar{\nu}\,\bar{\nu}^{*} - \beta\,\beta^{*}}, \tag{5.54}$$

$$\bar{A}_v \triangleq \frac{\nu\,\bar{\nu} - \beta\,\bar{\beta}}{\bar{\nu}\,\bar{\nu}^{*} - \beta\,\beta^{*}} \tag{5.55}$$

and

$$\tilde{A}_{vb} \triangleq \bar{D}_{vb} - A_v\,D_{vb} - \bar{A}_v\,D^{*}_{vb}. \tag{5.56}$$

Now, according to the element types used in a particular system, the gross coefficients A_i, $\bar{A}_i$ and $\tilde{A}_{ib}$ of (5.51)-(5.53) and A_v, $\bar{A}_v$ and $\tilde{A}_{vb}$ of (5.54)-(5.56) have to satisfy certain relationships to fulfil condition (5.28). The more element types used to describe a certain system, the more those relationships will be.

The impact of using the gross coefficients is obvious. Instead of dealing directly with the numerous coefficients α, $\bar{\alpha}$, etc., while searching for a proper adjoint system representation, we only investigate conditions on fewer gross coefficients. Moreover, in the absence of group terms, $\tilde{A}_{ib}$ and $\tilde{A}_{vb}$ are automatically zero and only $\tilde{A}_i$, $\bar{A}_i$, A_v and $\bar{A}_v$ are left for study.

We illustrate the use of the gross coefficients by the following examples.

5.4.2 Example 5.1

For the electronic system described in Table 5.1, and using (5.38), (5.49) and (5.50), equation (3.76) is written, for node elements, as

$$\frac{\partial f}{\partial V_j} = \hat{I}_j \tag{5.57}$$

and

$$\frac{\partial f}{\partial V_j^*} = \hat{\bar{I}}_j = A_i \hat{I}_j + \bar{A}_i \hat{I}_j^* + \tilde{A}_{ij} \tag{5.58}$$

and, for line elements, as

$$Y_t \frac{\partial f}{\partial I_t} = \hat{I}_t - Y_t \hat{U}_t \tag{5.59}$$

and

$$Y_t^* \frac{\partial f}{\partial I_t^*} = \hat{\bar{I}}_t - Y_t^* \hat{\bar{U}}_t = A_i \hat{I}_t + \bar{A}_i \hat{I}_t^* + \tilde{A}_{it} - Y_t^* (A_v \hat{U}_t + \bar{A}_v \hat{U}_t^* + \tilde{A}_{vt}). \tag{5.60}$$

Now, for a real function f, we have

$$\frac{\partial f}{\partial V_j} = \left(\frac{\partial f}{\partial V_j^*} \right)^* \tag{5.61}$$

and

$$\frac{\partial f}{\partial I_t} = \left(\frac{\partial f}{\partial I_t^*} \right)^* \tag{5.62}$$

so that (5.57) and (5.58) are consistent if

$$A_i = 0, \; \overline{A}_i = 1 \text{ and } \tilde{A}_{ij} = 0. \tag{5.63}$$

Using (5.63), equations (5.59) and (5.60) are also consistent if

$$A_v = 0, \; \overline{A}_v = 1 \text{ and } \tilde{A}_{vt} = \tilde{A}_{it} = 0. \tag{5.64}$$

Under conditions (5.63) and (5.64), either (5.57) or (5.58) can be used to define the adjoint node elements. Also, either (5.59) or (5.60) can be used to define the adjoint line elements.

In terms of the adjoining coefficients α, $\overline{\alpha}$, etc., it is obvious from (5.51)-(5.56) that

$$\overline{\alpha} = \alpha^*, \; \overline{\beta} = \beta^*, \; \overline{\xi} = \xi^* \text{ and } \overline{\nu} = \nu^*, \tag{5.65}$$

$$\overline{D}_{ib} = D_{ib}^* \text{ and } \overline{D}_{vb} = D_{vb}^* \tag{5.66}$$

are sufficient to satisfy (5.28). Note that (5.66) is an alternative condition to (3.37).

5.4.3 Example 5.2

For the power system described in Table 5.3, and following a similar procedure to that of Example 5.1 for the different element types of the system, it is a straightforward to show that, for a real function f, conditions (5.65) and (5.66) are also sufficient to satisfy (5.28).

5.4.4 Example 5.3

Consider, again, the electronic system of Table 5.1. Let

$$f = V_k, \qquad (5.67)$$

where V_k is a certain complex node voltage. With no group terms and using (5.38), (5.49) and (5.50), equation (3.76) is written, for the node element $j \neq k$, as

$$0 = \hat{I}_j \qquad (5.68)$$

and

$$0 = \hat{\overline{I}}_j = A_i \hat{I}_j + \overline{A}_i \hat{I}_j^* \qquad (5.69)$$

which requires no restrictions on A_i or $\overline{A}_i$, and for the node element k, as

$$1 = \hat{I}_k \qquad (5.70)$$

and

$$0 = \hat{\overline{I}}_k = A_i \hat{I}_k + \overline{A}_i \hat{I}_k^* \qquad (5.71)$$

which requires

$$A_i = -\overline{A}_i. \qquad (5.72)$$

Also, for line elements, we write

$$0 = \hat{I}_t - Y_t \hat{U}_t \qquad (5.73)$$

and

$$0 = A_i \hat{I}_t + \overline{A}_i \hat{I}_t^* - Y_t^*(A_v \hat{U}_t + \overline{A}_v \hat{U}_t^*) \qquad (5.74)$$

which requires

$$A_i Y_t = A_v Y_t^* \text{ for all } t. \qquad (5.75)$$

Hence

$$A_i = A_v = 0. \qquad (5.76)$$

Also

$$\overline{A}_i = \overline{A}_v. \qquad (5.77)$$

Conditions (5.72), (5.76) and (5.77) are simply

$$A_i = A_v = \overline{A}_i = \overline{A}_v = 0 \tag{5.78}$$

or, from (5.51)-(5.56)

$$\xi\ \overline{\xi} = \alpha\ \overline{\alpha}, \tag{5.79}$$

$$\nu\ \overline{\nu} = \beta\ \overline{\beta}, \tag{5.80}$$

$$\overline{\xi}^*\ \overline{\alpha} = \xi\ \alpha^* \tag{5.81}$$

and

$$\overline{\nu}^*\ \overline{\beta} = \nu\ \beta^*. \tag{5.82}$$

Observe that any member of the family of adjoining coefficients satisfying (5.79)-(5.82) can lead to the required sensitivities of V_k. In particular, the member

$$\alpha = \beta = 1 \tag{5.83}$$

and

$$\overline{\alpha} = \overline{\beta} = \xi = \overline{\xi} = \nu = \overline{\nu} = 0, \tag{5.84}$$

or

$$\overline{\xi} = \overline{\nu} = 1 \tag{5.85}$$

and

$$\overline{\alpha} = \alpha = \overline{\beta} = \beta = \xi = \nu = 0 \tag{5.86}$$

may be used. For the member (5.83) and (5.84), the adjoint system is defined, using (5.68), (5.70) and (5.73), as

$$\hat{I}_j = 0 \text{ for } j \neq k, \tag{5.87}$$

$$\hat{I}_k = 1 \tag{5.88}$$

and

$$\hat{I}_t = Y_t\ \hat{V}_t, \tag{5.89}$$

while, for the member (5.85) and (5.86), the adjoint system is defined as

$$\hat{I}_j = 0 \text{ for } j \neq k, \tag{5.90}$$

$$\hat{I}_k = 1 \tag{5.91}$$

and

$$\hat{I}_t = Y_t^* \hat{V}_t. \tag{5.92}$$

5.4.5 Discussion

As illustrated by the foregoing examples, the technique described in this section allows direct selection of proper adjoining coefficients for a given system and for a certain function. In this respect, sensitivities of some of the complex functions of practical interest can be obtained directly by appropriate adjustment of these coefficients.

On the other hand, the adjoining coefficients play an important role in the adjoint network formulation. The freedom acquired by defining a family of possible adjoining coefficients can be utilized to alter the modelling of the adjoint elements. In some cases, it is possible to achieve certain modelling for a particular adjoint element.

In the next section, more freedom in selecting the adjoining coefficients is afforded by considering the more general case of functional adjoining coefficients.

5.5 FUNCTIONAL ADJOINING COEFFICIENTS

In the analysis so far, we have considered the case of constant adjoining coefficients in (3.31). Since these adjoining coefficients are basically multipliers of zero quantities, the restriction of constant adjoining coefficients can be relaxed.

5.5.1 Cases of Formulation

In fact, the adjoining coefficients can be functions of the basic variables $\underset{\sim}{w}_b$ of (3.40). Moreover, since

$$\tau(\overline{\underset{\sim}{w}})\ \delta h(\overline{\underset{\sim}{w}}) = h(\overline{\underset{\sim}{w}})\ \delta\tau(\overline{\underset{\sim}{w}}) = 0, \tag{5.93}$$

where $\tau(\overline{\underset{\sim}{w}})$ stands for any of adjoining coefficients, $\overline{\underset{\sim}{w}}$ denotes a vector of the basic variables $\overline{\underset{\sim}{w}}_b$ and

$$h(\overline{\underset{\sim}{w}}) = 0 \tag{5.94}$$

represents any of the Tellegen or group terms, the adjoining coefficients are not required to be perturbed in (3.39). Hence, the sensitivity expressions derived so far are still valid even when the adjoining coefficients are functions of the basic variables.

On the other hand, the adjoining coefficients can be also functions of the adjoint variables $\underset{\sim}{\hat{w}}_b$ of (5.59). The case when the set of adjoining coefficients α, $\overline{\alpha}$, β, $\overline{\beta}$, ξ, $\overline{\xi}$, ν and $\overline{\nu}$ are functions of the adjoint variables usually results in nonlinear adjoint equations to be solved. The case when the adjoining coefficients Γ_k and $\overline{\Gamma}_k$ are linear functions of the adjoint variables is indeed of particular interest.

We consider the case in which the adjoining coefficients Γ_k and $\overline{\Gamma}_k$ are linear functions of the adjoint variables $\underset{\sim}{\hat{w}}_b$ contained in the kth group term in the forms

$$\Gamma_k = \gamma_{k0} + \sum_{b \in B_k} (\gamma_{kv}^{b} \hat{V}_b + \gamma_{kv}^{b*} \hat{V}_b^{*} + \gamma_{ki}^{b} \hat{I}_b + \gamma_{ki}^{b*} \hat{I}_b^{*}) \tag{5.95}$$

and

$$\overline{\Gamma}_k = \overline{\gamma}_{k0} + \sum_{b\in B_k} (\overline{\gamma}_{kv}^{b} \hat{V}_b + \overline{\gamma}_{kv}^{b*} \hat{V}_b^* + \overline{\gamma}_{ki}^{b} \hat{I}_b + \overline{\gamma}_{ki}^{b*} \hat{I}_b^*), \quad (5.96)$$

where B_k is the set of elements forming the kth group term, and as indicated before, the coefficients γ_{k0}, γ_{kv}^{b}, etc., are in general functions of the basic variables $\underset{\sim}{w}_b$.

It is straightforward to show that the forms (5.95) and (5.96) still lead to a linear, although less sparse, adjoint system to be solved. In the resulting form of adjoint system, the diagonal matrices $\underset{\sim}{\Phi}_{ij}$ and $\underset{\sim}{\Psi}_{ij}$, i,j = 1, 2 of (3.126) and (3.127) are replaced by the equivalent matrices $\underset{\sim}{\Phi}_{ij}^{e}$ and $\underset{\sim}{\Psi}_{ij}^{e}$, respectively.

In general, the matrices $\underset{\sim}{\Phi}_{ij}^{e}$ and $\underset{\sim}{\Psi}_{ij}^{e}$ are no longer diagonal matrices. The more adjoint variables appearing in (5.95) and (5.96), the more will be the off diagonal elements of $\underset{\sim}{\Phi}_{ij}^{e}$ and $\underset{\sim}{\Psi}_{ij}^{e}$.

Since the number and type of group terms to be considered in a particular problem are entirely dictated by the type of the system and the function f, we shall not proceed towards general derivations for different systems and different classes of functions. Instead, we illustrate by a simple example the concepts stated in this section.

5.5.2 Example 5.4

Consider the simple 2-bus system of Fig. 5.1. The system consists of a load, a slack generator and three transmission elements. Required data in pu for the problem is shown. Table 5.4 shows the currents and voltages of different elements resulting from the a.c. load flow solution.

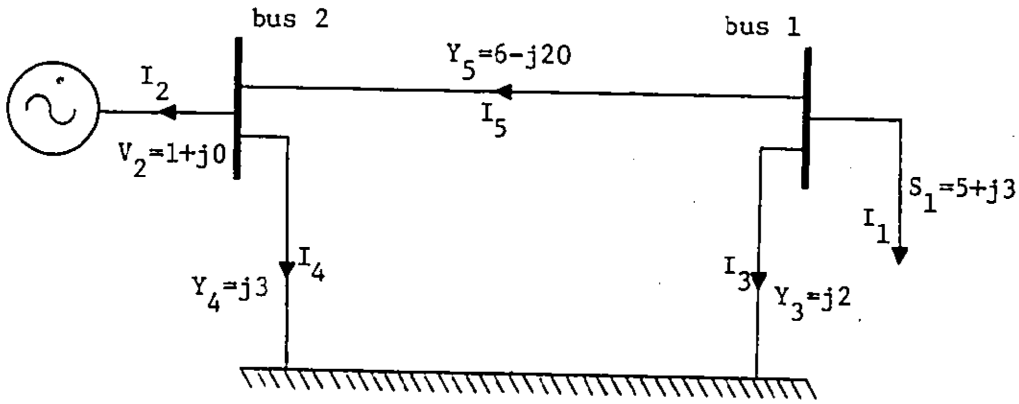


Fig. 5.1 2-bus system of Example 5.4

TABLE 5.4

SOLUTION OF EXAMPLE 5.4

b	I_b	V_b
1	5.2623-j5.5411	0.7352-j0.2041
2	-5.6705+j1.0706	1.0+j0.0
3	0.4082+j1.4705	0.7352-j.2041
4	0.0+j3.0	1.0+j0.0
5	-5.6705+j4.0706	-0.2648-j0.2041

Suppose we are interested in the sensitivities of the complex load bus voltage. The adjoining coefficients may be set according to the special case described in Chapter 4 in which we may define the two real functions

$$f_1 = \text{Re}\,\{V_1\} = (V_1 + V_1^*)/2$$

and

$$f_2 = \text{Im}\{V_1\} = j(V_1^* - V_1)/2.$$

The sensitivities of f_1 and f_2 are obtained in the same way described in Chapter 4. The adjoint matrix of coefficients and the RHS vectors for both f_1 and f_2 are shown in Table 5.5.

TABLE 5.5

ADJOINT SYSTEM OF EXAMPLE 5.4 WITH CONSTANT ADJOINING COEFFICIENTS

Adjoint Matrix of Coefficients	RHS Vector $f = \text{Re}\{V_1\}$	RHS Vector $f = \text{Im}\{V_1\}$
$\begin{bmatrix} 1.2972 & -6.0 & 9.1581 & -20.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ -26.8419 & 20.0 & 10.7283 & -6.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}$	$\begin{bmatrix} -0.5 \\ 0.0 \\ 0.0 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.0 \\ 0.0 \\ 0.5 \\ 0.0 \end{bmatrix}$

Alternatively, we may utilize the functional adjoining coefficients to obtain the sensitivities of the complex function

$$f = V_1,$$

directly, while altering the modelling of the adjoint system. For simplicity, we let

$$\xi = \bar{\xi} = \nu = \bar{\nu} = 0$$

and

$$\Gamma_k = \bar{\Gamma}_k = 0 \quad \text{for all } k \neq 1,$$

where we have considered the group terms

$$V_1 - V_2 - V_5 = 0$$

and

$$V_1^* - V_2^* - V_5^* = 0,$$

adjoined via coefficients Γ_1 and $\overline{\Gamma}_1$, respectively.

With various adjoint elements modelled according to (3.61), it is a straightforward to show that the consistent selection of the adjoining coefficients requires, for example,

$$\alpha = \beta = 1,$$

$$\overline{\alpha} = \overline{\beta} = I_1/[V_1^*(Y_3 + Y_5)],$$

$$\Gamma_1 = -\hat{I}_1 - 1/(\overline{\beta\beta}^* - 1)$$

and

$$\overline{\Gamma}_1 = -\overline{\beta}\,\hat{I}_1^* - \overline{\beta}/(\overline{\beta\beta}^* - 1).$$

Observe that this selection of the functional adjoining coefficients leads to modelling the load element in the adjoint system as a voltage source in the form

$$\hat{V}_1 = 1/[(\overline{\beta\beta}^* - 1)(Y_3 + Y_5)].$$

The derivatives of the complex load bus voltage shown in Table 5.6 are calculated from (3.56) using the solution of the resulting simple adjoint network.

5.6 CONCLUSIONS

A unified theoretical consistency study which allows proper selection of the adjoining coefficients introduced in the generalized version of Chapter 3 has been presented.

TABLE 5.6

DERIVATIVES OF V_1 OF EXAMPLE 5.4

b	Associated Derivatives		
1	$\frac{df}{dS_1} = -0.0348+j0.0366$	and	$\frac{df}{dS_1^*} = -0.0535-j0.0794$
2	$\frac{df}{dV_2} = 1.5248-j0.0462$	and	$\frac{df}{dV_2^*} = 0.7896+j0.1579$
3	$\frac{df}{dY_3} = -0.0311-j0.0462$	and	$\frac{df}{dY_3^*} = -0.0203+j0.0213$
4	$\frac{df}{dY_4} = 0.0$	and	$\frac{df}{dY_4^*} = 0.0$
5	$\frac{df}{dY_5} = -0.0080+j0.0231$	and	$\frac{df}{dY_5^*} = -0.0022-j0.0127$

The freedom acquired by exploiting both constant and functional adjoining coefficients has been investigated so that complex function sensitivities for different systems of different element types may be evaluated using a properly defined adjoint system.

The theoretical foundation of consistent modelling of different adjoint elements has been established by deriving suitable consistency criteria. These consistency criteria may be used to handle the more general branch modelling of power networks as distinct from that of typical electronic circuits.

6

COMPLEX ANALYSIS OF POWER NETWORKS: SPECIAL TOPICS

6.1 INTRODUCTION

The use of the conjugate notation has facilitated the derivation and subsequent formulation of the required sensitivity expressions in the Tellegen's theorem-based approach described in Chapter 3. The utilization of the same notation may lead to a special methodology for complex analysis of power networks regarding both the power flow solution and subsequent sensitivity calculations by other approaches.

In this chapter, some new concepts are introduced based upon which several special topics in complex analysis of power networks are presented and discussed.

6.2 SOLUTION OF POWER FLOW EQUATIONS USING TELLEGEN'S THEOREM

In this section, an important application of the approach presented in Chapters 3 and 4 is considered. We describe a new method for solving the load flow problem using Tellegen's theorem. Although the method can be described based on the generalized version of Chapter 3 where complex function sensitivities may be obtained, the method finds more useful features based on the simplified verison presented in Chapter 4.

6.2.1 Adjoint Matrix vs Jacobian Matrix

The adjoint matrix of coefficients of (4.41), although of the same size and sparsity as the Jacobian matrix of the load flow analysis by the Newton-Raphson method, is different from it.

The comparison between the adjoint matrix of coefficients of (4.41) and the Jacobian matrix of the load flow problem is indeed of particular interest. As has been pointed out in Chapter 4, our adjoint matrix of coefficients is more fundamental than the Jacobian matrix. Practically speaking, most of the elements of the adjoint matrix represent line conductances and susceptances which are constants representing basic data of the problem. On the other hand, the elements of the Jacobian matrix reflect mainly partial derivatives of bus powers w.r.t. bus voltages. These elements are voltage dependent and they have to be recalculated whenever the bus voltages are altered.

We remark that the adjoint matrix of coefficients has been built up to accommodate functions of general network variables, including both bus and line variables, which are represented in the RHS of adjoint equations. On the other hand, the Jacobian matrix, whose transpose is used as the adjoint matrix in the methods based on Lagrange multiplier approach, can only accommodate functions expressed in terms of bus variables which appear as dependent (or unknown) variables in the load flow equations.

6.2.2 Tellegen's Theorem and Load Flow Analysis

Since the sensitivity calculations are performed at a local load flow solution, the availability of the Jacobian matrix from the last load flow iteration is obviously advantageous. In this respect, it appears that a real challenge for Tellegen's theorem is its possible contribution in the solution of the load flow problem itself so that this simple and mostly constant adjoint matrix of coefficients of (4.41) may be used in both load flow analysis and subsequent sensitivity calculations.

The solution of the load flow problem by the proposed method based on Tellegen's theorem is performed simply by defining a number of real functions representing the unknown variables in power flow equations and treating them by successive forward and backward substitutions using the LU factors of the adjoint matrix of (4.41) at different load flow iterations.

6.2.3 Tellegen's Theorem-Based Method vs Newton-Raphson Method

Consider the set of real power flow equations in the general form

$$\underset{\sim}{f}(\underset{\sim}{\overline{V}}) = \underset{\sim}{\overline{S}}, \qquad (6.1)$$

where $\underset{\sim}{\overline{V}}$ is a vector of unknown real variables. Using k to denote the iteration number, the Newton-Raphson method incorporates the following main steps.

(i) Set $k \leftarrow 0$.

(ii) Calculate

$$\underset{\sim}{\overline{S}}^k = \underset{\sim}{f}(\underset{\sim}{\overline{V}}^k), \qquad (6.2)$$

where $\underset{\sim}{\overline{V}}^0$ is assumed.

(iii) Evaluate the Jacobian matrix

$$\underset{\sim}{J}^{k} = \left(\frac{\partial \overline{\underset{\sim}{S}}^{T}}{\partial \overline{\underset{\sim}{V}}} \right)_{k}^{T}. \tag{6.3}$$

(iv) Solve the set of linear equations

$$\underset{\sim}{J}^{k} \, \delta \overline{\underset{\sim}{V}}^{k} = \delta \overline{\underset{\sim}{S}}^{k}, \tag{6.4}$$

where

$$\delta \overline{\underset{\sim}{S}}^{k} = \overline{\underset{\sim}{S}}_{(scheduled)} - \overline{\underset{\sim}{S}}^{k}. \tag{6.5}$$

(v) Calculate

$$\overline{\underset{\sim}{V}}^{k+1} = \overline{\underset{\sim}{V}}^{k} + \delta \overline{\underset{\sim}{V}}^{k}. \tag{6.6}$$

(vi) If convergence is attained stop, otherwise set $k \leftarrow k+1$ and go to (ii).

On the other hand, the Tellegen's theorem-based method incorporates the following steps.

(i) Set $k \leftarrow 0$.

(ii) Calculate $\overline{\underset{\sim}{S}}^{k}$ from (6.2).

(iii) Evaluate those elements of the adjoint matrix $\underset{\sim}{T}^{k}$ of (4.41) which are required to be updated.

(iv) Using the LU factors of $\underset{\sim}{T}^{k}$, solve the sets of linear adjoint equations

$$\underset{\sim}{T}^{k} \, \hat{\underset{\sim}{V}}_{m}^{k} = \hat{\underset{\sim}{b}}_{m}^{k}, \tag{6.7}$$

where m denotes different elements of the vector $\overline{\underset{\sim}{V}}$.

(v) Calculate

$$\overline{V}_m^{k+1} = \overline{V}_m^k + \left(\frac{d\overline{V}_m}{d\overline{S}}\right)_k^T \delta\overline{S}^k, \tag{6.8}$$

where the total derivatives of $\overline{V}_m$ w.r.t. $\overline{S}$ are obtained, as described in Chapter 4, from corresponding sensitivity expressions.

(vi) If convergence is attained stop, otherwise set $k \leftarrow k+1$ and go to (ii).

6.2.4 Discussion

We notice that, although the Tellegen's theorem-based method enjoys less storage and a smaller number of recalculated elements of the adjoint matrix in each iteration, it requires several forward and backward substitutions to update those variables w.r.t. which convergence is not attained in one iteration.

Since both methods are based on first-order changes of the power flow equations for the exact a.c. load flow model, one can show that both methods create the same sequence of solution points, hence, they have the same rate of convergence.

6.2.5 Example 6.1

For the simple 2-bus power system of Example 5.4, equations (4.41) have, from Table 4.4, the form

$$\begin{bmatrix} 6 - \sigma_1 & 18 - \sigma_2 \\ -18 - \sigma_2 & 6 + \sigma_1 \end{bmatrix} \begin{bmatrix} \hat{V}_{11} \\ \hat{V}_{12} \end{bmatrix} = - \begin{bmatrix} \mathrm{Re}\{\partial f/\partial V_1\} \\ \mathrm{Im}\{\partial f/\partial V_1\} \end{bmatrix},$$

where

$$\sigma = \sigma_1 + j\,\sigma_2 = S_1/V_1^2$$

and f is given by

$$f \overset{\Delta}{=} V_{11} = \frac{1}{2}(V_1 + V_1^*)$$

and

$$f \overset{\Delta}{=} V_{12} = \frac{j}{2}\,(V_1^* - V_1).$$

Table 6.1 shows the results at successive iterations obtained from the Tellegen's theorem-based method described before. The initial value of V_1 is

$$V_1^0 = 1 + j\,0.$$

It can be shown that these results are identical to those obtained by applying the Newton-Raphson method.

