DISTRIBUTIVE LATTICES IN A LOCALIC TOPOS

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ABSTRACT

In this thesis we examine various properties of bounded distributive lattices in the topos of sheaves on a locale. We prove that **BooSh$\mathcal{L}$**, the category of Boolean algebras in **Sh$\mathcal{L}$**, is a reflective subcategory of **$\mathcal{D}$Sh$\mathcal{L}$**, the category of bounded distributive lattices in **Sh$\mathcal{L}$**. Injective distributive lattices in **Sh$\mathcal{L}$** are discussed, and two methods of constructing the injective hull of any lattice in **$\mathcal{D}$Sh$\mathcal{L}$** are described. We characterize indecomposable injectives in **$\mathcal{D}$Sh$\mathcal{L}$** and show that they are exactly the prime bounded distributive lattices. Simple lattices in **Sh$\mathcal{L}$** are described and characterized in terms of the points of $\mathcal{L}$. We examine cogenerating sets in **$\mathcal{D}$Sh$\mathcal{L}$** and the relationships among simple, prime and cogenerating objects in the category. Finally, we consider the initial object $2_{\mathcal{L}}$ of **$\mathcal{D}$Sh$\mathcal{L}$**, when it is complete and when a cogenerator; we then prove that any locale is isomorphic to the locale of congruences of $2_{\mathcal{L}}$.

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CHAPTER 1: **BooSh$\mathcal{L}$** IS A REFLECTIVE SUBCATEGORY OF **$\mathcal{D}$Sh$\mathcal{L}$**.

In this chapter we will define the internal congruence lattice $\mathcal{L}A$, for each $A \in$ **$\mathcal{D}$Sh$\mathcal{L}$**, in terms of the external lattice ConA of all congruences on A and show that ConA is a locale and $\mathcal{L}A$ is a bounded distributive lattice in **Sh$\mathcal{L}$**. We then define a map $\nabla : A \longrightarrow \mathcal{L}A$ which we use to prove that **BooSh$\mathcal{L}$** is a reflective subcategory of **$\mathcal{D}$Sh$\mathcal{L}$**.

It is useful first to recall from universal algebra the definition of a congruence Θ on an algebra A as an equivalence relation which is also a subalgebra of $A \times A$. Then a congruence Θ on an algebra $A \in$ Sh$\mathcal{L}$ is a subsheaf $\Theta \subseteq A \times A$ such that, for each place $U \in \mathcal{L}$, ΘU is a congruence on the algebra AU.

We describe ConA, for any algebra $A \in$ Sh$\mathcal{L}$ and hence for bounded distributive lattices, as follows: for $\Theta_i \in$ ConA, meet is given by intersection, with $(\bigwedge_i \Theta_i)U = \bigcap_i \Theta_i U$. Join is given by $(\bigvee_i \Theta_i)U \doteq \bigvee_i \Theta_i U$, that is, the sheaf reflection of the presheaf $\bigvee_i \Theta_i U$, where the latter join is in the congruence lattice of AU.

1.1 Proposition: For A any algebra in $\mathrm{Sh}\mathcal{L}$, ConA is a complete meet-continuous lattice.

We check that join and meet as given produce subsheaves and congruences on AU. For $U \rightsquigarrow \bigcap_i \Theta_i U$, note that the intersection of subsheaves $\Theta_i \subseteq A \times A$ is itself a subsheaf of $A \times A$, while the intersection of congruences is a congruence, so $\bigcap_i \Theta_i U$ is a subsheaf of $AU \times AU$ and a congruence on AU for each $U \in \mathcal{L}$. For $U \rightsquigarrow \bigvee_i \Theta_i U$, we need only that $\bigvee_i \Theta_i U$ is a subpresheaf of $A \times A$, i.e. a contravariant functor from $\mathcal{L}$ into the category of algebras, with an appropriate restriction map.

Let $\bigvee_i \Theta_i V$ be the congruence on AV generated by the up-directed union of the congruences $\Theta_i V$ on AV. Then a homomorphism $r: AU \longrightarrow AV$ induces a map $\bar{r}$ from the congruence lattice of AU to that of AV. Since taking inverse images under $\bar{r}$ preserves meets, $\bar{r}$ itself preserves joins and takes $\bigvee_i \Theta_i U$ to $\bigvee_i \Theta_i V$. Then, because each component of the subpresheaf belongs to the congruence lattice on AU, the subsheaf generated by the subpresheaf is a congruence on AU.

Now, meet is given by intersection, so ConA is complete in the usual lattice-theoretic sense.

ConA is also meet-continuous, i.e. for $\Phi, \Theta_i \in \mathrm{ConA}$, $\Phi \wedge \bigvee_i \Theta_i = \bigvee_i \Phi \wedge \Theta_i$, as follows: at $U \in \mathcal{L}$, $(\Phi \wedge \bigvee_i \Theta_i)U = \Phi U \cap (\bigvee_i \Theta_i)U \doteq \Phi U \cap \bigvee_i (\Theta_i U)$. Since $\bigvee_i (\Theta_i U)$ is a presheaf and ΦU is perforce a presheaf, we require the sheaf reflection of their intersection. Then we have $\Phi U \cap \bigvee_i (\Theta_i U) = \bigvee_i (\Phi U \cap \Theta_i U)$, because ΦU and $\Theta_i U$, for each i, belong to the congruence lattice of AU for each $U \in \mathcal{L}$. This congruence lattice is algebraic and therefore meet-continuous. We see that we now have the presheaf which generates the sheaf $(\bigvee_i \Phi \wedge \Theta_i)U$, from the definition of join in ConA. Hence ConA is complete and meet-continuous for any algebra $A \in \mathrm{Sh}\mathcal{L}$.

<u>Corollary</u>: <u>For $A \in \mathcal{D}\mathrm{Sh}\mathcal{L}$, ConA is a locale</u>.

The congruence lattice of AU for each $U \in \mathcal{L}$ is complete for A a distributive lattice in $\mathrm{Sh}\mathcal{L}$, therefore we have that $U \rightsquigarrow \bigvee_i \Theta_i U$ is a subpresheaf. Then for $\Phi, \Theta_i \in \mathrm{Con}A$, $(\Phi \wedge \bigvee_i \Theta_i)U \doteq \bigvee_i (\Phi U \cap \Theta_i U)$, which generates the sheaf $(\bigvee_i \Phi \wedge \Theta_i)U$. So ConA is distributive over arbitrary joins and hence is a locale.

We now define $(\mathcal{I}A)U = \mathrm{Con}(A|U)$ for each $U \in \mathcal{L}$, and proceed to show that this is a sheaf with values in $\mathcal{D}$. Note that, then, if we view $(\Omega^{A\times A})U$ as the set of all subsheaves of $(A \times A)|U$, $\mathcal{I}A \subseteq \Omega^{A\times A}$, since $\Theta \in (\mathcal{I}A)U = \mathrm{Con}(A|U)$ is a subsheaf of $A|U \times A|U = (A \times A)|U$.

<u>1.2 Proposition</u>: <u>$\mathcal{I}A \in \mathcal{D}\mathrm{Sh}\mathcal{L}$</u>.

We show first that $\mathcal{I}A$ defines a sheaf, then that it indeed belongs to $\mathcal{D}\mathrm{Sh}\mathcal{L}$.

For $V \leq U \in \mathcal{L}$, we have the restriction map

$$\mathrm{Con}(A|U) \longrightarrow \mathrm{Con}(A|V)$$
$$\Theta \rightsquigarrow \Theta|V$$

where $(\Theta|V)W = \Theta W$ for $W \leq V$. This is clearly a subpresheaf of $\Omega^{A \times A}$, and hence automatically is separating. To show that $\mathcal{I}A$ is patching, take $U = \bigvee_{i \in I} U_i$ and $\Theta_i \in (\mathcal{I}A)U_i = \mathrm{Con}(A|U_i)$ with $\Theta_i|U_i \wedge U_k = \Theta_k|U_i \wedge U_k$. Since $\mathrm{Con}(A|U_i) \subseteq (\Omega^{A \times A})U_i$, there exists a unique $\Theta \in (\Omega^{A \times A})U$ with $\Theta|U_i = \Theta_i$, because $\Omega^{A \times A}$ is a sheaf. This "patched together" Θ is the subsheaf of $A|U \times A|U$ such that, for any $a,b \in AV$ and $V \leq U$, $(a,b) \in \Theta V$ iff $(a|V \wedge U_i, b|V \wedge U_i) \in \Theta_i(V \wedge U_i)$ for all $i \in I$.

To see that Θ is the patching element required to make $\mathcal{I}A$ into a subsheaf of $\Omega^{A \times A}$, we need to check that each ΘV for $V \leq U$ is a congruence on AV. Clearly, $(a|V \wedge U_i, a|V \wedge U_i) \in \Theta_i(V \wedge U_i)$ for each $i \in I$, since each Θ_i is reflexive, and so ΘV is reflexive for all $V \leq U$. Similarly, since each Θ_i is symmetric, if $(a,b) \in \Theta V$ then $(a|V \wedge U_i, b|V \wedge U_i) \in \Theta_i(V \wedge U_i)$, for each $i \in I$. Then $(b|V \wedge U_i, a|V \wedge U_i) \in \Theta_i(V \wedge U_i)$ and therefore $(b,a) \in \Theta V$, so ΘV is symmetric. And ΘV is transitive since $(a,b) \in \Theta V$, $(b,c) \in \Theta V$ means that $(a|V \wedge U_i, b|V \wedge U_i) \in \Theta_i(V \wedge U_i)$ and

$(b|V \wedge U_i, c|V \wedge U_i) \in \Theta_i(V \wedge U_i)$ for each $i \in I$ and $V \leq U$. But $\Theta_i(V \wedge U_i)$ is transitive, and so $(a|V \wedge U_i, c|V \wedge U_i) \in \Theta_i(V \wedge U_i)$ and $(a,c) \in \Theta V$.

Finally, $\Theta_i U_i$ is a sublattice of $AU_i \times AU_i$ for all $i \in I$, which implies that ΘV is a sublattice of $AV \times AV$ and hence a congruence on AV. Thus Θ is indeed the patching element required to make $\mathcal{C}A$ a subsheaf of $\Omega^{A \times A}$.

