# Finite Group Actions on the Four-Dimensional Sphere

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By

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## Abstract

Smith theory provides powerful tools for understanding the geometry of singular sets of group actions on spheres. In this thesis, tools from Smith theory and spectral sequences are brought together to study the singular sets of elementary abelian groups acting locally linearly on $S^4$. It is shown that the singular sets of such actions are homeomorphic to the singular sets of linear actions. A short review of the literature on group actions on $S^4$ is included.

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# Contents

# 1 Introduction

The study of group actions on manifolds brings together many different techniques. Tools from the algebra of the group itself, from the algebraic topology of the space, from the geometry of low dimensional manifolds, from representation theory, from moduli spaces and other domains are brought together to answer questions about which groups act on a space, how those groups can be classified, in which ways a group can act, how to distinguish between group actions, what geometric information is preserved by a group action, and others. Manifolds of dimension four, and in particular the 4-sphere, are well-suited for study, because their geometry and topology are both structured (due to constructions such as the intersection form) and exotic.

Here the focus will be on the singular sets of group actions on 4-spheres. Smith theory, a tool for decoding information on the homology or cohomology of the fixed point sets of groups, will be the main theory exploited to extract information about these sets.

The body of this paper will be divided into three sections. In the first section, I will address preliminary concerns, such as definitions and foundational theorems. In the second, the literature regarding group actions on the 4-sphere will be surveyed. In the third section, I will prove the main result:

**Theorem 1.** *The singular sets of locally linear or smooth orientation preserving actions of elementary abelian groups on the four-dimensional sphere are homeomorphic to the singular sets of the standard linear actions.*

# 2 Introductory Material

## 2.1 Basic definitions and conventions

Before we begin, it is necessary to state certain definitions and conventions which we will be using throughout.

**Definition 1.** *A group $G$ is said to act on an set $\mathbf{X}$ if there is a homomorphism from $G$ to $\mathrm{Aut}(\mathbf{X})$, the group of bijections from $\mathbf{X}$ to itself.*

In the case where the set $\mathbf{X}$ has added structure, such as if $\mathbf{X}$ is a smooth manifold, then a group $G$ acts on $\mathbf{X}$ if there is a homomorphism from $G$ to the group of bijective self-maps which preserve that structure, in this case diffeomorphisms. If a group $G$ acts on a space $\mathbf{X}$, then $\mathbf{X}$ is called a $G$-space.

**Definition 2.** *A group action is effective if the homomorphism*

$$G \to \mathrm{Aut}(\mathbf{X})$$

*is injective. In other words, no elements of $G$ act as the identity on $\mathbf{X}$*

It will not be necessary to consider ineffective group actions. This is because if a group acts ineffectively, it factors through an effective one.

**Definition 3.** *A group action is free if no non-identity elements of the group fix any point of the space.*

**Definition 4.** *The stabilizer (or isotropy group) of a point $p \in \mathbf{X}$, $G_p$, is the subgroup of $G$ which fixes $p$. The stabilizer of a subset $\mathbf{Y} \subset \mathbf{X}$, $G_{\mathbf{Y}}$ is the subgroup of $G$ which fixes every point of $\mathbf{Y}$.*

$$G_p = \{g \in G \,|\, g \cdot p = p\}$$

$$G_{\mathbf{Y}} = \{g \in G \quad | \quad \forall p \in \mathbf{Y}, \quad g \cdot p = p\}$$

For every subgroup $H$ of $G$ acting on $\mathbf{X}$, the fixed point set of $H$ will be denoted $\mathbf{X}^H$.

$$\mathbf{X}^H = \{x \in \mathbf{X} \,|\, \forall h \in H,\, h \cdot x = x\}$$

**Definition 5.** $\bigcup_{\{1\} \neq H \subset G} \mathbf{X}^H$ *is called the singular set of the group action. It is the set of all points which are fixed by some non-identity element of the group.*

**Definition 6.** *A group action is pseudo-free if its singular set is a union of isolated points.*

**Definition 7.** *The orbit of a point $p \in \mathbf{X}$ is the set $\bigcup_{g \in G}\{g \cdot p\}$, and will be denoted $G \cdot p$. The orbit of a subset $\mathbf{Y} \subset \mathbf{X}$, $G \cdot \mathbf{Y}$, is the union $\bigcup_{p \in \mathbf{Y}} G \cdot p$.*

There is a bijection between the orbit of a point and the cosets of the isotropy subgroup of that point: for a point $y = g \cdot x \in G \cdot x$, assign the coset $gG_x$. Thus, for $G$ a finite group,

$$|G \cdot p| = |G/G_p|$$

The stabilizers of points in the same orbit are conjugate, with $G_p = g^{-1}G_{g \cdot p}g$. If $h \in G_p$, $(ghg^{-1}) \cdot g \cdot p = g \cdot p$, so $ghg^{-1} \in G_{g \cdot p}$, and $h \in g^{-1}G_{g \cdot p}g$. If $h \in g^{-1}G_{g \cdot p}g$, $h \cdot p = g^{-1} \cdot g' \cdot g \cdot p$ with $g' \in G_{g \cdot p}$, so $h \cdot p = p$, which is to say $h \in G_p$.

**Definition 8.** *A map between $G$-sets $\phi : \mathbf{X} \to \mathbf{Y}$ is $G$-equivariant if $g \cdot \phi(x) = \phi(g \cdot x)$ for every $g \in G$ and every $x \in X$.*

**Definition 9.** *Two spaces are $G$-equivalent if there exists an equivariant homeomorphism between them.*

There is a bijective map (in the case of finite groups, actually a homeomorphism)

$$\begin{aligned} \alpha \colon G/G_x &\to G \cdot x \\ gG_x &\mapsto g \cdot x \end{aligned}$$

This map is equivariant, therefore $G/G_x$ and $G \cdot x$ are $G$-equivalent spaces. We say an orbit which is equivalent to $G/H$ is of type $G/H$.

**Definition 10.** *Let $\mathbf{X}$ and $\mathbf{Y}$ be sets on which $G$ acts. $G$ acts diagonally on the product $\mathbf{X} \times \mathbf{Y}$. We define the equivalence relation*

$$(x, y) \sim (g \cdot x, g \cdot y), \ \forall g \in G$$

*The set of equivalence classes of this relation is $\mathbf{X} \times_G \mathbf{Y}$.*

**Definition 11.** *Let $P \subset X$ be an orbit of type $G/H$. A tube about $P$ is an equivariant homeomorphism into an open neighborhood of $P$*

$$\phi \colon G \times_H A \to X$$

*where $A$ is a space on which $G$ acts.*

**Definition 12.** *For $x \in \mathbf{S} \subset \mathbf{M}$, a smooth manifold, such that $G_x S = S$, $S$ is a slice at $x$ if the map*

$$\begin{aligned} G \times_{G_x} S &\to M \\ (g, s) &\mapsto g \cdot s \end{aligned}$$

*is a a tube about $G \cdot x$.*

Equivalently, a slice at $x$ is defined as a subspace $S$ of $X$ with the following properties:

1. $S$ is closed in $G \cdot S$
2. $G \cdot S$ is an open neighborhood of $G \cdot x$

3. $G_x \cdot S = S$

4. If $g \cdot S \cap S$ is non-empty, then $g$ is an element of $G_x$

**Definition 13.** *Let* $\mathbf{P}$ *be an orbit of type* $G/H$*, and let* $V$ *be a Euclidean space on which* $H$ *acts orthogonally. A linear tube about* $P$ *is a* $G$*-equivariant embedding onto a neighborhood of* $P$ *of the form*

$$\phi : G \times_H V \to M$$

**Definition 14.** $S$ *is a linear slice at* $x$ *if* $S$ *is a slice at* $x$ *and the canonically associated tube is* $G$*-equivalent to a linear tube.*

**Definition 15.** *The action of* $G$ *on* $\mathbf{X}$ *is locally smooth if there is a linear tube about each orbit.*

**Definition 16.** *Given a ring* $R$*, the group ring* $R[G] = \{\sum_{g\in G} a_g g \mid a_g \in R\}$*. For two elements of* $R[G]$*,* $r = \sum_{g\in G} a_g g$ *and* $s = \sum_{g\in G} b_g g$

$$\begin{aligned} r + s &= \textstyle\sum_{g\in G}(a_g + b_g)g \\ rs &= \textstyle\sum_{g\in G}\sum_{g_1 g_2 = g} a_{g_1} b_{g_2} g \end{aligned}$$

If $\mathbf{X}$ is a set on which $G$ acts, $\mathbb{Z}[\mathbf{X}]$ is a $\mathbb{Z}[G]$ module which admits a decomposition as $\mathbb{Z}[\mathbf{X}] \cong \times\mathbb{Z}[G/G_{x_i}]$, where the direct sum is indexed by the orbits of the $G$ action, and each $x_i$ is a representative of a distinct orbit.

## 2.2 The Projection and the Transfer homomorphisms

In this section, information is taken from Bredon's *Introduction to Compact Transformation Groups* [Bre72], in chapter III, sections 1 to 4. Definitions and results in this section use simplicial homology, and also assume that the order of the group $G$ is invertible in the coefficient ring. The main theorem of a paper by Illman [Ill78] is that smooth $G$-manifolds $M$ are triangularizable in a $G$-equivariant way, and that this triangularization is in a sense unique, i.e.

1. There exists a regular simplicial $G$-complex $K$ and an equivariant triangulation map

$$h : K \to \mathbf{M}$$

2. If $h_K : K \to \mathbf{M}$ and $h_L : L \to \mathbf{M}$ are equivariant triangulations of $\mathbf{M}$, then there are equivariant subdivisions $K'$ of $K$ and $L'$ of $L$ such that $K'$ and $L'$ are equivariantly isomorphic.

Understanding group actions on a manifold provides rich information about the structures of both the group and the manifold itself. For example, it is a well-known result (see, for example, Cartan and Eilenberg's Homological Algebra) that groups which act freely on an $n$-dimensional sphere have periodic cohomology, with period $n+1$. What this means and why this is so will be covered in section 1.4.

There is a clear relationship between the action of the group and the algebraic properties of the manifold it acts upon. The most elementary such relationship involves two homomorphisms in the homology or cohomology of the manifold. These homomorphisms are called transfer, and projection.

**Definition 17.** *A simplicial complex is regular if given elements $g_1, g_2, \dots, g_n$ of $G$ and two simplices, $(v_1, v_2, \dots, v_n)$ and $(g_1 \cdot v_1, g_2 \cdot v_2, \dots, g_n \cdot v_n)$, there is an element $g \in G$ such that $g \cdot v_i = g_i \cdot v_i$ for all $i$.*

Not all simplicial complexes are regular, but a simplicial complex can be made regular by passing to the second barycentric subdivision.

Suppose we are given a regular simplicial complex structure on $M$, with associated chain complex $C(M)$ over a ring $R$ (for which no divisor of the order of $G$ is a zero divisor). Since $G$ acts on $M$, it acts naturally as an endomorphism of $C$, and so $C$ can be considered as a $R[G]$-complex instead of a $R$-complex.

Projection is straightforward to understand. The action of $G$ on $M$ creates an obvious map between $C(M)$ and the quotient space $C(M/G)$, which sends any element $c$ to the orbit $[c]$. This function induces a homomorphism $\pi_* : H_*(M) \to H_*(M/G)$. This is the projection homomorphism.