6.3 COMPLEX SOLUTION OF POWER FLOW EQUATIONS

A variety of iterative numerical techniques for solving the load flow problem have been described (Stott 1974). The mode of formulation and solution used in these techniques is based on direct implementation of the method used. While Gauss-Seidel and other non derivative-based methods are described in the complex mode, the Newton-Raphson method has been described in the real mode. Using the conjugate notation, the Newton-Raphson method, however, can be interpreted formally in terms of first-order changes of problem variables. In this section, we invoke this interpretation to describe the Newton-Raphson method in the more compact complex mode, and we utilize some theoretical derivations given

TABLE 6.1

LOAD FLOW SOLUTION OF EXAMPLE 5.4 USING TELLEGEN'S THEOREM

Quantity	Iteration			
	1	2	3	4
δP_1	5.0	0.3776	0.0598	0.0032
δQ_1	1.0	1.1328	0.1794	0.0096
dV_{11}/dP_1	-0.0169	-0.0540	-0.0784	-0.0876
dV_{11}/dQ_1	-0.0562	-0.0701	-0.1028	-0.1152
δV_{11}	-0.1404	-0.0998	-0.0231	-0.0014
dV_{12}/dP_1	-0.0449	-0.0438	-0.0431	-0.0428
dV_{12}/dQ_1	0.0169	0.0173	0.0183	0.0186
δV_{12}	-0.2079	0.0030	0.0007	0.0000
V_{11}	0.8596	0.7598	0.7366	0.7352
V_{12}	-0.2079	-0.2048	-0.2041	-0.2041

in Chapter 5 to relate analytical aspects of the resulting form of equations to those of other familiar forms.

This section is concerned mainly with the fundamental formulation and resulting elimination technique. All expected subsequent improvements regarding efficient sparsity programmed ordered elimination (Tinney and Walker 1967), however, can follow.

6.3.1 Problem Formulation

The power network performance equations (Stagg and El-Abiad 1968) are written, using the bus frame of reference, in the admittance form

$$\underset{\sim}{Y}_T \underset{\sim}{V}_M = \underset{\sim}{I}_M, \tag{6.9}$$

where

$$\underset{\sim}{Y}_T = \underset{\sim}{Y}_{T1} + j\, \underset{\sim}{Y}_{T2} \tag{6.10}$$

is the bus admittance matrix of the network,

$$\underset{\sim}{V}_M = \underset{\sim}{V}_{M1} + j\, \underset{\sim}{V}_{M2} \tag{6.11}$$

is a column vector of the bus voltages, and

$$\underset{\sim}{I}_M = \underset{\sim}{I}_{M1} + j\, \underset{\sim}{I}_{M2} \tag{6.12}$$

is a vector of bus currents.

The bus loading equations are also written in the matrix form

$$\underset{\sim}{E}_M^* \underset{\sim}{I}_M = \underset{\sim}{S}_M^*, \tag{6.13}$$

where $\underset{\sim}{E}_M$ is a diagonal matrix of components of $\underset{\sim}{V}_M$ in corresponding order, i.e.,

$$\underset{\sim}{E}_M \underset{\sim}{v} = \underset{\sim}{V}_M, \tag{6.14}$$

$\underset{\sim}{v}$ is defined as

$$\underset{\sim}{v} \triangleq \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \tag{6.15}$$

and $\underset{\sim}{S}_M$ is a vector of the bus powers given by

$$\underset{\sim}{S}_M \triangleq \underset{\sim}{P}_M + j\underset{\sim}{Q}_M. \tag{6.16}$$

Substituting (6.9) into (6.13), we get

$$\underset{\sim}{E}_M^* \underset{\sim}{Y}_T \underset{\sim}{V}_M = \underset{\sim}{S}_M^*. \tag{6.17}$$

The system of nonlinear equations (6.17) represents the typical load flow problem.

We write (6.17) in the perturbed form

$$\underset{\sim}{K}^S \, \delta \underset{\sim}{V}_M + \underset{\sim}{\overline{K}}^S \, \delta \underset{\sim}{V}_M^* = \delta \underset{\sim}{S}_M^*, \tag{6.18}$$

where $\delta \underset{\sim}{V}_M$, $\delta \underset{\sim}{V}_M^*$ and $\delta \underset{\sim}{S}_M^*$ represent first-order changes of $\underset{\sim}{V}_M$, $\underset{\sim}{V}_M^*$ and $\underset{\sim}{S}_M^*$, respectively,

$$\underset{\sim}{K}^S \triangleq \underset{\sim}{E}_M^* \, \underset{\sim}{Y}_T \tag{6.19}$$

and $\underset{\sim}{\overline{K}}^S$ is a diagonal matrix of components of $\underset{\sim}{I}_M$, i.e.,

$$\underset{\sim}{\overline{K}}^S \, \underset{\sim}{v} = \underset{\sim}{I}_M. \tag{6.20}$$

The form (6.18) rigorously represents a set of linear equations to be solved in the Newton-Raphson iterative method. The form (6.18) and related forms will be used while bearing in mind that the equation corresponding to the slack bus may be eliminated.

6.3.2 Newton-Raphson Iteration in Complex Mode

The familiar form of the Newton-Raphson iteration in the real mode is obtained by separating (6.18) into real and imaginary parts and collecting the terms, appropriately, using the perturbed forms of (6.11) and (6.16), to get

$$\begin{bmatrix} (\underset{\sim}{K}_1^S + \underset{\sim}{\overline{K}}_1^S) & (- \underset{\sim}{K}_2^S + \underset{\sim}{\overline{K}}_2^S) \\ - (\underset{\sim}{K}_2^S + \underset{\sim}{\overline{K}}_2^S) & (- \underset{\sim}{K}_1^S + \underset{\sim}{\overline{K}}_1^S) \end{bmatrix} \cdot \begin{bmatrix} \delta \underset{\sim}{V}_{M1} \\ \delta \underset{\sim}{V}_{M2} \end{bmatrix} = \begin{bmatrix} \delta \underset{\sim}{P}_M \\ \delta \underset{\sim}{Q}_M \end{bmatrix}, \tag{6.21}$$

where we have set

$$\underset{\sim}{K}^S = \underset{\sim}{K}_1^S + j \, \underset{\sim}{K}_2^S \tag{6.22}$$

and

$$\bar{K}^S = \bar{K}_1^S + j\,\bar{K}_2^S \,. \tag{6.23}$$

The 2n x 2n matrix of coefficients in (6.21), n denoting the number of buses in the power network, constitutes the Jacobian matrix of the load flow problem.

On the other hand, equation (6.18) can be written in the consistent form

$$\begin{bmatrix} K^S & \bar{K}^S \\ \bar{K}^{S*} & K^{S*} \end{bmatrix} \begin{bmatrix} \delta V_M \\ \delta V_M^* \end{bmatrix} = \begin{bmatrix} \delta S_M^* \\ \delta S_M \end{bmatrix} . \tag{6.24}$$

From Theorems 5.1 and 5.2 it can be shown that the matrix of coefficients of (6.24) has the same rank as that of (6.21) and the system of equations (6.24) is consistent if and only if the system (6.21) is consistent.

Now, the system of complex equations (6.24) is equivalent to the more compact system of complex equations

$$\tilde{K}^S\, \delta V_M = \tilde{d}^S, \tag{6.25}$$

where we have defined

$$\tilde{K}^S = \bar{K}^{S*} - K^{S*}\,(\bar{K}^S)^{-1}\,K^S \tag{6.26}$$

and

$$\tilde{d}^S = \delta S_M - K^{S*}\,(\bar{K}^S)^{-1}\,\delta S_M^* . \tag{6.27}$$

In the jth iteration of the Newton-Raphson method in the complex mode, we solve the system of equations (6.25) with

$$\delta V_M = V_M^{j+1} - V_M^j, \tag{6.28}$$

$$\delta S_M^* = S_{M(\text{scheduled})}^* - K^S\, V_M^j \tag{6.29}$$

and the matrices $\underset{\sim}{K}^S$ and $\underset{\sim}{\overline{K}}^S$ are calculated at $\underset{\sim}{V}_M^j$.

6.3.3 Sparsity Considerations

A trade off between the direct use of forms (6.21) and (6.25) must take into account the sparsity of the matrix of coefficients. While the matrix $\underset{\sim}{K}^S$ of (6.22) has the same sparsity as the bus admittance matrix $\underset{\sim}{Y}_T$, the matrix of coefficients $\underset{\sim}{\tilde{K}}^S$ of (6.25) is as sparse as the matrix

$$\underset{\sim}{Y}_{TT} \triangleq \underset{\sim}{Y}_T \, \underset{\sim}{Y}_T. \tag{6.30}$$

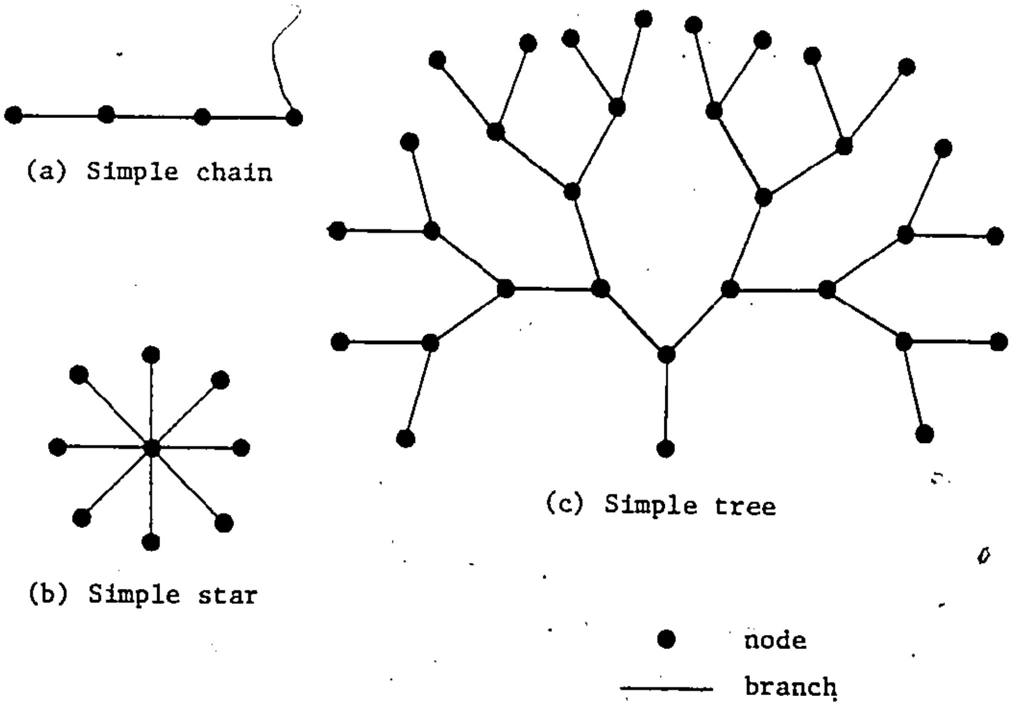
In other words, the advantage of the direct use of the compact nxn complex matrix $\underset{\sim}{\tilde{K}}^S$ rather than the 2nx2n real matrix of coefficients of (6.21) may be restricted by the relative sparsity of the matrices $\underset{\sim}{Y}_T$ and $\underset{\sim}{Y}_{TT}$, the factor which obviously depends on the graph of the network. To illustrate this point, we consider, in Fig. 6.1, three special structures for which the sparsity coefficients (Brameller, Allan and Hamam 1976) of $\underset{\sim}{Y}_T$ and $\underset{\sim}{Y}_{TT}$ are compared.

6.3.4 A Conjugate Elimination Technique

In this section, we present an alternative approach to the problem. Instead of applying the ordinary elimination methods to the more dense matrix of coefficients $\underset{\sim}{\tilde{K}}^S$ of (6.25), we use a special technique in order to handle, directly, the original form (6.18).

In order to facilitate the derivations, we introduce the following notation. First, we define the term

$$\{k_{ij}, \overline{k}_{ij}\} \, x_j \triangleq k_{ij} \, x_j + \overline{k}_{ij} \, x_j^*, \tag{6.31}$$



Network	Sparsity Coefficient $\underset{\sim}{Y}_T$	Sparsity Coefficient $\underset{\sim}{Y}_{TT}$
(a)	$1 - (3n - 2)/n^2$	$1 - (5n - 6)/n^2$
(b)	$1 - (3n - 2)/n^2$	0
(c)	$1 - (3n - 2)/n^2$	$1 - (6n - 8)/n^2$

n = number of nodes = order of $\underset{\sim}{Y}_T$ or $\underset{\sim}{Y}_{TT}$

Fig. 6.1 Sparsity coefficients of $\underset{\sim}{Y}_T$ and $\underset{\sim}{Y}_{TT}$ for simple networks

where k_{ij} and $\overline{k}_{ij}$ stand, for example, for general elements of the matrices $\underset{\sim}{K}^S$ and $\overline{\underset{\sim}{K}}^S$. We call the set of elements a of {a,b} the basic set and the set of elements b the conjugate set. Then we state the following basic rules which can be easily verified.

Rule 1

$$\{k_{ij}, \overline{k}_{ij}\}\ x_j = \{\overline{k}_{ij}, k_{ij}\}\ x_j^*. \tag{6.32}$$

Rule 2

$$(\{k_{ij}, \overline{k}_{ij}\}\ x_j)^* = \{k_{ij}^*, \overline{k}_{ij}^*\}\ x_j^* = \{\overline{k}_{ij}^*, k_{ij}^*\}\ x_j. \tag{6.33}$$

Rule 3

$$\mu\ \{k_{ij}, \overline{k}_{ij}\}\ x_j = \{\mu\ k_{ij}, \mu\ \overline{k}_{ij}\}\ x_j = \{k_{ij}, \overline{k}_{ij}\}\ (\mu\ x_j), \tag{6.34}$$

where μ is a complex scalar.

Rule 4

$$\{k_{ij},\overline{k}_{ij}\}\ x_j + \mu\ \{k_{\ell j},\overline{k}_{\ell j}\}\ x_j = \{(k_{ij}+\mu\ k_{\ell j}), (\overline{k}_{ij}+\mu\ \overline{k}_{\ell j})\}\ x_j. \tag{6.35}$$

The above notation may be exploited in developing suitable methods for solving systems of the form (6.18). Here, we invoke this notation to describe a technique which allows the forward Gaussian elimination process to be directly applied to the form (6.18).

The system of equations (6.18) is written, using (6.31), as

$$\sum_{j=1}^{n} \{k_{ij}, \overline{k}_{ij}\}\ x_j = b_i,\ i = 1, \ldots, n, \tag{6.36}$$

where x_j and b_i are elements of $\underset{\sim}{x} = \delta \underset{\sim}{V}_M$ and $\underset{\sim}{b} = \delta \underset{\sim}{S}_M^*$, respectively. Since

$$\overline{k}_{ij} = 0,\ \text{for } i \neq j, \tag{6.37}$$

equations (6.36) can be written as

$$\{k_{ii}, \overline{k}_{ii}\}\ x_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} \{k_{ij}, 0\}\ x_j = b_i,\ i = 1, \ldots, n. \tag{6.38}$$

We assume that the order of elimination has been taken into account by applying suitable permutations to (6.38).

At the first iteration, we write the first equation (6.38) as

$$\{k_{11}, \bar{k}_{11}\}\ x_1 + \sum_{j=2}^{n} \{k_{1j}, 0\}\ x_j = b_1 \tag{6.39}$$

or, using (6.33),

$$\{\bar{k}_{11}^*, k_{11}^*\}\ x_1 + \sum_{j=2}^{n} \{0, k_{1j}^*\}\ x_j = b_1^*. \tag{6.40}$$

Multiplying (6.40) by $\mu_1^{(1)*}$, where

$$\mu_i^{(1)} \triangleq - k_{1i}^*/\bar{k}_{ii} \tag{6.41}$$

and adding to (6.39), we get, using (6.35),

$$\{0, \bar{k}_{11}+\mu_1^{(1)*} k_{11}^*\}\ x_1 + \sum_{j=2}^{n} \{k_{1j}, \mu_1^{(1)*} k_{1j}^*\}\ x_j = b_1+\mu_1^{(1)*} b_1^* \tag{6.42}$$

or, using (3.33),

$$\{\bar{k}_{11}^* + \mu_1^{(1)} k_{11}, 0\}\ x_1 + \sum_{j=2}^{n} \{\mu_1^{(1)} k_{1j}, k_{1j}^*\}\ x_j = b_1^* + \mu_1^{(1)} b_1. \tag{6.43}$$

Multiplying equation i of (6.38), i = 2, ..., n, by $\mu_i^{(1)}$ of (6.41) and adding to (6.43), we get

$$\sum_{j=1}^{n} \{k_{1j}^{(1)}, 0\}\ x_j = b_1^{(1)}, \tag{6.44}$$

where

$$k_{11}^{(1)} = \bar{k}_{11}^* + \sum_{i=1}^{n} \mu_i^{(1)} k_{i1}, \tag{6.45}$$

$$k_{1j}^{(1)} = \sum_{i=1}^{n} \mu_i^{(1)} k_{ij},\ j = 2, \ldots, n \tag{6.46}$$

and

$$b_1^{(1)} = b_1^* + \sum_{i=1}^{n} \mu_i^{(1)} b_i. \tag{6.47}$$

Equation (6.44) together with equations 2, 3, ..., n of (6.38) represent a set of equations ready for applying the first iteration of a forward Gaussian elimination to the matrix $\underset{\sim}{K}^{S(1)}$ which is obtained by replacing the elements of the first row of $\underset{\sim}{K}^{S}$ by the elements of (6.45) and (6.46). Observe that we have evacuated the conjugate set of the first equation.

In general, at the mth iteration and with $k_{ij}^{(m-1)}$ and $b_i^{(m-1)}$ denoting the current elements of $\underset{\sim}{K}^{S}$ and $\underset{\sim}{b}$, respectively, we replace the elements of the mth row of $\underset{\sim}{K}^{S}$ by the elements

$$k_{mm}^{(m)} = \bar{k}_{mm}^{*} + \sum_{i=m}^{n} \mu_i^{(m)} k_{im}^{(m-1)} \tag{6.48}$$

and

$$k_{mj}^{(m)} = \sum_{i=m}^{n} \mu_i^{(m)} k_{ij}^{(m-1)}, \; j = m+1, \ldots, n \tag{6.49}$$

and we replace $b_m^{(m-1)}$ by

$$b_m^{(m)} = b_m^{(m-1)*} + \sum_{i=m}^{n} \mu_i^{(m)} b_i^{(m-1)}, \tag{6.50}$$

where

$$\mu_i^{(m)} \triangleq - k_{mi}^{(m-1)*}/\bar{k}_{ii}. \tag{6.51}$$

We shall call the special elimination process described by (6.48) -(6.50) the conjugate elimination in which the coefficients of the conjugate variables are successively eliminated.

6.3.5 Example 6.2

A tableau representation of the combined elimination process is shown in Table 6.2 for n=3, and corresponding numerical results are

TABLE 6.2

THE COMBINED ELIMINATION TECHNIQUE

Iteration No.	Basic Tableau			Conjugate Tableau			b	Type of Elimination
	k_{11}	k_{12}	k_{13}	$\bar{k}_{11}$	0	0	b_1	Original
0	k_{21}	k_{22}	k_{23}	0	$\bar{k}_{22}$	0	b_2	tableau
	k_{31}	k_{32}	k_{33}	0	0	$\bar{k}_{33}$	b_3	
	$k_{11}^{(1)}$	$k_{12}^{(1)}$	$k_{13}^{(1)}$	0	0	0	$b_1^{(1)}$	Conjugate
1a	k_{21}	k_{22}	k_{23}	0	$\bar{k}_{22}$	0	b_2	elimination
	k_{31}	k_{32}	k_{33}	0	0	$\bar{k}_{33}$	b_3	
	$k_{11}^{(1)}$	$k_{12}^{(1)}$	$k_{13}^{(1)}$	0	0	0	$b_1^{(1)}$	Gaussian
1b	0	$k_{22}^{(1)}$	$k_{23}^{(1)}$	0	$\bar{k}_{22}$	0	$b_2^{(1)}$	forward
	0	$k_{32}^{(1)}$	$k_{33}^{(1)}$	0	0	$\bar{k}_{33}$	$b_3^{(1)}$	elimination
	$k_{11}^{(1)}$	$k_{12}^{(1)}$	$k_{13}^{(1)}$	0	0	0	$b_1^{(1)}$	Conjugate
2a	0	$k_{22}^{(2)}$	$k_{23}^{(2)}$	0	0	0	$b_2^{(2)}$	elimination
	0	$k_{32}^{(1)}$	$k_{33}^{(1)}$	0	0	$\bar{k}_{33}$	$b_3^{(1)}$	
	$k_{11}^{(1)}$	$k_{12}^{(1)}$	$k_{13}^{(1)}$	0	0	0	$b_1^{(1)}$	Gaussian
2b	0	$k_{22}^{(2)}$	$k_{23}^{(2)}$	0	0	0	$b_2^{(2)}$	forward
	0	0	$k_{33}^{(2)}$	0	0	$\bar{k}_{33}$	$b_3^{(2)}$	elimination
	$k_{11}^{(1)}$	$k_{12}^{(1)}$	$k_{13}^{(1)}$	0	0	0	$b_1^{(1)}$	Conjugate
3a	0	$k_{22}^{(2)}$	$k_{23}^{(2)}$	0	0	0	$b_2^{(2)}$	elimination
	0	0	$k_{33}^{(3)}$	0	0	0	$b_3^{(3)}$	

shown in Table 6.3 where the solution of the arbitrary system of equations

$$x_1 - jx_2 + 2x_3 + 2x_1^* = 5$$
$$jx_1 - x_2 + jx_3 - x_2^* = j$$
$$2x_1 + jx_2 - x_3 + x_3^* = 0$$

TABLE 6.3

ELIMINATION TABLEAU OF EXAMPLE 6.2

Iteration No.	Basic Tableau			Elements of Conjugate Tableau	$\tilde{b}$
	1	-j	2	2	5
0	j	-1	j	-1	j
	2	j	-1	1	0
	7	j5	0	0	-3
1	0	-2/7	j	-1	j10/7
	0	-j3/7	-1	1	6/7
	7	j5	0	0	-3
2	0	12/7	j9/2	0	j24/7
	0	0	-17/8	1	0
	7	j5	0	0	-3
3	0	12/7	j9/2	0	j24/7
	0	0	-225/136	0	0

is investigated. The backward substitution results in the solution

$$\underset{\sim}{x} = \begin{bmatrix} 1 \\ j2 \\ 0 \end{bmatrix}.$$

6.3.6 Complex Formulation of Practical Systems

In the system of equations of form (6.18), it is assumed that all buses other than the slack bus are of the same type, namely a load bus type for which the active and reactive powers are known. In the following, we present a special technique of formulation which allows the generator-type buses, as well, to be represented while preserving the complex mode of (6.18).