Now we need only show that the restriction maps $Con(A|U) \longrightarrow Con(A|V)$ for $V \leq U$ are bounded lattice homomorphisms. Let $\alpha: Con(A|U) \longrightarrow Con(A|V)$ be a restriction map taking $\Theta \rightsquigarrow \Theta|V$, where $(\Theta|V)W = \Theta W$ for $W \leq V \leq U$, and let Θ, $\Phi \in Con(A|U)$. Taking the component at W, we have that α preserves meets: $(\alpha(\Theta \wedge \Phi))W = ((\Theta \wedge \Phi)|V)W = (\Theta \wedge \Phi)W = \Theta W \cap \Phi W = (\Theta|V)W \cap (\Phi|V)W = (\alpha(\Theta) \wedge \alpha(\Phi))W$. Joins are preserved in an analoguous fashion. Further, $(\alpha(\top))W = (\top|V)W = \top W$ where $\top$ is the top of $Con(A|U)$, and $(\alpha(\bot))W = (\bot|V)W = \bot W$ where $\bot$ is the bottom of $Con(A|U)$. So we have that $(\mathcal{C}A)U = Con(A|U)$ defines a sheaf with values in $\mathcal{D}$, or, equivalently, a bounded distributive lattice in $Sh\mathcal{L}$.

We next define a map $\nabla: A \longrightarrow \mathcal{C}A$ as having components $AU \longrightarrow (\mathcal{C}A)U = Con(A|U)$ for each $U \in \mathcal{L}$, given for $a \in AU$ and $V \leq U$ by $a \rightsquigarrow \nabla_a V = \{(x,y) \mid x \vee (a|V) = y \vee (a|V)\} \subseteq AV \times AV$.

<u>1.3 Proposition:</u> <u>$\nabla: A \longrightarrow \mathcal{C}A$ is an embedding in $\mathcal{D}Sh\mathcal{L}$.</u>

We require, for $V \le U \in \mathcal{L}$, that the following square commute:

$$\begin{array}{ccc} AU & \xrightarrow{\nabla_U} & (\mathcal{L}A)U \\ \downarrow & & \downarrow \\ AV & \xrightarrow[\nabla_V]{} & (\mathcal{L}A)V \end{array}$$

that is, $\nabla_U(a)|V = \nabla_V(a|V)$ for $a \in AU$. Now $\nabla_V(a|V)W = \nabla_{a|V}W$ $= \{(x,y) | x \vee (a|V)|W = y \vee (a|V)|W\}$, for $W \le V$. But $(a|V)|W$ $= a|W$, so $\nabla_V(a|V)W = \{(x,y) | x \vee (a|W) = y \vee (a|W)\}$. On the other hand, $(\nabla_U(a)|V)W = \nabla_a W = \{(x,y) | x \vee a|W = y \vee a|W\}$ for $W \le V$. Hence the two are equal, and ∇ is a natural transformation between the presheaves and thus a sheaf map.

To show that the mapping $\nabla : A \longrightarrow \mathcal{L}A$ is a lattice embedding, we first show that it is a lattice homomorphism. Meets are preserved, for if $(x,y) \in \nabla_a V \cap \nabla_b V$, then $x \vee a|V$ $= y \vee a|V$ and $x \vee b|V = y \vee b|V$. Then $(x \vee a|V) \wedge (x \vee b|V)$ $= (y \vee a|V) \wedge (y \vee b|V)$, that is, $x \vee (a|V \wedge b|V)$ $= y \vee (a|V \wedge b|V)$, so $x \vee ((a \wedge b)|V) = y \vee ((a \wedge b)|V)$. Hence (x,y) also belongs to $\nabla_{a\wedge b}V$. For the reverse inclusion, take $(x,y) \in \nabla_{a\wedge b}V$, so $x \vee ((a \wedge b)|V) =$ $y \vee ((a \wedge b)|V)$. Taking the join with $a|V$ on both sides of this equality, we have that $x \vee (((a \wedge b) \vee a)|V) =$ $y \vee (((a \wedge b) \vee a)|V)$, then by absorption $x \vee a|V =$ $y \vee a|V$. Taking the join with $b|V$ rather than $a|V$ yields $(x,y) \in \nabla_b V$ instead of $(x,y) \in \nabla_a V$. So $(x,y) \in \nabla_a V \cap \nabla_b V$.

To show that ∇ preserves joins, we first use the preceeding result to see that, for $a \leq b$, $\nabla_a V \cap \nabla_b V = \nabla_{a \wedge b} V = \nabla_a V$. It is clear then that $\nabla_a V \subseteq \nabla_{a \vee b} V$ and $\nabla_b V \subseteq \nabla_{a \vee b} V$. Conversely, let $(x,y) \in \nabla_{a \vee b} V$. Now, $(x, x \vee a|V) \in \nabla_a V$ for all $x \in AV$. By hypothesis, $(x \vee a|V, y \vee a|V) \in \nabla_b V$. Composing, we have that $(x,y) \in \nabla_a V \circ \nabla_b V \circ \nabla_a V \subseteq \nabla_a V \vee \nabla_b V \doteq (\nabla_a \vee \nabla_b)V$.

By definition, $\nabla_0 V = \{(x,y) | x \vee 0|V = y \vee 0|V\} = \{(x,y) | x = y\}$, which is the bottom $\perp_V$ of the congruence lattice $(\mathcal{L}A)V$. Similarly, $\nabla_e V = \{(x,y) | x \vee e|V = y \vee e|V\} = \{(x,y) | e|V = e|V\}$, which is the top $\top_V$ of $(\mathcal{L}A)V$. Hence the morphism $\nabla : A \longrightarrow \mathcal{L}A$ is a lattice homomorphism.

To see that ∇ is a monomorphism, we show that, for all $V \leq U$ in $\mathcal{L}$ and $a,b \in AU$, $\nabla_a V = \nabla_b V$ iff $a = b$. Taking $\nabla_a V = \nabla_b V$, we note that since $0 \vee a|V = a|V \vee a|V$, $(0,a) \in \nabla_a V$. Then by hypothesis, $0 \vee b|V = a|V \vee b|V$, which implies that $a \leq b$. Similarly, $b \leq a$, hence we have equality. The reverse implication is trivial, and so ∇ is a monomorphism.

We now define $\Delta : A \longrightarrow \mathcal{L}A$ componentwise, for $V \leq U \in \mathcal{L}$ and $a \in AU$, as $AU \longrightarrow Con(A|U)$, given by $a \rightsquigarrow \Delta_a V = \{(x,y) | x \wedge a|V = y \wedge a|V\} \subseteq AV \times AV$.

<u>1.4 Proposition</u>: <u>$\Delta : A \longrightarrow \mathcal{L}A$ is a dual lattice embedding</u>.

Begin by forming the dual A^* of bounded distributive lattice A by interchanging meets and joins as well as top and bottom elements. Clearly, A^* is again a bounded distributive lattice, and $(A^*)U = (AU)^*$. A subsheaf of $|A| \times |A|$ is a subalgebra of $A \times A$ iff it is a subalgebra of $(A \times A)^* = A^* \times A^*$, so $\mathcal{L}(A^*) = \mathcal{L}A \in \mathcal{B}Sh\mathcal{L}$. Then $\Delta : A \longrightarrow \mathcal{L}A$ is equivalent to $\nabla^* : A^* \longrightarrow \mathcal{L}(A^*)$, where $\vee^*$ is in the dual lattice A^*: $a \rightsquigarrow \nabla^*_a V = \{(x,y) \mid x \vee^* a|V = y \vee^* a|V\} = \{(x,y) \mid x \wedge a|V = y \wedge a|V\} = \Delta_a V$. This gives $\Delta_a \cap \Delta_b = \nabla^*_a \cap \nabla^*_b = \nabla^*_{a\wedge^* b} = \Delta_{a\vee b}$ and $\Delta_a \vee \Delta_b = \nabla^*_a \vee \nabla^*_b = \nabla^*_{a\vee^* b} = \Delta_{a\wedge b}$. Clearly, $\Delta_0 V$ is the top of the congruence lattice $(\mathcal{L}A^*)V$, while $\Delta_e V$ is the bottom, so $\Delta : A \longrightarrow \mathcal{L}A$ is a dual lattice homomorphism. It is a monomorphism by the same argument that ∇ is, hence we have a dual lattice embedding.

<u>Remark</u>: Let Θ_{ab} be the smallest congruence on $A|U$ containing (a,b) for $a,b \in AU$. Then $\nabla_a = \Theta_{0a}$ as follows: $(0,a) \in \nabla_a V$ since $0 \vee a|V = a|V = a|V \vee a|V$. For the reverse inclusion, if $(0,a) \in \Theta$ for any congruence Θ, then $(x, x \vee a|V) \in \Theta V$ for any $x \in AV$, also $(y, y \vee a|V) \in \Theta V$. Let $(x,y) \in \nabla_a V$, that is, $x \vee a|V = y \vee a|V$, then by transitivity, $(x,y) \in \Theta V$, and in particular, $(x,y) \in \Theta_{0a} V$.

Using dual arguments, we also have $\Delta_a = \Theta_{ae}$.

In the category $\mathbb{E}$ns it is true that every bounded distributive lattice A is contained in a Boolean algebra BA, called its <u>Boolean envelope</u>, such that A generates BA as a Boolean algebra. Further, the embedding of A into BA is essential in $\mathfrak{D}$, is an epimorphism in $\mathfrak{D}$, and is the reflection map from $\mathfrak{D}$ to Boo. We show here that the same holds true for $\mathfrak{D}$Sh$\mathcal{L}$ and BooSh$\mathcal{L}$.

In order to prove that Boolean envelopes exist in $\mathfrak{D}$Sh$\mathcal{L}$, we shall use the embedding $\nabla : A \longrightarrow \mathcal{B}A$. In fact, the desired Boolean envelope of A will be given by the Boolean part $\mathcal{B}A$ of $\mathcal{I}A$, that is, the sublattice of $\mathcal{I}A$ consisting, for each $U \in \mathcal{L}$, of the complemented elements of $(\mathcal{I}A)U$. We want to show first that $\mathcal{B}A$ is generated by all ∇_a, Δ_a where $a \in AU$ and $U \in \mathcal{L}$.

<u>1.5 Lemma</u>: <u>For $a \in AU$ and $U \in \mathcal{L}$, $\nabla_a \cap \Delta_a = \bot$ and $\nabla_a \vee \Delta_a = \top$.</u>

Take $(x,y) \in \nabla_a V \cap \Delta_a V$ for $V \le U \in \mathcal{L}$, $a \in AU$. Then perform the following calculation:

$$\begin{aligned} x &= x \wedge (x \vee a|V) = x \wedge (y \vee a|V) = (x \wedge y) \vee (x \wedge a|V) \\ &= (x \wedge y) \vee (y \wedge a|V) = y \wedge ((x \wedge y) \vee a|V) \\ &= y \wedge ((x \vee a|V) \wedge (y \vee a|V)) = y \wedge (y \vee a|V) = y. \end{aligned}$$

So $(x,y) \in \bot_V$, where $\bot_V$ is the bottom of $(\mathcal{I}A)V = \mathrm{Con}(A|V)$. The reverse inclusion is trivial, so $\nabla_a \cap \Delta_a = \bot$.