The transfer is a little more tricky. Consider the homomorphism $\sigma : C(M) \to C(M)$ defined as $c \mapsto \sum_{g \in G} gc$.

**Proposition 1.** *$\sigma$ and $\pi$ have the same kernel.*

*Proof.* Let $s$ be a simplex of $K/G$, with simplices $s_1, \dots, s_n$ of $K$ in the preimage of $s$ under $\pi$. By the definition of $\pi$, $\{s_1, \dots, s_n\}$ is an orbit, so there are $n = |G/G_{s_1}|$ such simplices.

We may restrict the argument to chains with simplices which are elements of a common orbit. Otherwise, if

$$c = \sum_{k=1}^{m} \sum \alpha_i t_{k_i}$$

where the sum is taken over $m$ distinct orbits $G \cdot t_{k_i}$,

$$\pi c = \sum_{k=1}^{m} \sum \alpha_i t_k$$

where each of the $s_k$ is a distinct generator of $K/G$. Then for $\pi c$ to be 0, it would require that $\pi \sum \alpha_i t_{k_i} = 0$ for every $k$.

So let $c = \sum \alpha_i s_i$. Then $\pi c = \sum \alpha_i s = 0$ if and only if $\sum \alpha_i = 0$. Also, since $G$ is transitive on the $s_i$, $\sigma c = \sum_{g \in G} g \cdot \sum \alpha_i s_i = m \sum s_i$.

Now $\pi \sigma c = \pi m \sum s_i = mns$, but also $\pi \sigma c = |G| \sum \alpha_i s$, so we know that $m = |G_{s_1}| \sum \alpha_i$.

Now suppose $\pi c = 0$. Then $\sum \alpha_i = 0$, so $m = 0$ and therefore $\sigma c = 0$. Similarly if $\sigma c = 0$, $m = 0$ and therefore $\sum \alpha_i = 0$, so $\pi c = 0$. $\square$

Since $\sigma$ and $\pi$ have the same kernel, their images are isomorphic. Therefore there is an injective homomorphism $\mu : C(M/G) \hookrightarrow C(M)$ which induces a homomorphism of homologies $\mu_* : H(M/G) \to H(M)$. That homomorphism is the transfer.

The transfer and projection have the following properties:

- $\pi_* \mu_* = |G|$
- $\mu_* \pi_* = \sigma_* = \sum_{g \in G} g_*$
- $\mu_* : H(M/G) \to H(M)^G$ is an isomorphism
- $\pi_* : H(M)^G \to H(M/G)$ is an isomorphism

The behaviour of the transfer generalizes for a subgroup H of G in the following way:

$$\mu_{G/H} : C(M/G) \to C(M/H)$$

is defined by the commutative diagram

$$\begin{array}{ccc} C(M/G) & \xrightarrow{\mu_{G/H}} & C(M/H) \\ \approx\big\downarrow & & \approx\big\downarrow \\ \sigma_G C(M) & \longrightarrow & \sigma_H C(M) \end{array}$$

It induces a homomorphism of homologies

$$(\mu_{G/H})_* : H(M/G) \to H(M/H)$$

with the following property.

$$(\pi_{G/H})_*(\mu_{G/H})_* = \frac{|G|}{|H|}$$

## 2.3 Smith Theory

Another relationship between a group and the homology of a space upon which it acts is explained by Smith theory, which deals predominantly with aiding in the computation of the homology of fixed point sets and orbit spaces. Within Smith theory we find the first important result in dealing with group actions on $S^4$: singular sets of group actions on $S^4$ will consist of collections of lower dimensional spheres. Information in this part is taken, unless otherwise specified, again from Bredon's book [Bre72], in sections

We begin by building up the basics necessary for understanding Smith theory.

Let $G$ be a group of prime order p, and $g$ a generator of $G$. Let $\sigma = \sum_{j=0}^{p-1} g^j \in Z[G]$, $\tau = 1 - g$. Let $\rho = \tau^j$, and $\bar{\rho} = \tau^{p-j}$.

Then there is an exact sequence of chain complexes with coefficients in $\mathbb{Z}/p$:

$$0 \to \bar{\rho}C(M) \times C(M^G) \xrightarrow{i} C(M) \xrightarrow{\rho} \rho C(M) \to 0$$

This sequence is exact. If $c$ is a chain in $C(M^G)$, $\rho c = \tau^{p-j}c = 0$, since $\tau c = 0$, and if $c$ is a chain complex in $\bar{\rho}C(K)$, $\rho\bar{\rho}c = 0$, since $\rho\bar{\rho} = \bar{\rho}\rho = \tau^p = 0$. If $c$ is a chain in the kernel of $\rho$, but not in $C(M^G)$, $c$ has the form

$$\sum_{j=1}^{n} m_j g^j d,$$

where $d$ is not a simplex in $M^G$. This chain corresponds uniquely to the element of the field $\mathbb{Z}/pG = \Lambda$,

$$\sum_{j=1}^{n} m_j g^j.$$

Therefore one need only show that the corresponding sequence is exact:

$$0 \to \bar{\rho}\Lambda \xrightarrow{i} \Lambda \xrightarrow{\rho} \rho\Lambda \to 0$$

Since $\Lambda$ is a vector space, $i$ is clearly injective and $\rho$ is clearly surjective, it only needs to be shown that $dim(\rho\Lambda) + \dim(\bar{\rho}\Lambda) = p$. This follows naturally from the fact that the kernel of $\tau$ as an endomorphism of $\mathbb{Z}/pG$ is one-dimensional (generated by $\sigma$), and therefore that $dim(\tau\Lambda) = dim(\Lambda) - 1$. Then since $\sigma = \tau^{p-1}$, and $ker(\tau) \subset \tau^j\Lambda$ for all $j$, it is clear that $dim(\tau^{j+1}\Lambda) = dim(\tau(\tau^j\Lambda)) = dim(\tau^j\Lambda) - 1$. Therefore $dim(\rho\Lambda) = p - j$, and $dim(\bar{\rho}\Lambda) = p - (p - j) = j$.

**Theorem 1.** *Defining $H^\rho(M, \mathbb{Z}/p) = H(\rho C(M))$ (the Smith special homology group), the exact sequence of chain complexes above induces a long exact sequence in homology. This long exact sequence,*

$$\ldots \xrightarrow{\delta_*} H_j^{\bar{\rho}}(M) \times H_j(M^G) \xrightarrow{i_*} H_j(M) \xrightarrow{\rho_*} H_j^{\rho}(M) \xrightarrow{\delta_*} H_{j-1}^{\bar{\rho}}(M) \times H_{j-1}(M^G) \to \ldots$$

*can be written as an exact triangle:*

$$\begin{tikzcd} & H(M) \arrow[dl, "\rho_*"] & \\ H^\rho(M) \arrow[rr, "\delta_*"'] & & H^{\bar{\rho}}(M) \times H(M^G) \arrow[ul, "i_*"'] \end{tikzcd}$$

*$i_*$ and $\rho_*$ have degree 0 and $\delta_*$ has degree $-1$.*

**Theorem 2.** *There is an exact triangle of homologies*

$$\begin{tikzcd} & H^{\tau^j}(M) \arrow[dl, "\tau_*"] & \\ H^{\tau^{j+1}}(M) \arrow[rr] & & H^{\sigma}(M) \arrow[ul, "i_*"'] \end{tikzcd}$$

*with the horizontal map having degree $-1$, and $\tau_*$ and $i_*$ having degree 0.*

The tools of Smith theory allow us to calculate bounds and constraints on the Euler characteristic of the fixed point set of a group action.

**Theorem 3.** *Suppose that $M$ is a manifold for which $C(M)$ is a finitely dimensional regular $\mathbb{Z}/p$-complex. Then for any integer $n \geq 0$ and any $\rho = \tau^i$, $1 \leq i \leq p-1$,*

$$\mathrm{rk}(H_n^\rho(M)) + \sum_{i \geq n} \mathrm{rk}(H_i(M^G)) \leq \sum_{i \geq n} \mathrm{rk}(H_i(M))$$

This theorem is proven by noting that the exact sequence

$$H_{j+1}^\rho(M) \to H_j^{\bar{\rho}}(M) \times H_j(M^G) \to H_j(M)$$

gives rise to the inequality:

$$\mathrm{rk}(H_j^{\bar{\rho}}(M) \times H_j(M^G)) = \mathrm{rk}(H_j^{\bar{\rho}}(M)) + \mathrm{rk}(H_j(M^G)) \leq \mathrm{rk}(H_{j+1}^\rho(M)) + \mathrm{rk}(H_j(M)).$$

Similarly, the exact sequence

$$H_{j+1}^{\bar{\rho}}(M) \to H_j^\rho(M) \times H_j(M^G) \to H_j(M)$$

gives rise to the inequality:

$$\mathrm{rk}(H_j^\rho(M) \times H_j(M^G)) = \mathrm{rk}(H_j^\rho(M)) + \mathrm{rk}(H_j(M^G)) \leq \mathrm{rk}(H_{j+1}^{\bar{\rho}}(M)) + \mathrm{rk}(H_j(M)).$$

Manipulation of these inequalities gives the desired result.

**Theorem 4.** *Let $M$ be a manifold with $C(M)$ a finite-dimensional regular $\mathbb{Z}/p$-complex whose homology has finite rank over $\mathbb{Z}/p$. Then*

$$\chi(M) + (p-1)\chi(M^G) = p\chi(M^*).$$

*In particular,*

$$\chi(M) \equiv \chi(M^G) \qquad \text{mod } p$$

Most important for us among the results of Smith theory is its application to group actions on spheres.

**Theorem 5.** *If $M$ is a mod $p$ homology $n$-sphere, then $M^G$ is a mod $p$ homology $r$-sphere for some $-1 \leq r \leq n$. If $p$ is odd, then $n - r$ is even.*

Since $p$-groups are solvable, and $M^G = (M^H)^(G/H)$ for any normal subgroup $H$, it will be possible to perform induction on the order of $G$.

Suppose $G = \mathbb{Z}/p$. Then by theorem 3,

$$\sum_{i \geq 0} \mathrm{rk}(H_i(M^G)) \leq \sum_{i \geq 0} \mathrm{rk}(H_i(M)) \text{ i.e.}$$

$$\mathrm{rk}(H(M^G)) \leq \mathrm{rk}(H(M)) = 2.$$

It follows from 4 that $\mathrm{rk}(H(M^G)) \neq 1$, and that if $p$ is odd,

$$\chi(M) = 1 + (-1)^n,$$
$$\chi(M^G) = 1 + (-1)^r$$

implies that $(-1)^{n-r} \equiv 1 \bmod p$, so that $n - r$ is even.

Now suppose $G$ is a $p$-group of order $p^m$, for $m \geq 2$, and the theorem is true for all $p$-groups of order less than $p^m$. $G$ has a non-trivial proper normal subgroup $H$, and by the induction hypothesis, $K^H$ is a mod $p$ homology $s$-sphere for some $-1 \leq s \leq n$. Since $(M^H)^{G/H} = M^G$ and $G/H$ is a $p$-group of order less than $p^m$ acting on $M^H$, $M^G$ is a mod $p$ homology $r$-sphere for some $-1 \leq r \leq s \leq n$. Also, if $p$ is odd, $n - s$ is even, and $s - r$ is even, so $n - r$ is even.