Consider the equation of (6.18) corresponding to a generator bus g. We define the complex quantity

$$\tilde{S}_g^0 \triangleq P_g + j\,|V_g|\,. \tag{6.52}$$

hence

$$\delta\tilde{S}_g^0 = \delta P_g + j\,\delta|V_g|\,. \tag{6.53}$$

Since

$$2P_g = V_g\,I_g^* + V_g^*\,I_g, \tag{6.54}$$

then

$$2\delta P_g = V_g\,\delta I_g^* + I_g^*\,\delta V_g + V_g^*\,\delta I_g + I_g\,\delta V_g^*. \tag{6.55}$$

Using (6.9), we write I_g in the form

$$I_g = \underset{\sim}{y}_g^T\,\underset{\sim}{V}_M, \tag{6.56}$$

where $\underset{\sim}{y}_g^T$ represents the corresponding row of the bus admittance matrix $\underset{\sim}{Y}_T$, hence

$$\delta I_g = \underset{\sim}{y}_g^T\,\delta\underset{\sim}{V}_M. \tag{6.57}$$

Also,

$$\delta|V_g| = \delta(V_g V_g^*)^{1/2} = (V_g\, \delta V_g^* + V_g^*\, \delta V_g)/(2|V_g|). \qquad (6.58)$$

Using (6.55)-(6.58) it is straightforward to show that $\delta \tilde{S}_g^0$ of (6.53) is given by

$$\delta \tilde{S}_g^0 = \underset{\sim}{k}_g^{0T}\, \delta \underset{\sim}{V}_M + \overline{\underset{\sim}{k}}_g^{0T}\, \delta \underset{\sim}{V}_M^*, \qquad (6.59)$$

where $\underset{\sim}{k}_g^{0T}$ which replaces the row of $\underset{\sim}{K}^S$ of (6.18) corresponding to the generator bus g has elements defined as

$$k_{gj}^0 \triangleq V_g^*\, Y_{gj}/2, \; j \neq g \qquad (6.60)$$

and

$$k_{gg}^0 \triangleq j\, V_g^*/(2|V_g|) + (V_g^*\, Y_{gg} + I_g^*)/2, \qquad (6.61)$$

Y_{ij} denoting elements of $\underset{\sim}{Y}_T$, and $\overline{\underset{\sim}{k}}_g^{0T}$ which replaces the row of $\overline{\underset{\sim}{K}}^S$ of (6.18) corresponding to the generator bus g has elements defined as

$$\overline{k}_{gj}^0 \triangleq V_g\, Y_{gj}^*/2, \; j \neq g \qquad (6.62)$$

and

$$\overline{k}_{gg}^0 \triangleq j\, V_g/(2|V_g|) + (V_g\, Y_{gg}^* + I_g)/2. \qquad (6.63)$$

The above formulation results in an equation of (6.38) for i = g of the form

$$\{k_{gg}^0, \overline{k}_{gg}^0\}\; x_g + \sum_{\substack{j=1 \\ j\neq g}}^{n} \{k_{gj}^0, \overline{k}_{gj}^0\}\; x_j = b_g^0, \qquad (6.64)$$

where b_g^0 stands for $\delta \tilde{S}_g^0$.

In order to prepare the original conjugate tableau of (6.18) to be suitable for applying the technique described in the previous subsection, we multiply equation i, i≠g, of (6.38) by the factor

$$\mu_i^0 \triangleq -\overline{k}_{gi}^0/\overline{k}_{ii} \qquad (6.65)$$

and sum all together to (6.64) to obtain, putting $\bar{k}_{gg} = \bar{k}^0_{gg}$,

$$\{k_{gg}, \bar{k}_{gg}\}\, x_g + \sum_{\substack{j=1 \\ j\neq g}}^{n} \{k_{gj}, 0\}\, x_j = b_g, \tag{6.66}$$

where

$$k_{gj} \triangleq k^0_{gj} + \sum_{\substack{i=1 \\ i\neq g}}^{n} \mu^0_i\, k_{ij}, \; j = 1, 2, .., n \tag{6.67}$$

and

$$b_g \triangleq b^0_g + \sum_{\substack{i=1 \\ i\neq g}}^{n} \mu^0_i\, b_i. \tag{6.68}$$

Equation (6.66), hence, represents the gth equation of (6.38).

Note that $|V_g|$ of (6.52) could be replaced, for example, by $|V_g|^2$. Moreover, one could equally well replace (Van Ness and Griffin 1961) the elements of $\delta \underset{\sim}{V}_M$ and $\delta \underset{\sim}{V}^*_M$, namely, δV_i and δV^*_i, $i = 1, \ldots, n$ by the relative quantities $\delta V_i/|V_i|$ and $\delta V^*_i/|V_i|$, respectively. In this case, the elements k_{ij} and $\bar{k}_{ij}$ of the ith row of the coefficient matrices are replaced by $|V_j|k_{ij}$ and $|V_j|\bar{k}_{ij}$, respectively.

6.4 METHOD OF COMPLEX LAGRANGE MULTIPLIERS WITH APPLICATIONS

In the real case, the Lagrangian approach has been successfully applied to power system analysis and design problems (Director and Sullivan 1978) where Lagrange multipliers obtained by solving a set of adjoint equations are used to relate first-order changes of a real function and control variables.

In some cases, the set of equality constraints is described basically in the compact complex form. Moreover, first-order change of a complex function may be required. The application of the Lagrangian approach requires separation of real and imaginary parts of the equality constraints as well as the function of interest which may alter the ease and compactness of formulation.

The study presented in this section exploits the conjugate notation to describe and formulate the Lagrangian approach in the complex form so that complex functions and constraints may be directly handled. The complex formulation of the Lagrangian approach and some important applications to power system sensitivity analysis are presented successively in this section.

6.4.1 The Complex Lagrangian Concept

We consider, as before, a complex function f of a set of complex variables $\underset{\sim}{\zeta}$ and their complex conjugate $\underset{\sim}{\zeta}^*$. We write

$$\underset{\sim}{\zeta} = \begin{bmatrix} \underset{\sim}{\zeta}_x \\ \underset{\sim}{\zeta}_u \end{bmatrix} , \tag{6.69}$$

where the variables $\underset{\sim}{\zeta}$ have been classified into n_x state variables $\underset{\sim}{\zeta}_x$ and n_u control variables $\underset{\sim}{\zeta}_u$. The state and control variables are related through the set of n_x complex equality constraints

$$\underset{\sim}{h}\,(\underset{\sim}{\zeta},\underset{\sim}{\zeta}^*) = \underset{\sim}{0} . \tag{6.70}$$

The first-order change of f is written, using (3.14), in the form

$$\delta f = [\ \underset{\sim}{f}^T_{\zeta x}\ \overline{\underset{\sim}{f}}^T_{\zeta x}\ ] \begin{bmatrix} \delta\underset{\sim}{\zeta}_x \\ \\ \delta\underset{\sim}{\zeta}^*_x \end{bmatrix} + [\ \underset{\sim}{f}^T_{\zeta u}\ \overline{\underset{\sim}{f}}^T_{\zeta u}\ ] \begin{bmatrix} \delta\underset{\sim}{\zeta}_u \\ \\ \delta\underset{\sim}{\zeta}^*_u \end{bmatrix}, \qquad (6.71)$$

where $\underset{\sim}{f}_{\zeta x}$, $\overline{\underset{\sim}{f}}_{\zeta x}$, $\underset{\sim}{f}_{\zeta u}$ and $\overline{\underset{\sim}{f}}_{\zeta u}$ stand for $(\partial f/\partial\underset{\sim}{\zeta}_x)$, $(\partial f/\partial\underset{\sim}{\zeta}^*_x)$, $(\partial f/\partial\underset{\sim}{\zeta}_u)$ and $(\partial f/\partial\underset{\sim}{\zeta}^*_u)$, respectively.

We write (6.70) in the perturbed form

$$\delta\underset{\sim}{h}\ (\underset{\sim}{\zeta},\underset{\sim}{\zeta}^*) = \underset{\sim}{0} \qquad (6.72)$$

or

$$[\underset{\sim}{H}_{\zeta x}\ \overline{\underset{\sim}{H}}_{\zeta x}] \begin{bmatrix} \delta\underset{\sim}{\zeta}_x \\ \\ \delta\underset{\sim}{\zeta}^*_x \end{bmatrix} + [\underset{\sim}{H}_{\zeta u}\ \overline{\underset{\sim}{H}}_{\zeta u}] \begin{bmatrix} \delta\underset{\sim}{\zeta}_x \\ \\ \delta\underset{\sim}{\zeta}^*_u \end{bmatrix} = \underset{\sim}{0}, \qquad (6.73)$$

where $\underset{\sim}{H}_{\zeta x}$, $\overline{\underset{\sim}{H}}_{\zeta x}$, $\underset{\sim}{H}_{\zeta u}$ and $\overline{\underset{\sim}{H}}_{\zeta u}$ stand for $(\partial\underset{\sim}{h}^T/\partial\underset{\sim}{\zeta}_x)^T$, $(\partial\underset{\sim}{h}^T/\partial\underset{\sim}{\zeta}^*_x)^T$, $(\partial\underset{\sim}{h}^T/\partial\underset{\sim}{\zeta}_u)^T$ and $(\partial\underset{\sim}{h}^T/\partial\underset{\sim}{\zeta}^*_u)^T$, respectively.

Using the complex conjugate of (6.73), we may write

$$\begin{bmatrix} \delta\underset{\sim}{\zeta}_x \\ \\ \delta\underset{\sim}{\zeta}^*_x \end{bmatrix} = - \begin{bmatrix} \underset{\sim}{H}_{\zeta x} & \overline{\underset{\sim}{H}}_{\zeta x} \\ \\ \overline{\underset{\sim}{H}}^*_{\zeta x} & \underset{\sim}{H}^*_{\zeta x} \end{bmatrix}^{-1} \begin{bmatrix} \underset{\sim}{H}_{\zeta u} & \overline{\underset{\sim}{H}}_{\zeta u} \\ \\ \overline{\underset{\sim}{H}}^*_{\zeta u} & \underset{\sim}{H}^*_{\zeta u} \end{bmatrix} \begin{bmatrix} \delta\underset{\sim}{\zeta}_u \\ \\ \delta\underset{\sim}{\zeta}^*_u \end{bmatrix} . \qquad (6.74)$$

Using the theorems of Chapter 5, it can be shown that the inverted matrix in (6.74) has full rank if and only if the system of equations (6.72) represents $2n_x$ independent conditions.

Using (6.74), δf of (6.71) is written in the form

$$\delta f = \left\{ [\ \underset{\sim}{f}^T_{\zeta u}\ \overline{\underset{\sim}{f}}^T_{\zeta u}\ ] - [\ \underset{\sim}{\lambda}^T\ \overline{\underset{\sim}{\lambda}}^T\ ] \begin{bmatrix} \underset{\sim}{H}_{\zeta u} & \overline{\underset{\sim}{H}}_{\zeta u} \\ \overline{\underset{\sim}{H}}^*_{\zeta u} & \underset{\sim}{H}^*_{\zeta u} \end{bmatrix} \right\} \begin{bmatrix} \delta \underset{\sim}{\zeta}_u \\ \delta \underset{\sim}{\zeta}^*_u \end{bmatrix} , \quad (6.75)$$

where

$$\begin{bmatrix} \underset{\sim}{H}^T_{\zeta x} & \overline{\underset{\sim}{H}}^{*T}_{\zeta x} \\ \overline{\underset{\sim}{H}}^T_{\zeta x} & \underset{\sim}{H}^{*T}_{\zeta x} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\lambda} \\ \overline{\underset{\sim}{\lambda}} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{f}_{\zeta x} \\ \overline{\underset{\sim}{f}}_{\zeta x} \end{bmatrix} . \quad (6.76)$$

Hence, the total formal derivatives of f are given, from (6.75), by

$$\frac{df}{d\underset{\sim}{\zeta}_u} = \underset{\sim}{f}_{\zeta u} - \underset{\sim}{H}^T_{\zeta u}\,\underset{\sim}{\lambda} - \overline{\underset{\sim}{H}}^{*T}_{\zeta u}\,\overline{\underset{\sim}{\lambda}} \quad (6.77)$$

and

$$\frac{df}{d\underset{\sim}{\zeta}^*_u} = \overline{\underset{\sim}{f}}_{\zeta u} - \overline{\underset{\sim}{H}}^T_{\zeta u}\,\underset{\sim}{\lambda} - \underset{\sim}{H}^{*T}_{\zeta u}\,\overline{\underset{\sim}{\lambda}} . \quad (6.78)$$

The complex Lagrange multipliers $\underset{\sim}{\lambda}$ and $\overline{\underset{\sim}{\lambda}}$ of (6.77) and (6.78) are obtained by solving the set of complex adjoint equations (6.76).

Note that in the case when the function f and constraints $\underset{\sim}{h}$ are all pure real, the application of (3.19) results in the complex conjugate relationships $\overline{\underset{\sim}{f}}_{\zeta x} = \underset{\sim}{f}^*_{\zeta x}$ and $\overline{\underset{\sim}{H}}_{\zeta x} = \underset{\sim}{H}^*_{\zeta x}$ and (6.76) reduces to a system of n_x complex equations in the real variables $(\underset{\sim}{\lambda} + \overline{\underset{\sim}{\lambda}})$. The solution of this system of equations is then substituted into (6.77) and (6.78) which form a complex conjugate pair since, from (3.19), $\overline{\underset{\sim}{f}}_{\zeta u} = \underset{\sim}{f}^*_{\zeta u}$ and $\overline{\underset{\sim}{H}}_{\zeta u} = \underset{\sim}{H}^*_{\zeta u}$.

We have stated the Lagrangian approach in the complex form and derived the corresponding adjoint equations to be solved for the

Lagrange multipliers so that the required formal derivatives (6.77) and (6.78) may be obtained. In the following two subsections, we consider some applications of the complex Lagrangian approach in power system analysis and design.

6.4.2 Application to Power Network Analysis

The complex Lagrangian approach described in the previous section can be applied, for example, to power network sensitivity calculations. The set of complex equality constraints (6.70) may represent the power flow equations of the form (6.17) or

$$\mathbf{h} = \mathbf{S}_M^* - \mathbf{E}_M^* \, \mathbf{Y}_T \, \mathbf{V}_M = \mathbf{0}. \tag{6.79}$$

The vectors $\boldsymbol{\zeta}_x$ and $\boldsymbol{\zeta}_u$ of (6.69) are defined as

$$\boldsymbol{\zeta}_x \triangleq \begin{bmatrix} \mathbf{V}_L \\ S_n \end{bmatrix} \tag{6.80}$$

and

$$\boldsymbol{\zeta}_u \triangleq \begin{bmatrix} \mathbf{S}_L \\ V_n \end{bmatrix}, \tag{6.81}$$

where we have classified, for simplicity, the buses as load-type buses of voltages $\mathbf{V}_L$ and powers $\mathbf{S}_L$ and a slack bus of voltage V_n and power S_n. We write (6.79) in the corresponding partitioned form

$$\begin{bmatrix} \mathbf{h}_L \\ h_n \end{bmatrix} = \begin{bmatrix} \mathbf{S}_L^* \\ S_n^* \end{bmatrix} - \begin{bmatrix} \mathbf{E}_L^* & \mathbf{0} \\ \mathbf{0} & V_n^* \end{bmatrix} \begin{bmatrix} \mathbf{Y}_{LL} & \mathbf{Y}_{LN} \\ \mathbf{Y}_{LN}^T & Y_{nn} \end{bmatrix} \begin{bmatrix} \mathbf{V}_L \\ V_n \end{bmatrix}, \tag{6.82}$$

where the symmetric bus admittance matrix has been partitioned into $\mathbf{Y}_{LL}$, $\mathbf{Y}_{LN}$, $\mathbf{Y}_{LN}^T$ and Y_{nn} of appropriate dimensions.

The matrices $(\partial \underset{\sim}{h}^T/\partial \underset{\sim}{\zeta}_x)$, $(\partial \underset{\sim}{h}^T/\partial \underset{\sim}{\zeta}_x^*)$, $(\partial \underset{\sim}{h}^T/\partial \underset{\sim}{\zeta}_u)$ and $(\partial \underset{\sim}{h}^T/\partial \underset{\sim}{\zeta}_u^*)$ are given, respectively, by

$$\frac{\partial \underset{\sim}{h}^T}{\partial \underset{\sim}{\zeta}_x} = \begin{bmatrix} \dfrac{\partial \underset{\sim}{h}_L^T}{\partial \underset{\sim}{V}_L} & \dfrac{\partial h_n}{\partial \underset{\sim}{V}_L} \\ \dfrac{\partial \underset{\sim}{h}_L^T}{\partial S_n} & \dfrac{\partial h_n}{\partial S_n} \end{bmatrix}, \tag{6.83}$$

$$\frac{\partial \underset{\sim}{h}^T}{\partial \underset{\sim}{\zeta}_x^*} = \begin{bmatrix} \dfrac{\partial \underset{\sim}{h}_L^T}{\partial \underset{\sim}{V}_L^*} & \dfrac{\partial h_n}{\partial \underset{\sim}{V}_L^*} \\ \dfrac{\partial \underset{\sim}{h}_L^T}{\partial S_n^*} & \dfrac{\partial h_n}{\partial S_n^*} \end{bmatrix}, \tag{6.84}$$

$$\frac{\partial \underset{\sim}{h}^T}{\partial \underset{\sim}{\zeta}_u} = \begin{bmatrix} \dfrac{\partial \underset{\sim}{h}_L^T}{\partial \underset{\sim}{S}_L} & \dfrac{\partial h_n}{\partial \underset{\sim}{S}_L} \\ \dfrac{\partial \underset{\sim}{h}_L^T}{\partial V_n} & \dfrac{\partial h_n}{\partial V_n} \end{bmatrix} \tag{6.85}$$

and

$$\frac{\partial \underset{\sim}{h}^T}{\partial \underset{\sim}{\zeta}_u^*} = \begin{bmatrix} \dfrac{\partial \underset{\sim}{h}_L^T}{\partial \underset{\sim}{S}_L^*} & \dfrac{\partial h_n}{\partial \underset{\sim}{S}_L^*} \\ \dfrac{\partial \underset{\sim}{h}_L^T}{\partial V_n^*} & \dfrac{\partial h_n}{\partial V_n^*} \end{bmatrix}, \tag{6.86}$$

Using (6.82)-(6.86) the matrices $\underset{\sim}{H}_{\zeta x}$, $\overline{\underset{\sim}{H}}_{\zeta x}$, $\underset{\sim}{H}_{\zeta u}$ and $\overline{\underset{\sim}{H}}_{\zeta u}$ of (6.73) are given, respectively, by

$$\underset{\sim}{H}_{\zeta x} = \begin{bmatrix} -(\underset{\sim}{E}_L^* \underset{\sim}{Y}_{LL}) & \underset{\sim}{0} \\ -(V_n^* \underset{\sim}{Y}_{LN}^T) & 0 \end{bmatrix}, \tag{6.87}$$

$$\overline{\underset{\sim}{H}}_{\zeta x} = \begin{bmatrix} -\operatorname{diag}\{\underset{\sim}{I}_L\} & \underset{\sim}{0} \\ \underset{\sim}{0} & 1 \end{bmatrix}, \tag{6.88}$$

$$\underset{\sim}{H}_{\zeta u} = \begin{bmatrix} \underset{\sim}{0} & -(\underset{\sim}{E}_L^* \underset{\sim}{Y}_{LN}) \\ \underset{\sim}{0} & -V_n^* Y_{nn} \end{bmatrix} \tag{6.89}$$

and

$$\overline{\underset{\sim}{H}}_{\zeta u} = \begin{bmatrix} \underset{\sim}{1} & \underset{\sim}{0} \\ \underset{\sim}{0} & -I_n \end{bmatrix}, \tag{6.90}$$

where the bus currents

$$\underset{\sim}{I}_M = \begin{bmatrix} \underset{\sim}{I}_L \\ I_n \end{bmatrix} \tag{6.91}$$

are given by (6.9).

Hence, for a given function f with the formal derivatives $\underset{\sim}{f}_{\zeta x}$ and $\overline{\underset{\sim}{f}}_{\zeta x}$, the adjoint system of equations (6.76) is formed using (6.87) and (6.88) and solved for the Lagrange multipliers $\underset{\sim}{\lambda}$ and $\overline{\underset{\sim}{\lambda}}$. The total formal derivatives of f w.r.t. the control variables are then calculated from (6.77) and (6.78) using (6.89) and (6.90).

We remark that the choice of $\underset{\sim}{V}_L$ and S_n as the only control variables $\underset{\sim}{\zeta}_u$ has been made for simplicity. We could equally well define other control variables, e.g., line admittances. Note also that the voltage-controlled buses or generator-type buses can be included, as before, by defining complex state variables, e.g.,

$$\zeta_x^g \triangleq Q_g + j\,\delta_g \tag{6.92}$$

and control variables, e.g.,

$$\zeta_u^g \triangleq P_g + j\,|V_g| . \tag{6.93}$$

The modification required to include these control and state variables can be performed in a straightforward manner.

6.4.3 The Element-Local Lagrangian Technique

In this section, we consider an important application of the complex Lagrangian concept stated before. This application is concerned with the Tellegen's theorem-based approach presented in Chapter 3 for sensitivity calculations in electrical networks. By this approach, only two state variables $\underset{\sim}{x}_b$ and two control variables $\underset{\sim}{u}_b$ are defined for each element. The function f must be expressed solely in terms of the $\underset{\sim}{x}_b$ and $\underset{\sim}{u}_b$. In some cases, however, the function f may be expressed basically in terms of the $\underset{\sim}{x}_b$ and $\underset{\sim}{u}_b$ as well as other dependent variables $\underset{\sim}{\rho}_b$ which, by themselves, are functions of $\underset{\sim}{x}_b$ and $\underset{\sim}{u}_b$. The variables $\underset{\sim}{\rho}_b$ may be related to $\underset{\sim}{x}_b$ and $\underset{\sim}{u}_b$ through a set of complicated equality constraints so that the direct expression of $\underset{\sim}{\rho}_b$ in terms of $\underset{\sim}{x}_b$ and $\underset{\sim}{u}_b$ may be difficult or impossible.

In what follows, we show how the complex Lagrangian concept stated before can be applied to handle any number of the complex dependent variables $\underset{\sim}{\rho}_b$ in terms of which the function f may be expressed.

We assume that the $n_{\rho b}$ variables $\underset{\sim}{\rho}_b$ associated with element b are related to the element variables $\underset{\sim}{z}_b$ by the set of $n_{\rho b}$ equality constraints

$$\underset{\sim}{h}_b\,(\underset{\sim}{x}_b,\ \underset{\sim}{u}_b,\ \underset{\sim}{\rho}_b) = \underset{\sim}{0} \tag{6.94}$$

and we denote by δf_b the change in f due to changes in $\underset{\sim}{x}_b$, $\underset{\sim}{u}_b$ and $\underset{\sim}{\rho}_b$, hence

$$\delta f = \sum_b \delta f_b. \tag{6.95}$$

Now, we apply the element-local Lagrangian concept as follows. We write δf_b as

$$\delta f_b = \underset{\sim}{f}_{xb}^T\, \delta \underset{\sim}{x}_b + \underset{\sim}{f}_{ub}^T\, \delta \underset{\sim}{u}_b + \underset{\sim}{f}_{\rho b}^T\, \delta \underset{\sim}{\rho}_b, \tag{6.96}$$

where $\underset{\sim}{f}_{xb}$, $\underset{\sim}{f}_{ub}$ and $\underset{\sim}{f}_{\rho b}$ denote $\partial f/\partial \underset{\sim}{x}_b$, $\partial f/\partial \underset{\sim}{u}_b$ and $\partial f/\partial \underset{\sim}{\rho}_b$, respectively. Also, we write $\delta \underset{\sim}{h}_b$ as

$$\delta \underset{\sim}{h}_b = \underset{\sim}{H}_{xb}\, \delta \underset{\sim}{x}_b + \underset{\sim}{H}_{ub}\, \delta \underset{\sim}{u}_b + \underset{\sim}{H}_{\rho b}\, \delta \underset{\sim}{\rho}_b = \underset{\sim}{0}, \tag{6.97}$$

where $\underset{\sim}{H}_{xb}$, $\underset{\sim}{H}_{ub}$ and $\underset{\sim}{H}_{\rho b}$ stand for $(\partial \underset{\sim}{h}_b^T/\partial \underset{\sim}{x}_b)^T$, $(\partial \underset{\sim}{h}_b^T/\partial \underset{\sim}{u}_b)^T$ and $(\partial \underset{\sim}{h}_b^T/\partial \underset{\sim}{\rho}_b)^T$, respectively. Hence

$$\delta \underset{\sim}{\rho}_b = -\, \underset{\sim}{H}_{\rho b}^{-1}\, (\underset{\sim}{H}_{xb}\, \delta \underset{\sim}{x}_b + \underset{\sim}{H}_{ub}\, \delta \underset{\sim}{u}_b), \tag{6.98}$$

where $\underset{\sim}{H}_{\rho b}$ is a full rank matrix.

Substituting (6.98) into (6.96), we get

$$\delta f_b = (\underset{\sim}{f}_{xb}^T - \underset{\sim}{\lambda}_{\rho b}^T\, \underset{\sim}{H}_{xb})\, \delta \underset{\sim}{x}_b + (\underset{\sim}{f}_{ub}^T - \underset{\sim}{\lambda}_{\rho b}^T\, \underset{\sim}{H}_{ub})\, \delta \underset{\sim}{u}_b, \tag{6.99}$$

where the element-local Lagrange multipliers $\underset{\sim}{\lambda}_{\rho b}$ are obtained by solving

$$\underset{\sim}{H}_{\rho b}^T\, \underset{\sim}{\lambda}_{\rho b} = \underset{\sim}{f}_{\rho b}. \tag{6.100}$$

Equation (6.99) instead of (3.54) expresses δf_b. We therefore define the adjoint network by setting

$$\hat{\underset{\sim}{\eta}}_{bx} = \underset{\sim}{f}_{xb} - \underset{\sim}{H}^T_{xb}\, \underset{\sim}{\lambda}_{\rho b}, \qquad (6.101)$$

hence, from (3.52), (6.95) and (6.99)

$$\delta f = \sum_b (\underset{\sim}{f}_{ub} - \hat{\underset{\sim}{\eta}}_{bu} - \underset{\sim}{H}^T_{ub}\, \underset{\sim}{\lambda}_{\rho b})^T \,\delta \underset{\sim}{u}_b, \qquad (6.102)$$

from which

$$\frac{df}{d\underset{\sim}{u}_b} = \underset{\sim}{f}_{ub} - \hat{\underset{\sim}{\eta}}_{bu} - \underset{\sim}{H}^T_{ub}\, \underset{\sim}{\lambda}_{\rho b}, \qquad (6.103)$$

which gives the required formal derivatives of f w.r.t. the complex control variables $\underset{\sim}{u}_b$.