Now consider $\nabla_a V \vee \Delta_a V$. Since $(0|V, a|V) \in \nabla_a V$ and

$(a|V,e|V) \in \Delta_a V$, we have that $(0|V,a|V),(a|V,e|V) \in \nabla_a V \vee \Delta_a V$. This implies that $(x,x \vee a|V),(x \vee a|V,e|V) \in \nabla_a V \vee \Delta_a V$, and hence $(x,e|V) \in \nabla_a V \vee \Delta_a V$ for all $x \in A|V$. The same is then true for $y \in A|V$. Therefore, composing, $(x,y) \in \nabla_a V \vee \Delta_a V$ for all x and y in $A|V$, that is, $\top \subseteq \nabla_a \vee \Delta_a$. As the opposite inclusion is trivial, we have the required result. Hence $\nabla_a V$ and $\Delta_a V$ are complements in $(\mathcal{I}A)U$ for all $V \leq U \in \mathcal{L}$.

<u>1.6 Lemma</u>: <u>$\Theta_{ab} = \Delta_a \cap \nabla_b$ for $a \leq b$</u>.

For $a,b \in AU$ as given, we have the pair of equations $a \vee b = b = b \vee b$ and $a \wedge a = a = a \wedge b$, which imply respectively that $(a,b) \in \nabla_b U$ and $(a,b) \in \Delta_a U$. Hence $(a,b) \in (\Delta_a \cap \nabla_b)U$, and therefore $\Theta_{ab} \subseteq \Delta_a \cap \nabla_b$. Conversely, given a congruence Θ with $(a,b) \in \Theta U$, we have that $(y \vee a|V, y \vee b|V) \in \Theta V$ for $V \leq U$ and $y \in AV$. Taking $(x,y) \in \Delta_a V \cap \nabla_b V$ for any $V \leq U$, we have that $x \wedge a|V = y \wedge a|V$ and $x \vee b|V = y \vee b|V$. Performing the following calculation shows that $(x,y) \in \Theta V$:

$$x = x \wedge (x \vee b|V) = x \wedge (y \vee b|V) \underset{\Theta V}{\equiv} x \wedge (y \vee a|V)$$
$$= (x \wedge y) \vee (x \wedge a|V) = (x \wedge y) \vee (y \wedge a|V)$$
$$= y \wedge (x \vee a|V) \underset{\Theta V}{\equiv} y \wedge (x \vee b|V) = y \wedge (y \vee b|V) = y.$$

So we have $\Delta_a \cap \nabla_b \subseteq \Theta$, and in particular, $\Delta_a \cap \nabla_b \subseteq \Theta_{ab}$.

<u>1.7 Lemma</u>: <u>For $\Theta \in (\mathcal{I}A)U$, $U \in \mathcal{L}$, $\Theta U = \bigvee \Theta_{ab} U$, where</u>

<u>$(a,b) \in \Theta U$ and $a \leq b$.</u>

Let $\Theta \in (\mathcal{I}A)U$ and $(a,b) \in \Theta U$ with $a \leq b$. Then $\Theta_{ab} \subseteq \Theta$, implying $\Theta_{ab}U \subseteq \Theta U$, which in turn implies $\bigvee(\Theta_{ab}U) \subseteq \Theta U$, where the join is taken over all pairs $(a,b) \in \Theta U$ with $a \leq b$. For the reverse inclusion, take $(c,d) \in \Theta U$. Then $(c,d) \in \Theta_{cd}U$, which implies that $(c \wedge d, d)$ and $(c, c \wedge d) \in \bigvee(\Theta_{ab}U)$ over $(a,b) \in \Theta U$ with $a \leq b$, and thus (c,d) is an element of the join.

<u>1.8 Proposition</u>: <u>$\Theta \in (\mathcal{B}A)U$ implies that for some $U = \bigvee U_i$, $\Theta | U_i$ is, for each $i \in I$, a finite join of congruences $\Delta_a \cap \nabla_b$ on $A|U_i$, with $a \leq b$ in AU_i.</u>

It is enough to prove this for $U = E$, the top of $\mathcal{L}$, since applying it to $\downarrow U$ produces the general result. Let $\Theta \in (\mathcal{B}A)E \subseteq (\mathcal{I}A)E = \mathrm{Con}(A|E) = \mathrm{Con}A$. Then Θ is a congruence on A, and, since it belongs to $\mathcal{B}A$, Θ has a complement Φ such that $\Theta \wedge \Phi = \perp$ and $\Theta \vee \Phi = \top$. Now $E = \bigvee U_i$ for all $U_i \in \mathcal{L}$, and since at any U, $(\Theta \vee \Phi)U \doteq \Theta U \vee \Phi U$, we have $(\Theta U_i) \vee (\Phi U_i) = \top_{U_i}$, the top of the congruence lattice of AU_i for each i, where the join is in the congruence lattice of AU_i. Again we may take the case $U_i = E$; the result can be applied to $\downarrow U_i$. Thus we have Θ and Φ as congruences on A with $\Theta \wedge \Phi = \perp$ and $\Theta \vee \Phi = \top$ in the congruence lattice of AE.

From the lemma we know that $\Theta E = \bigvee(\Delta_a E \cap \nabla_b E)$, for $(a,b) \in \Theta E$ and $a \leq b$. Then the fact that $\Theta E \vee \Phi E = \top$, combined with the compactness of $\top$ in the congruence lattice of AE, shows that $\bigvee(\Delta_{a_k} E \cap \nabla_{b_k} E) \vee \Phi E = \top$ for finitely many $(a_k, b_k) \in \Theta E$, $a_k \leq b_k$. Intersecting with ΘE, we get $(\Theta E \cap \Phi E) \vee (\Theta E \cap \bigvee_{k=1}^{n} (\Delta_{a_k} E \cap \nabla_{b_k} E)) = \Theta E$, and hence $\Theta E = \bigvee_{k=1}^{n} (\Delta_{a_k} E \cap \nabla_{b_k} E)$ in the congruence lattice of AE.

We now want to show that $\Theta|U_i$ is equal to a finite join of congruences on $A|U_i$, that is $\Theta|U = \bigvee_{k=1}^{n} (\Delta_{a_k} U \cap \nabla_{b_k} U)$ in $\mathrm{Con}(A|U)$, for all $U \in \mathcal{L}$. We have that $\Theta E \cap \Phi E = \perp$, hence $\Theta|U \cap \Phi|U = \perp$ on $A|U$; also, $\Theta E \vee \Phi E = \bigvee_{k=1}^{n} (\Delta_{a_k} E \cap \nabla_{b_k} E) \vee \Phi E = \top$, and hence $\Theta U \vee \Phi U = \bigvee_{k=1}^{n} (\Delta_{a_k} U \cap \nabla_{b_k} U) \vee \Phi U = \top$ on $A|U$, where $\bigvee(\Delta_{a_k} U \cap \nabla_{b_k} U) \leq \Theta|U$. Intersecting the equality with $\Theta|U$ gives $\Theta|U = \bigvee_{k=1}^{n} (\Delta_{a_k} U \cap \nabla_{b_k} U)$, as required.

Having already proved that $\Delta_a = (\nabla_a)'$, we may now write $\Theta|U_i = \bigvee_{k=1}^{n} ((\nabla_{a_k})' \cap \nabla_{b_k})$ for each $\Theta \in (\mathcal{B}A)U$ and $i \in I$. Hence the image of A under ∇ generates $\mathcal{B}A$ as a Boolean algebra, and $\mathcal{B}A$ is indeed the Boolean envelope of A.

<u>1.9 Proposition:</u> <u>$\nabla : A \to \mathcal{B}A$ is an epimorphism in $\mathcal{D}Sh\mathcal{L}$</u>.

Let $A, B \in \mathcal{D}Sh\mathcal{L}$ and $f, g : \mathcal{B}A \to B$ be homomorphisms in $\mathcal{D}Sh\mathcal{L}$ so that the following square commutes:

$$\begin{array}{ccc} A & \xrightarrow{\nabla} & \mathcal{L}A \\ {\scriptstyle\nabla}\downarrow & & \downarrow{\scriptstyle g} \\ \mathcal{L}A & \xrightarrow[f]{} & B \end{array}$$

i.e., $f\nabla = g\nabla$.

We want to show $f = g$, that is, for $\Theta \in (\mathcal{L}A)U$ and $U \in \mathcal{L}$, $f_U(\Theta) = g_U(\Theta)$. Let $U = \bigvee U_i$ where $\Theta|U_i = \bigvee_{k=1}^{n}(\Delta a_k \cap \nabla b_k)$ for suitable $a_k \leq b_k$ in AU_i. Since f_U is a homomorphism, $f_U(\Theta)|U_i = f_U(\Theta|U_i) = f_{U_i}(\bigvee_{k=1}^{n}(\Delta a_k \cap \nabla b_k)) = \bigvee_{k=1}^{n}(f_{U_i}(\Delta a_k) \cap f_{U_i}(\nabla b_k))$. By hypothesis, $f_{U_i}(\nabla b_k) = g_{U_i}(\nabla b_k)$, and, taking the complement of $f_{U_i}(\nabla a_k) = g_{U_i}(\nabla a_k)$, we also have $f_{U_i}(\Delta a_k) = g_{U_i}(\Delta a_k)$. Hence $\bigvee_{k=1}^{n}(f_{U_i}(\Delta a_k) \cap f_{U_i}(\nabla b_k))$ $= \bigvee_{k=1}^{n}(g_{U_i}(\Delta a_k) \cap g_{U_i}(\nabla b_k) = g_{U_i}(\Theta|U_i)$, which is equal to $g_U(\Theta)|U_i$ since g is also a homomorphism. So $f_U(\Theta)|U_i = g_U(\Theta)|U_i$ for all $i \in I$, hence $f_U(\Theta) = g_U(\Theta)$ for each $U \in \mathcal{L}$, and finally, $f = g$ as required.

Recall that a monomorphism $h: A \longrightarrow B$ is called essential if, for any map $g: C \longrightarrow A$, the composition hg is monic implies that g itself is a monomorphism.