## 2.4 Group cohomology

### 2.4.1 Classifying spaces

Before beginning, we will recall a few concepts about fibre bundles.

Given a fibering

$$\begin{array}{ccc} \mathbf{F} & \rightarrow & \mathbf{E} \\ & & \downarrow \\ & & \mathbf{B} \end{array}$$

and a smooth function $f : \mathbf{X} \rightarrow \mathbf{B}$, there is a fibre bundle, $f^*(\mathbf{E}) = \{(x, y) \in \mathbf{X} \times \mathbf{E} \,|\, \pi(y) = f(x)\}$, which is induced by $f$ in the sense that the following diagram commutes

$$\begin{array}{ccc} \mathbf{F} & \rightarrow & \mathbf{F} \\ \downarrow & & \downarrow \\ f^*(\mathbf{E}) & \rightarrow & \mathbf{E} \\ \downarrow & & \downarrow \\ \mathbf{X} & \rightarrow & \mathbf{B} \end{array}$$

For group cohomology, we wish to study a very specific type of fibre bundle: the principal $G$-bundle, defined by the fibering

$$\begin{array}{ccc} G & \to & \mathbf{E} \\ & & \downarrow \\ & & \mathbf{B} \end{array}$$

Now we may define the classifying space of a group $G$, $\mathbf{B}G$.

**Theorem 6.** *Given a group $G$ there exists a classifying space $\mathbf{B}G$, and a principal $G$-bundle*

$$\begin{array}{ccc} G & \to & \mathbf{E}G \\ & & \downarrow \\ & & \mathbf{B}G \end{array}$$

*with the property that if*

$$\begin{array}{ccc} G & \to & \mathbf{E} \\ & & \downarrow \\ & & \mathbf{X} \end{array}$$

*is a principal $G$-bundle, then there is a function $f : \mathbf{X} \to \mathbf{B}G$, unique up to homotopy, such that the pullback of $f$ on the fibering over $\mathbf{B}G$ is the fibering over $\mathbf{X}$, which is to say $f^*(\mathbf{E}G) = E$.*

The classifying space has the special property that the set of homotopy classes of maps from any space $\mathbf{X}$ into $\mathbf{B}G$ identifies the set of isomorphism classes of principal $G$-bundles over $\mathbf{X}$.

Now we define the cohomology of the group, $H^*(G)$ to be the cohomology of the classifying space $H^*(\mathbf{B}G)$.

The cohomology of the group can also be defined by means of resolutions of the group ring $\mathbb{Z}G$. Given a projective resolution $F$ of $\mathbb{Z}$ over $\mathbb{Z}G$

$$\ldots \to F_2 \to F_1 \to \mathbb{Z}G \to \mathbb{Z}$$

(for example, the chain complex of $\mathbf{E}G$), define $H^*(G)$ to be the cohomology of the resolution. This definition is independent of the choice of resolution, up to isomorphism (see pages 24 and 33 in [Bro94a]).

The inclusion $H \hookrightarrow G$ induces the restriction map in group cohomology, defined as

$$(\mathrm{res}^G_H)^*\colon H^*(G) \to H^*(H)$$

.

As with any cohomology theory, group cohomology has a cup product

$$\cup\colon H^*(G) \otimes H^*(G) \to H^*(G)$$

defining a pairing

$$\cup\colon H^i(G) \otimes H^j(G) \to H^{i+j}(G)$$

.

This cup product is natural with respect to the restriction map, which is to say

$$(\mathrm{res}^G_H)^*(a \cup b) = (\mathrm{res}^G_H)^* a \cup (\mathrm{res}^G_H)^* b$$

**Theorem 7.** *For $p$ an odd prime, $H^*(\mathbb{Z}/p;\mathbb{Z}/p) = E(\nu) \otimes (\mathbb{Z}/p[\eta])$, where $E(\nu)$ is an exterior algebra on a generator of dimension 1, and $\mathbb{Z}/p[\eta]$ is a polynomial algebra on a generator of dimension 2.*

**Theorem 8.** *For $p$ an odd prime, $H^*((\mathbb{Z}/p)^2;\mathbb{Z}/p) = E(\nu_1,\nu_2)\otimes(\mathbb{Z}/p[\eta_1,\eta_2])$, where each $\nu_i$ is a generator of dimension 1, and each $\eta_i$ is a generator of dimension 2.*

**Theorem 9.** *For $p = 2$, $H^*((\mathbb{Z}/p)^n;\mathbb{Z}/p) = \mathbb{Z}/p[\eta_1,\ldots,\eta_n]$, where each $\eta_i$ is a generator of dimension 1.*

**Theorem 10** ( [Lew68])**.** *The integral cohomology of the group $\mathbb{Z}/p\times\mathbb{Z}/p$ is $P(\alpha,\beta)\otimes\mathbb{Z}[\mu]$ where the degree of $\alpha$ and $\beta$ in $H(G)$ is two, and the degree of $\mu$ is three. When $p$ is 2, there is the relation $\mu^2 = \alpha\beta^2 + \alpha^2\beta$.*

## 2.5 Introduction to spectral sequences

### 2.5.1 What is a spectral sequence

Spectral sequences are a means of computing the homological information of a fibration (or fibre bundle, if that is your preferred construction)

It is standard to think of a spectral sequence as a book with infinitely many pages. On each page, there is a grid composed of points and arrows. Each point is an abelian group, labeled $E^r_{p,q}$. The letter $r$ identifies on which page, and the letters $p$ and $q$ on which spot of the grid the term resides. Each arrow is a homomorphism. The composition of subsequent arrows is a zero map. In other words, each arrow is a differential, and the lines defined by following arrows are chain complexes. This tells us how to move one page ahead: the abelian group which holds the $(p,q)$ spot on page $r+1$ is the homology (or cohomology) of the chain complex on page $r$, at that spot.

Hatcher, in his (as of this date) unpublished book *Spectral Sequences* [Hat] on page 8 (page 24 for the cohomological version), states the following theorem which makes precise the definition of the spectral sequence.

**Theorem 11.** *Let $F \to X \to B$ be a fibration with path connected base $B$. If the fundamental group of $B$ acts trivially on the homology (resp. cohomology) of the fibre, $H_*(F)$ (resp. $H^*(F)$), then there is a spectral sequence $\{E^r_{p,q}, \mathrm{d}_r\}$ (resp. $\{E_r^{p,q}, \mathrm{d}_r\}$) with the following properties:*

1. $\mathrm{d}_r : E^r_{p,q} \to E^r_{p-r,q+r+1}$,
2. $E^{r+1} = \mathrm{Ker}\, \mathrm{d}_r / \mathrm{Im}\, \mathrm{d}_r$,
3. $E^{\infty}_{p,n-p}$ *is isomorphic to to the successive quotients* $F^p_n / F^{p-1}_n$ *in the filtration of* $H_n(X)$, $0 \subset F^0_n \subset \ldots \subset F^n_n = H_n(X)$,
4. $E^2_{p,q} \cong H_p(B; H_q(F))$.

*(The properties in cohomology spectral sequences are analogous.)*

In the case where the fundamental group $\pi_1$ of $B$ does not act trivially on the cohomology of the fibre, it is necessary to use local coefficients. In other words, we use the cohomology of the chain complex $Hom_{\mathbb{Z}[\pi_1]}(C(\widetilde{B}), H^*(F))$, where $\widetilde{B}$ is the universal cover of $B$, and $H^*(F)$ is considered as a left $\mathbb{Z}[\pi_1]$ module by fiber transport.

### 2.5.2 The Borel construction

Let $\mathbf{R}$ be a ring, and $X$ a $G$-CW complex, with chain complex $C_*(X;\mathbf{R})$.

Each $C_i(X;\mathbf{R})$ is isomorphic to a direct sum of permutation modules.

$$C_i(X;\mathbf{R}) \cong \times_{\sigma \in X^{(i)}/G} \mathbf{R}_{\sigma_i} \otimes_{G_{\sigma_i}} \mathbf{R}G$$

Where $X^{(i)}/G$ is the set of orbits of the action of $G$ on the $i$-skeleton of $X$, and $\mathbf{R}_{\sigma_i}$ is the sign representation of $G$'s action on the cell $\sigma$.

The Borel $G$-equivariant cohomology of the space $X$, $H^*_G(X;\mathbf{R})$ is defined as the cohomology of the twisted product $X \times_G EG = (X \times G)/G$, where the action of $G$ on $X \times EG$ is diagonal (the Borel construction). This twisted product defines a fibration.

$$\begin{array}{ccc} X & \hookrightarrow & X \times_G EG \\ & & \downarrow \\ & & BG \end{array}$$

This fibration is associated to a spectral sequence, which relates the cohomology of the group to the cohomology of the singular set, the fixed point set, and of the Borel construction. For example, if the space $X$ is a free $G$-CW complex, the fibration

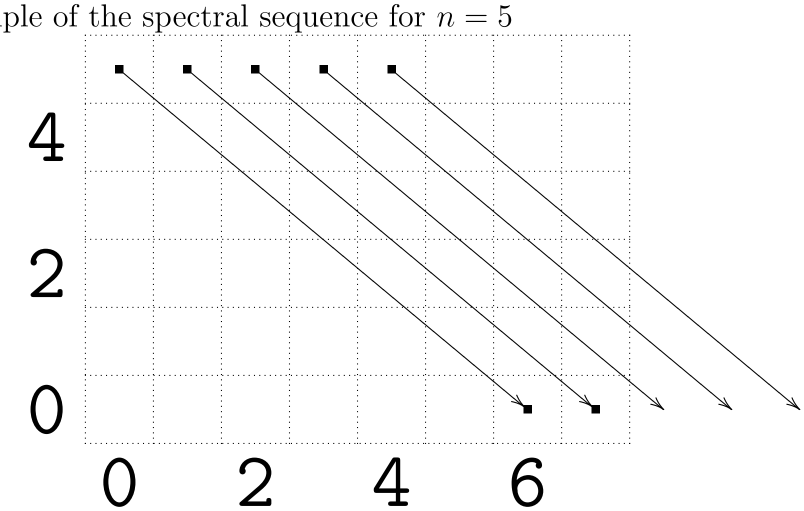
$$\begin{array}{ccc} X & \hookrightarrow & X \times_G EG \\ & & \downarrow \\ & & X/G \end{array}$$

has contractible fibre, and therefore the Borel construction $X \times_G EG$, and the orbit space $X/G$, are homotopy equivalent and have equal cohomology groups.