6.5 CONCLUSIONS

In this chapter, some new aspects of power network analysis have been presented.

A new method for solving the load flow problem has been described. The method exploits the useful features of the simplified version of the Tellegen's theorem-based method for sensitivity calculations described in Chapter 4. The method employs a simple and mostly constant adjoint matrix and provides the same rate of convergence as the Newton-Raphson method. In this method, sensitivities of the dependent variables are readily available at the load flow solution without performing an additional adjoint analysis. Moreover, most of the intermediate computations do not involve trigonometric function evaluations.

A suitable technique for solving, in the complex mode, the load flow problem by the Newton-Raphson method has been presented. The advantage of retaining the compact complex mode of the power flow equations has been achieved via a proper elimination technique by which the non-familiar form of the resulting complex equations can be directly handled.

The far reaching consequences gained by using the compact conjugate notation have been exploited in formulating the Lagrangian approach in the complex form. First-order changes and formal derivatives of complex functions of interest subject to general complex equality constraints can be evaluated, directly, while keeping the original compact complex mode of formulation. Some important applications to power network sensitivity analysis have been studied. The possibility of defining a general number of states associated with a particular branch in the approach of Chapter 3 has been afforded by describing an element-local Lagrangian technique.

7

GENERALIZED POWER NETWORK SENSITIVITIES: A COMPLEX ADJOINT VERSION

7.1 INTRODUCTION

The study of Chapters 3, 4 and 5 shows that the adjoint network approaches based on Tellegen's theorem with suitable extensions provide the flexibility of defining and modelling each adjoint network element associated with the corresponding element of the original network. The adjoint matrix of coefficients, although of the same size and sparsity as the Jacobian matrix of the original load flow problem, has to be calculated at the load flow solution.

The other class of adjoint network approaches (Dommel and Tinney 1968, Fischl and Wasley 1978) provides the advantage of using the transpose of the Jacobian matrix of the load flow problem as an adjoint matrix of coefficients. These approaches, however, handle functions of non-bus quantities through some transformations.

When describing adjoint network approaches which exploit the Jacobian of the load flow problem, the sensitivity expressions for different elements have been derived according to the mode of formulation used, e.g., polar or cartesian. Different forms of sensitivity expressions have been presented for different applications. A unified sensitivity study for this class of adjoint network approaches is performed in this chapter.

The conjugate notation, which describes first-order changes of general complex functions in terms of formal derivatives w.r.t. complex system variables, provides a useful tool for describing a generalized adjoint network sensitivity approach. As presented in this chapter, generalized sensitivity expressions are easily derived, compactly described and effectively used subject to any mode of formulation. The adjoint matrix of coefficients is always the transpose of the Jacobian of the original load flow problem and, regardless of the formulation, these generalized sensitivity expressions can be used.

To illustrate the concepts developed, two examples of the simplest 2-bus sample power system are employed. The formulas derived, however, are general and can be directly programmed for a general power system of practical size.

7.2 BASIC FORMULATION

The system of equations (6.17) is written in the perturbed form

$$\underset{\sim}{K}^S \, \delta \underset{\sim}{V}_M + \overline{\underset{\sim}{K}}^S \, \delta \underset{\sim}{V}_M^* = \delta \underset{\sim}{S}_M^* - \underset{\sim}{E}_M^* \, \delta \underset{\sim}{Y}_T \, \underset{\sim}{V}_M, \tag{7.1}$$

where, again, $\delta \underset{\sim}{V}_M$, $\delta \underset{\sim}{V}_M^*$, $\delta \underset{\sim}{S}_M^*$ and $\delta \underset{\sim}{Y}_T$ represent first-order changes of $\underset{\sim}{V}_M$, $\underset{\sim}{V}_M^*$, $\underset{\sim}{S}_M^*$ and $\underset{\sim}{Y}_T$, respectively, and $\underset{\sim}{K}^S$ and $\overline{\underset{\sim}{K}}^S$ are given, respectively, by (6.19) and (6.20).

We write (7.1) in the form

$$\underset{\sim}{K}^S \, \delta \underset{\sim}{V}_M + \overline{\underset{\sim}{K}}^S \, \delta \underset{\sim}{V}_M^* = \underset{\sim}{d}^S, \tag{7.2}$$

where we have defined

$$\underset{\sim}{d}^S \triangleq \delta \underset{\sim}{S}_M^* - \underset{\sim}{E}_M^* \, \delta \underset{\sim}{Y}_T \, \underset{\sim}{V}_M. \tag{7.3}$$

Note that for constant $\underset{\sim}{Y}_T$, $\underset{\sim}{d}^S$ of (7.3) is simply $\delta \underset{\sim}{S}_M^*$ and (7.2) rigorously represents a set of linear equations to be solved by the Newton-Raphson method.

The form (7.2) must be adjusted for practical considerations. The equation of (7.2) corresponding to the slack bus is replaced by

$$\underset{\sim}{k}_n^T \, \delta \underset{\sim}{V}_M + \overline{\underset{\sim}{k}}_n^T \, \delta \underset{\sim}{V}_M^* = \delta V_n^*, \tag{7.4}$$

where we have assigned the last bus, namely the nth bus, as a slack bus,

$$\underset{\sim}{k}_n = \underset{\sim}{0} \tag{7.5}$$

and

$$\overline{\underset{\sim}{k}}_n = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} . \tag{7.6}$$

Also, for generator-type buses and using (6.52)-(6.56),

$$\delta I_g = \underset{\sim}{y}_g^T \, \delta \underset{\sim}{V}_M + \underset{\sim}{V}_M^T \, \delta \underset{\sim}{y}_g \tag{7.7}$$

and (6.58), ~~it~~ is straightforward to show that $\delta \tilde{S}_g^{0*}$, written as $\delta \tilde{S}_g^*$, is given by

$$\delta \bar{S}_g^* = \underset{\sim}{k}_g^T \, \delta \underset{\sim}{V}_M + \overline{\underset{\sim}{k}}_g^T \, \delta \underset{\sim}{V}_M^* + V_g^* \, \underset{\sim}{V}_M^T \, \delta \underset{\sim}{y}_{g/2} + V_g \, \underset{\sim}{V}_M^{*T} \, \delta \underset{\sim}{y}_{g/2}^*, \tag{7.8}$$

where

$$\underset{\sim}{k}_g \triangleq (V_g^*/2) \, \underset{\sim}{y}_g + [\underset{\sim}{y}_g^{*T} \, \underset{\sim}{V}_M^*/2 - jV_g^*/(2|V_g|)] \, \underset{\sim}{\mu}_g \tag{7.9}$$

and

$$\overline{\underset{\sim}{k}}_g \triangleq (V_g/2) \, \underset{\sim}{y}_g^* + [\underset{\sim}{y}_g^T \, \underset{\sim}{V}_{M/2} - j \, V_g/(2|V_g|)] \, \underset{\sim}{\mu}_g \tag{7.10}$$

and where $\underset{\sim}{\mu}_g$ is a column vector of unity gth element and zero other elements. Using (7.8), the equation of (7.2) corresponding to the gth bus is replaced by

$$\underset{\sim}{k}_g^T \, \delta \underset{\sim}{V}_M + \overline{\underset{\sim}{k}}_g^T \, \delta \underset{\sim}{V}_M^* = d_g, \tag{7.11}$$

where

$$d_g = \delta P_g - j\, \delta|V_g| - V_g^* \underset{\sim}{V}_M^T \, \delta \underset{\sim}{y}_g/2 - V_g \, \underset{\sim}{V}_M^{*T} \, \delta \underset{\sim}{y}_g^*/2. \tag{7.12}$$

We write (7.2), including (7.4) for the slack bus and (7.12) for generator buses, in the form

$$\underset{\sim}{K} \, \delta \underset{\sim}{V}_M + \overline{\underset{\sim}{K}} \, \delta \underset{\sim}{V}_M^* = \underset{\sim}{d}. \tag{7.13}$$

7.3 MODES OF FORMULATION

In the previous section, we have considered the complex formulation of power system equations. We shall exploit this formulation to derive compact forms of sensitivity expressions. In this section, we investigate, via suitable transformations, the relationship between the complex formulation and other formulations. This investigation provides the possibility of formulating the adjoint equations to be solved in the same mode as the original load flow problem. Hence, the available Jacobian of the load flow may be used in solving the adjoint system.

7.3.1 Transformation for Rectangular Formulation

We define the transformation matrix

$$\underset{\sim}{L}^q \triangleq \begin{bmatrix} \underset{\sim}{L}_1 & \underset{\sim}{L}_1^* \\ \underset{\sim}{L}_2 & \underset{\sim}{L}_2^* \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \underset{\sim}{1}^n & \underset{\sim}{1}^n \\ -\underset{\sim}{j}^n & \underset{\sim}{j}^n \end{bmatrix}, \tag{7.14}$$

hence

$$(\underset{\sim}{L}^q)^{-1} = \begin{bmatrix} \underset{\sim}{1}^n & \underset{\sim}{j}^n \\ \underset{\sim}{1}^n & -\underset{\sim}{j}^n \end{bmatrix}, \tag{7.15}$$

n denoting the number of buses in the power network. It follows, using (3.8) and (6.11), that

$$\begin{bmatrix} \underset{\sim}{V}_{M1} \\ \underset{\sim}{V}_{M2} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{L}_1 & \underset{\sim}{L}_1^* \\ \underset{\sim}{L}_2 & \underset{\sim}{L}_2^* \end{bmatrix} \begin{bmatrix} \underset{\sim}{V}_M \\ \underset{\sim}{V}_M^* \end{bmatrix}, \tag{7.16}$$

hence

$$\begin{bmatrix} \delta\underset{\sim}{V}_{M1} \\ \delta\underset{\sim}{V}_{M2} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{L}_1 & \underset{\sim}{L}_1^* \\ \underset{\sim}{L}_2 & \underset{\sim}{L}_2^* \end{bmatrix} \begin{bmatrix} \delta\underset{\sim}{V}_M \\ \delta\underset{\sim}{V}_M^* \end{bmatrix}. \tag{7.17}$$

Using the perturbed form (7.17), it is straightforward to show that (7.13) can be written in the form

$$\begin{bmatrix} (\underset{\sim}{K}_1 + \overline{\underset{\sim}{K}}_1) & (-\underset{\sim}{K}_2 + \overline{\underset{\sim}{K}}_2) \\ -(\underset{\sim}{K}_2 + \overline{\underset{\sim}{K}}_2) & (-\underset{\sim}{K}_1 + \overline{\underset{\sim}{K}}_1) \end{bmatrix} \begin{bmatrix} \delta\underset{\sim}{V}_{M1} \\ \delta\underset{\sim}{V}_{M2} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{d}_1 \\ -\underset{\sim}{d}_2 \end{bmatrix}, \tag{7.18}$$

where we have set

$$\underset{\sim}{K} = \underset{\sim}{K}_1 + j\,\underset{\sim}{K}_2, \tag{7.19}$$

$$\overline{\underset{\sim}{K}} = \overline{\underset{\sim}{K}}_1 + j\,\overline{\underset{\sim}{K}}_2, \tag{7.20}$$

and

$$\underset{\sim}{d} = \underset{\sim}{d}_1 + j\,\underset{\sim}{d}_2. \tag{7.21}$$

The 2n x 2n matrix of coefficients in (7.18), denoted by $\underset{\sim}{K}^{crt}$, constitutes the well-known Jacobian matrix of the load flow problem in rectangular form. Moreover, writing (7.13) in the form

$$[\underset{\sim}{K} \quad \overline{\underset{\sim}{K}}] \begin{bmatrix} \delta\underset{\sim}{V}_M \\ \delta\underset{\sim}{V}_M^* \end{bmatrix} = \underset{\sim}{d}, \tag{7.22}$$

it follows that

$$[\underset{\sim}{K}\ \overline{\underset{\sim}{K}}] = [\underset{\sim}{K}^q\ \overline{\underset{\sim}{K}^q}]\begin{bmatrix} \underset{\sim}{L}_1 & \underset{\sim}{L}_1^* \\ \underset{\sim}{L}_2 & \underset{\sim}{L}_2^* \end{bmatrix}, \tag{7.23}$$

where $\underset{\sim}{K}^q$ and $\overline{\underset{\sim}{K}^q}$ are formed directly from the Jacobian of (7.18) as

$$\underset{\sim}{K}^q = (\underset{\sim}{K}_1 + \overline{\underset{\sim}{K}}_1) + j(\underset{\sim}{K}_2 + \overline{\underset{\sim}{K}}_2) \tag{7.24}$$

and

$$\overline{\underset{\sim}{K}^q} = (-\underset{\sim}{K}_2 + \overline{\underset{\sim}{K}}_2) - j(-\underset{\sim}{K}_1 + \overline{\underset{\sim}{K}}_1)\ . \tag{7.25}$$

Observe that (7.23) relates the Jacobian of the complex formulation (7.13) to the Jacobian of the rectangular formulation (7.18).

7.3.2 Transformation for Polar Formulation

For the polar formulation, we set

$$V_i = |V_i| \angle \delta_i\ ,\ i = 1, \ldots, n, \tag{7.26}$$

where V_i are elements of $\underset{\sim}{V}_M$,

$$|\underset{\sim}{V}| \triangleq \begin{bmatrix} |V_1| \\ \vdots \\ |V_n| \end{bmatrix} \tag{7.27}$$

and

$$\underset{\sim}{\delta} \triangleq \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_n \end{bmatrix}. \tag{7.28}$$

Then, we define the transformation matrix

$$\underset{\sim}{L}^{p} \triangleq \begin{bmatrix} \underset{\sim}{L}_{\delta} & \underset{\sim}{L}_{\delta}^{*} \\ \underset{\sim}{L}_{v} & \underset{\sim}{L}_{v}^{*} \end{bmatrix}, \tag{7.29}$$

where $\underset{\sim}{L}_{\delta}$, $\underset{\sim}{L}_{\delta}^{*}$, $\underset{\sim}{L}_{v}$ and $\underset{\sim}{L}_{v}^{*}$ are diagonal matrices whose elements represent the formal partial derivatives $\partial\delta_i/\partial V_i$, $\partial\delta_i/\partial V_i^*$, $\partial|V_i|/\partial V_i$ and $\partial|V_i|/\partial V_i^*$, respectively, hence

$$\underset{\sim}{L}_{\delta} \triangleq \text{diag } \{L_{\delta i}\} \tag{7.30}$$

and

$$\underset{\sim}{L}_{v} \triangleq \text{diag } \{L_{vi}\}, \tag{7.31}$$

where

$$L_{\delta i} = -j/(2\ V_i) \tag{7.32}$$

and

$$L_{vi} = V_i^*/(2|V_i|). \tag{7.33}$$

The inverse of $\underset{\sim}{L}^{p}$ is given by

$$(\underset{\sim}{L}^{p})^{-1} = \begin{bmatrix} \tilde{\underset{\sim}{L}}_{\delta} & \tilde{\underset{\sim}{L}}_{v} \\ \tilde{\underset{\sim}{L}}_{\delta}^{*} & \tilde{\underset{\sim}{L}}_{v}^{*} \end{bmatrix}. \tag{7.34}$$

where $\tilde{\underset{\sim}{L}}_{\delta}$, $\tilde{\underset{\sim}{L}}_{\delta}^{*}$, $\tilde{\underset{\sim}{L}}_{v}$ and $\tilde{\underset{\sim}{L}}_{v}^{*}$ are diagonal matrices whose elements are the partial derivatives $\partial V_i/\partial\delta_i$, $\partial V_i^*/\partial\delta_i$, $\partial V_i/\partial|V_i|$ and $\partial V_i^*/\partial|V_i|$, respectively, hence

$$\tilde{\underset{\sim}{L}}_{\delta} \triangleq \text{diag } \{\tilde{L}_{\delta i}\} \tag{7.35}$$

and

$$\tilde{\underset{\sim}{L}}_{v} \triangleq \text{diag } \{\tilde{L}_{vi}\}, \tag{7.36}$$

where

$$\tilde{L}_{\delta i} = jV_i \qquad (7.37)$$

and

$$\tilde{L}_{vi} = V_i/|V_i|. \qquad (7.38)$$

Similarly to (7.17), we may write

$$\begin{bmatrix} \delta\mathbf{\delta} \\ \delta|\mathbf{V}| \end{bmatrix} = \begin{bmatrix} \mathbf{L}_\delta & \mathbf{L}_\delta^* \\ \mathbf{L}_v & \mathbf{L}_v^* \end{bmatrix} \begin{bmatrix} \delta\mathbf{V}_M \\ \delta\mathbf{V}_M^* \end{bmatrix}. \qquad (7.39)$$

Using the perturbed form (7.39), it is straightforward to show that (7.13) can also be written in the form

$$\begin{bmatrix} \mathbf{K}_1^p & \overline{\mathbf{K}}_1^p \\ -\mathbf{K}_2^p & -\overline{\mathbf{K}}_2^p \end{bmatrix} \begin{bmatrix} \delta\mathbf{\delta} \\ \delta|\mathbf{V}| \end{bmatrix} = \begin{bmatrix} \mathbf{d}_1 \\ -\mathbf{d}_2 \end{bmatrix}, \qquad (7.40)$$

where we have set

$$\mathbf{K}^p = \mathbf{K}_1^p + j\,\mathbf{K}_2^p \qquad (7.41)$$

and

$$\overline{\mathbf{K}}^p = \overline{\mathbf{K}}_1^p + j\,\overline{\mathbf{K}}_2^p \qquad (7.42)$$

and where the matrices $\mathbf{K}^p$ and $\overline{\mathbf{K}}^p$ are related to $\mathbf{K}$ and $\overline{\mathbf{K}}$ through the relationship

$$[\mathbf{K} \;\; \overline{\mathbf{K}}] = [\mathbf{K}^p \;\; \overline{\mathbf{K}}^p] \begin{bmatrix} \mathbf{L}_\delta & \mathbf{L}_\delta^* \\ \mathbf{L}_v & \mathbf{L}_v^* \end{bmatrix}. \qquad (7.43)$$

The 2n x 2n matrix of coefficients in (7.40), denoted by $\mathbf{K}^{plr}$, constitutes the well-known Jacobian matrix of the load flow problem in polar form.

Observe that (7.43) relates the Jacobian of the complex formulation (7.13) to the Jacobian of the polar formulation (7.40), where $\underset{\sim}{K}^p$ and $\underset{\sim}{\overline{K}}^p$ are formed directly from the Jacobian of (7.40).

At the end of this section, we illustrate the foregoing concepts by two simple examples.

7.3.3 Example 7.1

Consider, first, the 2-bus sample power system of Fig. 7.1 which

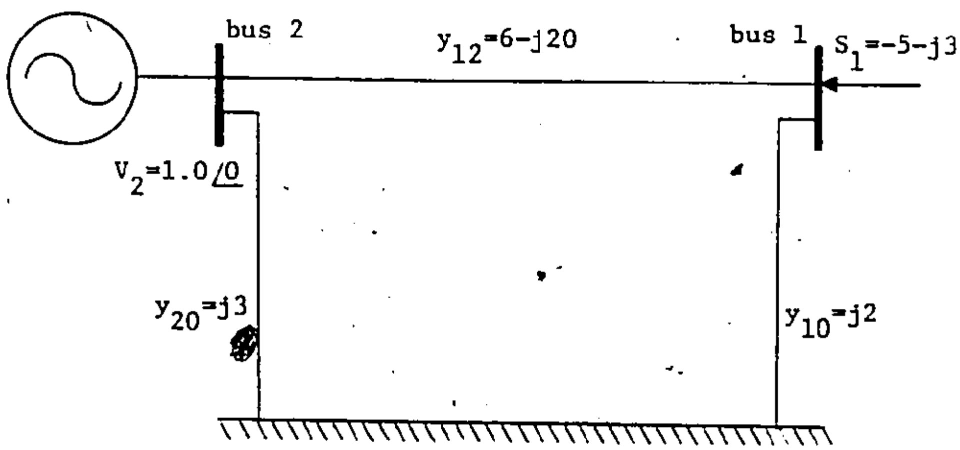


Fig. 7.1 2-bus load-slack sample power system

has been considered in Example 5.4. The solution of the load flow equations (6.17) is given by

$$V_1 = 0.7352 - j\,0.2041$$

and

$$S_2 = 5.6705 + j\,1.0706.$$

Note that S_2 is the injected power at bus 2. The matrices $\underset{\sim}{K}$ and $\underset{\sim}{\overline{K}}$ of (7.22) are given by

$$\underset{\sim}{K} = \begin{bmatrix} (8.0852 - j\ 12.0097) & (-8.4934 + j\ 13.4802) \\ 0 & 0 \end{bmatrix}$$

and

$$\underset{\sim}{\overline{K}} = \begin{bmatrix} (-5.2623 + j\ 5.5411) & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence, using cartesian coordinates, the matrix of coefficients of (7.18) has, using (7.19) and (7.20), the form

$$\underset{\sim}{K}^{crt} = \begin{bmatrix} 2.8229 & -8.4934 & 17.5508 & -13.4802 \\ 0 & 1 & 0 & 0 \\ 6.4686 & -13.4802 & -13.3475 & 8.4934 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is the Jacobian of the load flow problem in cartesian coordinates when the slack bus equations are included.

For the polar formulation, the matrices $\tilde{\underset{\sim}{L}}_\delta$ and $\tilde{\underset{\sim}{L}}_v$ of (7.34) are given by

$$\tilde{\underset{\sim}{L}}_\delta = \begin{bmatrix} (0.2041 + j\ 0.7352) & 0 \\ 0 & j \end{bmatrix}$$

and

$$\tilde{\underset{\sim}{L}}_v = \begin{bmatrix} (0.9636 - j\ 0.2675) & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence, using (7.29), (7.34) and (7.43), the matrices $\underset{\sim}{K}^P$ and $\underset{\sim}{\overline{K}}^P$ are given by

$$\underset{\sim}{K}^{p} = \begin{bmatrix} (13.4802 + j\,8.4934) & (-13.4802 - j\,8.4934) \\ 0 & -j \end{bmatrix}$$

and

$$\underset{\sim}{\overline{K}}^{p} = \begin{bmatrix} (-1.9745 - j\,9.8031) & (-8.4934 + j\,13.4802) \\ 0 & 1 \end{bmatrix},$$

from which the matrix of coefficients of (7.40) has the form

$$\underset{\sim}{K}^{plr} = \begin{bmatrix} 13.4802 & -13.4802 & -1.9745 & -8.4934 \\ 0 & 0 & 0 & 1 \\ -8.4934 & 8.4934 & 9.8031 & -13.4802 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

which is the Jacobian of the load flow problem in polar coordinates when the slack bus equations are included.

7.3.4 Example 7.2

Now, consider the 2-bus sample power system of Fig. 7.2 which consists of a generator bus and a slack bus. The solution of the load flow equations (6.17) is given by

$$\delta_1 = -0.1995 \text{ rad},$$

$$Q_1 = -1.9929$$

and

$$S_2 = 4.2742 - j\,1.7131.$$

The matrices $\underset{\sim}{K}$ and $\underset{\sim}{\overline{K}}$ of (7.22) are given by

$$\underset{\sim}{K} = \begin{bmatrix} (2.3920 - j\,9.4199) & (-4.4300 + j\,8.2864) \\ 0 & 0 \end{bmatrix}$$

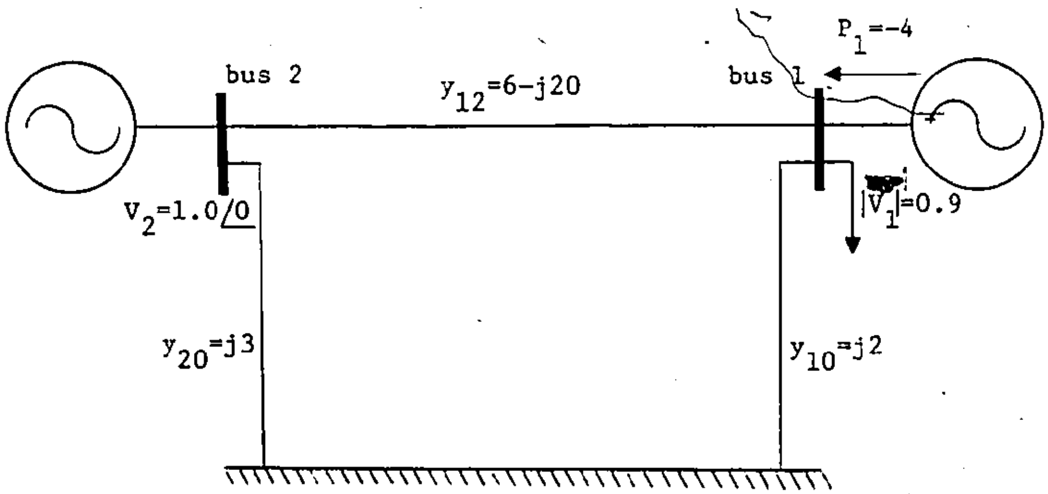


Fig. 7.2 2-bus generator-slack sample power system

and

$$\bar{\underset{\sim}{K}} = \begin{bmatrix} (2.1938 + j\,8.4398) & (-4.4300 - j\,8.2864) \\ 0 & 1 \end{bmatrix}.$$

Hence, using cartesian coordinates, the matrix of coefficients of (7.18) has, using (7.19) and (7.20), the form

$$\underset{\sim}{K}^{crt} = \begin{bmatrix} 4.5858 & -8.8600 & 17.8597 & -16.5729 \\ 0 & 1 & 0 & 0 \\ 0.9802 & 0 & -0.1982 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

which is the Jacobian of the load flow problem in cartesian coordinates when the slack bus equations are included.