1.10 Proposition: $\nabla: A \longrightarrow \mathcal{L}A$ is an essential embedding.

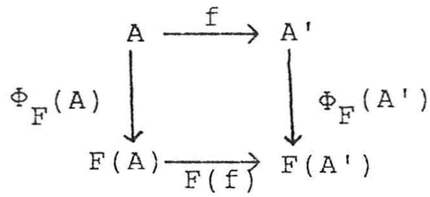
Let $h: \mathcal{L}A \longrightarrow C$ be a homomorphism in $\mathfrak{D}Sh\mathcal{L}$, for $C \in \mathfrak{D}Sh\mathcal{L}$, such that $h\nabla$ is monic. We want to show that h

itself is monic. Since $\mathcal{L}A$ is Boolean and h is a homomorphism in $\Omega Sh\mathcal{L}$, the image of $\mathcal{L}A$ under h is Boolean. We may therefore assume that C is Boolean and h is a Boolean homomorphism, with components $h_U(\bot) = \bot_U$ and $h_U(\top) = \top_U$ for each $U \in \mathcal{L}$. Also for Θ in $(\mathcal{L}A)U$, $h_U(\Theta)' = h_U(\Theta')$, where Θ' denotes the complement of Θ. We show, for each $\Theta \in (\mathcal{L}A)U$, that $h_U(\Theta) = \bot_U$ implies that $\Theta = \bot$, for all $U \in \mathcal{L}$.

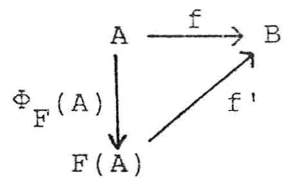
Let $U = \bigvee U_i$ so that $\Theta|U_i = \bigvee_{k=1}^{n} (\Delta_{a_k} \cap \nabla_{b_k})$ with $a_k \leq b_k$ in AU_i. Then for each U_i, $h_U(\Theta)|U_i = h_{U_i}(\Theta|U_i) = h_{U_i}(\bigvee_{k=1}^{n} (\Delta_{a_k} \cap \nabla_{b_k})) = \bigvee_{k=1}^{n} (h_{U_i}(\Delta_{a_k}) \cap h_{U_i}(\nabla_{b_k})) = \bigvee_{k=1}^{n} \bot_{U_i} = \bot_{U_i}$. Since h is a homomorphism and $\Delta_{a_k} = (\nabla_{a_k})'$, we use a well-known property of Boolean algebras to get $h_{U_i}(\nabla_{b_k}) \leq h_{U_i}(\nabla_{a_k})$ for each $i \in I$, so $h_{U_i}(\nabla_{b_k}) = h_{U_i}(\nabla_{b_k} \cap \nabla_{a_k})$. Again, each h_{U_i} is a homomorphism, so $\nabla_{b_k} = \nabla_{a_k} \cap \nabla_{b_k}$, which implies that $\nabla_{b_k} \subseteq \nabla_{a_k}$. Finally, since ∇ is also a monomorphism, $b_k \leq a_k$. But by hypothesis, $a_k \leq b_k$, so we have equality, and $\Theta = \bot$ for each $U \in \mathcal{L}$, and h is therefore monic.

Let B be a subcategory of A. Then B is a <u>reflective subcategory</u> of A if there exists a functor $F: A \longrightarrow B$ such that, for all objects $A \in \mathbf{A}$, there exists a map $\Phi_F(A): A \longrightarrow F(A)$ which satisfies two conditions --

(1) for each $f: A \longrightarrow A'$ the following square commutes:



and (2) for each object $B \in \mathbf{B}$ and map $f: A \longrightarrow B$ in **A**, there exists a map $f': F(A) \longrightarrow B$, so that the following triangle commutes:



To establish that **BooSh$\mathcal{L}$** is a reflective subcategory of $\mathfrak{B}$Sh$\mathcal{L}$, we first require:

<u>1.11 Lemma:</u> <u>For $h: A \longrightarrow B$ a homomorphism in $\mathfrak{B}$Sh$\mathcal{L}$, A and B in $\mathfrak{B}$Sh$\mathcal{L}$, there exists a Boolean homomorphism $\tilde{h}: \mathcal{L}A \longrightarrow \mathcal{L}B$.</u>

Let $\Phi \in (\mathcal{I}A)U = \mathrm{Con}(A|U)$ and $\Theta \in (\mathcal{I}B)U = \mathrm{Con}(B|U)$ for each $U \in \mathcal{L}$. Then there exists a map $\tilde{h}: \mathcal{I}A \longrightarrow \mathcal{I}B$ so that $\tilde{h}_U(\Phi) \subseteq \Theta$ iff $\Phi \subseteq (h|U \times h|U)^{-1}(\Theta)$, which is in turn true iff $h_U^2(\Phi) \subseteq \Theta$. Hence $\tilde{h}_U(\Phi)$ is the congruence on $B|U$ generated by all $(h_V(a), h_V(c))$ for $(a,c) \in \Phi V$ and $V \leq U$. We note that, as a left adjoint to a meet-preserving map, $\tilde{h}_U$ preserves arbitrary joins. Also, $\tilde{h}_U(\bot)$ is the congruence on

$B|U$ generated by $(h_V(a), h_V(a))$ for $a \in AV$, all $V \leq U$, so $\tilde{h}_U$ preserves the bottom of $(\mathcal{L}A)U$. Since $\tilde{h}_U(\top)$ is the congruence on $B|U$ which contains $(h_V(0), h_V(e)) = (0,e)$, we have that $\tilde{h}_U(\top)$ preserves the top of $(\mathcal{L}A)U$. By an earlier remark, $\tilde{h}_U(\nabla_a) = \tilde{h}_U(\Theta_{0a}) = \Theta_{h_U(0)h_U(a)} = \Theta_{0h_U(a)} = \nabla_{h_U(a)}$ for all $a \in AU$, and dually $\tilde{h}_U(\Delta_a) = \Delta_{h_U(a)}$. We see that $\tilde{h}_U$ preserves meets as follows: $\tilde{h}_U(\nabla_a) \wedge \tilde{h}_U(\nabla_b) = \nabla_{h_U(a)} \cap \nabla_{h_U(b)} = \nabla_{h_U(a) \cap h_U(b)} = \tilde{h}_U(\nabla_a \cap \nabla_b)$. So $\tilde{h}$ is a lattice homomorphism.

To prove that $\tilde{h}_U$ is a homomorphism from $(\mathcal{B}A)U$ to $(\mathcal{B}B)U$, we need to show that $\tilde{h}_U(\Delta_a \cap \nabla_b) = \Delta_{h_U(a)} \cap \nabla_{h_U(b)}$ for $a \leq b$. Since $\tilde{h}$ preserves meets, $\tilde{h}_U(\nabla'_a) \cap \tilde{h}_U(\nabla_b) = \nabla'_{h_U(a)} \cap \nabla_{h_U(b)} = \Delta_{h_U(a)} \cap \nabla_{h_U(b)}$. Thus if $\Phi \in (\mathcal{B}A)U$ then $\tilde{h}_U(\Phi) \in (\mathcal{B}B)U$.

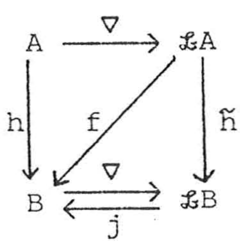
To see that $\tilde{h}$ is a Boolean homomorphism, it remains only to show that it preserves complements. Take $\Phi, \Theta \in (\mathcal{B}A)U$ such that $\Phi \cap \Theta = \bot$ and $\Phi \vee \Theta = \top$. Then $\tilde{h}_U(\Phi) \vee \tilde{h}_U(\Theta) = \tilde{h}_U(\Phi \vee \Theta) = \tilde{h}_U(\top) = \top$. To show that $\tilde{h}_U(\Phi) \wedge \tilde{h}_U(\Theta) = \bot$, it is enough to assume $\Phi = \bigvee(\Delta_{a_k} \cap \nabla_{b_k}), \Theta = \bigvee(\Delta_{c_j} \cap \nabla_{d_j})$. Since $\Phi \cap \Theta = \bot$, we have that $\Delta_{a_k} \cap \nabla_{b_k} \cap \Delta_{c_j} \cap \nabla_{d_j} = 0$. Then $\tilde{h}_U(\Phi) \wedge \tilde{h}_U(\Theta) = \tilde{h}_U(\bigvee_{k,j} 0) = \bot$.

1.12 Lemma: If $A \in$ BooSh$\mathcal{L}$, then $\nabla : A \to \mathfrak{L}A$ is an isomorphism.

For any $a \in AU$, $\nabla : AU \to (\mathcal{I}A)U = \mathrm{Con}(A|U)$ is a homomorphism in $\mathfrak{D}$, hence it preserves complements if they exist. Here AU is Boolean, so $\nabla_{a'} = (\nabla_a)' = \Delta_a$ for all $a \in AU$. Hence each Δ_a has a preimage under ∇, and thus ∇ is onto. Because ∇ is already a monomorphism, this means that A is isomorphic to $\mathfrak{L}A$.

1.13 Proposition: $\nabla : A \to \mathfrak{L}A$ is the reflection map which makes BooSh$\mathcal{L}$ a reflective subcategory of $\mathfrak{D}$Sh$\mathcal{L}$.

Consider $A \in \mathfrak{D}$Sh$\mathcal{L}$, $B \in$ BooSh$\mathcal{L}$, and $h : A \to B$. Take ∇ as before and $\tilde{h} : \mathfrak{L}A \to \mathfrak{L}B$. Let us define a map $f : \mathfrak{L}A \to B$ as the composite of $\tilde{h} : \mathfrak{L}A \to \mathfrak{L}B$ and $j : \mathfrak{L}B \to B$, where j is the inverse of $b \rightsquigarrow \nabla_b$, which we know exists due to Lemma 1.12. We now have the following diagram:



For $a \in AU$, $f_U(\nabla_a) = j_U\tilde{h}_U(\nabla_a) = j_U(\nabla_{h_U(a)}) = h_U(a)$. Since $\tilde{h}$ and j are Boolean homomorphisms, so is f. That f is unique is easily seen. Suppose there exists a Boolean homomorphism $g: \mathcal{L}A \longrightarrow B$ so that $g\nabla = h$. Then, for $U \in \mathcal{L}$, $g_u(\nabla_a) = h(a) = f_U(\nabla_a)$. Also, since g and f are Boolean, $g_U(\Delta_a) = f_U(\Delta_a)$. The g_U and f_U coincide on all $\bigvee(\Delta_{a_k} \cap \nabla_{b_k})$ for a_k, $b_k \in AU$ and by Proposition 1.8 are thus equal.