**Lemma 1** ( [McC02])**.** *Let $M$ be a 4-manifold with locally linear $G$-action, and let $\Sigma$ be its singular set. Then $H^i_G(M) \cong H^i_G(\Sigma)$ for $i > 4$.*

**Theorem 12.** *A group $G$ which acts freely on a finite $G$-CW complex with the homology of an $n$-sphere, $X^n$, has periodic cohomology with period dividing $n+1$.*

The proof of this follows from the inspection of the spectral sequence. Below is an example of the spectral sequence for $n = 5$



Clearly, there is only one possible differential, $d_{n+1}$. Since $G$ is acting freely on $S^n$, two facts are known. The first is that $H_G^*(X) = H^*(X/G)$, and the second is that $\operatorname{rk} H^i(X/G) = 0$ for all $i > n$. A basic fact about spectral sequences is that the cohomology of the fibration can be determined by successive filtrations of the $E_r^{p,q}$, which in this case translates to $H_G^i(X) = \oplus_{p+q=i} E_\infty^{p,q}$, from which it is deduced that $E_\infty^{p,q} = 0$ for all $p+q \geq n+1$. Since $E_\infty^{p,q} = H^p(G; H^0(S^n))/\mathrm{im} d_{n+1}$, it follows that $d_{n+1} : E_2^{p,n} \to E_2^{p+n+1,0}$ is an isomorphism for all $p > 0$. In other words, $H^p(G; H^n(S^n))$ is isomorphic to $H^{p+n+1}(G; H^0(S^n))$, which clearly implies that $G$ is periodic with periodicity dividing $n+1$.

# 3 Literature review

The fixed point set of a $\mathbb{Z}/p$ action on $S^3$ is either empty or an unknotted $S^1$ [MB84]. This is the solution to the Smith conjecture.

The Smith conjecture is false when generalized to higher dimensions. Giffen [Gif66] produces a family of counterexamples in dimensions greater than four using the branched cover of twist-spun knots. In dimension four, Giffen produces counterexamples for odd order transformations.

Gordon, in [Gor74], and Sumners, in [Sum75], each provide independent proofs that the Smith conjecture is also false in dimension four when looking at actions of groups of even order. Together, these provide conclusive evidence that not all group actions on higher dimensional spheres are equivalent to linear actions.

Fintushel, in 1976 [Fin76], gave a classification of effective, locally smooth circle actions on homotopy 4-spheres up to weak equivalence (two $G$-manifolds $M_1$ and $M_2$ are weakly equivalent if there is a homeomorphism $H\colon M_1 \to M_2$ such that, for $g \in G$ and $x \in M_1$, $H(g \cdot x) = \theta(g)H(x)$, for some automorphism $\theta$ of $G$.

Fintushel identifies orbit data called "admissible systems of data" as either

1. $\{M^*\}$, where there are no exceptional orbits;
2. $\{M^*, \alpha\}$, where $\alpha \in \mathbb{Z}$, if there is an exceptional orbit of type $\mathbb{Z}/\alpha$;
3. $\{(M^*, E^* \cup F^*), \alpha, \beta\}$, where $\alpha, \beta \in \mathbb{Z}$, $\alpha$ and $\beta$ are relatively prime, if there is one exceptional orbit of type $\mathbb{Z}/\alpha$ and one of type $\mathbb{Z}/\beta$

Here $E^*$ is the image of the exceptional orbits, and $F^*$ is the boundary of the orbit space.

Then the actions are classified via their admissible data.

**Theorem 1** ( [Fin76]). *The association of orbit data to a locally smooth $S^1$ action on a homotopy 4-sphere induces a bijective correspondence between admissible systems of data and weak equivalence classes of locally smooth $S^1$ actions on homotopy 4-spheres.*

As a corollary, Fintushel notes that if $(S, K)$ if a knotted pair, the circle action associted with $\{(S^3, K), \alpha, \beta\}$ does not extend to larger compact connected Lie groups.

Peter Sie Pao, also in 1976 [Pao78], showed that there are infinitely many non-linear circle actions on the four-dimensional sphere. Furthermore, an affirmative answer to the Poincaré conjecture means his results imply that those non-linear actions, along with the linear actions, are the only $S^1$ actions possible on $S^4$.

The main theorem of this paper is that if $M$ is a $S^1$-manifold and a homotopy 4-sphere with orbit space either $S^3$ or $D^3$, $M$ is actually $S^4$. With this proven, every non-trivial knot $K \subset S^3$ results in an admissible system of data $\{(S^3, K), \alpha, \beta\}$ corresponding to an action with total space $S^4$ which is not extendable to an action of a larger compact connected Lie group.

In this article it is also proven that a homotopy 4-sphere $M$ admits a locally smooth circle action if and only if there is a locally flat imbedding $S^2 \hookrightarrow M$ such that the knot complement of the image of the imbedding is fibred over $S^1$ with finite cyclic structure group.

Fintushel and Stern, in 1981 [FS81], construct a free involution on $S^4$ for which the orbit space is not $s$-cobordant to the real projective 4-space, and therefore the action is not equivalent to the antipodal map. In order to do so, they build up a system by which one can distinguish exotic orbit spaces of involutions. The exotic involutions can be distinguished due to the following proposition:

**Proposition 1.** *Let $T$ be a free involution on a homotopy* 4*-sphere whose quotient is s-cobordant to $RP^4$ and which desuspends to an involution $t$ on a homology* 3*-sphere $\Sigma^3$. Then there is an almost-framing $\mathfrak{F}$ for $\Sigma^3/t$ such that* $\mu(\Sigma^3/t; \mathfrak{F}) + \frac{1}{2}\alpha(\Sigma^3, t) \cong \pm 1 (mod\ 16)$.

The exotic involution is then constructed by starting with a Brieskorn homology 3-sphere, $\Sigma(3, 5, 19)$. Once it is determined that the Brieskorn 3-

sphere is the boundary of a contractible manifold $U^4$ with double cover $S^4$, they use the free involution $t$ coming from the circle action on $\Sigma(3,5,19)$, isotopic to the identity, to construct the desired involution $T$ on $U^4 \cup_t U^4$, diffeomorphic to $S^4$.

Kwasik and Schultz, in 1990 [KS90], seek to extend results on pseudofree actions of cyclic groups on higher dimensional spheres. The main theorem here is the analog of the "topological twisted desuspension theorem", which states

**Theorem 2.** *Let $\Phi$ be a pseudofree topological action of $\mathbb{Z}/2n$ on $S^{2k}$ with no fixed points ($k \geq 3$). Then there is an invariant*

$$\sigma(\Phi) = \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/n])$$

*such that $\Phi$ is a twisted desuspension if and only if $\sigma(\Phi) = 0$.*

In dimension four, this theorem is replaced by the "weak topological twisted desuspension theorem", which states

**Theorem 3.** *Let $\Phi$ be a strictly pseudofree action on $S^4$, and let $S_0$ denote the singular set of $\Phi$. Then there is an invariant $\sigma(\Phi) \in \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/n])$ such that if $\sigma(\Phi) = 0$,*

1. *There is a homology 3-sphere $\Sigma^3$ with free $\mathbb{Z}/2n$ action and an equivariant embedding of $\Sigma^3 \times I$ (with nontrivial action of $\mathbb{Z}/2n$ on $I$) in $S^4 \setminus S_0$ such that the inclusion of $\Sigma^3$ in $S^4 \setminus S_0$ induces an isomorphism in integral homology;*

2. *Let $W$ be the equivariant unit $H^+$-cobordism of $\Sigma^3$ to itself defined in [KS88]. Then $S^4 \setminus S_0$ is equivariantly homeomorphic to*

$$(\Sigma^3 \times I \cup (\cup_{i=1}^{\infty} W \times ((\mathbb{Z}/2n)/(\mathbb{Z}/n)) \times \{i\}))$$

*modulo the identifications implied by merging $\delta_- W \times ((\mathbb{Z}/2n)/(\mathbb{Z}/n)) \times \{0\}$ with $\Sigma^3 \times \{-1, 1\}$ and $\delta_- W \times ((\mathbb{Z}/2n)/(\mathbb{Z}/n)) \times \{i\}$ with $\delta_+ W \times ((\mathbb{Z}/2n)/(\mathbb{Z}/n)) \times \{i-1\}$ for $i > 0$.*

This weak version of the twisted desuspension theorem is in fact the strongest possible. The authors demonstrate this by providing counterexamples when any of the hypotheses are lifted.

The authors also classify strictly pseudofree topological $\mathbb{Z}/2n$ actions with trivial $\tilde{K}_0$ invariant on the four sphere. The classification is similar to the classification in higher dimensions:

**Theorem 4.** *Within a given local type and simple homotopy type there are exactly two distinct equivalence classes of strictly pseudofree topological $\mathbb{Z}/2n$ actions on $S^{2k}$ with vanishing $\tilde{K}_0$ invariant.*

Edmonds [Edm10] also investigates pseudofree group actions on spheres. Progressing from a result by Kulkarni, Edmonds proves that dihedral groups do not act pseudofreely and locally linearly on $S^n$ for $n \cong 0 \bmod 4$. In the same year, Hambleton shows in [Ham11] that finite dihedral groups do not act pseudofreely and locally linearly on any even dimensional spheres $S^{2k}$ with $k > 1$.

Milnor, in 1957 [Mil57], listed all groups which act freely and orthogonally on $S^3$:

1. The trivial group 1,
2. $Q_{8n}$, with presentation $\{x, y : \ x^2 = (xy)^2 = y^{2n}\}$,
3. $P_{48}$, with presentation $\{x, y : \ x^2 = (xy)^3 = y^4,\ x^4 = 1\}$,
4. $P_{120}$, with presentation $\{x, y : \ x^2 = (xy)^3 = y^5,\ x^4 = 1\}$,
5. The generalized dihedrals $D_{2^k(2n+1)}$ with presentation $\{x, y : \ x^{2^k} = 1,\ y^{2n+1} = 1,\ xyx^{-1} = y^{-1}\}$, $k \geq 2$, $n \geq 1$,
6. $P'_{8 \cdot 3^k}$, with presentation $\{x, y, z : \ x^2 = (xy)^2 = y^2,\ zyz^{-1} = xy,\ z^{3^k} = 1\}$, $k \geq 1$.

Zimmermann and Mecchia, in 2006 [MZ06], then in 2011 [MZ11], investigate the groups which can act on $S^4$. In the first paper, they determined which non-Abelian finite simple and non-solvable groups can act on the four sphere.

**Theorem 5** ( [MZ06])**.** *The only finite non-Abelian simple groups admitting an action on a homology 4-sphere are $\mathbf{A}_5 \cong PSL(2,5)$ and $\mathbf{A}_6 \cong PSL(2,9)$.*

**Theorem 6** ( [MZ06])**.** *Let $G$ be a finite nonsolvable group acting orientation-preservingly on a homology 4-sphere. Then either:*

1. *$G$ contains a normal subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$, with factor group isomorphic to either $\mathbf{A}_5$ or $\mathbf{S}_5$;*
2. *$G$ is isomorphic to $\mathbf{A}_5$ or $\mathbf{S}_5$; or*

3. *$G$ contains a subgroup of index at most two isomorphic to one of the following groups:*
   - $\mathbf{A}_5 \times C$, *where $C$ is either dihedral or cyclic;*
   - *the central product* $\mathbf{A}_5^* \times_{\mathbb{Z}/2\mathbb{Z}} \mathbf{A}_5^*$, *where* $\mathbf{A}_5^* = SL(2,5)$*;*
   - *the central product* $\mathbf{A}_5^* \times_{\mathbb{Z}/2\mathbb{Z}} C$, *where $C$ is a solvable group which acts freely and orientation preservingly on $S^3$.*

In the later paper, Zimmermann and Mecchia fully determine which finite groups act on $S^4$.