For the polar formulation, the matrices $\tilde{L}_\delta$ and $\tilde{L}_v$ of (7.34) are given by

$$\tilde{L}_\delta = \begin{bmatrix} (0.1784 + j\,0.8822) & 0 \\ 0 & j \end{bmatrix}$$

and

$$\tilde{L}_v = \begin{bmatrix} (0.9802 - j\,0.1982) & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence, using (7.29), (7.34) and (7.43), the matrices K^p and $\overline{K}^p$ are given by

$$K^p = \begin{bmatrix} 16.5729 & -16.5729 \\ 0 & -j \end{bmatrix}$$

and

$$\overline{K}^p = \begin{bmatrix} 0.9556 - j\,1.0 & -8.8600 \\ 0 & 1 \end{bmatrix},$$

from which the matrix of coefficients of (7.40) has the form

$$K^{plr} = \begin{bmatrix} 16.5729 & -16.5729 & 0.9556 & -8.8600 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

which is the Jacobian of the load flow problem in polar coordinates when the slack bus equations are included.

7.4 SENSITIVITY CALCULATIONS

In this section, we derive the required sensitivity expressions using the compact complex form (7.13). We exploit the relationships derived in the previous section to provide the flexibility of solving the resulting adjoint system of equations in other modes of formulation.

7.4.1 Adjoint System for a Standard Complex Form

We write (7.13) in the form

$$\begin{bmatrix} \underset{\sim}{K} & \underset{\sim}{\overline{K}} \\ \underset{\sim}{\overline{K}}^* & \underset{\sim}{K}^* \end{bmatrix} \begin{bmatrix} \delta \underset{\sim}{V}_M \\ \delta \underset{\sim}{V}_M^* \end{bmatrix} = \begin{bmatrix} \underset{\sim}{d} \\ \underset{\sim}{d}^* \end{bmatrix}. \tag{7.44}$$

As shown before, the matrix of coefficients of (7.44), denoted by $\underset{\sim}{K}^{cmp}$, has the same rank as that of (7.18) and the system of equations (7.44) is consistent if and only if the system (7.18) is consistent.

For a real function f, we may, using (3.14), write

$$\delta f = [\hat{\underset{\sim}{\mu}}^T \quad \hat{\underset{\sim}{\mu}}^{*T}] \begin{bmatrix} \delta \underset{\sim}{V}_M \\ \delta \underset{\sim}{V}_M^* \end{bmatrix} + \delta f_\rho, \tag{7.45}$$

where we have defined

$$\hat{\underset{\sim}{\mu}} \triangleq \frac{\partial f}{\partial \underset{\sim}{V}_M} \tag{7.46}$$

and used

$$\frac{\partial f}{\partial \underset{\sim}{V}_M} = \left(\frac{\partial f}{\partial \underset{\sim}{V}_M^*} \right)^*, \tag{7.47}$$

δf_ρ denoting the change in f due to changes in other variables in terms of which f may be explicitly expressed. Hence, from (7.44)

$$\delta f = [\hat{\underset{\sim}{\mu}}^T \quad \hat{\underset{\sim}{\mu}}^{*T}] \begin{bmatrix} \underset{\sim}{K} & \underset{\sim}{\overline{K}} \\ \underset{\sim}{\overline{K}}^* & \underset{\sim}{K}^* \end{bmatrix}^{-1} \begin{bmatrix} \underset{\sim}{d} \\ \underset{\sim}{d}^* \end{bmatrix} + \delta f_\rho \tag{7.48}$$

or

$$\delta f = [\hat{\underset{\sim}{V}}^T, \quad \hat{\underset{\sim}{V}}^{*T}] \begin{bmatrix} \underset{\sim}{d} \\ \underset{\sim}{d}^* \end{bmatrix} + \delta f_\rho , \tag{7.49}$$

where

$$\begin{bmatrix} \underset{\sim}{K}^T & \underset{\sim}{\overline{K}}^{*T} \\ \underset{\sim}{\overline{K}}^T & \underset{\sim}{K}^{*T} \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{V}} \\ \hat{\underset{\sim}{V}}^* \end{bmatrix} = \begin{bmatrix} \hat{\underset{\sim}{\mu}} \\ \hat{\underset{\sim}{\mu}}^* \end{bmatrix} \tag{7.50}$$

or, simply,

$$[\underset{\sim}{K}^T \quad \underset{\sim}{\overline{K}}^{*T}] \begin{bmatrix} \hat{\underset{\sim}{V}} \\ \hat{\underset{\sim}{V}}^* \end{bmatrix} = \hat{\underset{\sim}{\mu}} \, . \tag{7.51}$$

Hence, the first-order change of the real function f and corresponding gradients can be evaluated by solving (7.50) and substituting into (7.49).

7.4.2 Adjoint System for Rectangular Formulation

Similarly to (7.45), we may write, using the rectangular formulation

$$\delta f = [\hat{\underset{\sim}{\mu}}_r^T \ \hat{\underset{\sim}{\mu}}_s^T] \begin{bmatrix} \delta \underset{\sim}{V}_{M1} \\ \delta \underset{\sim}{V}_{M2} \end{bmatrix} + \delta f_\rho , \tag{7.52}$$

where we have defined

$$\hat{\underset{\sim}{\mu}}_r \triangleq \frac{\partial f}{\partial \underset{\sim}{V}_{M1}} \tag{7.53}$$

and

$$\hat{\underset{\sim}{\mu}}_s \triangleq \frac{\partial f}{\partial \underset{\sim}{V}_{M2}} . \tag{7.54}$$

Hence, from (7.18)

$$\delta f = [\hat{\underset{\sim}{V}}_r^T \; \hat{\underset{\sim}{V}}_s^T] \begin{bmatrix} \underset{\sim}{d}_1 \\ -\underset{\sim}{d}_2 \end{bmatrix} + \delta f_\rho , \tag{7.55}$$

where

$$\begin{bmatrix} (\underset{\sim}{K}_1+\overline{\underset{\sim}{K}}_1)^T & -(\underset{\sim}{K}_2+\overline{\underset{\sim}{K}}_2)^T \\ (-\underset{\sim}{K}_2+\overline{\underset{\sim}{K}}_2)^T & (-\underset{\sim}{K}_1+\overline{\underset{\sim}{K}}_1)^T \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{V}}_r \\ \hat{\underset{\sim}{V}}_s \end{bmatrix} = \begin{bmatrix} \hat{\underset{\sim}{\mu}}_r \\ \hat{\underset{\sim}{\mu}}_s \end{bmatrix} . \tag{7.56}$$

Observe that the matrix of coefficients of (7.56) is the transpose of the Jacobian matrix of the load flow problem in the rectangular form (7.18).

7.4.3 Theorem 7.1

(a) The solution vectors $\hat{\underset{\sim}{V}}_r$ and $\hat{\underset{\sim}{V}}_s$ of the adjoint system of equations (7.56) are given by

$$\hat{\underset{\sim}{V}}_r = 2\text{Re}\{\hat{\underset{\sim}{V}}\}$$

and

$$\hat{\underset{\sim}{V}}_s = 2\text{Im}\{\hat{\underset{\sim}{V}}\}.$$

where $\hat{\underset{\sim}{V}}$ is given from (7.50).

(b) The RHS vectors $\hat{\underset{\sim}{\mu}}_r$ and $\hat{\underset{\sim}{\mu}}_s$ of the adjoint system of equations (7.56) are given by

$$\hat{\underset{\sim}{\mu}} = \underset{\sim}{L}_1^T \hat{\underset{\sim}{\mu}}_r + \underset{\sim}{L}_2^T \hat{\underset{\sim}{\mu}}_s ,$$

where $\hat{\underset{\sim}{\mu}}$ is given by (7.46) and $\underset{\sim}{L}_1$ and $\underset{\sim}{L}_2$ are given by (7.14).

Proof

Comparing (7.49) and (7.55), and using (7.21), we get

$$\hat{V} = (\hat{V}_r + j\hat{V}_s)/2. \tag{7.57}$$

From (7.57), the first part of the theorem is proved. Now, multiplying (7.56) from the left by the transpose of L^q of (7.14) and using the relation

$$2\begin{bmatrix} (K_1 + \bar{K}_1)^T & -(K_2 + \bar{K}_2)^T \\ (-K_2 + \bar{K}_2)^T & (-K_1 + \bar{K}_1)^T \end{bmatrix} = \begin{bmatrix} K^{qT} & (K^{q*})^T \\ \bar{K}^{qT} & (\bar{K}^{q*})^T \end{bmatrix} \cdot \begin{bmatrix} 1 & j \\ 1 & -j \end{bmatrix}, \tag{7.58}$$

it follows from (7.23) and (7.57) that

$$\begin{bmatrix} K^T & \bar{K}^{*T} \\ \bar{K}^T & K^{*T} \end{bmatrix} \begin{bmatrix} \hat{V} \\ \hat{V}^* \end{bmatrix} = \begin{bmatrix} L_1^T & L_2^T \\ L_1^{*T} & L_2^{*T} \end{bmatrix} \begin{bmatrix} \hat{\mu}_r \\ \hat{\mu}_s \end{bmatrix}, \tag{7.59}$$

hence, from (7.50)

$$\begin{bmatrix} \hat{\mu} \\ \hat{\mu}^* \end{bmatrix} = \begin{bmatrix} L_1^T & L_2^T \\ L_1^{*T} & L_2^{*T} \end{bmatrix} \begin{bmatrix} \hat{\mu}_r \\ \hat{\mu}_s \end{bmatrix} \tag{7.60}$$

or, simply

$$\hat{\mu} = [L_1^T \;\; L_2^T] \begin{bmatrix} \hat{\mu}_r \\ \hat{\mu}_s \end{bmatrix} \quad \blacksquare \tag{7.61}$$

The relationship (7.61) could also be derived by applying, formally, the chain rule of differentiation using the definitions (7.46), (7.53) and (7.54).

Observe that equation (7.57) relates the solution of the adjoint system (7.56) to that of (7.51), and equation (7.61) relates the RHS of (7.56) to that of (7.51).

7.4.4 Adjoint System for Polar Formulation

Using the polar formulation, we may write

$$\delta f = [\hat{\underset{\sim}{\mu}}_{\delta}^{T} \ \hat{\underset{\sim}{\mu}}_{V}^{T}] \begin{bmatrix} \delta\underset{\sim}{\delta} \\ \delta|\underset{\sim}{V}| \end{bmatrix} + \delta f_{p}, \tag{7.62}$$

where we have defined

$$\hat{\underset{\sim}{\mu}}_{\delta} \triangleq \frac{\partial f}{\partial \underset{\sim}{\delta}} \tag{7.63}$$

and

$$\hat{\underset{\sim}{\mu}}_{V} \triangleq \frac{\partial f}{\partial |\underset{\sim}{V}|} . \tag{7.64}$$

Hence, from (7.40)

$$\delta f = [\hat{\underset{\sim}{V}}_{\delta}^{T} \ \ \hat{\underset{\sim}{V}}_{V}^{T}] \begin{bmatrix} \underset{\sim}{d}_1 \\ -\underset{\sim}{d}_2 \end{bmatrix} + \delta f_{\rho}, \tag{7.65}$$

where

$$\begin{bmatrix} \underset{\sim}{K}_1^{PT} & -\underset{\sim}{K}_2^{PT} \\ \overline{\underset{\sim}{K}}_1^{PT} & -\overline{\underset{\sim}{K}}_2^{PT} \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{V}}_{\delta} \\ \hat{\underset{\sim}{V}}_{V} \end{bmatrix} = \begin{bmatrix} \hat{\underset{\sim}{\mu}}_{\delta} \\ \hat{\underset{\sim}{\mu}}_{V} \end{bmatrix} . \tag{7,66}$$

The matrix of coefficients of (7.66) is the transpose of the Jacobian matrix of the load flow problem in the polar form.

7.4.5 Theorem 7.2

(a) The solution vectors $\hat{\underset{\sim}{V}}_{\delta}$ and $\hat{\underset{\sim}{V}}_{V}$ of the adjoint system of equations

(7.66) are given by

$$\hat{\underset{\sim}{V}}_\delta = 2\mathrm{Re}\{\hat{\underset{\sim}{V}}\}$$

and

$$\hat{\underset{\sim}{V}}_v = 2\mathrm{Im}\{\hat{\underset{\sim}{V}}\},$$

where $\hat{V}$ is given from (7.50).

(b) The RHS vectors $\hat{\underset{\sim}{\mu}}_\delta$ and $\hat{\underset{\sim}{\mu}}_v$ of the adjoint system of equations (7.66) are given by

$$\underset{\sim}{\hat{\mu}} = \underset{\sim}{L}_\delta^T \hat{\underset{\sim}{\mu}}_\delta + \underset{\sim}{L}_v^T \hat{\underset{\sim}{\mu}}_v^*,$$

where $\underset{\sim}{\hat{\mu}}$ is given by (7.46) and $\underset{\sim}{L}_\delta$ and $\underset{\sim}{L}_v$ are given by (7.30) and (7.31).

<u>Proof</u>

Comparing (7.49) and (7.65), and using (7.21), we get

$$\hat{\underset{\sim}{V}} = (\hat{\underset{\sim}{V}}_\delta + j\,\hat{\underset{\sim}{V}}_v)/2 \ . \tag{7.67}$$

From (7.67), the first part of the theorem is proved. Now, multiplying (7.66) from left by the transpose of $\underset{\sim}{L}^P$ of (7.29) and using the relation

$$2\begin{bmatrix} \underset{\sim}{K}_1^{PT} & -\underset{\sim}{K}_2^{PT} \\ \overline{\underset{\sim}{K}}_1^{PT} & -\overline{\underset{\sim}{K}}_2^{PT} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{K}^{PT} & (\underset{\sim}{K}^{P*})^T \\ \overline{\underset{\sim}{K}}^{PT} & (\overline{\underset{\sim}{K}}^{P*})^T \end{bmatrix} \begin{bmatrix} \underset{\sim}{1} & \underset{\sim}{j} \\ \underset{\sim}{1} & -\underset{\sim}{j} \end{bmatrix}, \tag{7.68}$$

it follows from (7.43) and (7.67) that

$$\begin{bmatrix} \underset{\sim}{K}^T & \overline{\underset{\sim}{K}}^{*T} \\ \overline{\underset{\sim}{K}}^T & \underset{\sim}{K}^{*T} \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{V}} \\ \hat{\underset{\sim}{V}}^* \end{bmatrix} = \begin{bmatrix} \underset{\sim}{L}_\delta^T & \underset{\sim}{L}_v^T \\ \underset{\sim}{L}_\delta^{*T} & \underset{\sim}{L}_v^{*T} \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{\mu}}_\delta \\ \hat{\underset{\sim}{\mu}}_v \end{bmatrix}, \tag{7.69}$$

hence, from (7.50)

$$\begin{bmatrix} \underset{\sim}{\hat{\mu}} \\ \underset{\sim}{\hat{\mu}}^{*} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{L}_{\delta}^{T} & \underset{\sim}{L}_{V}^{T} \\ \underset{\sim}{L}_{\delta}^{*T} & \underset{\sim}{L}_{V}^{*T} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\hat{\mu}}_{\delta} \\ \underset{\sim}{\hat{\mu}}_{V} \end{bmatrix} , \tag{7.70}$$

or, simply

$$\underset{\sim}{\hat{\mu}} = [\underset{\sim}{L}_{\delta}^{T} \quad \underset{\sim}{L}_{V}^{T}] \begin{bmatrix} \underset{\sim}{\hat{\mu}}_{\delta} \\ \underset{\sim}{\hat{\mu}}_{V} \end{bmatrix} \tag{7.71}$$

Again, the relationship (7.71) could also be derived by applying, formally, the chain rule of differentiation using the definitions (7.46), (7.63) and (7.64).

Equation (7.67) relates the solution of the adjoint system (7.66) to that of (7.51), and equation (7.71) relates the RHS of (7.66) to that of (7.51).

7.4.6 Remarks

We remark that using (7.57) or (7.67), the adjoint system can be formulated and solved in a convenient mode, preferably the same formulation as the original load flow problem, while the first-order change of f and corresponding gradients are derived compactly using the adjoint variables $\underset{\sim}{\hat{V}}$. On the other hand, the relations (7.61) and (7.71) allow the use of more elegant formal derivatives which, in many cases, facilitate the formulation. For example, consider the function

$$f = \sigma|V_i - V_j|^2 = \sigma(V_i - V_j)(V_i^{*} - V_j^{*}), \tag{7.72}$$

where V_i and V_j are the ith and jth components of $\underset{\sim}{V}_M$, respectively, and σ is real. Note that f of (7.72) may represent, for example, the power loss in line ij. For the polar formulation, $\hat{\mu}_\delta$ and $\hat{\mu}_v$ of (7.66) are calculated as follows. The ith and jth components of $\hat{\mu}_\delta$ and $\hat{\mu}_v$ are given by (all other components are zero)

$$\hat{\mu}_{\delta i} = \sigma[-2\ (|V_i| \cos \delta_i - |V_j| \cos \delta_j)\ |V_i| \sin \delta_i$$
$$+2\ (|V_i| \sin \delta_i - |V_j| \sin \delta_j)\ |V_i| \cos \delta_i],$$
$$\hat{\mu}_{\delta j} = \sigma[2\ (|V_i| \cos \delta_i - |V_j| \cos \delta_j)\ |V_j| \sin \delta_j$$
$$-2\ (|V_i| \sin \delta_i - |V_j| \sin \delta_j)\ |V_j| \cos \delta_j],$$

$$\hat{\mu}_{vi} = \sigma[2\ (|V_i| \cos \delta_i - |V_j| \cos \delta_j) \cos \delta_i$$
$$+2\ (|V_i| \sin \delta_i - |V_j| \sin \delta_j) \sin \delta_i],$$

$$\hat{\mu}_{vj} = \sigma[2\ (|V_i| \cos \delta_i - |V_j| \cos \delta_j) \cos \delta_j$$
$$-2\ (|V_i| \sin \delta_i - |V_j| \sin \delta_j) \sin \delta_j].$$

Alternatively, one may calculate

$$\hat{\mu} = \sigma \begin{bmatrix} 0 \\ \vdots \\ (V_i^* - V_j^*) \\ \vdots \\ -(V_i^* - V_j^*) \\ \vdots \\ 0 \end{bmatrix}$$

and use (7.70) to obtain $\hat{\mu}_v$ and $\hat{\mu}_\delta$, where $(L^{pT})^{-1}$ is the transpose of $(L^p)^{-1}$ of (7.34). In this example, the derivation of the formal derivatives is clearly easier.

We also remark that other forms of power flow equations can be handled in a similar way. The previous theorems can be easily generalized for other formulations provided that transformations similar to (7.14) and (7.29) are defined.

At the end of this section, we illustrate the foregoing concepts by the two simple examples considered in the previous section.

7.4.7 Example 7.3

For the first system, consider the function

$$f = |V_1|^2 = V_1 V_1^*.$$

From (7.46),

$$\hat{\underset{\sim}{\mu}} = \begin{bmatrix} V_1^* \\ 0 \end{bmatrix} = \begin{bmatrix} 0.7352 + j0.2041 \\ 0 \end{bmatrix}$$

and (7.51) has the solution

$$\hat{\underset{\sim}{V}} = \begin{bmatrix} 0.0562 + j0.0892 \\ 1.6788 + j0.0 \end{bmatrix}.$$

Also, for the polar formulation, we have from (7.63) and (7.64)

$$\hat{\underset{\sim}{\mu}}_\delta = \underset{\sim}{0}$$

and

$$\hat{\underset{\sim}{\mu}}_v = \begin{bmatrix} 2|V_1| \\ 0 \end{bmatrix} = \begin{bmatrix} 1.5261 \\ 0 \end{bmatrix}$$

and (7.66) has the solution

$$\hat{\underset{\sim}{V}}_\delta = \begin{bmatrix} 0.1123 \\ 3.3577 \end{bmatrix}$$

and

$$\hat{\underset{\sim}{V}}_v = \begin{bmatrix} 0.1783 \\ 0 \end{bmatrix}.$$

Note that the $\hat{V}_{\delta}$ and $\hat{V}_{v}$ obtained for the polar formulation and $\hat{V}$ satisfy (7.67).

7.4.8 Example 7.4

For the second system, consider the function

$$f = \delta_1 = \tan^{-1} \left[\frac{V_1 + V_1^*}{j(V_1^* - V_1)}\right].$$

From (7.46),

$$\hat{\mu} = \begin{bmatrix} -j/(2V_1) \\ 0 \end{bmatrix} = \begin{bmatrix} 0.1101 - j0.5445 \\ 0 \end{bmatrix}$$

and (7.51) has the solution

$$\hat{V} = \begin{bmatrix} 0.0302 - j0.0288 \\ 0.2673 + j0.5 \end{bmatrix}.$$

Also, for the polar formulation, we have from (7.63) and (7.64)

$$\hat{\mu}_{\delta} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$\hat{\mu}_{v} = 0$$

and (7.66) has the solution

$$\hat{V}_{\delta} = \begin{bmatrix} 0.0603 \\ 0.5346 \end{bmatrix}$$

and

$$\hat{V}_{v} = \begin{bmatrix} -0.0577 \\ 1.0 \end{bmatrix}.$$

Note that the $\hat{V}_{\delta}$ and $\hat{V}_{v}$ obtained for the polar formulation and $\hat{V}$ satisfy (7.67).

7.5 GRADIENT EVALUATION

In the previous section, we have derived the adjoint systems in different modes of formulation and investigated the relationships between the corresponding excitation and solution vectors. The first-order change δf can be calculated from (7.49), (7.55) and (7.65). In this section, we derive and tabulate standard expressions for the derivatives of f w.r.t. control variables.

7.5.1 General Derivation

We consider the buses to be ordered in the same way as in Chapter 3, i.e., subscripts $\ell=1, 2, \ldots, n_L$ identify load buses, $g = n_L+1, \ldots, n_L + n_G$ identify generator buses and $n = n_L + n_G+1$ identifies the slack bus.