This Boolean homomorphism $\tilde{h}: \mathcal{L}A \longrightarrow B$ makes **BooSh$\mathcal{L}$** a reflective subcategory of $\mathcal{B}$Sh$\mathcal{L}$, with reflection map ∇.

CHAPTER 2: INJECTIVES AND INJECTIVE HULLS IN $\mathfrak{D}$SH$\mathcal{L}$.

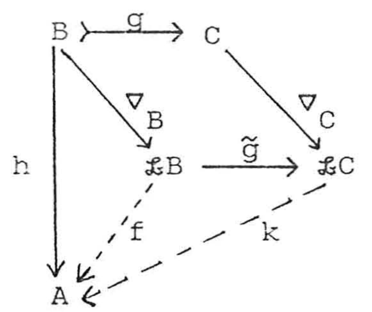
We will now consider injectives in $\mathfrak{D}$Sh$\mathcal{L}$, describe the injective hull of any object in the category, and then characterize the indecomposable injectives.

Recall from general category theory the characterisation of injectivity: an object A is injective in a specified category iff, for objects B and C in the category, for any morphism $h:B \longrightarrow A$ and any monomorphism $g:B \rightarrowtail C$, there exists a morphism $f:C \longrightarrow A$ such that $fg = h$. An injective hull of an object A is an essential injective extension of A.

2.1 Lemma: A $\in$ **BooSh$\mathcal{L}$** is injective in **BooSh$\mathcal{L}$** iff it is injective in $\mathfrak{D}$Sh$\mathcal{L}$.

Let A $\in$ **BooSh$\mathcal{L}$** be injective in $\mathfrak{D}$Sh$\mathcal{L}$, $h:B \longrightarrow A$ be a Boolean homomorphism, and $g:B \rightarrowtail C$ be a monomorphism in BooSh$\mathcal{L}$. Since Boolean homomorphisms are lattice homomorphisms and any monomorphism in **BooSh$\mathcal{L}$** is monic in $\mathfrak{D}$Sh$\mathcal{L}$, there exists in $\mathfrak{D}$Sh$\mathcal{L}$ a map $f:C \longrightarrow A$ so that $fg = h$. **BooSh$\mathcal{L}$** being a full subcategory of $\mathfrak{D}$Sh$\mathcal{L}$, by Proposition 1.13, this gives the required mapping in **BooSh$\mathcal{L}$**, making A injective in **BooSh$\mathcal{L}$**.

For the converse, let $A \in \mathbf{BooSh}\mathcal{L}$ be injective in $\mathbf{BooSh}\mathcal{L}$, with $h:B \longrightarrow A$ a homomorphism in $\mathfrak{D}\mathbf{Sh}\mathcal{L}$ and $g:B \rightarrowtail C$ a monomorphism in $\mathfrak{D}\mathbf{Sh}\mathcal{L}$. Then we have the following diagram, where $\mathfrak{B}B$ and $\mathfrak{B}C$ are the Boolean envelopes of B and C respectively:



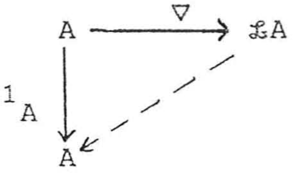
Since $\nabla_B : B \longrightarrow \mathfrak{B}B$ is the Boolean reflection map of B (Prop. 1.13), there exists a map $f:\mathfrak{B}B \longrightarrow A$ such that $f\nabla_B = h$. Now, $\tilde{g}\nabla_B = \nabla_C g$, which is monic, and, since ∇_B is essential, $\tilde{g}$ is itself a monomorphism. A is injective in $\mathbf{BooSh}\mathcal{L}$, so there exists a map $k:\mathfrak{B}C \longrightarrow A$, which composes with $\nabla_C : C \longrightarrow \mathfrak{B}C$ to make A injective in $\mathfrak{D}\mathbf{Sh}\mathcal{L}$.

<u>Remark</u>: Applying this lemma to $\mathbf{2}_{\mathcal{L}}$ one obtains that $\mathbf{2}_{\mathcal{L}}$ is an injective Boolean algebra iff it is an injective bounded distributive lattice in $\mathbf{Sh}\mathcal{L}$. Now, these assertions may be regarded as the Boolean Ultrafilter Theorem (BUT) and as the Prime Ideal Theorem (PIT) for distributive lattices, respectively, and hence we have - as in ZF Set Theory - that BUT holds iff PIT does, in any $\mathbf{Sh}\mathcal{L}$.

<u>2.2 Proposition</u>: <u>$A \in \mathfrak{D}\mathrm{Sh}\mathcal{L}$ is injective in $\mathfrak{D}\mathrm{Sh}\mathcal{L}$ iff A is complete Boolean.</u>

Let A be a complete Boolean algebra in $\mathrm{Sh}\mathcal{L}$. From BALT 1.9, we know that a Boolean algebra in $\mathrm{Sh}\mathcal{L}$ is complete iff it is injective as a Boolean algebra. Then by the above lemma, A is also injective in $\mathfrak{D}\mathrm{Sh}\mathcal{L}$.

Conversely, let A be injective in $\mathfrak{D}\mathrm{Sh}\mathcal{L}$. This produces the following diagram:



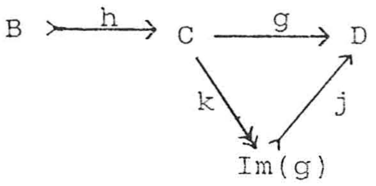
Then, since the essential monomorphism $\nabla : A \to \mathfrak{B}A$ has a left inverse, it is an isomorphism, and hence A is isomorphic to its Boolean envelope. By the lemma, A is injective in $\mathrm{BooSh}\mathcal{L}$, and then by the result quoted above, it is complete.

The next lemma uses the result from BALT (Proposition 1.10) that $B \in \mathrm{BooSh}\mathcal{L}$ has as its injective hull $[\,] : B \to \mathfrak{n}B$.

<u>2.3 Lemma</u>: <u>$A \to \mathfrak{B}A \to \mathfrak{n}(\mathfrak{B}A)$ is an essential monomorphism in $\mathfrak{D}\mathrm{Sh}\mathcal{L}$.</u>

From Proposition 1.10 we know that $\nabla : A \to \mathfrak{B}A$ is an essential monomorphism in $\mathfrak{D}\mathrm{Sh}\mathcal{L}$, and from BALT 1.8, $[\,] : B \to \mathfrak{n}B$ is an essential monomorphism in $\mathrm{BooSh}\mathcal{L}$. We show that essential monomorphisms in $\mathrm{BooSh}\mathcal{L}$ are essential in $\mathfrak{D}\mathrm{Sh}\mathcal{L}$ as well.

Let $B, C \in \mathbf{BooSh}\mathcal{L}$, $D \in \mathfrak{D}\mathbf{Sh}\mathcal{L}$, $h: B \rightarrowtail C$ be essential monic in $\mathbf{BooSh}\mathcal{L}$, and $g: C \rightarrow D$ be a lattice homomorphism in $\mathbf{Sh}\mathcal{L}$. Now, g has an epi-mono factorization, with $g = jk$, giving the following diagram:



Let gh be monic - then kh is, for if $khm = khn$, then $jkhm = jkhn$, i.e. $ghm = ghn$. Since gh is monic, $m = n$. But since k is an onto map from a Boolean algebra, it is a Boolean homomorphism, and since h is essential in $\mathbf{BooSh}\mathcal{L}$, k is also a monomorphism. Now, j is monic, hence $g = jk$ is, thus proving that $h: B \rightarrow C$ is an essential monomorphism in $\mathfrak{D}\mathbf{Sh}\mathcal{L}$ as well as in $\mathbf{BooSh}\mathcal{L}$.

Hence $B \rightarrow \mathfrak{N}B$, for Boolean B, is essential in $\mathfrak{D}\mathbf{Sh}\mathcal{L}$, and $A \rightarrow \mathcal{B}A \rightarrow \mathfrak{N}(\mathcal{B}A)$ is an essential monomorphism.

<u>2.4 Proposition</u>: <u>$\mathfrak{D}\mathbf{Sh}\mathcal{L}$ has injective hulls</u>.

From BALT 1.10 we know that $[\,]: B \rightarrow \mathfrak{N}B$ is the injective hull of any $B \in \mathbf{BooSh}\mathcal{L}$. Then for $A \in \mathfrak{D}\mathbf{Sh}\mathcal{L}$, $\mathcal{B}A \in \mathbf{BooSh}\mathcal{L}$, and $[\,]: \mathcal{B}A \rightarrow \mathfrak{N}(\mathcal{B}A)$ is the injective hull of $\mathcal{B}A$. By the lemma, $A \rightarrow \mathfrak{N}(\mathcal{B}A)$ is essential; combined with Lemma 2.1 this shows that $\mathfrak{N}(\mathcal{B}A)$ is an essential injective extension of A, that is, an injective hull of A.

This describes the injective hull of $A \in \mathfrak{D}\mathbf{Sh}\mathcal{L}$ as a

Boolean algebra of certain ideals of congruences on A. Alternatively, we may use the following lemmata to produce the injective hull of A by a construct of a simpler type.

<u>2.5 Lemma</u>: <u>$\mathfrak{J}B \cong \mathcal{C}B$ for $B \in \mathrm{BooSh}\mathcal{L}$</u>.

Let $B \in \mathrm{BooSh}\mathcal{L}$ and, for $U \in \mathcal{L}$, $\Theta \in (\mathcal{C}B)U = \mathrm{Con}(B|U)$. Define $f_U : (\mathcal{C}B)U \longrightarrow (\mathfrak{J}B)U$, that is $f_U : \mathrm{Con}(B|U) \longrightarrow \mathrm{Id}(B|U)$, by $\Theta \rightsquigarrow J$, where J is given at $W \leq U$ as $JW = \{x \in BW | (x,0) \in \Theta W\}$. We first prove that this defines a sheaf map, then that it is one-one, onto, and order-preserving.