**Theorem 7** ( [MZ11])**.** *A finite group $G$ which admits an orientation-preserving action on a homology 4-sphere is isomorphic to one of the following:*

1. *a subgroup of the Weyl group* $W = (\mathbb{Z}/2)^4 \rtimes \mathbf{S}_5$ *;*
2. $\mathbf{A}_5$, $\mathbf{S}_5$, $\mathbf{A}_6$, *or,* $\mathbf{S}_6$ *;*
3. *an orientation-preserving subgroup of* $O(3)\times O(2)$ *(and hence of* $SO(5)$*);*
4. *a subgroup of* $SO(4)$*, or a* 2*-fold extension of such a group.*

*And if $G$ is a non-solvable group, it may be a subgroup of $O(4)$ instead of a (possibly twofold extension) of a subgroup of $SO(4)$.*

As a result, the groups which act on $S^4$ are either subgroups of $SO(5)$, or twofold extensions of subgroups of $O(4)$. This also gives a complete classification of subgroups of $SO(5)$.

Because of the Smith theory results stating that the $p$-group actions on spheres have spheres as fixed point sets, it is worthwhile to attempt to relate some of the tools from real representations to tools in group actions. Dotzel developed some tools to study these ”spherical representations” (see [Dot82], [DH81b] and tom Dieck's book Transformation Groups [tD87]).

Dotzel [Dot81a] begins his investigation with elementary Abelian $p$-groups.

**Theorem 8** (Borel's equation)**.** *For $G$ an elementary Abelian $p$-group, acting on a mod $p$ homology $n$-sphere $X$, with each fixed point set $X^H$ a mod $p$ homology sphere of dimension $n(H)$*

$$n - n(H) = \sum (n(H) - n(G))$$

*where the sum is over all subgroups of co-rank 1.*

For $K \leqslant G$, let $K/p = \{g \in G | \, pg \in K\}$. Also let $A_H = \{K \leqslant G | \, K \geqslant H, G/K \text{ cyclic}\}$.

**Theorem 9** ( [Dot81a]). *Let $G$ be any finite Abelian $p$-group, acting cellularly on a finite CW-complex $X$ such that for $H \leqslant G$, $H \neq 0$, and $X^H$ is a mod $p$ homology $n(H)$-sphere. Assume that there is an $n > 0$ such that $H_*(X; \mathbb{Z}/p) = 0$, for $* \neq 0$. For any subgroup $H$ of $G$ assume the following identity holds for the action of $G/H$ on $X^H$:*

$$n(H) - n(G) = \sum (n(K) - n(K/p))$$

*with the sum running over $A_H$. Assume that $n(K) - n(K/p)$ is even if $p$ is 2 and $|G/K| > 2$, and $n - n(G)$ is even if $p$ is odd.*

*Then $H_n(X; \mathbb{Z}/p) = \mathbb{Z}/p + F$, with $F$ free over $\mathbb{Z}/p[G]$.*

Dotzel also proves the converse result.

**Theorem 10** ( [Dot81a]). *Let $G$ be a finite Abelian $p$-group acting cellularly on a finite CW-complex $X$ such that each $X^H$ is a $\mathbb{Z}/p$ homology $n(H)$-sphere, for $H \neq 0$. Suppose $H_i(X; \mathbb{Z}/p) = 0$ for $i \neq n$, and $H_n(X; \mathbb{Z}/p) = \mathbb{Z}/p + F$, with $F$ free over $\mathbb{Z}/p[G]$, for some integer $n$.*

*Then $n - n(G) = \sum (n(K) - n(K/p))$, where the sum is taken over $A_0$.*

Dotzel extends his results from finite Abelian $p$-groups to finite Abelian groups in 1987 [Dot87a].

Let $G$ be a finite Abelian group acting on a finite CW-complex $X$ which has the $\mathbb{Z}(\mathcal{P}$ homology of $S^n$, where $\mathcal{P}$ is the set of primes dividing $|G|$. For each $p \in \mathcal{P}$, let $G(p)$ be the set of all Sylow $p$-subgroups of $G$. Let $H$ be a $p$-subgroup of $G$, so that the fixed point set $X^H$ has the $\mathbb{Z}/p$ homology of a $n(H, G)$-sphere.

The function $n(-, G) : \{H \leqslant G | \, H \text{ is a } p\text{-subgroup of } G\} \to \mathbb{Z}$ satisfies the following Borel-Smith conditions [tD87, p. 210]:

1. The Borel formula: if $H \leqslant K$ are both $p$-subgroups of $G$ and $K/H = \mathbb{Z}/p \times \mathbb{Z}_p$, then $n(H, G) - n(K, G) = \sum (n(K', G) - n(K, G))$, with the sum being over all $H \leqslant K' \leqslant K$ such that $K'/H = \mathbb{Z}/p$.
2. If $H \leqslant K$ are $p$-subgroups of $G$, then $n(K, G) \leq n(H, G)$.
3. If $H \leqslant K$ are $p$-subgroups of $G$ with $K/H = \mathbb{Z}/p$, and $p$ is odd, then $n(H, G) - n(K, G)$ is even.

4. If $H \leqslant K$ are 2-subgroups of $G$ such that $K/H = \mathbb{Z}/4\mathbb{Z}$, $K'/H = \mathbb{Z}/2\mathbb{Z}$, then $n(H,G) - n(K',G)$ is even.

Dotzel shows that under certain conditions, such a dimension function is realized by a real represention.

**Theorem 11** ( [Dot87a]). *Let $G$ be a finite Abelian group and suppose $N(-,G)$ is a non-negative integer valued function defined on the $p$-subgroups of $G$ for all $p \mid |G|$, satisfying the previous conditions along with the condition that $N(e,G) - N(H,G)$ is even if $H$ is a 2-subgroup of $G$. Then there exists a real representation $V$ of $G$ such that for any $p$-subgroup $H$ of $G$, $p \mid |G|$, dim $V^H = N(H,G)$. Furthermore, if $\bar{V}$ is another such representation of $G$ then for all subgroups $H$ of $G$,*

$$\textit{dim } V^H \equiv \textit{dim } \bar{V}^H \qquad (\textit{mod } 2)$$

**Corollary 1** ( [Dot87a]). *Let $G$ be a finite Abelian group and suppose the 2-Sylow subgroup of $G$ is cyclic. If $N(-,G)$ is a non-negative integer valued function defined on the $p$-subgroups of $G$, $p \mid |G|$, satisfying only conditions 1-4, then there exists a real representation $R$ such that for any $p$-subgroup $H$ of $G$, dim $V^H = N(H,G) + 1$.*

McCooey lists necessary and sufficient conditions for a locally linear action on $S^4$ to be topologically concordant to a linear action.

**Theorem 12** ( [McC07]). *A locally linear $\mathbb{Z}/p \times \mathbb{Z}/p$ action on $S^4$ is topologically concordant to a linear action if and only if the Kervaire-Arf invariant $c \in \mathbb{Z}/2$ vanishes. Therefore there are at most two concordance classes of $\mathbb{Z}/p \times \mathbb{Z}/p$ actions on $S^4$.*

**Definition 1.** *Two functions $f, g : M \to N$ are topologically concordant if there is a function $F : M \times I \to N \times I$ such that $F|_{M\times 0} = f$, and $F|_{M\times 1} = g$.*

**Definition 2.** *Two actions $\phi$ and $\psi$ on $S^4$ are concordant if there is a locally linear action $\Psi$ on $S^4 \times I$ such that $\Psi|_{S^4\times 0} = \phi$, $\Psi|_{S^4\times 1} = \psi$, and $\mathrm{Fix}(g, S^4 \times I)$ is homeomorphic to $\mathrm{Fix}(g, S^4) \times I$. This definition is stronger than the previous definition of concordance.*

**Definition 3.** *The space of quadratic forms on a $\mathbb{Z}/2$ vector space of dimension $2n$ has $2^{2n}$ elements, and every quadratic form on that vector space is the orthogonal sum of these elements. The Arf invariant of a quadratic form over $\mathbb{Z}/2$ is the value appearing most often among these summands.*

**Definition 4.** *The Kervaire-Arf invariant of a manifold of dimension $4n+2$ is the Arf invariant of the framing of the middle dimensional $\mathbb{Z}/2$ homology.*

McCooey proves his result by generalizing the proof of the analogous result for $G = \mathbb{Z}/p$.

**Theorem 13** ( [McC07]). *Every $G = \mathbb{Z}/p$ action on $S^4$ which fixes a $S^2$ is concordant to a linear action.*

This is proved by determining that a certain obstruction to surgery vanishes. The pair Fix$G$ is a slice knot in $S^4$, so the pair $(S^4, \mathrm{Fix}G)$ bounds a pair $(B^5, B^3)$. If it is possible to perform surgery on the pair $(B^5, N(B^3)$ in order to produce the pair consisting of a standard 3-disc within a 5-disc, then the concordance of the actions has been constructed. Calculating the obstructions to this surgery, we can determine that the obstructions vanish, thereby proving the theorem.

Furuta [Fur89], De Michelis [DM89], and Buchdahl, Kwasik, and Schultz [BKS90] each proved independantly that manifolds with the homology of $S^4$ do not admit smooth or locally linear group actions fixing only one point.

Furuta uses gauge theory to prove that orientation preserving smooth actions can not fix only one point. The author shows that the fixed point set of the action can be defined by a certain gauge-transformation equivariant equation on the set of group invariant connections over the principal $SO(3)$ bundle resulting from the quotient $P/(\pm 1)$ (where $P$ is the principal $SU(2)$ bundle with Chern class $c_2 = -1$). Furuta then shows that the equation can be perturbed to have a one dimensional manifold as solution, with the same ends as $\mathrm{Fix}\, G$. Since a one dimensional manifold cannot have only one end, the fixed point set cannot consist of only one point.

Buchdahl and co-authors, and De Michelis, both prove essentially the same result.

**Theorem 14.** *Let $\Sigma^4$ be a closed integral homology 4-sphere. Then $\Sigma^4$ admits no locally linear finite group actions with exactly one fixed point. Moreover, a discrete fixed point set for such an action is either empty or consists of two points.*

In the Buchdahl article, this is proven by splitting the problem into two cases. If the group $G$ contains a normal $p$-group $P$, then Smith theory tells us that $P$ must either a $S^0$ or a $S^2$ , and therefore that $G/P$ must act on that lower dimensional sphere, fixing either nothing or a $S^0$. If $G$ does not contain a normal $p$-group $P$, it is shown that it must either be isomorphic to the alternating group $\mathbf{A}_5$ or the group $\mathbf{A}_5^2$. The authors proceed to prove that neither group acts on $S^4$ in the desired way.

There is a large literature concerning smooth or locally-linear finite group actions on 4-dimensional manifolds, but in this survey we have restricted attention to work concerning actions on the 4-sphere.

# 4 The shape of singular sets of elementary abelian group actions on $\mathbf{S}^4$

## 4.1 Groups of type $\mathbb{Z}/p \times \mathbb{Z}/p$, for $p$ odd

In the case of linear actions, it is possible to choose the group's generators ($s$, and $t$) such that they will have matrix representations ($S$ and $T$) as

$$S = \begin{bmatrix} cos(\frac{2\pi}{p}) & -sin(\frac{2\pi}{p}) & 0 & 0 & 0 \\ sin(\frac{2\pi}{p}) & cos(\frac{2\pi}{p}) & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & cos(\frac{2\pi}{p}) & -sin(\frac{2\pi}{p}) & 0 \\ 0 & 0 & sin(\frac{2\pi}{p}) & cos(\frac{2\pi}{p}) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

In this case, it is clear that the singular set is two entwined spheres $\mathbf{S}_s^2$ and $\mathbf{S}_t^2$, touching only at the poles $p_1 = (0,0,0,0,1)$ and $p_2 = (0,0,0,0,-1)$.