The vector d of (7.22) is now partitioned into subvectors associated with the sets of load, generator and slack buses of appropriate dimension in the form

$$d = \begin{bmatrix} d_L \\ d_G \\ d_n \end{bmatrix}, \tag{7.73}$$

where d_L has elements d_ℓ given, from (7.3), by

$$d_\ell = \delta S_\ell^* - V_\ell^* V_M^T \delta y_\ell, \tag{7.74}$$

y_ℓ representing the corresponding row of the bus admittance matrix Y_T,

$\underset{\sim}{d}_G$ has elements d_g given by (7.12) and d_n is δV_n^* from (7.4). Also, the vector $\hat{\underset{\sim}{V}}$ of (7.49) is partitioned correspondingly in the form

$$\hat{\underset{\sim}{V}} = \begin{bmatrix} \hat{\underset{\sim}{V}}_L \\ \hat{\underset{\sim}{V}}_G \\ \hat{V}_n \end{bmatrix} . \tag{7.75}$$

Note that the above formulation leads to expressing the vector $\underset{\sim}{d}$ solely in terms of variations in control variables. The gradients can be obtained by writing (7.49) in the form

$$\delta f = \hat{\underset{\sim}{V}}_L^T \; \underset{\sim}{d}_L + \hat{\underset{\sim}{V}}_G^T \; \underset{\sim}{d}_G + \hat{V}_n d_n + \left(\frac{\partial f}{\partial \underset{\sim}{\rho}}\right)^T \delta \underset{\sim}{\rho}$$

$$+ \hat{\underset{\sim}{V}}_L^{*T} \; \underset{\sim}{d}_L^* + \hat{\underset{\sim}{V}}_G^{*T} \; \underset{\sim}{d}_G^* + \hat{V}_n^* d_n^* + \left(\frac{\partial f}{\partial \underset{\sim}{\rho}}\right)^{*T} \delta \underset{\sim}{\rho}^* . \tag{7.76}$$

The first term of (7.76) is given, using (7.74), by

$$\hat{\underset{\sim}{V}}_L^T \, \underset{\sim}{d}_L = \sum_{\ell=1}^{n_L} \hat{V}_\ell d_\ell$$

$$= \sum_{\ell=1}^{n_L} (\hat{V}_\ell \; \delta S_\ell^*) - \sum_{\ell=1}^{n_L} \sum_{m=1}^{n} (\hat{V}_\ell \; V_\ell^* \; V_m \; \delta Y_{\ell m}) , \tag{7.77}$$

where $Y_{\ell m}$ is an element of $\underset{\sim}{Y}_T$, which is assumed, without loss of generality, to be a symmetric admittance matrix, or

$$\hat{\underset{\sim}{V}}_L^T \; \underset{\sim}{d}_L = \sum_{\ell=1}^{n_L} (\hat{V}_\ell \; \delta S_\ell^*) + \sum_{\ell=1}^{n_L} \sum_{\substack{m=1 \\ m \neq \ell}}^{n} \hat{V}_\ell \; V_\ell^* \; (V_m - V_\ell) \; \delta y_{\ell m}$$

$$- \sum_{\ell=1}^{n} (\hat{V}_\ell \; V_\ell^* \; V_\ell \; \delta y_{\ell 0}) , \tag{7.78}$$

where $y_{\ell m}$ denotes the admittance of line ℓm connecting load bus ℓ with bus m (=ℓ, g or n), and $y_{\ell 0}$ is the shunt admittance at bus ℓ. The second term of (7.76) is given, using (7.12) by

$$\hat{\underset{\sim}{V}}_G^T \underset{\sim}{d}_G = \sum_{g=n_L+1}^{n-1} \hat{V}_g \, d_g$$

$$= \sum_{g=n_L+1}^{n-1} \hat{V}_g \, (\delta P_g - j\, \delta|V_g|)$$

$$- \sum_{g=n_L+1}^{n-1} \sum_{m=1}^{n} \hat{V}_g \, \mathrm{Re}\{ V_g^* V_m \, \delta Y_{gm}\} \quad (7.79)$$

or

$$\hat{\underset{\sim}{V}}_G^T \underset{\sim}{d}_G = \sum_{g=n_L+1}^{n-1} \hat{V}_g(\delta P_g - j\delta|V_g|) + \sum_{g=n_L+1}^{n-1} \sum_{\substack{m=1 \\ m\neq g}}^{n} \hat{V}_g \, \mathrm{Re}\{V_g^*(V_m - V_g)\, \delta y_{gm}\}$$

$$- \sum_{g=n_L+1}^{n-1} \hat{V}_g \, \mathrm{Re}\{V_g^* V_g \, \delta y_{g0}\} , \quad (7.80)$$

where y_{gm} denotes the admittance of line gm connecting generator bus g with bus m (=ℓ, g or n), and y_{g0} is the shunt admittance at bus g. The third term of (7.76) is given, using (7.4) by

$$\hat{V}_n \, d_n = \hat{V}_n \, \delta V_n^* . \quad (7.81)$$

The fourth term of (7.76) is simply the first-order change of f due to changes in other variables $\underset{\sim}{\rho}$ in terms of which the function f may be explicitly expressed.

Equations (7.78), (7.80) and (7.81) provide useful information for gradient evaluation since they provide direct expressions in terms of variations in control variables of interest. The derivatives of the function f w.r.t. the control variables are obtained as follows, where we temporarily assume that $\underset{\sim}{\rho}$ does not contain such control variables.

7.5.2 Derivatives of Real Function w.r.t. Bus-Type Variables

From (7.78) and its complex conjugate, the derivatives of f w.r.t. the demand S_ℓ and S_ℓ^* at load bus ℓ is given by

$$\frac{df}{dS_\ell} = \hat{V}_\ell^* , \qquad (7.82)$$

and

$$\frac{df}{dS_\ell^*} = \hat{V}_\ell . \qquad (7.83)$$

From (7.80) and its complex conjugate, the derivatives of f w.r.t. the real generated power P_g and the voltage magnitude $|V_g|$ at generator bus g are given by

$$\frac{df}{d\tilde{S}_g} = \hat{V}_g^* \qquad (7.84)$$

and

$$\frac{df}{d\tilde{S}_g^*} = \hat{V}_g , \qquad (7.85)$$

where $\tilde{S}_g$ is given by

$$\tilde{S}_g = P_g + j|V_g| . \qquad (7.86)$$

From (7.81) and its complex conjugate, the derivatives of f w.r.t. the slack bus voltage V_n and V_n^* are given by

$$\frac{df}{dV_n} = \hat{V}_n^* \qquad (7.87)$$

and

$$\frac{df}{dV_n^*} = \hat{V}_n . \tag{7.88}$$

In practice, the phase angle of the slack bus voltage is set to zero as a reference angle. Hence, the slack bus has only one practical real control variable.

7.5.3 Derivatives of Real Function w.r.t. Line Variables

The derivatives of f w.r.t. line control variables y_{ij} can be obtained from (7.78) and (7.80) and their complex conjugate as follows. For $y_{\ell\ell'}$, between load buses ℓ and ℓ', we have from (7.78) and its complex conjugate

$$\frac{df}{dy_{\ell\ell'}} = (\hat{V}_\ell V_\ell^* - \hat{V}_{\ell'} V_{\ell'}^*)(V_{\ell'} - V_\ell) \tag{7.89}$$

and

$$\frac{df}{dy_{\ell\ell'}^*} = (\hat{V}_\ell^* V_\ell - \hat{V}_{\ell'}^* V_{\ell'})(V_{\ell'}^* - V_\ell^*) . \tag{7.90}$$

For $y_{\ell 0}$ between load bus ℓ and ground, we have from (7.78) and its complex conjugate

$$\frac{df}{dy_{\ell 0}} = - \hat{V}_\ell V_\ell^* V_\ell \tag{7.91}$$

and

$$\frac{df}{dy_{\ell 0}^*} = - \hat{V}_\ell^* V_\ell V_\ell^* . \tag{7.92}$$

For $y_{gg'}$ between generator buses g and g', we have from (7.80) and its complex conjugate

$$\frac{df}{dy_{gg'}} = (\hat{V}_{g1} V_g^* - \hat{V}_{g'1} V_{g'}^*) (V_{g'} - V_g) \qquad (7.93)$$

and

$$\frac{df}{dy_{gg'}^*} = (\hat{V}_{g1} V_g - \hat{V}_{g'1} V_{g'}) (V_{g'}^* - V_g^*) , \qquad (7.94)$$

where

$$\hat{V}_m = \hat{V}_{m1} + j\hat{V}_{m2} \qquad (7.95)$$

and m is a bus index. For y_{g0} between generator bus g and ground, we have from (7.80),

$$\frac{df}{dy_{g0}} = \frac{df}{dy_{g0}^*} = - \hat{V}_{g1} V_g^* V_g . \qquad (7.96)$$

For $y_{\ell g}$ between load bus ℓ and generator bus g, we have from (7.78) and (7.80) and their complex conjugate

$$\frac{df}{dy_{\ell g}} = (\hat{V}_{g1} V_g^* - \hat{V}_\ell V_\ell^*) (V_\ell - V_g) \qquad (7.97)$$

and

$$\frac{df}{dy_{\ell g}^*} = (\hat{V}_{g1} V_g - \hat{V}_\ell^* V_\ell) (V_\ell^* - V_g^*) . \qquad (7.98)$$

For $y_{\ell n}$ between load bus ℓ and the slack bus n, we have from (7.78) and its complex conjugate

$$\frac{df}{dy_{\ell n}} = \hat{V}_\ell V_\ell^* (V_n - V_\ell) \qquad (7.99)$$

and

$$\frac{df}{dy_{\ell n}^*} = \hat{V}_\ell^* V_\ell (V_n^* - V_\ell^*) . \qquad (7.100)$$

Finally, for y_{gn} between generator bus g and the slack bus n, we have from (7.80) and its complex conjugate

$$\frac{df}{dy_{gn}} = \hat{V}_{g1}\ V_g^*\ (V_n - V_g) \qquad (7.101)$$

and

$$\frac{df}{dy^*_{gn}} = \hat{V}_{g1}\ V_g\ (V_n^* - V_g^*)\ . \qquad (7.102)$$

7.5.4 Special Considerations

If $\underset{\sim}{\rho}$ of (7.76) contains some of the above control variables, the partial derivatives of f w.r.t. appropriate control variables must be added to the expressions obtained.

When any of the control variables u_k is a function of some real design variables, we write

$$\delta u_k = \sum_i \frac{\partial u_k}{\partial \zeta_{ki}} \Delta \zeta_{ki}, \qquad (7.103)$$

where ζ_{ki} is the ith design variable associated with u_k and $\Delta\zeta_{ki}$ denotes the change in ζ_{ki}. Hence,

$$\frac{df}{d\zeta_{ki}} = \frac{df}{du_k} \frac{\partial u_k}{\partial \zeta_{ki}}\ . \qquad (7.104)$$

Equations (7.82)-(7.85), (7.87)-(7.94) and (7.96)-(7.102) compactly define the required formal derivatives of the real function f w.r.t. complex control variables. In practice, gradients w.r.t. real and imaginary parts of the defined control variables are of direct interest. These gradients are simply obtained from

$$\frac{df}{du_{k1}} = 2\mathrm{Re}\{\frac{df}{du_k}\} \qquad (7.105)$$

and

$$\frac{df}{du_{k2}} = -2\mathrm{Im}\{\frac{df}{du_k}\}, \quad (7.106)$$

where the complex control variable u_k is given by

$$u_k = u_{k1} + j\, u_{k2}. \quad (7.107)$$

Table 7.1 summarizes the derived expressions of real function gradients w.r.t. real control variables of practical interest.

At the end of this section, we apply the formulas derived for gradient evaluation to the 2-bus examples considered before.

7.5.5 Example 7.5

Using the values of $\hat{V}$ obtained, we have for the first system

$$\frac{df}{dP_1} = 2\hat{V}_{11} = 0.1123,$$

$$\frac{df}{dQ_1} = 2\hat{V}_{12} = 0.1783,$$

$$\frac{df}{dV_{21}} = 2\hat{V}_{21} = 3.3577,$$

$$\frac{df}{B_{10}} = 2|V_1|^2\, \hat{V}_{12} = 0.1038,$$

$$\frac{df}{dG_{12}} = 2\mathrm{Re}\ \{\hat{V}_1 V_1^* \ (V_2-V_1)\} = -0.0192$$

and

$$\frac{df}{dB_{12}} = -2\mathrm{Im}\ \{\hat{V}_1 V_1^* \ (V_2-V_1)\} = -0.0502,$$

TABLE 7.1

DERIVATIVES OF A REAL FUNCTION f W.R.T. CONTROL VARIABLES

Variable	Description	Derivative
P_ℓ	demand real power	$2\ \hat{V}_{\ell 1}$
Q_ℓ	demand reactive power	$2\ \hat{V}_{\ell 2}$
P_g	generator real power	$2\ \hat{V}_{g1}$
$\|V_g\|$	generator bus voltage magnitude	$2\ \hat{V}_{g2}$
V_{n1}	real component of slack bus voltage	$2\ \hat{V}_{n1}$
$G_{\ell\ell'}$	conductance between two load buses	$2\ \mathrm{Re}\{(\hat{V}_\ell V_\ell^* - \hat{V}_{\ell'} V_{\ell'}^*)(V_{\ell'} - V_\ell)\}$
$B_{\ell\ell'}$	susceptance between two load buses	$-2\ \mathrm{Im}\{(\hat{V}_\ell V_\ell^* - \hat{V}_{\ell'} V_{\ell'}^*)(V_{\ell'} - V_\ell)\}$
$G_{\ell 0}$	load shunt conductance	$-2\ \|V_\ell\|^2 \hat{V}_{\ell 1}$
$B_{\ell 0}$	shunt susceptance of a load bus	$2\ \|V_\ell\|^2\ \hat{V}_{\ell 2}$
$G_{gg'}$	conductance between two generator buses	$2\ \mathrm{Re}\{(\hat{V}_{g1} V_g^* - \hat{V}_{g'1} V_{g'}^*)(V_{g'} - V_g)\}$
$B_{gg'}$	susceptance between two generator buses	$-2\ \mathrm{Im}\{(\hat{V}_{g1} V_g^* - \hat{V}_{g'1} V_{g'}^*)(V_{g'} - V_g)\}$
G_{g0}	shunt conductance of a generator bus	$-2\ \|V_g\|^2\ \hat{V}_{g1}$
B_{g0}	shunt susceptance of a generator bus	0
$G_{\ell g}$	conductance between a load and a generator buses	$2\ \mathrm{Re}\{(\hat{V}_{g1} V_g^* - \hat{V}_\ell V_\ell^*)(V_\ell - V_g)\}$
$B_{\ell g}$	susceptance between a load and a generator buses	$-2\ \mathrm{Im}\{(\hat{V}_{g1} V_g^* - \hat{V}_\ell V_\ell^*)(V_\ell - V_g)\}$

TABLE 7.1 (continued)

DERIVATIVES OF A REAL FUNCTION f W.R.T. CONTROL VARIABLES

Variable	Description	Derivative
$G_{\ell n}$	conductance between a load and slack buses	$2\ \mathrm{Re}\{\hat{V}_\ell V_\ell^*(V_n - V_\ell)\}$
$B_{\ell n}$	susceptance between a load and slack buses	$-2\ \mathrm{Im}\{\hat{V}_\ell V_\ell^*(V_n - V_\ell)\}$
G_{gn}	conductance between a generator and slack buses	$2\hat{V}_{g1}\ \mathrm{Re}\{V_g^*(V_n - V_g)\}$
B_{gn}	susceptance between a generator and slack buses	$-2\ \hat{V}_{g1}\ \mathrm{Im}\{V_g^* V_n\}$

where $G_{mm'}$ and $B_{mm'}$ denote, respectively, the conductance and susceptance of line mm' connecting buses m and m', m'=0 denotes the ground.

7.5.6 Example 7.6

Also, for the second system, we have

$$\frac{df}{dP_1} = 2\hat{V}_{11} = 0.0603,$$

$$\frac{df}{d|V_1|} = 2\hat{V}_{12} = -0.0577,$$

$$\frac{df}{dV_{21}} = 2\hat{V}_{21} = 0.5346,$$

$$\frac{df}{dB_{10}} = 0.0,$$

$$\frac{df}{dG_{12}} = 2\hat{V}_{11} \, Re\{V_1^* \, (V_2 - V_1)\} = 0.0044$$

and

$$\frac{df}{dB_{12}} = -2\hat{V}_{11} \, Im\{V_1^* V_2\} = -0.0108.$$

The gradients obtained can be easily checked by small perturbations about the base-case values. Note that the derivatives of the function f w.r.t. nonexisting parameters, e.g., G_{10}, can be evaluated as well.

7.6 SENSITIVITY OF COMPLEX FUNCTIONS

In the previous sections, we have derived the required sensitivity expressions and gradients for a general real function. The relationships between different modes of formulation have been investigated and expressions relating the RHS and solution vector of corresponding adjoint systems have been derived.

The sensitivities of a general complex function can be obtained using the previous formulas derived, simply, by considering the real and imaginary parts separately. In this case, only the RHS of the adjoint system of equations has to be changed. In other words, only one forward and one backward substitutions are required for each real function, provided that the LU factors of the formed matrix of coefficients are stored and that the base-case point remains unchanged.

In this section, we show how the compact complex formulation can be exploited to formulate the adjoint system corresponding to a general complex function and to derive the required sensitivities. The relationships between different modes of formulation are, again, investigated for the complex function case.

7.6.1 General Derivation

For a complex function f, we may write, using (3.14)

$$\delta f = [\hat{\mu}^T \;\; \hat{\bar{\mu}}^T] \begin{bmatrix} \delta V_M \\ \delta V_M^* \end{bmatrix} + \delta f_\rho, \tag{7.108}$$

where we have defined

$$\hat{\mu} \triangleq \frac{\partial f}{\partial V_M} \tag{7.109}$$

and

$$\hat{\bar{\mu}} \triangleq \frac{\partial f}{\partial V_M^*}, \tag{7.110}$$

δf_ρ being the change in f due to changes in other variables in terms of which f may be explicitly expressed. Hence, from (7.44)

$$\delta f = [\hat{\mu}^T \;\; \hat{\bar{\mu}}^T] \cdot \begin{bmatrix} K & \bar{K} \\ \bar{K}^* & K^* \end{bmatrix}^{-1} \begin{bmatrix} d \\ d^* \end{bmatrix} + \delta f_\rho \tag{7.111}$$

or

$$\delta f = [\hat{V}^T \; \hat{\bar{V}}^T] \begin{bmatrix} d \\ d^* \end{bmatrix} + \delta f_\rho, \tag{7.112}$$

where

$$\begin{bmatrix} K^T & \bar{K}^{*T} \\ \bar{K}^T & K^{*T} \end{bmatrix} \begin{bmatrix} \hat{V} \\ \hat{\bar{V}} \end{bmatrix} = \begin{bmatrix} \hat{\mu} \\ \hat{\bar{\mu}} \end{bmatrix} \tag{7.113}$$

which represents the adjoint system of equations to be solved. The first-order change of the complex function f can be evaluated by solving

(7.113) and substituting into (7.112).

7.6.2 Theory of Adjoint Relationships

The relationships between the adjoint solution of different modes of formulation are derived as follows. Let

$$f = f_1 + j\, f_2 , \tag{7.114}$$

hence

$$\delta f = \delta f_1 + j\, \delta f_2 \tag{7.115}$$

and let $\hat{\underset{\sim}{V}}{}_r^1$ and $\hat{\underset{\sim}{V}}{}_s^1$ be the solution of the adjoint system (7.56) using cartesian coordinates for the real function f_1. Similarly, let $\hat{\underset{\sim}{V}}{}_r^2$ and $\hat{\underset{\sim}{V}}{}_s^2$ be the solution of (7.56) for the real function f_2. Hence, using (7.55) and (7.112), one may write

$$\hat{\underset{\sim}{V}}{}^T \underset{\sim}{d}' + \hat{\bar{\underset{\sim}{V}}}{}^T \underset{\sim}{d}^* = (\hat{\underset{\sim}{V}}{}_r^{1T}\, \underset{\sim}{d}_1 - \hat{\underset{\sim}{V}}{}_s^{1T}\, \underset{\sim}{d}_2) + j(\hat{\underset{\sim}{V}}{}_r^{2T}\, \underset{\sim}{d}_1 - \hat{\underset{\sim}{V}}{}_s^{2T}\, \underset{\sim}{d}_2), \tag{7.116}$$

hence, from (7.21)

$$\hat{\underset{\sim}{V}} = (\hat{\underset{\sim}{V}}{}_r^1 - \hat{\underset{\sim}{V}}{}_s^2)/2 + j(\hat{\underset{\sim}{V}}{}_s^1 + \hat{\underset{\sim}{V}}{}_r^2)/2 \tag{7.117}$$

and

$$\hat{\bar{\underset{\sim}{V}}} = (\hat{\underset{\sim}{V}}{}_r^1 + \hat{\underset{\sim}{V}}{}_s^2)/2 + j(-\hat{\underset{\sim}{V}}{}_s^1 + \hat{\underset{\sim}{V}}{}_r^2)/2. \tag{7.118}$$

Equations (7.117) and (7.118) relate the solutions of the adjoint system (7.56) for both f_1 and f_2 to the solution of (7.113) for the complex function f.

Similarly, let $\hat{\underset{\sim}{V}}{}_\delta^1$ and $\hat{\underset{\sim}{V}}{}_v^1$ be the solution of the adjoint system (7.66) using polar coordinates for the real function f_1. Also, let $\hat{\underset{\sim}{V}}{}_\delta^2$ and $\hat{\underset{\sim}{V}}{}_v^2$ be the solution of (7.66) for the real function f_2. Hence, using (7.65) and (7.112), one may write

$$\hat{\underset{\sim}{V}}^T \underset{\sim}{d} + \hat{\bar{\underset{\sim}{V}}}^T \underset{\sim}{d}^* = (\hat{\underset{\sim}{V}}_\delta^{1T} \underset{\sim}{d}_1 - \hat{\underset{\sim}{V}}_v^{1T} \underset{\sim}{d}_2) + j(\hat{\underset{\sim}{V}}_\delta^{2T} \underset{\sim}{d}_1 - \hat{\underset{\sim}{V}}_v^{2T} \underset{\sim}{d}_2), \quad (7.119)$$

hence, from (7.21)

$$\hat{\underset{\sim}{V}} = (\hat{\underset{\sim}{V}}_\delta^1 - \hat{\underset{\sim}{V}}_v^2)/2 + j(\hat{\underset{\sim}{V}}_v^1 + \hat{\underset{\sim}{V}}_\delta^2)/2 \quad (7.120)$$

and

$$\hat{\bar{\underset{\sim}{V}}} = (\hat{\underset{\sim}{V}}_\delta^1 + \hat{\underset{\sim}{V}}_v^2)/2 + j(-\hat{\underset{\sim}{V}}_v^1 + \hat{\underset{\sim}{V}}_\delta^2)/2. \quad (7.121)$$

Equations (7.120) and (7.121) relate the solutions of the adjoint system (7.66) for both f_1 and f_2 to the solution of (7.113) for the complex function f.

7.6.3 Gradient Calculations

For gradient calculations, we proceed as before and use the partitioned forms (7.73), (7.75) and

$$\hat{\bar{\underset{\sim}{V}}} = \begin{bmatrix} \hat{\bar{\underset{\sim}{V}}}_L \\ \hat{\bar{\underset{\sim}{V}}}_G \\ \hat{\bar{V}}_n \end{bmatrix}, \quad (7.122)$$

and we write (7.49) in the form

$$\delta f = \hat{\underset{\sim}{V}}_L^T \underset{\sim}{d}_L + \hat{\underset{\sim}{V}}_G^T \underset{\sim}{d}_G + \hat{V}_n d_n + (\frac{\partial f}{\partial \underset{\sim}{\rho}})^T \delta \underset{\sim}{\rho}$$

$$+ \hat{\bar{\underset{\sim}{V}}}_L^T \underset{\sim}{d}_L^* + \hat{\bar{\underset{\sim}{V}}}_G^T \underset{\sim}{d}_G^* + \hat{\bar{V}}_n d_n^* + (\frac{\partial f}{\partial \underset{\sim}{\rho}^*})^T \delta \underset{\sim}{\rho}^*. \quad (7.123)$$

The first, second and third terms of (7.123) are given by (7.78), (7.80) and (7.81), respectively. The fifth term of (7.123) is given, using (7.74), by

$$\hat{\bar{V}}_L^T \, d_L^* = \sum_{\ell=1}^{n_L} (\hat{\bar{V}}_\ell \, \delta S_\ell) + \sum_{\ell=1}^{n_L} \sum_{\substack{m=1 \\ m\neq\ell}}^{n} \hat{\bar{V}}_\ell \, V_\ell (V_m^* - V_\ell^*) \, \delta y_{\ell m}^*$$

$$- \sum_{\ell=1}^{n} \hat{\bar{V}}_\ell \, V_\ell \, V_\ell^* \, \delta y_{\ell 0}^* . \qquad (7.124)$$

The sixth term of (7.123) is given, using (7.12), by

$$\hat{\bar{V}}^T \, d_G^* = \sum_{g=n_L+1}^{n-1} \hat{\bar{V}}_g (\delta P_g + j\delta |V_g|) + \sum_{g=n_L+1}^{n-1} \sum_{\substack{m=1 \\ m\neq g}}^{n} \hat{\bar{V}}_g \, \mathrm{Re}\{V_g^* (V_m - V_g) \, \delta y_{gm}\}$$

$$- \sum_{g=n_L+1}^{n-1} \hat{\bar{V}}_g \, \mathrm{Re}\{V_g^* \, V_g \, \delta y_{g0}\} . \qquad (7.125)$$

The seventh term of (7.123) is given, using (7.4) by

$$\hat{\bar{V}}_n \, d_n^* = \hat{\bar{V}}_n \, \delta V_n . \qquad (7.126)$$

Equations (7.78), (7.80), (7.81), (7.124)-(7.126) provide useful information for gradient evaluation of the complex function f w.r.t. the control variables of interest. Under the assumption that ρ does not contain such control variables, the derivatives of the complex function f are obtained as follows.