To show that f is a sheaf map, we require that the following diagram commute, for all $V \leq U \in \mathcal{L}$:

$$\begin{array}{ccc} \mathrm{Con}(B|U) & \xrightarrow{f_U} & \mathrm{Id}(B|U) \\ \downarrow & & \downarrow \\ \mathrm{Con}(B|V) & \xrightarrow{f_V} & \mathrm{Id}(B|V) \end{array}$$

Now, $(f_U(\Theta)|V)W = \{x \in BW | (x,0) \in \Theta W,\ W \leq U\}|V$, which equals $\{x \in BW | (x,0) \in \Theta W\}$ for $W \leq V$; on the other hand, $(f_V(\Theta|V))|W = \{x \in BW | (x,0) \in (\Theta|V)W\} = \{x \in BW | (x,0) \in \Theta W\}$, for $W \leq V$. Thus $f_U(\Theta)|V = f_V(\Theta|V)$ for all $V \leq U$, and f is a sheaf map.

To see that f is one-one, note that $(a,b) \in \Theta U$ iff $(a \wedge b, a \vee b) \in \Theta U$ iff $(a' \wedge b, 0) \in \Theta U$. So $(a,b) \in \Theta U$ iff $a' \wedge b \in JW$ for all $W \leq U$ in $\mathcal{L}$, and thus J is completely determined by Θ.

Let I be an ideal of $B|U$ and define Θ on $B|U$ by $\Theta W = \{(a,b) \mid a+b \in IW\}$, for $W \leq U \in \mathcal{L}$, where $a+b$ is the usual symmetric difference $(a' \wedge b) \vee (a \wedge b')$. It is a standard computation in Boolean algebra that ΘW is a congruence on BW; we must show that $W \rightsquigarrow \Theta W$ is a subsheaf of $B|U \times B|U$.

$W \rightsquigarrow \Theta W$ is a subpresheaf of $B|U \times B|U$ because the restriction homomorphism preserves symmetric difference, and hence it is separating. To show that it is patching, let $U = \bigvee U_i$ and $(a_i, b_i) \in \Theta U_i \subseteq BU_i \times BU_i$, with $(a_i, b_i)|U_i \wedge U_k = (a_k, b_k)|U_i \wedge U_k$. This means that $a_i|U_i \wedge U_k = a_k|U_i \wedge U_k$ and $b_i|U_i \wedge U_k = b_k|U_i \wedge U_k$. But $a_i, b_i \in BU_i$, which is a sheaf, hence there exists $a \in BU$ with $a|U_i = a_i$ and $b \in BU$ with $b|U_i = b_i$. We claim that $(a,b) \in \Theta U$, that is, $a+b \in IU$. Now, $a_i + b_i \in IU_i$ by definition, so $(a+b)|U_i \in IU_i$; I is itself a sheaf, so indeed $a+b \in IU$, and $(a,b) \in \Theta U$.

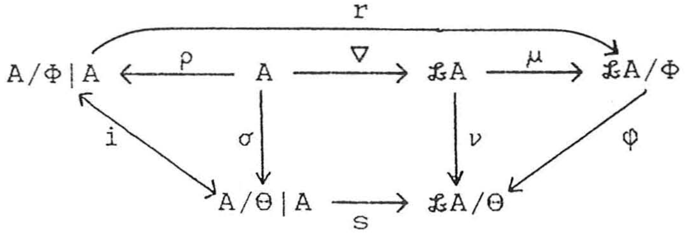
Finally, $\{a \mid (a,0) \in \Theta W\} = \{a \mid a+0 \in IW\} = \{a \mid (a' \wedge 0) \vee (a \wedge 1) \in IW\} = IW$, and so $I = f_U(\Theta)$ and f is onto as well as one-one.

It remains only to show that f is an order-preserving map; then it is an isomorphism. Let $\Theta_1 \leq \Theta_2$ be congruences on $B|U$ and $x \in J_1 W$ for $W \leq U$, i.e. $(x,0) \in \Theta_1 W$. But since $\Theta_1 W \subseteq \Theta_2 W$ for all $W \leq U \in \mathcal{L}$, clearly $(x,0) \in \Theta_2 W$ and $x \in J_2 W$. On the other hand, let $I_1 \subseteq I_2$ be ideals of $B|U$. Then take $(a,b) \in \Theta_1 W$, that is, $a + b \in I_1 W$. But then $a + b \in I_2 W$, so $\Theta_1 W \subseteq \Theta_2 W$. Hence f preserves order, and $\mathfrak{F}B \cong \mathcal{I}B$.

<u>Remark</u>: A stronger version of this result, for **Ens**, was published in 1952 by J. Hashimoto[10]. He proved that IdB ≅ ConB where B is a generalised Boolean lattice, that is, a relatively complemented lattice with a zero. We, however, do not require the stronger result.

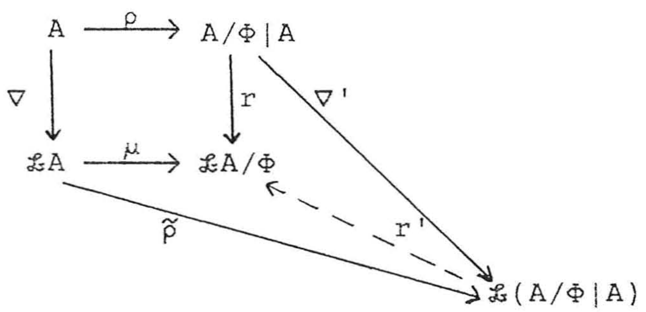
<u>2.6 Lemma</u>: <u>For $A \in \mathcal{B}\mathrm{Sh}\mathcal{L}$, $\mathcal{I}(\mathcal{B}A) \cong \mathcal{I}A$.</u>

Let $A \in \mathcal{B}\mathrm{Sh}\mathcal{L}$ and $\mathcal{B}A$ be its Boolean envelope. Let Θ and Φ be congruences on $\mathcal{B}A$, then for $\Theta|A = \Phi|A$, we claim that $\Theta = \Phi$. Without loss of generality, we may take $\Phi \subseteq \Theta$. We have the following diagram:



Since $\Theta|A = \Phi|A$, $A/\Theta|A \cong A/\Phi|A$. The maps μ, ν, ρ, σ are the appropriate canonical homomorphisms, and ∇ is the essential embedding of A into $\mathcal{B}A$. The maps r and s are induced homomorphisms.

In order to show that $\Theta = \Phi$, we require that $\mathcal{B}A/\Theta \cong \mathcal{B}A/\Phi$, i.e., that ϕ is an isomorphism. First we show that $r: A/\Phi|A \to \mathcal{B}A/\Phi$ is the Boolean envelope of $A/\Phi|A$. Consider the following diagram:



Now, r is a homomorphism with Boolean image, and ∇' is the reflection map from $\mathbf{DSh}\mathcal{L}$ to $\mathbf{BooSh}\mathcal{L}$, so there exists a map $r': \mathcal{B}(A/\Phi|A) \to \mathcal{B}A/\Phi$ such that $\nabla' r' = r$. Since $\mathrm{Ker}(\mu\nabla) = \Phi|A$ and $\mu\nabla = r\rho$, we have that $\mathrm{Ker}(r\rho) = \Phi|A$. But ρ being the quotient homomorphism, $\Phi|A = \mathrm{Ker}(\rho)$, and so r is a monomorphism; since ∇' is essential, this makes r' monic as well.

It remains to show that $\mathrm{Im}(r') = \mathcal{B}A/\Phi$. Now, $\nabla^{-1}\mu^{-1}(\mathrm{Im}(r')) = \rho^{-1}r^{-1}(\mathrm{Im}(r'))$; since $\mathrm{Im}(r) \subseteq \mathrm{Im}(r')$, $\rho^{-1}r^{-1}(\mathrm{Im}(r')) = \rho^{-1}(A/\Phi|A) = A$, which implies that $\mathrm{Im}(\nabla) \subseteq \mu^{-1}(\mathrm{Im}(r'))$. Then, since $\mu^{-1}(\mathrm{Im}(r'))$ is a Boolean subalgebra of $\mathcal{B}A$, and $\mathcal{B}A$ is generated by A, it is clear that

$\mu^{-1}(\mathrm{Im}(r')) = \mathcal{B}A$. Finally, since μ is onto, we have $\mathrm{Im}(r') = \mu(\mathcal{B}A) = \mathcal{B}A/\Phi$. So r' is an isomorphism from $\mathcal{B}(A/\Phi|A)$ to $\mathcal{B}A/\Phi$, and $r : A/\Phi|A \longrightarrow \mathcal{B}A/\Phi$ is the Boolean envelope of $A/\Phi|A$.

Returning to the first diagram, we have that $s : A/\Theta|A \longrightarrow \mathcal{B}A/\Theta$ is also a Boolean envelope. Hence $s = \varphi r$ is monic, which implies, since r is essential, that φ is a monomorphism. Similarly, we can construct a monomorphism $\alpha : \mathcal{B}A/\Theta \longrightarrow \mathcal{B}A/\Phi$, hence φ is left invertible. Since r and s are essential, so is φ; therefore, φ is an isomorphism and $\mathcal{B}A/\Phi \cong \mathcal{B}A/\Theta$. Since $\Phi \subseteq \Theta$, this gives $\Theta \subseteq \Phi$ and hence equality. Thus we have a one-one map from $\mathrm{Con}(\mathcal{B}A)$ to $\mathrm{Con}A$, given by restriction. Applying this to $\downarrow U$ for each $U \in \mathcal{L}$ and using the fact that $(\mathcal{B}A)|U = \mathcal{B}(A|U)$, we obtain a monomorphism from $\mathcal{L}(\mathcal{B}A)$ to $\mathcal{L}A$.

Now we let $\Theta \in \mathrm{Con}A$ and show that there exists a congruence Φ on $\mathcal{B}A$ such that $\Phi|A = \Theta$. Let i and j be the essential embeddings of A into $\mathcal{B}A$ and A/Θ into $\mathcal{B}(A/\Theta)$ respectively. Then the canonical homomorphism $\nu : A \longrightarrow A/\Theta$ induces a homomorphism $f : \mathcal{B}A \longrightarrow \mathcal{B}(A/\Theta)$, giving this diagram:

$$\begin{array}{ccc} A & \xrightarrow{\ i\ } & \mathcal{B}A \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle f} \\ A/\Theta & \xrightarrow[\ j\ ]{} & \mathcal{B}(A/\Theta) \end{array}$$

Let $\Phi = \mathrm{Ker}(f)$ and let $b, c \in AV$ for $V \in \mathcal{L}$. Then $\Phi|A = \Theta$, since $(b,c) \in \Theta V$ iff $(i_V(b), i_V(c)) \in \Theta V$, as follows: $(i_V(b), i_V(c)) \in \Phi V$ iff $f_V i_V(b) = f_V i_V(c)$, which is true iff $j_V \nu_V(b) = j_V \nu_V(c)$, which, since j is a monomorphism, is true iff $\nu_V(b) = \nu_V(c)$, in turn true iff $(b,c) \in \Theta V$.