The same is true in the locally linear case. By Smith Theory, we know that the fixed point set of the group must be an $S^0$ or an embedded $S^2$. If $G$ fixes an $S^2$, then $G$ has a faithful representation into the group $SO(2)$, described by the group's action on the tangent space at any fixed point. This is absurd, however, so we conclude that $G$ fixes a pair of points. Then by diagonalizing the representation of $G$ into $SO(4)$ determined by $G$'s action on the tangent space of one of those fixed points, we can find generators $s$ and $t$ which have the form of rotation matrices:

$$S = \begin{bmatrix} cos(\frac{2\pi}{p}) & -sin(\frac{2\pi}{p}) & 0 & 0 \\ sin(\frac{2\pi}{p}) & cos(\frac{2\pi}{p}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & cos(\frac{2\pi}{p}) & -sin(\frac{2\pi}{p}) \\ 0 & 0 & sin(\frac{2\pi}{p}) & cos(\frac{2\pi}{p}) \end{bmatrix}$$

It is clear that the fixed point sets of generators $a$ and $b$ are embedded 2-spheres meeting in a pair of points.

## 4.2 $G = \mathbb{Z}/2 \times \mathbb{Z}/2$

### 4.2.1 $G$ acts linearly on $S^4$

There are three types of singular sets, depending on the number of fixed dimensions for each generator.

In the first case,

$$S = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$ST = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This is a pair of $S^2$ meeting at their poles. This singular set arrangement will be labeled ”type A”.

In the second case,

$$S = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$ST = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This is three $S^2$, all intersecting in a single circle. This singular set will be labeled "type B".

In the third case,

$$S = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$ST = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

This is a disjoint union of a 2-sphere and two 0-spheres. This singular set will be labeled "type C".

### 4.2.2 $G$ acts locally linearly and preserving orientation on $S^4$

**Theorem 13.** *The locally linear action of $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ has only three possible singular sets, and each is homeomorphic to a singular set of a linear action.*

We know from Smith theory that the fixed point set of any generator must be some homology sphere of dimension at most four. Since the actions must be orientation preserving, we also know that neither fixed point set can be empty, in which case the orbit space would have the homology of $\mathbb{R}P^4$ and thus be non-orientable. Nor can either generator fix an $S^1$ or an $S^3$, in which case examining the tangential representation of the generator at any fixed point would have to have determinant $-1$, and thus be orientation reversing there. Thus any generator must fix either a homology $S^0$ or $S^2$.

After choosing a pair of generators $s$ and $t$ for $\mathbb{Z}/2 \times \mathbb{Z}/2$, we can label the three possible fixed point sets, $F_s$, $F_t$, and $F_{st}$. Then each individual possibility for a singular set arrangement can be considered.

Up to a renaming of the group elements, there are four possibilities for the fixed point sets in $\Sigma$.

| $F_a$ | $F_b$ | $F_{ab}$ |
|---|---|---|
| $S^0$ | $S^0$ | $S^0$ |
| $S^0$ | $S^0$ | $S^2$ |
| $S^0$ | $S^2$ | $S^2$ |
| $S^2$ | $S^2$ | $S^2$ |

We can exclude certain arrangements of fixed point sets.

**Step 1.** *If there are two $S^2$, they must intersect.*

Supposing they are disjoint, $t$ acts on the manifold $X = S^4 \setminus F_s$, which by Alexander duality has $H_0(X) = H_1(X) = \mathbb{Z}$, $H_i(X) = 0$ for all $i$ greater than 1. By a result of Smith theory,

$$\sum_{i \geq r} \operatorname{rk} H_i(X^t) \leq \sum_{i \geq r} \operatorname{rk} H_i(X)$$

for all $r$. $X$ fails this criterion, since $\operatorname{rk} H_2(X) = 0$ and $\operatorname{rk} H_2(X^t) = 1$.

**Step 2.** *The singular set can not be three distinct pairs of isolated points.*

Consider $X = S^4 \setminus \Sigma$, a free $G$-space with $\mathbb{Z}/2$ homology computed using Alexander duality as

$$\begin{gathered} H^0(X) = \mathbb{Z}/2 \\ H^1(X) = \widetilde{H}_2(\Sigma) = 0 \\ H^2(X) = \widetilde{H}_1(\Sigma) = 0 \\ H^3(X) = \widetilde{H}_0(\Sigma) = (\mathbb{Z}/2)^5 \end{gathered}$$

And $H^3(X)$ has a $\mathbb{Z}/2[G]$-module structure, which is given by the short exact sequence

$$0 \to H^3(X) \to V \to \mathbb{Z}/2 \to 0$$

Where $V = \mathbb{Z}/2[G/\langle s\rangle] \times \mathbb{Z}/2[G/\langle t\rangle] \times \mathbb{Z}/2[G/\langle st\rangle]$. This exact sequence is obtained from the long exact sequence of relative homology

$$\ldots \to H_1(S^4) \to H_1(S^4, \Sigma) \to H_0(\Sigma) \to H^0(S^4) \to 0$$

Here $H_1(S^4) = 0$, $H_1(S^4, \Sigma) \cong H^3(X)$ by excision, and $H_0(\Sigma) = V$, resulting in the desired short exact sequence.

Thus the spectral sequence of $E(X \times_G EG)$ on any page consists of two lines, $E_n^{0,q}$ and $E_n^{3,q}$. Then since the action of $G$ on $X$ is free, $H^i_G(X) \cong H^i(X/G)$, and since $H^i(X/G) = \times E_\infty^{p,i-p}$, and $H^5(X/G) = 0$, it suffices to show that one term of the direct sum $\times E_\infty^{p,i-p}$ is non-zero to achieve a contradiction.

**Lemma 2.** $E_\infty^{0,5} = H^5(G; \mathbb{Z}/2)/\operatorname{coker} d_4 \neq 0$, *which is to say that*

$$d_4 \colon H^1(G; H^3(X; \mathbb{Z}/2)) \to H^5(G; \mathbb{Z}/2)$$

*is not surjective.*

We will prove this by comparing the ranks of the domain and codomain. Since we have a short exact sequence of $\mathbb{Z}/2[G]$-modules involving $H^3(X)$, we can obtain a long exact sequence of cohomology

$$\begin{aligned} 0 \to H^0(G; H^3(X)) \to H^0(G; V) &\to H^0(G; \mathbb{Z}/2) \to \\ H^1(G; H^3(X)) &\to H^1(G; V) \to H^1(G; \mathbb{Z}/2) \to \ldots \end{aligned}$$

$H^0(G; \mathbb{Z}/2)$ has rank 1, and $H^1(G; V) = H^1(G; \mathbb{Z}/2[G/\langle s\rangle] \times \mathbb{Z}/2[G/\langle t\rangle] \times \mathbb{Z}/2[G/\langle st\rangle] = H^1(G/\langle s\rangle; \mathbb{Z}/2) \times H^1(G/\langle t\rangle; \mathbb{Z}/2) \times H^1(G/\langle st\rangle; \mathbb{Z}/2)$ has rank 3. So the kernel of the map

$$H^1(G; H^3(X)) \to H^1(G; V)$$

has rank at most 1, and the image of the map has rank at most 3, from which we conclude that $H^1(G; H^3(X))$ has rank at most 4.

But the rank of $H^5(G; \mathbb{Z}/2)$ is exactly 6, so $d_4$ could not possibly be surjective. Therefore $\Sigma$ cannot be three distinct pairs of points.

**Step 3.** *If two different group elements each fix a pair of isolated points, these points must all be distinct.*

Otherwise, a point in the intersection is an isolated global fixed point, so there is a free $\mathbb{Z}/2\times\mathbb{Z}/2$ action on the boundary of an open ball neighborhood of each of those fixed points, which is impossible (see, for example, [Smi42]).

**Step 4.** *If two distinct group elements each fix a $S^2$, these spheres must be distinct.*

Otherwise, $G$ acts freely on $X = S^4 \setminus \Sigma$, a homology circle, and this is impossible.

We are now left with the following possibilities, up to relabeling of group elements:

| $F_s$ | $F_t$ | $F_{st}$ |
|---|---|---|
| $S^2$ | $S^2$ | $S^0$ |
| $S^2$ | $S^2$ | $S^2$ |
| $S^0$ | $S^0$ | $S^2$ |

We are also much more constrained in how these fixed point sets may intersect. In the first instance, the two $S^2$ ($F_s$ and $F_t$) must intersect, and $F_s \cap F_t \subset F_{st}$, therefore $F_s \cap F_t$ must equal the $F_{st} = S^0$, so we have type A.

In the second instance, no two $S^2$ may intersect in only two points. If two $S^2$ (say $F_s$ and $F_t$) did intersect in two isolated points, then by observing the tangential representations of $s$ and $t$ at either point we see that these two points would be isolated fixed points of $st$, which contradicts the assumed arrangement. Therefore the 2-spheres must meet in more than two points, and must nevertheless meet, by Smith theory, in some $i$-dimensional sphere. Therefore they meet in a circle, and we have type B.

In the third instance, all the fixed point sets must be disjoint, as any point in the intersection would be a global fixed point, and therefore the two $S^0$ fixed point sets would have to intersect, a situation we have already excluded. The case of two copies of $S^0$ and a $S^2$ all disjoint is a singular set of type C.

## 4.3 $G = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$

### 4.3.1 $G$ acts linearly on $S^4$

There are two types of singular sets in the linear case.

In the first case, there are no global fixed points, and the group has generators:

$$S = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The singular set in this case is conceptually simple. Its components are four 2-spheres ($F_s$, $F_t$, $F_{st}$ and $F_u$) and three 0-spheres ($F_{su}$, $F_{tu}$, $F_{stu}$). The sets $F_s$, $F_t$, and $F_{st}$ meet each other in one circle, and meet $F_u$ in $F_{su}$, $F_{tu}$, and $F_{stu}$. In terms of the building blocks described in the previous section, there are three type A, three type C, and one type B. Call this arrangement type I.

| A | B | C |
|---|---|---|
| 3 | 1 | 3 |

In the second case, there are two global fixed points. A choice of generators would be:

$$S = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The singular set here consists of six 2-spheres, and one 0-sphere at the intersection of those six sets. In terms of the building blocks mentioned previously, there are three type A and four type B. Call this arrangement type II.

| A | B | C |
|---|---|---|
| 3 | 4 | 0 |

#### 4.3.2 $G$ acts locally linearly and preserving orientation on $S^4$

We may divide the problem into two cases based on whether there is a pair of global fixed points or not.