7.6.4 Derivatives of Complex Function w.r.t. Bus-Type Variables

From (7.78) and (7.124), the derivatives of f w.r.t. the demand S_ℓ and S_ℓ^* at load bus ℓ is given by

$$\frac{df}{dS_\ell} = \hat{\tilde{V}}_\ell \tag{7.127}$$

and

$$\frac{df}{dS_\ell^*} = \hat{V}_\ell . \tag{7.128}$$

From (7.80) and (7.125), the derivatives of f w.r.t. the generator control variables are given by

$$\frac{df}{d\tilde{S}_g} = \hat{\tilde{V}}_g \tag{7.129}$$

and

$$\frac{df}{d\tilde{S}_g^*} = \hat{V}_g , \tag{7.130}$$

where $\tilde{S}_g$ is given by (7.86). From (7.81) and (7.126), the derivatives of f w.r.t. the slack bus voltage V_n and V_n^* are given by

$$\frac{df}{dV_n} = \hat{\tilde{V}}_n \tag{7.131}$$

and

$$\frac{df}{dV_n^*} = \hat{V}_n . \tag{7.132}$$

7.6.5 Derivatives of Complex Function w.r.t. Line Variables

The derivatives of f w.r.t. line control variables y_{ij} can be obtained from (7.78), (7.80), (7.124) and (7.125) as follows. For $y_{\ell\ell'}$ between load buses ℓ and ℓ', we have from (7.78) and (7.124)

$$\frac{df}{dy_{\ell\ell'}} = (\hat{V}_\ell V_\ell^* - \hat{V}_{\ell'} V_{\ell'}^*)(V_{\ell'} - V_\ell) \tag{7.133}$$

and

$$\frac{df}{dy^*_{\ell\ell'}} = (\hat{\bar{V}}_\ell \, V_\ell - \hat{\bar{V}}_{\ell'} \, V_{\ell'}) \, (V^*_{\ell'} - V^*_\ell). \qquad (7.134)$$

For $y_{\ell 0}$ between load bus ℓ and ground, we have from (7.78) and (7.124)

$$\frac{df}{dy_{\ell 0}} = - \hat{V}_\ell \, V^*_\ell \, V_\ell \qquad (7.135)$$

and

$$\frac{df}{dy^*_{\ell 0}} = - \hat{\bar{V}}_\ell \, V_\ell \, V^*_\ell . \qquad (7.136)$$

For $y_{gg'}$ between generator buses g and g', we have from (7.80) and (7.125)

$$\frac{df}{dy_{gg'}} = \frac{1}{2} \, [(\hat{V}_g + \hat{\bar{V}}_g)V^*_g - (\hat{V}_{g'} + \hat{\bar{V}}_{g'})V^*_{g'}](V_{g'} - V_g) \qquad (7.137)$$

and

$$\frac{df}{dy_{gg'}} = \frac{1}{2} \, [(\hat{V}_g + \hat{\bar{V}}_g)V_g - (\hat{V}_{g'} + \hat{\bar{V}}_{g'})V_{g'}](V^*_{g'} - V^*_g). \qquad (7.138)$$

For y_{g0} between generator bus g and ground, we have from (7.80) and (7.125)

$$\frac{df}{dy_{g0}} = \frac{df}{dy^*_{g0}} = - \frac{1}{2} \, (\hat{V}_g + \hat{\bar{V}}_g) \, V^*_g V_g . \qquad (7.139)$$

For $y_{\ell g}$ between load bus ℓ and generator bus g, we have from (7.78), (7.80), (7.124) and (7.125)

$$\frac{df}{dy_{\ell g}} = [\frac{1}{2} \, (\hat{V}_g + \hat{\bar{V}}_g) \, V^*_g - \hat{V}_\ell \, V^*_\ell] \, (V_\ell - V_g) \qquad (7.140)$$

and

$$\frac{df}{dy^*_{\ell g}} = [\frac{1}{2}(\hat{V}_g + \hat{\bar{V}}_g) V_g - \hat{\bar{V}}_\ell V_\ell] (V^*_\ell - V^*_g). \qquad (7.141)$$

For $y_{\ell n}$ between load bus ℓ and the slack bus n, we have from (7.78) and (7.124)

$$\frac{df}{dy_{\ell n}} = \hat{V}_\ell V^*_\ell (V_n - V_\ell) \qquad (7.142)$$

and

$$\frac{df}{dy^*_{\ell n}} = \hat{\bar{V}}_\ell V_\ell (V^*_n - V^*_\ell). \qquad (7.143)$$

Finally, for y_{gn} between generator bus g and the slack bus n, we have from (7.80) and (7.125)

$$\frac{df}{dy_{gn}} = \frac{1}{2}(\hat{V}_g + \hat{\bar{V}}_g) V^*_g (V_n - V_g) \qquad (7.144)$$

and

$$\frac{df}{dy^*_{gn}} = \frac{1}{2}(\hat{V}_g + \hat{\bar{V}}_g) V_g (V^*_n - V^*_g) . \qquad (7.145)$$

7.6.6 Special Considerations

If $\underset{\sim}{\rho}$ of (7.123) contains any of the foregoing control variables, the partial derivatives of f w.r.t. appropriate control variables must be added to the expressions (7.127)-(7.145).

Equations (7.127)-(7.145) compactly define the required formal derivatives of the complex function f w.r.t. complex control variables. The gradients of f w.r.t. real and imaginary parts of the control variables are obtained using

$$\frac{df}{du_{k1}} = \frac{df}{du_k} + \frac{df}{du_k^*} \qquad (7.146)$$

and

$$\frac{df}{du_{k2}} = j\left(\frac{df}{du_k} - \frac{df}{du_k^*} \right), \qquad (7.147)$$

where u_k is given by (7.107). Expressions of forms (7.146) and (7.147) can be directly obtained from (7.127)-(7.145).

7.6.7 Example 7.7

Now, we consider the first 2-bus example and the complex function

$$f = V_1 = V_{11} + j\,V_{12}.$$

Using cartesian coordinates, the adjoint system solutions for V_{11} and V_{12} are given, respectively, by

$$\hat{\underset{\sim}{V}}{}_r^1 = \begin{bmatrix} 0.0883 \\ 2.3144 \end{bmatrix},$$

$$\hat{\underset{\sim}{V}}{}_s^1 = \begin{bmatrix} 0.1161 \\ 0.2041 \end{bmatrix},$$

$$\underset{\sim}{V}{}_r^2 = \begin{bmatrix} 0.0428 \\ 0.1117 \end{bmatrix}$$

and

$$\underset{\sim}{V}{}_s^2 = \begin{bmatrix} -0.0187 \\ 0.7352 \end{bmatrix},$$

hence, from (7.117) and (7.118),

$$\hat{V} = \begin{bmatrix} 0.0535 + j\,0.0794 \\ 0.7896 + j\,0.1579 \end{bmatrix}$$

and

$$\hat{\bar{V}} = \begin{bmatrix} 0.0348 - j\,0.0366 \\ 1.5248 - j\,0.0462 \end{bmatrix}.$$

The derivatives of f w.r.t. control variables are calculated, using the derived expressions, as follows. For S_1,

$$\frac{df}{dS_1} = \hat{\bar{V}}_1 = 0.0348 - j\,0.0366$$

and

$$\frac{df}{dS_1^*} = \hat{V}_1 = 0.0535 + j\,0.0794,$$

hence, from (7.146) and (7.147),

$$\frac{df}{dP_1} = 0.0883 - j\,0.0428$$

and

$$\frac{df}{dQ_1} = 0.1161 - j\,0.0187.$$

For V_2,

$$\frac{df}{dV_2} = \hat{\bar{V}}_2 = 1.5248 - j\,0.0462$$

and

$$\frac{df}{dV_2^*} = \hat{V}_2 = 0.7896 + j\,0.1579,$$

hence, from (7.146)

$$\frac{df}{dV_{21}} = 2.3144 + j\,0.1117.$$

For y_{10},

$$\frac{df}{dy_{10}} = - |V_1|^2 \hat{V}_1 = - 0.0311 - j\,0.0462$$

and

$$\frac{df}{dy_{10}^*} = - |V_1|^2 \hat{\hat{V}}_1 = - 0.0203 + j\,0.0213,$$

hence, from (7.146) and (7.147)

$$\frac{df}{dG_{10}} = - 0.0514 - j\,0.0249$$

and

$$\frac{df}{dB_{10}} = 0.0676 - j\,0.0109.$$

For y_{12},

$$\frac{df}{dy_{12}} = \hat{V}_1 V_1^* (V_2 - V_1) = - 0.0080 + j\,0.0231$$

and

$$\frac{df}{dy_{12}^*} = \hat{\hat{V}}_1 V_1 (V_2^* - V_1^*) = - 0.0022 - j\,0.0127,$$

hence, from (7.146) and (7.147)

$$\frac{df}{dG_{12}} = - 0.0102 + j\,0.0104$$

and

$$\frac{df}{dB_{12}} = - 0.0358 - j\,0.0059.$$

7.7 CONCLUSIONS

A unified study for the class of adjoint network approaches to power system sensitivity analysis which exploits the Jacobian matrix of the load flow solution has been presented. Generalized sensitivity expressions which are easily derived, compactly described and effectively used for calculating first-order changes and gradients of functions of interest have been obtained. These generalized sensitivity expressions are common to all modes of formulation, e.g., polar and cartesian.

A first step towards deriving these generalized sensitivity expressions has been performed where we have utilized a special complex notation to compactly describe the transformations relating different ways of formulating power network equations to a standard complex form. This special notation and the derived transformations have been used to effectively derive the required sensitivity expressions only by matrix manipulations.

The use of these generalized sensitivity expressions requires only the solution of an adjoint system of linear equations, the matrix of coefficients of which is simply the transpose of the Jacobian matrix of the load flow solution in any mode of formulation. These generalized sensitivity expressions are applicable to both real and complex modes of performance functions as well as the control variables defined in a particular study.

8

LAGRANGIAN VS TELLEGEN APPROACHES TO NETWORK SENSITIVITY ANALYSIS: A UNIFIED COMPREHENSIVE COMPARISON

8.1 INTRODUCTION

Two approaches, namely the Lagrange multiplier approach and Tellegen's theorem approach are widely used for sensitivity calculations in both electronic and power networks. Methods based on the two approaches have been described and applied on an individual basis. A combination of the two approaches has been proposed in Chapter 6.

The material presented in this chapter aims at investigating relationships between the two approaches. This investigation is accomplished by employing common bases of description and analysis through which the required aspects of comparison can be clearly stated.

8.2 BASIC FORMULATION

As stated in Chapter 2, we denote by f a single valued continuous real or complex function of n_x system state variables $\underset{\sim}{x}$ and n_u control variables $\underset{\sim}{u}$ which may be real or complex. We also denote by $\underset{\sim}{h}$ a set of n_x real or complex equality constraints relating $\underset{\sim}{x}$ to $\underset{\sim}{u}$.

The first-order change of f is written as

$$\delta f = \underset{\sim}{f}_x^T \, \delta \underset{\sim}{x} + \underset{\sim}{f}_u^T \, \delta \underset{\sim}{u}, \tag{8.1}$$

where $\underset{\sim}{f}_x$ and $\underset{\sim}{f}_u$ denote $\partial f/\partial \underset{\sim}{x}$ and $\partial f/\partial \underset{\sim}{u}$, respectively. Also, the first-order change of $\underset{\sim}{h}$ is written as

$$\delta \underset{\sim}{h} = \underset{\sim}{H}_x \, \delta \underset{\sim}{x} + \underset{\sim}{H}_u \, \delta \underset{\sim}{u} = \underset{\sim}{0}, \qquad (8.2)$$

where $\underset{\sim}{H}_x$ and $\underset{\sim}{H}_u$ stand for $(\partial \underset{\sim}{h}^T/\partial \underset{\sim}{x})^T$ and $(\partial \underset{\sim}{h}^T/\partial \underset{\sim}{u})^T$, respectively.

In the case of complex variables, $\underset{\sim}{x}$ and $\underset{\sim}{u}$ may contain complex conjugate pairs and $\underset{\sim}{f}_x$, $\underset{\sim}{f}_u$, $\underset{\sim}{H}_x$ and $\underset{\sim}{H}_u$ of (8.1) and (8.2) may represent formal partial derivatives w.r.t. the complex variables $\underset{\sim}{x}$ and $\underset{\sim}{u}$.

As described in previous chapters, when dealing with electrical networks, $\underset{\sim}{x}$ and $\underset{\sim}{u}$ may be classified into 2-component subvectors $\underset{\sim}{x}_b$ and $\underset{\sim}{u}_b$, respectively, associated with different element (branch) types. In general, $\underset{\sim}{x}_b$ and $\underset{\sim}{u}_b$ constitute node variables $\underset{\sim}{x}_m$ and $\underset{\sim}{u}_m$ and line variables $\underset{\sim}{x}_t$ and $\underset{\sim}{u}_t$. For example, $\underset{\sim}{x}_m$ may represent node voltages in a typical linear electronic network. In this case the components of $\underset{\sim}{x}_m$ are, e.g., V_{m1} and V_{m2} or V_m and V_m^*, where V_{m1} and V_{m2} are, respectively, the real and imaginary parts of V_m. In power networks $\underset{\sim}{x}_m$ and $\underset{\sim}{u}_m$ are further classified into vectors associated with load ($\underset{\sim}{x}_\ell$, $\underset{\sim}{u}_\ell$), generator ($\underset{\sim}{x}_g$, $\underset{\sim}{u}_g$) and slack generator ($\underset{\sim}{x}_n$, $\underset{\sim}{u}_n$) branches.

In general, we write

$$\underset{\sim}{x} = \{\underset{\sim}{x}_b\} = \{\underset{\sim}{x}_m, \underset{\sim}{x}_t\} \qquad (8.3)$$

and

$$\underset{\sim}{u} = \{\underset{\sim}{u}_b\} = \{\underset{\sim}{u}_m, \underset{\sim}{u}_t\}. \qquad (8.4)$$

In the above formulation, we have assumed that the number of state variables defined is $2n_B$, n_B denoting number of branches in the network. This assumption is made to simplify the comparison between Lagrange multiplier and Tellegen's theorem approaches performed in the following sections. Both of these approaches can, however, be applied for a general number of state variables (Bandler and El-Kady 1980g).

8.3 DESCRIPTION OF LAGRANGE MULTIPLIER APPROACH

In this approach, and as described in Chapter 2, we use (8.2) to write the first-order change δf of (8.1) in the form

$$\delta f = (\underset{\sim}{f}_u - \underset{\sim}{H}_u^T \underset{\sim}{\lambda})^T \delta \underset{\sim}{u}, \tag{8.5}$$

where $\underset{\sim}{\lambda}$ is a vector of the n_x Lagrange multipliers obtained by solving the adjoint equations

$$\underset{\sim}{H}_x^T \underset{\sim}{\lambda} = \underset{\sim}{f}_x. \tag{8.6}$$

Hence, from (8.5)

$$\frac{df}{d\underset{\sim}{u}} = \underset{\sim}{f}_u - \underset{\sim}{H}_u^T \underset{\sim}{\lambda}. \tag{8.7}$$

For use later, we now describe the approach in a slightly different way. We employ the classifications of (8.3) and (8.4) to define the element-local Lagrangian term as

$$\delta L_b \triangleq (\underset{\sim}{\lambda}^T \underset{\sim}{H}_{bx}) \delta \underset{\sim}{x}_b + (\underset{\sim}{\lambda}^T \underset{\sim}{H}_{bu}) \delta \underset{\sim}{u}_b, \tag{8.8}$$

where

$$\underset{\sim}{H}_x = [\underset{\sim}{H}_{1x} \ldots \underset{\sim}{H}_{n_B x}] \tag{8.9}$$

and

$$\underset{\sim}{H}_u = [\underset{\sim}{H}_{1u} \ldots \underset{\sim}{H}_{n_B u}], \tag{8.10}$$

$\underset{\sim}{H}_{bx}$ and $\underset{\sim}{H}_{bu}$ being $2n_B$x2 submatrices. We also define

$$\delta L \triangleq \sum_b \delta L_b, \tag{8.11}$$

hence, from (8.2) and (8.8)

$$\delta L = 0. \tag{8.12}$$

Using (8.8), (8.12) and

$$\delta f = \sum_b (\underset{\sim}{f}_{xb}^T \delta \underset{\sim}{x}_b + \underset{\sim}{f}_{ub}^T \delta \underset{\sim}{u}_b), \tag{8.13}$$

we may write, from (8.11)

$$\delta \mathcal{L} = \delta f - \sum_b [(f_{xb}^T - \lambda^T H_{bx}) \delta x_b + (f_{ub}^T - \lambda^T H_{bu}) \delta u_b]. \quad (8.14)$$

Observe that when λ of (8.14) satisfies (8.6), namely

$$H_{bx}^T \lambda = f_{xb}, \text{ for all } b, \quad (8.15)$$

then (8.14) reduces to

$$\delta \mathcal{L} = \delta f - \sum_b (f_{ub} - H_{bu}^T \lambda)^T \delta u_b, \quad (8.16)$$

hence, from (8.12)

$$\delta f = \sum_b (f_{ub} - H_{bu}^T \lambda)^T \delta u_b, \quad (8.17)$$

so that

$$\frac{df}{du_b} = f_{ub} - H_{bu}^T \lambda \quad (8.18)$$

which is a form of (8.7).

8.4 DESCRIPTION OF TELLEGEN'S THEOREM APPROACH

In this approach, the application of Tellegen's theorem results in the identity

$$\delta T = 0, \quad (8.19)$$

where

$$\delta T \triangleq \sum_b \delta T_b, \quad (8.20)$$

the element-local Tellegen term δT_b is defined as

$$\delta T_b \triangleq \hat{\eta}_{bx}^T \delta x_b + \hat{\eta}_{bu}^T \delta u_b \quad (8.21)$$

and the 2-component vectors $\hat{\eta}_{bx}$ and $\hat{\eta}_{bu}$ are linear functions of the formulated adjoint network current variables $\hat{I}_b$ and voltage variables $\hat{V}_b$

(and their complex conjugate). Using (8.21) and (8.13), we may write, from (8.20)

$$\delta T = \delta f - \sum_b [(f_{xb}^T - \hat{\eta}_{bx}^T)\, \delta x_b + (f_{ub}^T - \hat{\eta}_{bu}^T)\, \delta u_b]. \tag{8.22}$$

The adjoint network is defined by setting

$$\hat{\eta}_{bx} = f_{xb}, \tag{8.23}$$

hence (8.22) reduces to

$$\delta T = \delta f - \sum_b (f_{ub} - \hat{\eta}_{bu})^T\, \delta u_b. \tag{8.24}$$

Thus, from (8.19)

$$\frac{df}{du_b} = f_{ub} - \hat{\eta}_{bu}. \tag{8.25}$$

8.5 ANALOGY AND COMPARISON

In the last two sections, we have described, on a unified basis, both the Lagrange multiplier and Tellegen's theorem approaches to sensitivity calculations in electrical networks. In this section, we investigate the analogous features of the two approaches and state a general comparison between them.

First, we remark on the resemblance between the element-local Lagrangian term δL_b of (8.8) and the element-local Tellegen term δT_b of (8.21). We also remark on the resemblance between equation (8.12) formed to satisfy (8.2), namely, the network equations and equation (8.19) formed by applying Tellegen's theorem. The δf of (8.14) and (8.22) is expressed solely in terms of variations of control variables via defining, respectively, the adjoint systems (8.15) and (8.23). The solution of the adjoint network is then used to obtain the total derivatives $df/d\underset{\sim}{u}_b$ from (8.18) and (8.25), respectively.

In the Lagrange multiplier approach, the adjoint system of equations to be solved for the adjoint variables (Lagrange multipliers) $\underset{\sim}{\lambda}$ constitutes a $2n_B \times 2n_B$ matrix of coefficients. In general, when other state variables are defined the order of the matrix of coefficients is determined by the total number of state variables considered. On the other hand, the adjoint system of equations in the Tellegen's theorem approach represents a set of network equations and constitutes only a $2n \times 2n$ matrix of coefficients. The compactness of the adjoint system formulation in the Tellegen's theorem approach is afforded in essence by realizing, when formulating the adjoint equations, Kirchhoff's relations between the different adjoint variables which constitute a fictitious electrical network.

Assuming that the effort required is divided into formulation and solution parts of the adjoint system, we immediately see that the Tellegen's theorem approach sweeps the major effort into the formulation part and results in only 2n adjoint equations to be solved. In contrast, the Lagrange multiplier approach requires almost nothing to formulate the adjoint system which then constitutes n_x adjoint equations to be solved.

Note that if we formulate the vectors $\underset{\sim}{I}$ and $\underset{\sim}{V}$ to contain all branch current and voltage variables, respectively, and consider (Wu 1980) the perturbed relationships

$$\delta \underset{\sim}{I} = \underset{\sim}{H}_{ix}\, \delta \underset{\sim}{x} + \underset{\sim}{H}_{iu}\, \delta \underset{\sim}{u}, \tag{8.26}$$

$$\delta \underset{\sim}{V} = \underset{\sim}{H}_{vx}\, \delta \underset{\sim}{x} + \underset{\sim}{H}_{vu}\, \delta \underset{\sim}{u} = \underset{\sim}{A}^T\, \delta \underset{\sim}{V}_M, \tag{8.27}$$

$$\underset{\sim}{A}\ \delta \underset{\sim}{I} = \underset{\sim}{0} \tag{8.28}$$

and (8.1), where $\underset{\sim}{A}$ is a form of incidence matrix and $\underset{\sim}{V}_M$ contains node (bus) voltage variables, it is straightforward to show that a vector $\hat{\underset{\sim}{n}}_u$, which contains all the $\hat{\underset{\sim}{n}}_{bu}$ of (8.21), is given by

$$\hat{\underset{\sim}{n}}_u = \underset{\sim}{H}_{vu}^T \underset{\sim}{\lambda}_i + \underset{\sim}{H}_{iu}^T \underset{\sim}{\lambda}_v, \tag{8.29}$$

where $\underset{\sim}{\lambda}_i$ and $\underset{\sim}{\lambda}_v$ satisfy KCL and KVL, respectively, and the relationship

$$\underset{\sim}{H}_{vx}^T \underset{\sim}{\lambda}_i + \underset{\sim}{H}_{ix}^T \underset{\sim}{\lambda}_v = \underset{\sim}{f}_x. \tag{8.30}$$

8.6 CONCLUSIONS

The two widely used approaches to sensitivity calculations in electrical networks, namely the Lagrange multiplier and Tellegen's theorem approaches, have been described and compared. The description has been performed on a unified basis where we have defined and employed element-local terms in formulating the two approaches so that different aspects of comparison are clearly investigated. The resemblance in formulating and analyzing the adjoint systems of the two approaches has been discussed.

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CONCLUSIONS

The material presented in this thesis has laid the foundation of a new methodology for adjoint network approaches to sensitivity calculations performed in a wide variety of power system analysis and planning problems. This methodology overcomes modelling difficulties observed by previous workers and leads to new techniques for network sensitivity evaluation.

The new methodology utilizes a useful complex notation which allows the exact power system steady-state component models to be considered. The notation has been described, theoretically justified and practically applied to several problems involving power flow analysis as well as efficient and compact computation of power network sensitivities.

We have presented a new approach to sensitivity calculations in power networks. The approach exploits the powerful features of Tellegen's theorem in an augmented form. It allows power network sensitivities to be calculated based on the a.c. power flow model in general and without any approximations. The generalized version of Chapter 3 utilizes the special complex notation and the concept of basic/element variables and their mutual transformations so that the required sensitivities may be obtained for any chosen set of real and/or complex variables of practical interest. The flexibility of modelling the adjoint elements has been afforded via complex adjoining coefficients set to proper values. We have derived and tabulated general sensitivity expressions which are common to all relevant power system studies. The approach, although applied to power networks, has been justified as a general network approach.

A simplified version imbued with many desirable features has been described in Chapter 4. In this version the adjoining coefficients are set to particular values allowing the sensitivities of general real functions to be effectively calculated. This version enjoys a simple adjoint system and sensitivity expressions.

Introducing and applying the concepts of functional adjoining coefficients and appropriate group terms, a valuable capability of the adjoint approach has been exploited in Chapter 5. The derivatives of a complex function may be supplied via one adjoint analysis as in typical linear electronic circuits. A theoretical foundation has been established and a unified consistency study for proper selection of the adjoining coefficients has been performed.

The far-reaching consequences of the methodology have been exploited further in Chapter 6. We have utilized the useful features of the simplified version of Chapter 4 to describe a new method for solving the load flow problem. The method employs a sparse and mostly constant matrix of coefficients and provides the same rate of convergence as the Newton-Raphson method. It directly supplies sensitivities of the dependent variables at the load flow solution without further adjoint simulation.

The Newton-Raphson method for solving the load flow equations has been interpreted, formally, in terms of variations in complex variables and their complex conjugate and described in the compact complex form. We have employed a special elimination technique to handle the resulting form of equations.

The possibility of handling a general number of complex dependent variables defined in a particular problem has been afforded by developing a complex version of the Lagrange multiplier approach. Based on this complex version, we have stated a useful element-local Lagrangian concept which has led to a combined Tellegen/Lagrange technique. In this context, the assumption of only two states associated with a branch in the approach of Chapter 3 has been relaxed.