Finally, it is easily seen that we have an order isomorphism: $\Theta|A \subseteq \Phi|A$ iff $\Theta|A = \Theta|A \cap \Phi|A = (\Theta \cap \Phi)|A$, which is true iff $\Theta = \Theta \cap \Phi$, that is, $\Theta \subseteq \Phi$.

Hence $\mathcal{I}(\mathcal{S}A) \cong \mathcal{I}A$, by restriction, again by applying the above to $\downarrow U$ for each $U \in \mathcal{L}$.

We are now ready to describe an alternative construction of the injective hull for $A \in \mathcal{D}\mathbf{Sh}\mathcal{L}$. For any locale M in $\mathbf{Sh}\mathcal{L}$, we know that the equalizer of $id_M : M \to M$ and $()^{**} : M \to M$ is the Boolean algebra M^* of normal elements of M. For $A \in \mathcal{D}\mathbf{Sh}\mathcal{L}$, $\mathcal{I}A$ is a locale, by the corollary to Proposition 1.1. From the above lemmata, we have $\mathcal{I}A \cong \mathcal{I}(\mathcal{S}A) \cong \mathfrak{J}(\mathcal{S}A)$, hence the equalizer $(\mathcal{I}A)^*$ of $id_A : \mathcal{I}A \to \mathcal{I}A$ and $()^{**} : \mathcal{I}A \to \mathcal{I}A$ is isomorphic to the sheaf $\mathcal{N}(\mathcal{S}A)$ of normal ideals of $\mathcal{S}A$:

$$\begin{array}{ccccc}
(\mathcal{I}A)^* & \subseteq & \mathcal{I}A & \overset{id}{\underset{()^{**}}{\rightrightarrows}} & \mathcal{I}A \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
\mathcal{N}(\mathcal{S}A) & \subseteq & \mathfrak{J}(\mathcal{S}A) & \overset{id}{\underset{()^{**}}{\rightrightarrows}} & \mathfrak{J}(\mathcal{S}A)
\end{array}$$

Then, as in BALT(p.14, preprint), the map $\nabla : A \to \mathcal{I}A$ factors through $(\mathcal{I}A)^*$, and we may write $\nabla : A \to (\mathcal{I}A)^*$ as the injective hull of A.

A bounded distributive lattice A in **Sh**$\mathcal{L}$ is called <u>indecomposable</u> if it is non-trivial, and any isomorphism $A \cong B \times C$ in $\mathcal{D}$**Sh**$\mathcal{L}$ implies that either B or C is trivial.

<u>2.7 Proposition</u>: <u>The indecomposable injectives in $\mathcal{D}$**Sh**$\mathcal{L}$ are exactly the $\sigma_*(2)$ for the points $\sigma:\mathcal{L} \to 2$ of the locale $\mathcal{L}$.</u>

In **BooSh**$\mathcal{L}$ the indecomposable injectives are exactly these $\sigma_*(2)$ (BALT 2.1); from Lemma 2.1, the injectives in $\mathcal{D}$**Sh**$\mathcal{L}$ are those in **BooSh**$\mathcal{L}$. Hence the $\sigma_*(2)$ are certainly injective in $\mathcal{D}$**Sh**$\mathcal{L}$ and are, indeed, the only candidates for the indecomposable injectives. It remains to show that the $\sigma_*(2)$ are in fact indecomposable in $\mathcal{D}$**Sh**$\mathcal{L}$.

Let A be a bounded distributive lattice in **Sh**$\mathcal{L}$ of the type $\sigma_*(2)$ and suppose $A \cong B \times C$ for $B, C \in \mathcal{D}$**Sh**$\mathcal{L}$. Then, since A must be Boolean, B and C belong to **BooSh**$\mathcal{L}$, and $A \cong B \times C$ as Boolean algebras. By the result quoted above, then either B or C must be trivial, and hence A is indecomposable.

CHAPTER 3: OTHER PROPERTIES OF DISTRIBUTIVE LATTICES IN $\mathbf{Sh}\mathcal{L}$.

Having established some basic facts about injectives in $\mathfrak{D}\mathbf{Sh}\mathcal{L}$, we turn now to a consideration of other properties of distributive lattices in $\mathbf{Sh}\mathcal{L}$ and the relationships among them. We are interested, specifically, in prime and simple distributive lattices in $\mathbf{Sh}\mathcal{L}$, in cogenerators of $\mathfrak{D}\mathbf{Sh}\mathcal{L}$, and in the initial object $2_{\mathcal{L}}$ of $\mathfrak{D}\mathbf{Sh}\mathcal{L}$.

A bounded distributrive lattice $A \in \mathfrak{D}\mathbf{Sh}\mathcal{L}$ is called prime if, for any $\Theta, \Phi \neq \bot$ in ConA, $\Theta \cap \Phi \neq \bot$. Note that this means that the bottom element of the lattice ConA is prime in the usual set-theoretic sense.

3.1 Proposition: The prime $A \in \mathfrak{D}\mathbf{Sh}\mathcal{L}$ are exactly the indecomposable injectives.

Let $A \in \mathfrak{D}\mathbf{Sh}\mathcal{L}$ be prime. Then $\mathfrak{L}A$ is indecomposable as follows: suppose that it is decomposable, say $\mathfrak{L}A \cong C \times D$ for nontrivial $C, D \in \mathbf{BooSh}\mathcal{L}$. Then the projections $C \times D \longrightarrow C$ and $C \times D \longrightarrow D$ determine nontrivial congruences on $C \times D$ with trivial meet. Then $(\Theta|A) \cap (\Phi|A) = \bot$, hence $\Theta|A = \bot$ or $\Phi|A = \bot$; then $\Theta = \bot$ or $\Phi = \bot$, since B is an essential extension of A. But A is prime, giving a contradiction, which shows that $\mathcal{L}A$ is indecomposable. It then follows that the injective hull of $\mathcal{L}A$ is also indecomposable, hence $\mathcal{H}(\mathcal{L}A) \cong \sigma_*(2)$ for some $\sigma:\mathcal{L} \to 2$. However, $\sigma_*(2)$ has no proper sublattice, so $A \cong \sigma_*(2)$.

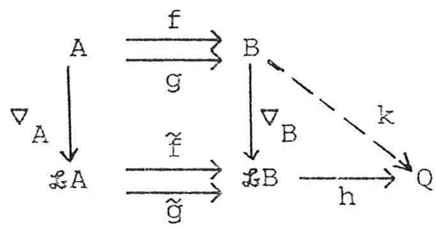
For the converse, let $A = \sigma_*(2)$, $\Theta \in \mathrm{Con}A$, and $\mathcal{F} \subseteq \mathcal{L}$ be the completely prime filter $\sigma^{-1}\{1\}$ associated with the point $\sigma:\mathcal{L} \to 2$. Then for $U \in \mathcal{F}$, $AU = (\sigma_* 2)U = 2$, and either ΘU is the identity on 2 or $(0,1) \in \Theta U$. If $\Theta \neq \perp_2$, then there exists $U \in \mathcal{F}$ with $(0,1) \in \Theta U$. Hence if Φ is any other nontrivial congruence on A and $(0,1) \in \Phi W$ for $W \in \mathcal{F}$, we have $(0,1) \in \Theta(U \wedge W) \cap \Phi(U \wedge W) = (\Theta \cap \Phi)(U \wedge W)$. But $U \wedge W \in \mathcal{F}$, and hence $\Theta \cap \Phi \neq \perp$.

Recall from general category theory that an object C in **A** is a <u>cogenerator</u> iff, for $f,g:A \to B$ distinct morphisms in **A**, there exists a morphism $h:B \to C$ in **A** such that $hf \neq hg$.

<u>3.2 Lemma</u>: <u>A set in **BooSh$\mathcal{L}$** cogenerates **BooSh$\mathcal{L}$** iff it cogenerates **$\mathcal{D}$Sh$\mathcal{L}$**.</u>

Let $\mathcal{X} \in$ **BooSh$\mathcal{L}$** be a cogenerating subset of **$\mathcal{D}$Sh$\mathcal{L}$**, and let $A,B \in$ **BooSh$\mathcal{L}$** with distinct Boolean homomorphisms $f,g:A \to B$. But <u>a priori</u>, A and B are bounded distributive lattices in **Sh$\mathcal{L}$**, and f and g are distinct lattice homomorphisms. Hence there exists $Q \in \mathcal{X}$ and a map $h:B \to Q$ with $hf \neq hg$. Thus $\mathcal{X}$ is a cogenerating set in **BooSh$\mathcal{L}$**.

Conversely, let $\mathcal{X} \in$ **BooSh**$\mathcal{L}$ be a cogenerating subset of **BooSh**$\mathcal{L}$, and let $A, B \in \mathfrak{D}$**Sh**$\mathcal{L}$ with distinct lattice homomorphisms $f, g: A \longrightarrow B$. Applying the map ∇ to A and B produces the commuting square in the following diagram, where $\tilde{f}$ and $\tilde{g}$ are the homomorphisms induced by f and g respectively. Q belongs to $\mathcal{X}$.



We see that $\tilde{f}$ and $\tilde{g}$ are distinct as follows: suppose $\tilde{f} = \tilde{g}$. Then $\tilde{f}\nabla_A = \tilde{g}\nabla_A$, that is, $\nabla_B f = \nabla_B g$. But ∇_B is monic, so $f = g$, which contradicts the original choice of f and g.

Now $\mathcal{X}$ is a cogenerating set in **BooSh**$\mathcal{L}$, so there exists $Q \in \mathcal{X}$ and a map $h: \mathfrak{B}B \longrightarrow Q$, with $h\tilde{f} \neq h\tilde{g}$. We then compose h with ∇_B to get the map $k: B \longrightarrow Q$. Since $h\tilde{f} \neq h\tilde{g}$, $h\tilde{f}\nabla_A \neq h\tilde{g}\nabla_A$. Then $h\nabla_B f \neq h\nabla_B g$, and, finally, $kf \neq kg$ for $k = h\nabla_B$. So $\mathcal{X}$ is a cogenerating set in $\mathfrak{D}$**Sh**$\mathcal{L}$.

<u>3.3 Proposition</u>: <u>The indecomposable injectives in $\mathfrak{D}$**Sh**$\mathcal{L}$ cogenerate $\mathfrak{D}$**Sh**$\mathcal{L}$ iff $\mathcal{L}$ is spatial</u>.