**Proposition 2.** *If there are global fixed points, then the singular set is a type II.*

If $x$ is one such fixed point, then consider the representation $(\mathbb{Z}/2)^3 \hookrightarrow SO(T_x) = SO(4)$. The corresponding matrices are all simultaneously diagonalizable, so the matrix group can be represented by the group generated by

$$S = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

As can be seen by enumerating the elements of this group, there are six elements which fix planes through the origin, $S$, $T$, $U$, $ST$, $TU$, and $STU$, and one element which acts freely, $SU$. These correspond, respectively, to six group elements which fix 2-spheres, and the element which fixes only the global fixed points. By enumerating the $(\mathbb{Z}/2)^2$ subgroups, we see that there are four in which the fixed planes of the generators meet in a line, and three in which the fixed planes intersect only at the origin. These correspond, respectively, to four type B arrangements, and three type A arrangements, all meeting together in two points, resulting in the desired singular set configuration.

**Proposition 3.** *If there are no global fixed points, then the singular set is of type* I.

**Step 1.** *There must exist a subgroup corresponding to a singular set of type either* A *or* B.

Supposing there were only subgroups of type C, their $S^2$ components must coincide, since they cannot be disjoint. For if they were not disjoint, and they did not coincide, they would have to meet in an $S^0$ or an $S^1$, resulting in an excluded $(\mathbb{Z}/2)^2$ singular set. Recall also that none of the elements which fix $S^0$ can share a point, and that only one element can fix a particular $S^2$.

We now have enough to derive a contradiction. Let $s$ and $t$ be generators of one $(\mathbb{Z}/2)^2$ subgroup, each fixing an $S^0$, and let $u$ be a generator of another, distinct from $s$ and $t$, and also fixing each an $S^0$. Consider the product $su$. If $F_{su} = S^0$, we have a $(\mathbb{Z}/2)^2$ subgroup ($\langle s, u\rangle$) which fixes three distinct pairs of points, which is impossible. If it fixes an $S^2$, then $su$ and $st$ fix the same $S^2$, and therefore $u = t$, which is impossible. Therefore there must be some $(\mathbb{Z}/2)^2$ subgroup which has a singular set of type A or B.

**Step 2.** *There must be a* $(\mathbb{Z}/2)^2$ *subgroup which has a singular set of type B.*

We start with a $(\mathbb{Z}/2)^2$ subgroup, generated by $s$ and $t$, which does not have a singular set of type C. If it is of type B, we are done. Otherwise, it is of type A. Any third generator, $u$, must act on this singular set, leaving the structure invariant. In particular, $u$ must act permute the points of $F_{st}$.

Now if $F_u = S^0$, disjoint from $F_s$ and $F_t$, then $F_u$ and $F_{st}$ are disjoint pairs of points, so the subgroup generated by $st$ and $u$ must have a singular set of type C, and therefore $stu$ fixes a $S^2$. Now given that we have picked a $u$ which fixes a $S^2$, there are four possibilities for the intersections $F_u \cap F_s$ and $F_u \cap F_t$, depending on whether the pairs intersect in $S^0$ or $S^1$. If $F_u \cap F_s = S^1$ or $F_u \cap F_t = S^1$, we are done. This leaves the possibility of $F_u \cap F_s = S^0$ and $F_u \cap F_t = S^0$. In this case, $F_{su} = F_u \cap F_s$, $F_{tu} = F_u \cap F_t$, and $F_{st}$ are three disjoint pairs of $S^0$, with $su$ and $tu$ generating a $(\mathbb{Z}/2)^2$ subgroup, which is impossible.

**Proposition 4.** *There is only one possible singular set of* $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ *without global fixed points built up from a singular set of type B.*

We will construct the singular set. Let the subgroup corresponding to the type B singular set, $\Sigma'$, be generated by elements $s$ and $t$.

**Step 1.** *Supposing $u$, a third generator for $(\mathbb{Z}/2)^3$, has $F_u = S^0$, then that $S^0$ can not be disjoint from $\Sigma'$.*

Otherwise, $F_u$, $F_{su}$, and $F_s$ would be arranged in a type C pattern, but so would $F_u$, $F_{tu}$, and $F_t$. The element $su$ acts freely on $F_u$, as must $tu$. This creates a contradiction: $st$ must fix $F_u$, but $F_u$ is disjoint from $F_{st}$.

Given that $F_u$ intersects $\Sigma'$, and therefore that $F_u \subset F_s$, $F_t$, or $F_{st}$, we can show that one of $F_{su}$, $F_{tu}$, or $F_{stu}$ is a $S^2$, by the same argument as used above in the $(\mathbb{Z}/2)^2$ case.

**Step 2.** *Assuming $F_u = S^2$, the action of $u$ on $\Sigma'$ determines the singular set for $G$.*

$F_u$ must, by earlier arguments, intersect each of the $S^2$ in $\Sigma'$. Since there are no global fixed points, $u$ must act freely on the circle at the intersection of $F_s$ and $F_t$. This strongly restricts the possible intersections of $F_u$ and any of the $S^2$ in $\Sigma'$. By the Jordan curve theorem, the circle $F_s \cap F_t$ divides each of the 2-spheres into an interior area and an exterior area. Suppose $F_u$ meets $F_s$ in a circle. Then that circle lies entirely in either the interior or the exterior area bounded by $F_s \cap F_t$. The group element $t$ fixes the bounding circle but acts on $F_s$, so $t$ switches the interior and exterior regions of $F_s$. As a result, $F_s \cap F_u$ would be translated from one area to the other, which is impossible. As a result, $F_u$ can not intersect any of those $S^2$ in a circle. Therefore it intersects each of them in a $S^0$, at which point we have achieved the singular set of type II. $F_s$, $F_t$, and $F_{st}$ form the type B pattern. $F_s$, $F_u$, and $F_{su}$, $F_t$, $F_u$, and $F_{tu}$, as well as $F_{st}$, $F_u$, and $F_{stu}$ form the three type A patterns. Last, $F_{su}$, $F_{stu}$, and $F_t$, $F_{su}$, $F_{tu}$, and $F_{st}$, as well as $F_{tu}$, $F_{stu}$, $F_s$ form the three type C patterns.

## 4.4 $G = (\mathbb{Z}/2)^4$

### 4.4.1 The linear case

There is exactly one $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$-subgroup within the diagonal matrices of $SO(5)$, and that is the one generated by the elements

$$S = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad V = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

The singular set for this action consists of 10 interlocked 2-spheres, being the fixed point sets of $S$, $T$, $U$, $V$, $ST$, $TU$, $UV$, $STU$, $TUV$, $STUV$. When described using the $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ blocks previously encountered, there are ten blocks of type I, and five blocks of type II.

## 4.5 $G$ acts locally linearly and preserving orientation on $S^4$

**Theorem 1.** *There is exactly one possible configuration of singular sets of locally linear $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$. Speficically, the singular set of an orientation preserving locally linear action of $G = (\mathbb{Z}/2)^4$ on $S^4$ has a singular set homeomorphic to the linear orthogonal action.*

We will show this to be true by constructing that singular set from the singular set of a $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ subgroup. We already know, from above, that there are only two possible configurations for these singular sets, which have been labeled types I and II.

**Step 1.** *There may not be* 15 $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ *subgroups corresponding to type* II *singular sets.*

Each subgroup corresponding to a type II singular set needs to have a distinct pair of global fixed points. For if $\Sigma_1'$ and $\Sigma_2'$ were the singular sets of two distinct $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ subgroups $H_1$ and $H_2$ which shared global fixed points $x_1$ and $x_2$, then the tangential representations of $H_1$ and $H_2$ at, say, $x_1$ would be simultaneously diagonalizable. Since there is only one $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ which fits in the diagonal of $SO(4)$, the elements of $H_1$ and $H_2$ must fix the same planes of $T_{x_1}$. As a result, $H_1$ and $H_2$ would have the same singular set, and therefore be equal. Therefore if $\Sigma_1'$ and $\Sigma_2'$ are distinct, they must have distinct global fixed points.

In each singular set of type II, the global fixed $S^0$ is the fixed point set of some element of the corresponding subgroup, therefore, if there are 15 distinct subgroups corresponding to type II singular sets, there must be 15 group elements each fixing a distinct $S^0$. This is clearly absurd, as at least one element of $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ must fix an $S^2$. Therefore there must be at least one subgroup which corresponds to a singular set of type I.

**Step 2.** *If there is a $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ subgroup corresponding to a singular set of type I, then the whole singular set of the entire group must follow the structure of the singular set of the linear model.*

Let $\Sigma'$ be a singular set of type I corresponding to a $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ subgroup with generators $s$, $t$, and $u$, such that $F_s \cap F_t = S^1$. Let $v$ be a generator of $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ so that $s$, $t$, $u$, $v$ together generate the whole group. The element $v$ acts on $\Sigma'$, and since this action must leave each of the $S^2$ components of $\Sigma'$ invariant, it must leave the $S^1$ at the intersection of $F_s$ and $F_t$ invariant.

However, $v$ can not act freely on that $S^1$. We recall, from the construction of the type I singular set, that $u$ already acts freely on $F_s \cap F_t$. If $v$ acted freely as well, $uv$ would fix the circle, and act on $S^4$ in a non-trivial manner, so we can conclude that $uv$ would fix a $S^2$ distinct from $F_s$, $F_t$, and $F_{st}$. The elements $suv$, $tuv$, and $stuv$ also would fix $F_s \cap F_t$, and therefore would also fix distinct 2-spheres meeting in $F_s \cap F_t$. So $s$, $t$, $uv$, $st$, $suv$, $tuv$, and $stuv$ would all fix distinct $S^2$ intersecting in the original $S^1$. But this is an enumeration of all the elements of the $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ subgroup generated by $s$, $t$, and $uv$, excluding the identity element, and the above pattern does not match either of the possible $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ singular sets (for example, there is no type A subpattern in this singular set). We conclude that $v$ does not act freely on $F_s \cap F_t$.

By the same token, $v$ can not fix that $S^1$. The argument follows precisely the preceding one, with $v$ in the place of $uv$.

As a result, $v$ must fix two points on the $S^1$. As a result, the subgroup generated by $s$, $t$ and $v$ has two global fixed points, and has a type II singular set. Therefore exactly one of $v$, $sv$, $tv$ or $stv$ must fix a $S^0$. If $v$ does not fix an $S^2$, then we can repeat the previous argument while switching the labels of $sv$ and $v$. As a result, we may assume that $v$ fixes an $S^2$ which intersects $F_s \cap F_t$ in exactly two points. From this point on the labeling of all elements of $G$ will remain fixed.

We can now proceed to construct $\Sigma$.

**Step 3.** *We can deduce the fixed point sets of each element of $G$, from the assumption that $stv$ fixes $S^0$.*

The argument is mirrored in the case where, instead of assuming $stv$ fixes a $S^0$, it is assumed that $sv$ or $tv$ fixes a $S^0$. In these cases, no part of the argument changes other than the permutation of certain elements of the subgroups.

The subgroup $\langle stu, tu, stv \rangle$ contains the type C corresponding to the subgroup generated by $\langle stu, tu \rangle$ and therefore has a singular set of type I. The type I singular set has three 0-spheres, which are accounted for in $F_{stu}$, $F_{tu}$, and $F_{stv}$. The rest are 2-spheres, from whis we can deduce that $uv$ fixes a $S^2$.

The subgroups $\langle t, tu, stv \rangle$ and $\langle s, su, stv \rangle$ also have singular sets of type I, since they contains the type C corresponding to the subgroups generated by $\langle tu, stv \rangle$ and $\langle su, stv \rangle$, respectively, from which we can deduce that $suv$ and $tuv$ fix $S^2$ as well.