The experience gained so far allowed us, in Chapter 7, to describe a generalized version of the class of methods of sensitivity calculations which exploit the Jacobian matrix, already available from the load flow solution, in formulating the adjoint equations. We have defined a standard compact complex form of the power flow equations and derived transformations to other forms. These transformations have been used to establish a theoretical foundation for standard relationships between the solutions of adjoint networks of different formulation. With these standard relationships, generalized sensitivity expressions for real functions have been derived and tabulated. The more general case of complex functions has also been studied and corresponding sensitivity formulas have been derived.

A unified comprehensive comparison between the Lagrangian and Tellegen's theorem methods of sensitivity evaluation has been presented in Chapter 8. It highlights some analytic features of the techniques employed in Chapters 3-7.

The simplicity of the examples employed throughout the thesis, we feel, contributes significantly to an easier understanding of the theory. Our sensitivity expressions are, however, general and applicable to networks of practical size.

The work presented in this thesis is believed to provide some promising research directions for the future in the general area of modelling and analysis of power networks. In particular, all previous Tellegen's theorem-based applications which have been performed, for lack of suitable methodology, based on approximated power flow models can be performed based on the exact a.c. power flow model. For instance, some transmission planning methods (Püttgen 1976) can be utilized for short term planning where actual control models are more important. Moreover, since our methodology affords a very convenient way of treating exact element models in power networks, applications to multi area structures can be done in a similarly compact and simple manner. Diakoptical-based sensitivity methods which exploit Tellegen's theorem to handle large power networks may be developed where the extended Tellegen's sum is partitioned to distinguish the elements associated with each part of the main network and common variables simulating the tie lines.

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APPENDIX A

DERIVATION OF π-EQUIVALENT NETWORKS FOR PHASE SHIFTING TRANSFORMERS

For the structure shown in Fig. 3.5, we have the relationships (Stagg and El-Abiad 1968)

$$I_p = \frac{1}{Z_t a a^*} (V_p - aV_q) \qquad (A.1)$$

and

$$I_q = \frac{1}{Z_t a} (aV_q - V_p), \qquad (A.2)$$

where I_p and I_q are, respectively, the injected currents into buses p and q, or simply

$$I_p = g^{\bullet} V_p + h^{\bullet} V_q \qquad (A.3)$$

and

$$I_q = \tilde{g}^{\bullet} V_p + \tilde{h}^{\bullet} V_q, \qquad (A.4)$$

where

$$g^{\bullet} = \frac{1}{Z_t a a^*}, \qquad (A.5)$$

$$h^{\bullet} = -\frac{1}{Z_t a^*}, \qquad (A.6)$$

$$\tilde{g}^{\bullet} = -\frac{1}{Z_t a} \qquad (A.7)$$

and

$$\tilde{h}^{\bullet} = \frac{1}{Z_t} . \tag{A.8}$$

Also, we have

$$I_p = I_A + I_B \tag{A.9}$$

and

$$I_q = - I_A + I_C, \tag{A.10}$$

where I_A, I_B and I_C are the branch currents of the π-network of Fig. 3.5.

We consider the general branch modelling of the π-network

$$I_A + \tilde{\alpha}^{\bullet} I_A^{*} = \alpha^{\bullet} (V_p - V_q) + \bar{\alpha}^{\bullet} (V_p^{*} - V_q^{*}), \tag{A.11}$$

$$I_B + \tilde{\beta}^{\bullet} I_B^{*} = \beta^{\bullet} V_p + \bar{\beta}^{\bullet} V_p^{*} \tag{A.12}$$

and

$$I_C + \tilde{\gamma}^{\bullet} I_C^{*} = \gamma^{\bullet} V_q + \bar{\gamma}^{\bullet} V_q^{*}. \tag{A.13}$$

Substituting (A.3)-(A.10) into (A.11)-(A.13), it is straightforward to show that the coefficients in (A.11)-(A.13) can be chosen to satisfy

$$\alpha^{\bullet} = -h^{\bullet} , \tag{A.14}$$

$$\beta^{\bullet} = g^{\bullet} + h^{\bullet} \tag{A.15}$$

and

$$\gamma^{\bullet} = \tilde{g}^{\bullet} + \tilde{h}^{\bullet} . \tag{A.16}$$

Also,

$$\bar{\alpha}^{\bullet} = - \tilde{\beta}^{\bullet} h^{\bullet *} , \tag{A.17}$$

$$\tilde{\alpha}^{\bullet} = \tilde{\beta}^{\bullet} , \tag{A.18}$$

APPENDIX B

DERIVATION OF STANDARD ELEMENT MATRICES

Loads

For a load the Jacobian $\underset{\sim}{J}_\ell$ using the set of element variables $\overline{\underset{\sim}{z}}_\ell$ of (3.80) is given by

$$J_\ell = \left[\begin{array}{cc|cc} V_\ell^*/[2(V_\ell V_\ell^*)^{1/2}] & V_\ell/[2(V_\ell V_\ell^*)^{1/2}] & 0 & 0 \\ -j/(2V_\ell) & j/(2V_\ell^*) & 0 & 0 \\ \hline I_\ell^*/2 & I_\ell/2 & V_\ell^*/2 & V_\ell/2 \\ -jI_\ell^*/2 & jI_\ell/2 & jV_\ell^*/2 & -jV_\ell/2 \end{array}\right], \qquad (B.1)$$

hence

$$\underset{\sim}{J}_\ell^{-1} = \left[\begin{array}{cc|cc} (V_\ell/V_\ell^*)^{1/2} & jV_\ell & 0 & 0 \\ (V_\ell^*/V_\ell)^{1/2} & -jV_\ell^* & 0 & 0 \\ \hline -I_\ell/(V_\ell V_\ell^*)^{1/2} & jI_\ell & 1/V_\ell^* & -j/V_\ell^* \\ -I_\ell^*/(V_\ell V_\ell^*)^{1/2} & -jI_\ell^* & 1/V_\ell & j/V_\ell \end{array}\right], \qquad (B.2)$$

so that

$$\overline{\underset{\sim}{\theta}}_{\ell i} = \begin{bmatrix} (\alpha V_\ell + \xi V_\ell^*)/(V_\ell V_\ell^*)^{1/2} & (\overline{\xi} V_\ell + \overline{\alpha} V_\ell^*)/(V_\ell V_\ell^*)^{1/2} \\ j(\alpha V_\ell - \xi V_\ell^*) & j(\overline{\xi} V_\ell - \overline{\alpha} V_\ell^*) \end{bmatrix}, \qquad (B.3)$$

$$\overline{\underset{\sim}{\theta}}_{\ell v} = \begin{bmatrix} -(\beta I_\ell + \nu I_\ell^*)/(V_\ell V_\ell^*)^{1/2} & -(\overline{\nu} I_\ell + \overline{\beta} I_\ell^*)/(V_\ell V_\ell^*)^{1/2} \\ j(\beta I_\ell - \nu I_\ell^*) & j(\overline{\nu} I_\ell - \overline{\beta} I_\ell^*) \end{bmatrix}, \qquad (B.4)$$

$$\underset{\sim}{M}_{11}^\ell \, \underset{\sim}{\Lambda}_i^\ell \, \underset{\sim}{w}_{\ell i} = \begin{bmatrix} \sum_k \lambda_{\ell k}[N_{i\ell}^k (V_\ell/V_\ell^*)^{1/2} + \overline{N}_{i\ell}^k (V_\ell^*/V_\ell)^{1/2}] \\ \sum_k j\lambda_{\ell k}[N_{i\ell}^k V_\ell - \overline{N}_{i\ell}^k V_\ell^*] \end{bmatrix}, \qquad (B.5)$$

$$\dot{\bar{\beta}} = (\dot{g}^{*} + \dot{h}^{*})\dot{\tilde{\beta}} \tag{A.19}$$

and

$$\dot{\bar{\gamma}} = (\dot{\tilde{g}}^{*} + \dot{\tilde{h}}^{*})\dot{\tilde{\gamma}}\,, \tag{A.20}$$

where $\dot{\tilde{\beta}}$, $\dot{\tilde{\beta}}^{*}$, $\dot{\tilde{\gamma}}$ and $\dot{\tilde{\gamma}}^{*}$ are related by

$$a^{*}\dot{\tilde{\gamma}}(\dot{\tilde{\beta}}\dot{\tilde{\beta}}^{*} - 1) = -a\dot{\tilde{\beta}}(\dot{\tilde{\gamma}}\dot{\tilde{\gamma}}^{*} - 1). \tag{A.21}$$

The choice

$$\dot{\tilde{\gamma}} = a \text{ and } \dot{\tilde{\beta}} = -a^{*} \tag{A.22}$$

provides the branch modelling shown in Fig. 3.5.

$$\underset{\sim}{M}_{12}^{\ell}\ \underset{\sim}{\Lambda}_{v}^{\ell}\ \underset{\sim}{w}_{\ell v} = \begin{bmatrix} \sum_k \lambda_{\ell k}[-N_{v\ell}^{k} I_{\ell}/(V_{\ell}V_{\ell}^{*})^{1/2} - \overline{N}_{v\ell}^{k} I_{\ell}^{*}/(V_{\ell}V_{\ell}^{*})^{1/2}] \\ \sum_k j\lambda_{\ell k}[N_{v\ell}^{k} I_{\ell} - \overline{N}_{v\ell}^{k}\ I_{\ell}^{*}] \end{bmatrix} . \quad (B.6)$$

Using the set of element variables $\tilde{\underset{\sim}{z}}_{\ell}$ of (3.81) the Jacobian is given by

$$\underset{\sim}{J}_{\ell} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline I_{\ell}^{*} & 0 & 0 & V_{\ell} \\ 0 & I_{\ell} & V_{\ell}^{*} & 0 \end{array}\right] , \quad (B.7)$$

hence

$$\underset{\sim}{J}_{\ell}^{-1} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & -I_{\ell}/V_{\ell}^{*} & 0 & 1/V_{\ell}^{*} \\ -I_{\ell}^{*}/V_{\ell} & 0 & 1/V_{\ell} & 0 \end{array}\right] , \quad (B.8)$$

so that

$$\overline{\underset{\sim}{\theta}}_{\ell i} = \begin{bmatrix} \alpha & \overline{\xi} \\ \xi & \overline{\alpha} \end{bmatrix} , \quad (B.9)$$

$$\overline{\underset{\sim}{\theta}}_{\ell v} = \begin{bmatrix} -\nu I_{\ell}^{*}/V_{\ell} & -\overline{\beta} I_{\ell}^{*}/V_{\ell} \\ -\beta I_{\ell}/V_{\ell}^{*} & -\overline{\nu} I_{\ell}/V_{\ell}^{*} \end{bmatrix} , \quad (B.10)$$

$$\underset{\sim}{M}_{11}^{\ell}\ \underset{\sim}{\Lambda}_{i}^{\ell}\ \underset{\sim}{w}_{\ell i} = \begin{bmatrix} \sum_k \lambda_{\ell k}\ N_{i\ell}^{k} \\ \sum_k \lambda_{\ell k}\ \overline{N}_{i\ell}^{k} \end{bmatrix} , \quad (B.11)$$

$$\underset{\sim}{M}_{12}^{\ell}\ \underset{\sim}{\Lambda}_{v}^{\ell}\ \underset{\sim}{w}_{\ell v} = \begin{bmatrix} \sum_k -\lambda_{\ell k}\ \overline{N}_{v\ell}^{k}\ I_{\ell}^{*}/V_{\ell} \\ \sum_k -\lambda_{\ell k}\ N_{v\ell}^{k}\ I_{\ell}/V_{\ell}^{*} \end{bmatrix} . \quad (B.12)$$

Generators

For a generator the Jacobian $\underset{\sim}{J}_g$ using the set of element variables $\underset{\sim}{\bar{z}}_g$ of (3.82) is given by

$$\underset{\sim}{J}_g = \left[\begin{array}{cc|cc} -j/(2V_g) & j/(2V_g^*) & 0 & 0 \\ -jI_g^*/2 & jI_g/2 & jV_g^*/2 & -jV_g/2 \\ \hline V_g^*/[2(V_g V_g^*)^{1/2}] & V_g/[2(V_g V_g^*)^{1/2}] & 0 & 0 \\ I_g^*/2 & I_g/2 & V_g^*/2 & V_g/2 \end{array}\right], \quad (B.13)$$

hence

$$\underset{\sim}{J}_g^{-1} = \left[\begin{array}{cc|cc} jV_g & 0 & V_g/(V_g V_g^*)^{1/2} & 0 \\ -jV_g^* & 0 & V_g^*/(V_g V_g^*)^{1/2} & 0 \\ \hline jI_g & -j/V_g^* & -I_g/(V_g V_g^*)^{1/2} & 1/V_g^* \\ -jI_g^* & j/V_g & -I_g^*/(V_g V_g^*)^{1/2} & 1/V_g \end{array}\right], \quad (B.14)$$

so that

$$\underset{\sim}{\bar{\theta}}_{gi} = \begin{bmatrix} j(\alpha V_g - \xi V_g^*) & j(\bar{\xi} V_g - \bar{\alpha} V_g^*) \\ 0 & 0 \end{bmatrix}, \quad (B.15)$$

$$\underset{\sim}{\bar{\theta}}_{gv} = \begin{bmatrix} j(\beta I_g - \nu I_g^*) & j(\bar{\nu} I_g - \bar{\beta} I_g^*) \\ j(-\beta/V_g^* + \nu/V_g) & j(-\bar{\nu}/V_g^* + \bar{\beta}/V_g) \end{bmatrix}, \quad (B.16)$$

$$\underset{\sim}{M}_{11}^g \, \underset{\sim}{\Lambda}_i^g \, \underset{\sim}{w}_{gi} = \begin{bmatrix} \sum_k j\lambda_{gk}[N_{ig}^k V_g - \bar{N}_{ig}^k V_g^*] \\ 0 \end{bmatrix}, \quad (B.17)$$

$$\underset{\sim}{M}_{12}^g \, \underset{\sim}{\Lambda}_v^g \, \underset{\sim}{w}_{gv} = \begin{bmatrix} \sum_k j\lambda_{gk}[N_{vg}^k I_g - \bar{N}_{vg}^k I_g^*] \\ \sum_k j\lambda_{gk}[-N_{vg}^k/V_g^* + \bar{N}_{vg}^k/V_g] \end{bmatrix}. \quad (B.18)$$

Using the set of element variables $\underset{\sim}{\tilde{z}}_g$ of (3.83) the Jacobian is given by

$$\underset{\sim}{J}_g = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline V_g^* & V_g & 0 & 0 \\ I_g^* & I_g & V_g^* & V_g \end{array}\right], \qquad (B.19)$$

hence

$$\underset{\sim}{J}_g^{-1} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ -V_g^*/V_g & 0 & 1/V_g & 0 \\ \hline 0 & 1 & 0 & 0 \\ (I_g V_g^*/V_g^2 - I_g^*/V_g) & -V_g^*/V_g & -I_g/V_g^2 & 1/V_g \end{array}\right], \qquad (B.20)$$

so that

$$\underset{\sim}{\overline{\theta}}_{gi} = \begin{bmatrix} (\alpha-\xi V_g^*/V_g) & (\overline{\xi}-\overline{\alpha} V_g^*/V_g) \\ 0 & 0 \end{bmatrix}, \qquad (B.21)$$

$$\underset{\sim}{\overline{\theta}}_{gv} = \begin{bmatrix} \nu(I_g V_g^*/V_g^2 - I_g^*/V_g) & \overline{\beta}(I_g V_g^*/V_g^2 - I_g^*/V_g) \\ \beta-\nu V_g^*/V_g & \overline{\nu}-\overline{\beta} V_g^*/V_g \end{bmatrix}, \qquad (B.22)$$

$$\underset{\sim}{M}_{11}^g \, \underset{\sim}{\Lambda}_i^g \, \underset{\sim}{w}_{gi} = \begin{bmatrix} \sum_k \lambda_{gk}[N_{ig}^k - \overline{N}_{ig}^k \, V_g^*/V_g] \\ 0 \end{bmatrix}, \qquad (B.23)$$

$$\underset{\sim}{M}_{12}^g \, \underset{\sim}{\Lambda}_v^g \, \underset{\sim}{w}_{gv} = \begin{bmatrix} \sum_k \lambda_{gk} \, \overline{N}_{vg}^k (I_g V_g^*/V_g^2 - I_g^*/V_g) \\ \sum_k \lambda_{gk}[N_{vg}^k - \overline{N}_{vg}^k \, V_g^*/V_g] \end{bmatrix}. \qquad (B.24)$$

<u>Slack Generator</u>

For the slack generator the Jacobian $\underset{\sim}{J}_n$ using the set of element variables $\underset{\sim}{\overline{z}}_n$ of (3.84) is given by

$$\underset{\sim}{J}_n = \left[\begin{array}{cc|cc} I_n^*/2 & I_n/2 & V_n^*/2 & V_n/2 \\ -jI_n^*/2 & jI_n/2 & jV_n^*/2 & -jV_n/2 \\ \hline V_n^*/[2(V_nV_n^*)^{1/2}] & V_n/[2(V_nV_n^*)^{1/2}] & 0 & 0 \\ -j/(2V_n) & j/(2V_n^*) & 0 & 0 \end{array}\right], \quad (B.25)$$

hence

$$\underset{\sim}{J}_n^{-1} = \left[\begin{array}{cc|cc} 0 & 0 & V_n/(V_nV_n^*)^{1/2} & jV_n \\ 0 & 0 & V_n^*/(V_nV_n^*)^{1/2} & -jV_n^* \\ \hline 1/V_n^* & -j/V_n^* & -I_n/(V_nV_n^*)^{1/2} & jI_n \\ 1/V_n & j/V_n & -I_n^*/(V_nV_n^*)^{1/2} & -jI_n^* \end{array}\right], \quad (B.26)$$

so that

$$\underset{\sim}{\overline{\theta}}_{ni} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (B.27)$$

$$\underset{\sim}{\overline{\theta}}_{nv} = \begin{bmatrix} (\beta/V_n^*+\nu/V_n) & (\overline{\nu}/V_n^*+\overline{\beta}/V_n) \\ j(-\beta/V_n^*+\nu/V_n) & j(-\overline{\nu}/V_n^*+\overline{\beta}/V_n) \end{bmatrix}, \quad (B.28)$$

$$\underset{\sim}{M}_{11}^n \, \underset{\sim}{\Lambda}_i^n \, \underset{\sim}{w}_{ni} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (B.29)$$

$$\underset{\sim}{M}_{12}^n \, \underset{\sim}{\Lambda}_n^n \, \underset{\sim}{w}_{nv} = \begin{bmatrix} \sum_k \lambda_{nk}[N_{vn}^k/V_n^*+\overline{N}_{vn}^k/V_n] \\ \sum_k j\lambda_{nk}[-N_{vn}^k/V_n^*+\overline{N}_{vn}^k/V_n] \end{bmatrix}, \quad (B.30)$$

Using the set of element variables $\tilde{\underset{\sim}{z}}_n$ of (3.85) the Jacobian is given by

$$\underset{\sim}{J}_{n} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] , \qquad \text{(B.31)}$$

hence

$$\underset{\sim}{J}_{n}^{-1} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & -0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] , \qquad \text{(B.32)}$$

so that

$$\overline{\underset{\sim}{\theta}}_{ni} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} , \qquad \text{(B.33)}$$

$$\overline{\underset{\sim}{\theta}}_{nv} = - \overline{\underset{\sim}{\Lambda}}_{v} = \begin{bmatrix} \beta & \overline{\nu} \\ \nu & \overline{\beta} \end{bmatrix} , \qquad \text{(B.34)}$$

$$\underset{\sim}{M}_{11}^{n} \underset{\sim}{\Lambda}_{i}^{n} \underset{\sim}{w}_{ni} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} ; \qquad \text{(B.35)}$$

$$\underset{\sim}{M}_{12}^{N} \underset{\sim}{\Lambda}_{v}^{N} \underset{\sim}{w}_{nv} = \begin{bmatrix} \sum_{k} \lambda_{nk} N_{vn}^{k} \\ \sum_{k} \lambda_{nk} \overline{N}_{vn}^{k} \end{bmatrix} . \qquad \text{(B.36)}$$

<u>Transmission Elements</u>

For a transmission element the Jacobian $\underset{\sim}{J}_t$ using the set of element variables $\overline{\underset{\sim}{z}}_t$ of (3.86) is given by

$$\underset{\sim}{J}_t = \left[\begin{array}{cc|cc} 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & -j/2 & j/2 \\ \hline -I_t/(2V_t^2) & -I_t^*/(2V_t^*)^2 & 1/(2V_t) & 1/(2V_t^*) \\ jI_t/(2V_t^2) & -jI_t^*/(2V_t^*)^2 & -j/(2V_t) & j/(2V_t^*) \end{array}\right] , \quad (B.37)$$

hence

$$\underset{\sim}{J}_t^{-1} = \left[\begin{array}{cc|cc} V_t/I_t & jV_t/I_t & -V_t^2/I_t & -jV_t^2/I_t \\ V_t^*/I_t^* & -jV_t^*/I_t^* & -V_t^{*2}/I_t^* & jV_t^{*2}/I_t^* \\ \hline 1 & j & 0 & 0 \\ 1 & -j & 0 & 0 \end{array}\right] , \quad (B.38)$$

so that

$$\underset{\sim}{\bar{\theta}}_{ti} = \begin{bmatrix} (\alpha V_t/I_t + \xi V_t^*/I_t^*) & (\bar{\xi} V_t/I_t + \bar{\alpha} V_t^*/I_t^*) \\ j(\alpha V_t/I_t - \xi V_t^*/I_t^*) & j(\bar{\xi} V_t/I_t - \bar{\alpha} V_t^*/I_t^*) \end{bmatrix} , \quad (B.39)$$

$$\underset{\sim}{\bar{\theta}}_{tv} = \begin{bmatrix} (\beta+\nu) & (\bar{\nu}+\bar{\beta}) \\ j(\beta-\nu) & j(\bar{\nu}-\bar{\beta}) \end{bmatrix} , \quad (B.40)$$

$$\underset{\sim}{M}_{11}^t \, \underset{\sim}{\Lambda}_i^t \, \underset{\sim}{w}_{ti} = \begin{bmatrix} \sum_k \lambda_{tk} \, [N_{it}^k \, V_t/I_t + \bar{N}_{it}^k \, V_t^*/I_t^*] \\ \sum_k j\lambda_{tk} \, [N_{it}^k \, V_t/I_t - \bar{N}_{it}^k \, V_t^*/I_t^*] \end{bmatrix} , \quad (B.41)$$

$$\underset{\sim}{M}_{12} \, \underset{\sim}{\Lambda}_v^t \, \underset{\sim}{w}_{tv} = \begin{bmatrix} \sum_k \lambda_{tk} [N_{vt}^k + \bar{N}_{vt}^k] \\ \sum_k j\lambda_{tk} [N_{vt}^k - \bar{N}_{vt}^k] \end{bmatrix} . \quad (B.42)$$

Using the set of element variables $\tilde{\underset{\sim}{z}}_t$ of (3.87) the Jacobian is given by

$$\underset{\sim}{J}_t = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -I_t/V_t^2 & 0 & 1/V_t & 0 \\ 0 & -I_t^*/V_t^{*2} & 0 & 1/V_t^* \end{array}\right] , \qquad (B.43)$$

hence

$$\underset{\sim}{J}_t^{-1} = \left[\begin{array}{cc|cc} V_t/I_t & 0 & -V_t^2/I_t & 0 \\ 0 & V_t^*/I_t^* & 0 & -V_t^{*2}/I_t^* \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] , \qquad (B.44)$$

so that

$$\underset{\sim}{\overline{\theta}}_{ti} = \begin{bmatrix} \alpha\ V_t/I_t & \overline{\xi}\ V_t/I_t \\ \xi\ V_t^*/I_t^* & \overline{\alpha}\ V_t^*/I_t^* \end{bmatrix} , \qquad (B.45)$$

$$\underset{\sim}{\overline{\theta}}_{tv} = \begin{bmatrix} \beta & \overline{\nu} \\ \nu & \overline{\beta} \end{bmatrix} , \qquad (B.46)$$

$$\underset{\sim}{M}_{11}^{t}\ \underset{\sim}{\Lambda}_i^t\ \underset{\sim}{w}_{ti} = \begin{bmatrix} \sum_k \lambda_{tk}\ N_{it}^k\ V_t/I_t \\ \sum_k \lambda_{tk}\ \overline{N}_{it}^k\ V_t^*/I_t^* \end{bmatrix} , \qquad (B.47)$$

$$\underset{\sim}{M}_{12}^{t}\ \underset{\sim}{\Lambda}_v^t\ \underset{\sim}{w}_{tv} = \begin{bmatrix} \sum_k \lambda_{tk}\ N_{vt}^k \\ \sum_k \lambda_{tk}\ \overline{N}_{vt}^k \end{bmatrix} . \qquad (B.48)$$

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