Since the indecomposable injectives of $\mathfrak{D}$**Sh**$\mathcal{L}$ are exactly those of **BooSh**$\mathcal{L}$, we use the lemma and the result from BALT (2.3) which states that the indecomposable injective Boolean algebras in **Sh**$\mathcal{L}$ cogenerate **BooSh**$\mathcal{L}$ iff $\mathcal{L}$ is spatial.

A bounded distributive lattice A in **Sh**$\mathcal{L}$ is called <u>simple</u> if A is nontrivial and, for any homomorphism $h: A \longrightarrow B$ in $\mathbf{D}$**Sh**$\mathcal{L}$, either h is a monomorphism or B is trivial. Clearly, this is equivalent to saying that for any $\Theta \in \mathrm{Con}A = (\mathcal{I}A)E$, either $\Theta = \bot$ or $\Theta = \top$. Hence, trivially, for Θ, $\Phi \in \mathcal{I}A$, if $\Theta \cap \Phi = \bot$ then either $\Theta = \bot$ or $\Phi = \bot$, so simple distributive lattices are prime. Note that this is also equivalent to saying that $A \in \mathbf{D}$**Sh**$\mathcal{L}$ is simple iff ConA is a 2-chain.

A point $\sigma: \mathcal{L} \longrightarrow \mathbf{2}$ is called <u>closed</u> iff the associated $S = \bigvee U$, over the U with $\sigma(U) = 0$, is maximal, which is true if and only if $\mathfrak{S} = \sigma^{-1}\{1\}$ is a minimal completely prime filter.

<u>3.4 Lemma</u>: <u>$A \in \mathbf{D}$**Sh**$\mathcal{L}$ is simple iff A is a simple Boolean algebra in **Sh**$\mathcal{L}$.</u>

If A is a simple Boolean algebra in **Sh**$\mathcal{L}$, then it is trivially true that it is a simple distributive lattice, since, for Boolean A, $\mathrm{Con}A \cong \mathrm{Id}A$.

Conversely, if A is a simple distributive lattice in **Sh**$\mathcal{L}$, it is prime and hence, by Proposition 2.8, an indecomposable injective, which makes it Boolean. Again, for Boolean A, $\mathrm{Con}A \cong \mathrm{Id}A$, and A is thus simple as a Boolean algebra.

Corollary: $A \in \mathcal{D}Sh\mathcal{L}$ is simple iff $A \cong \sigma_*(2)$ where $\sigma : \mathcal{L} \to 2$ is a closed point.

This is a direct result of the lemma and Proposition 3.5 of BALT (preprint), which states that the simple Boolean algebras in $Sh\mathcal{L}$ are exactly the $\sigma_*(2)$ for closed points σ of $\mathcal{L}$.

3.5 Proposition: The simple $A \in \mathcal{D}Sh\mathcal{L}$ cogenerate $\mathcal{D}Sh\mathcal{L}$ iff $\mathcal{L}$ is isomorphic to the topology of a T_1-space.

This is a direct result of our Lemmata 3.2, 3.4, and Proposition 3.6 of BALT (preprint), which states that the simple Boolean algebras in $Sh\mathcal{L}$ cogenerate $BooSh\mathcal{L}$ iff $\mathcal{L}$ is isomorphic to the topology of a T_1-space.

Consider now the initial distributive lattice $2_{\mathcal{L}}$ of $Sh\mathcal{L}$.

The preprint of BALT contains the proposition (4.1) that the initial Boolean algebra $2_{\mathcal{L}}$ in $Sh\mathcal{L}$ is complete iff $\mathcal{L}$ is a Stone algebra, that is, $U^* \vee U^{**} = E$ for all $U \in \mathcal{L}$. Since $2_{\mathcal{L}}$ has identical order structure in $\mathcal{D}Sh\mathcal{L}$ and in $BooSh\mathcal{L}$, we form, effortlessly,

3.6 Proposition: The initial bounded distributive lattice $2_{\mathcal{L}}$ is complete iff $\mathcal{L}$ is a Stone algebra.

Applying yet another result from BALT (3.3), that $\mathbf{2}_{\mathcal{L}}$ cogenerates **BooSh**$\mathcal{L}$ iff $\mathcal{L} \cong \mathbf{2}$, we get

<u>3.7 Proposition</u>: <u>$\mathbf{2}_{\mathcal{L}}$ cogenerates $\mathfrak{D}$**Sh**$\mathcal{L}$ iff $\mathcal{L} \cong \mathbf{2}$</u>.

From Lemma 3.2, $\mathbf{2}_{\mathcal{L}}$ cogenerates $\mathfrak{D}$**Sh**$\mathcal{L}$ iff it cogenerates **BooSh**$\mathcal{L}$.

<u>3.8 Lemma</u>: <u>$\mathfrak{Z}\mathbf{2}_{\mathcal{L}} \cong \Omega$</u>.

We have, for $U \in \mathcal{L}$, $(\mathfrak{Z}\mathbf{2}_{\mathcal{L}})U = \mathrm{Id}(\mathbf{2}_{\mathcal{L}}|U) = \mathrm{Id}(\mathbf{2}_{\downarrow U})$. Now, from BALT (preprint, 4.4) the map $S \rightsquigarrow (\mathbf{2}_{\mathcal{L}}|S)^{\#}$, $S \in \mathcal{L}$, is an order isomorphism $i: \mathcal{L} \xrightarrow{\sim} \mathrm{Id}(\mathbf{2}_{\mathcal{L}})$, where $(\mathbf{2}_{\mathcal{L}}|S)^{\#}$ is the ideal of $\mathbf{2}_{\mathcal{L}}$ given at $W \in \mathcal{L}$ as

$$(\mathbf{2}_{\mathcal{L}}|S)^{\#}W \doteq \begin{cases} (\mathbf{2}_{\mathcal{L}}|S)W, & \text{for } W \leq S \\ 0 \quad, & \text{for } W \not\leq S \end{cases}$$

We require $(\mathfrak{Z}\mathbf{2}_{\mathcal{L}})U \xrightarrow{\sim} \Omega U = \downarrow U$. Applying the order isomorphism to $\downarrow U$ in $\mathcal{L}$, $\downarrow U \xrightarrow{\sim} \mathrm{Id}(\mathbf{2}_{\downarrow U})$, but $\mathrm{Id}(\mathbf{2}_{\downarrow U}) = (\mathfrak{Z}(\mathbf{2}_{\downarrow U}))U = (\mathfrak{Z}(\mathbf{2}_{\mathcal{L}}|U))U = (\mathfrak{Z}\mathbf{2}_{\mathcal{L}})U$. So it is necessary only to show that this defines a sheaf map, i.e. that the following square commutes:

$$\begin{array}{ccc} \downarrow U & \xrightarrow{i_U} & \mathrm{Id}(\mathbf{2}_{\downarrow U}) \\ \downarrow & & \downarrow \\ \downarrow V & \xrightarrow{i_V} & \mathrm{Id}(\mathbf{2}_{\downarrow V}) \end{array} \qquad (V \leq U)$$

Since restriction in $\downarrow U$ is given, for $V \leq U$, by $S \rightsquigarrow S \wedge V$, this amounts to proving that $i_V(S \wedge V) = i_U(S)|V$, for $S \in \downarrow U$

and $V \leq U$ in $\mathcal{L}$. For $S \leq U$ and $W \leq V$ we have

$$i_V(S \wedge V)W \doteq \begin{cases} 2_{\mathcal{L}}W, & W \leq S \wedge V \\ 0, & \text{otherwise} \end{cases}$$

and

$$i_U(S)W \doteq \begin{cases} 2_{\mathcal{L}}W, & W \leq S \\ 0, & \text{otherwise} \end{cases}$$

which are the same since $W \leq S$ iff $W \leq S \wedge V$. Thus $i_V(S|V) = i_U(S)|V$ and we have a sheaf map $\downarrow U \rightarrow \mathrm{Id}(2_{\downarrow U})$, making $\mathfrak{Z}2_{\mathcal{L}} \cong \Omega$.

3.9 Proposition: Any locale $\mathcal{L}$ is isomorphic to the locale of congruences of the initial object of $\mathcal{D}\mathrm{Sh}\mathcal{L}$.

This is a direct result of the lemma and Lemma 2.5. which states that for Boolean $A \in \mathrm{Sh}\mathcal{L}$, $\mathfrak{Z}A \cong \mathfrak{C}A$. Thus we get $\mathrm{Con}(2_{\mathcal{L}}) = (\mathfrak{C}2_{\mathcal{L}})E \cong (\mathfrak{Z}2_{\mathcal{L}})E \cong \Omega E = \mathcal{L}$.

Corollary: For any locales $\mathcal{L}$ and $\mathcal{M}$, $\mathcal{D}\mathrm{Sh}\mathcal{L} \sim \mathcal{D}\mathrm{Sh}\mathcal{M}$ iff $\mathcal{L} \cong \mathcal{M}$.

Note that this observation can also be made as a consequence of a theorem of Borceux and van den Bossche[7] (p.120).

As noted in the preprint of BALT (p. 34), any equivalence between categories preserves initial objects, their quotients, and the associated congruences. The corollary then follows immediately.

BIBLIOGRAPHY

[1] R. Balbes and P. Dwinger, Distributive lattices. University of Missouri Press, 1975.

[2] B. Banaschewski, The BUT revisited, notes for talks at UNISA, Pretoria, July 1985 (unpublished).

[3] ...., An introduction to universal algebra, notes for talks at I.I.T. Kanpur, summer 1972 (unpublished).

[4] B. Banaschewski and K. Bhutani, Boolean algebras in a localic topos, preprint.

[5] ...., ..., Math. Proc. Cambridge Phil. Soc. (to appear).

[6] B. Banaschewski and G. Bruns, Injective hulls in the category of distributive lattices. J. Reine Angew. Math. 232 (1968): 102-109.

[7] F. Borceux and G. van den Bossche, Algebra in a localic topos with applications to ring theory. LNM 1038, Springer Verlag, Berlin Heidelberg New York Tokyo, 1983.

[8] G. Grätzer, Lattice theory: First concepts and distributive lattices. W.H. Freeman and Company, San Francisco, 1971.

[9] H. Herrlich and G. Strecker, Category theory. Heldermann Verlag, Berlin, 1979.

[10] J. Hashimoto, Ideal theory for lattices. Math. Japon. 2(1952):149-186.

[11] P.T. Johnstone, Topos theory. Academic Press, London, 1977.