Knowing that $uv$, $suv$ and $tuv$ fix $S^2$, it is clear that $\langle s, t, uv \rangle$ has a singular set of type II, since it contains six unique $S^2$: $F_s$, $F_t$, $F_{uv}$, $F_{st}$, $F_{suv}$, and $F_{tuv}$. We can conclude that the remaining group element, $stuv$, fixes a $S^0$.

At this point, we have enumerated all the elements of $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$:

| $S^2$ | $S^0$ |
|---|---|
| $F_s$ | $F_{su}$ |
| $F_t$ | $F_{tu}$ |
| $F_{st}$ | $F_{stu}$ |
| $F_u$ | $F_{stv}$ |
| $F_v$ | $F_{stuv}$ |
| $F_{uv}$ | |
| $F_{sv}$ | |
| $F_{tv}$ | |
| $F_{suv}$ | |
| $F_{tuv}$ | |

We can now list the types of each individual $(\mathbb{Z}/2)^3$ subgroup, as they are uniquely identified by the number of 2-spheres they contain. Therefore as a result of having determined the fixed point set of each element, we have also built each of the $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ building blocks of $\Sigma$. The subgroups corresponding to type I are

$$\langle s,\, t,\, u\rangle$$
$$\langle s,\, t,\, v\rangle$$
$$\langle s,\, u,\, v\rangle$$
$$\langle t,\, u,\, v\rangle$$
$$\langle s,\, t,\, uv\rangle$$
$$\langle sv,\, t,\, u\rangle$$
$$\langle stu,\, stv,\, tuv\rangle$$
$$\langle stu,\, stv,\, suv\rangle$$
$$\langle stu,\, tuv,\, suv\rangle$$
$$\langle stv,\, tuv,\, suv\rangle$$

And the subgroups corresponding to type II are

$$\langle st,\, u,\, v\rangle$$
$$\langle s,\, tu,\, v\rangle$$
$$\langle s,\, tv,\, u\rangle$$
$$\langle st,\, tu,\, v\rangle$$
$$\langle sv,\, tu,\, uv\rangle$$

The collection of fixed point sets of each element of $G$ is the same in both the linear and locally linear cases. Furthermore, these sets must all intersect in the same way in both cases, since, for all $g$ and $h$ in $G$, $\langle g, h\rangle$ will be contained in at least one $(\mathbb{Z}/2)^3$, for which the singular sets are the same regardless of whether the action is linear or locally linear. Therefore $F_g$ and $F_h$ will intersect in the same way in both cases. We can conclude that the unique singular set pattern for $(\mathbb{Z}/2)^4$ is the same in the locally linear and linear cases.

# 5 Conclusion

One corollary to the main result is that the Borel spectral sequence of an elementary abelian group acting locally linearly and orientation preservingly on $S^4$ collapses if and only if there are global fixed points.

**Proposition 1.** *If a group acts on $S^4$ with global fixed points, the Borel spectral sequence collapses.*

Let $x_0$ be a global fixed point of the action. We will compare the Borel spectral sequences which converge to $H^*_G(S^4)$ and to $H^*_G(x_0)$. By naturality, there is a map of spectral sequences induced by the restriction map on cohomology $\rho : H^*_G(S^4) \to H^*_G(x_0)$.

We examine the second page of both spectral sequences.

$$E_2^{p,q}(S^4 \times_G EG) = H^p(G; H^q(S^4))$$

Therefore the page consists of two lines: that corresponding to $H^0(S^4)$ coefficients, and that corresponding to $H^4(S^4)$ coefficients. On the other hand, $E_2^{p,q}(x_0 \times_G EG)$ has only one line, corresponding to $H^0(x_0)$ coefficients. It is clear that the bottom lines of both spectral sequences are isomorphic.

We use the naturality of the induced map of spectral sequences to show that the unique differential $\mathrm{d}_5$ of $E_2^{p,q}(S^4 \times_G EG)$ is the zero map. Let $\alpha \in H^p(G; H^4(S^4))$ be sent to $\beta = \mathrm{d}_5\alpha$. But $\mathrm{d}'_5\rho^* = 0$, since $\mathrm{d}'_5$, the differential from the $4^{th}$ line of $E_2^{p,q}(x_0 \times_G EG)$, is the zero map. By the naturality commutative diagram, $\rho^*\mathrm{d}_5\alpha = \mathrm{d}_5\rho^*\alpha$, so $\rho^*\mathrm{d}_5\alpha = 0$. But $\rho^*$ is an isomorphism on the bottom line, therefore $\mathrm{d}_5\alpha = 0$. This is true for any $\alpha$, so $\mathrm{d}_5$ is the zero map.

**Proposition 2.** *If $G$, an elementary abelian group, acts locally linearly on a homology 4-sphere $X$ without global fixed points, the spectral sequence of the fibration $X \hookrightarrow X \times_G EG \to BG$ does not collapse.*

The first thing to note is that if $G$ acts without global fixed points, it has a normal $\mathbb{Z}/2 \times \mathbb{Z}/2$ subgroup $H$ with singular set corresponding to type C.

Therefore it is enough to show that the spectral sequence of $X \times_H EH$ does not collapse, as there is a natural homomorphism $E(X \times_H EH) \to E(X \times_G EG)$.

We will compare the ranks of the direct sums $E_\infty^{5,0}(X \times_H EH) \oplus E_\infty^{1,4}(X \times_H EH)$ and $E_\infty^{0,5}(\Sigma \times_H EH \oplus E_\infty^{3,2}(\Sigma \times_H EH)$. As is already known (see for example [McC02]), $H^i_H(X) \cong H^i_H(\Sigma)$, for all $i \geq 5$, so the ranks must agree.

On the second page of the spectral sequence,

$$E_2^{p,q}(X \times_H EH) = H^p(H; \mathbb{Z}/2) = \mathbb{Z}/2[\eta_1, \eta_2]$$

And this is a polynomial algebra with no relations. So $E_2^{5,0}(X \times_H EH)$ has rank 6, and $E_2^{1,4}(X \times_H EH)$ has rank 2. If the spectral sequence does not collapse, the $E_2$ page is equal to the $E_\infty$ page, and the rank of $E_\infty^{5,0}(X \times_H EH) \oplus E_\infty^{1,4}(X \times_H EH)$ is 8.

The second page of the spectral sequence of $E\Sigma \times_H EH$ has only the two lines

$$E_2^{p,2}(\Sigma \times_H EH) = H^p(H; H^2(\Sigma))$$
$$E_2^{p,0}(\Sigma \times_H EH) = H^p(H; H^0(\Sigma))$$

Where $H^0(\Sigma) = (\mathbb{Z}/2)^5$, as $\Sigma$ has five connected components (two pairs of points each switched by a generator, and one 2-sphere left invariant by the group), and is represented as a $\mathbb{Z}/2[H]$-module as $\mathbb{Z}/2[H/\langle s\rangle] \oplus \mathbb{Z}/2[H/\langle t\rangle] \oplus \mathbb{Z}/2$, and $H^2(\Sigma; \mathbb{Z}/2) = \mathbb{Z}/2$.

By Shapiro's Lemma,

$$H^i(G; \mathbb{Z}/2[H/\langle s\rangle] \oplus \mathbb{Z}/2[H/\langle t\rangle] \oplus \mathbb{Z}/2) = H^i(\langle s\rangle; \mathbb{Z}/2) \oplus H^i(\langle t\rangle; \mathbb{Z}/2) \oplus H^i(H; \mathbb{Z}/2)$$

So if the spectral sequence for $\Sigma \times_H EH$ were to collapse, the rank of $E_\infty^{5,0}(\Sigma \times_H EH) \oplus E_\infty^{3,2}(\Sigma \times_H EH)$ would be 12. In the case where the spectral sequence does not collapse, it is necessary to determine the image and kernel of the differential $\mathrm{d}_3 : E_2^{2,2} \to E_2^{5,0}$.

**Lemma 1.** *The image of this differential has rank 3.*

Let $z$ be a generator of $H^0(H; H^2(\Sigma))$, and let $a$ be an element of $H^2(H; H^0(\Sigma))$. Cup product by $a$ is a map from $H^i(H; H^j(\Sigma))$ to $H^{i+2}(H; H^j(\Sigma))$, and the diagram

$$\begin{tikzcd}
H^0(H; H^2(\Sigma)) \arrow[r, "a\cup"] \arrow[d, "\mathrm{d}_3"'] & H^2(H; H^2(\Sigma) \arrow[d, "\mathrm{d}_3"'] \\
H^3(H; H^0(\Sigma)) \arrow[r, "a\cup"] & H^5(H; H^0(\Sigma))
\end{tikzcd}$$

is a commutative square. So $a \cup \mathrm{d}_3 z = \mathrm{d}_3(a \cup z)$, and each of the elements of $H^2(H; H^2(\Sigma))$ is obtained as $a \cup z$, where $a$ is an element which comes from the $H^2(H; \mathbb{Z}/2)$ component of $H^2(H; H^0(\Sigma))$, corresponding to the invariant sphere. Since $\mathrm{d}_3$ is not the zero map, $\mathrm{d}_3 z$ is not zero, and therefore neither are $a \cup \mathrm{d}_3 z$ for the $a \in H^2(H; \mathbb{Z}/2)$. Therefore the image of $\mathrm{d}_3 : E_2^{2,2} \to E_2^{5,0}$ has rank at least three, and since the rank of $H^2(H; H^2(\Sigma))$ is three, the rank of the image must be exactly three. The differential is in fact injective.

Similarly, the differential from $E_2^{3,2} \to E_2^{6,0}$ is injective.

Therefore the rank of the direct sum $E_\infty^{5,0}(\Sigma \times_H EH) \oplus E_\infty^{3,2}(\Sigma \times_H EH)$ is five. We can safely conclude that the spectral sequence of $X \times_H EH$ does not collapse.

As a consistency check, we can verify that the ranks match if the spectral sequence does not collapse. In this case $H^*(H; H^0(X))$ and $H^*(H; H^0(X))$ are polynomial algebras, and since the image of the generator of $H^0(H; H^4(X))$ is nonzero, and all cup products are injective, it is easy to show that every differential is injective, and that the rank of $E_\infty^{5,0}(X \times_H EH) \oplus E_\infty^{1,4}(X \times_H EH)$ is precisely the rank of the cokernel of the differential $\mathrm{d}_5 : E_2^{0,4} \to E_2^{5,0}$. The rank of the direct sums along the $i + j = 5$ line of the two $E_\infty$ pages agree if the spectral sequence of $E(X \times_H EH)$ does not collapse.

One question the main result might answer is whether the singular set of any locally linear $G$-action is $G$-homotopy equivalent to the standard linear action. One might also wonder how the result about the collapse of the Borel spectral sequence generalizes to other groups. Is it possible that the spectral sequence collapses if and only if no $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ subgroup has a singular set of type C? In this case it is not required that the 2-Sylow subgroups of $G$ be normal, nor is it required that the $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ subgroup be normal in the 2-Sylow subgroup. However, it may be possible to use the fact that a Borel spectral sequence for $G$ collapses if and only if the Borel spectral sequence for $P$ collapses, for $P$ a $p$-Sylow subgroup of $G$.

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