ALGEBRA

IN

A TOPOS OF SHEAVES

BY

©

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ALGEBRA

IN

A TOPOS OF SHEAVES

to

My Wife Mahnaz

and

My Parents

تقدیم به همسرم مهناز
و به پدر و مادرم.

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ABSTRACT

In this thesis, we undertake the study of some classical set-based algebraic concepts in a topos-theoretic setting. Actually, the topoi we are particularly interested in are the Grothendieck topoi.

The main topics from Universal Algebra considered here are injectivity, equational compactness and tensor products.

After proving some general results about the above notions, we show that, for any set $\mathcal{H}$ of quasi-equations and an <u>arbitrary</u> Grothendieck topos $\mathbf{E}$, $\mathcal{M}od(\mathcal{H},\mathbf{E})$ has enough injectives iff $\mathcal{M}od\,\mathcal{H}$ has. Also that, for a noetherian locale $\mathcal{L}$, pure homomorphisms, equational compactness and the existence of equationally compact hulls are characterized here the same way as in Ens. Finally, we consider the notion of bimorphisms for algebras in a topos and prove, among other things, the counterpart of a result for algebras in Ens that tensor products and Universal bimorphisms are equivalent for suitable categories of algebras.

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TABLE OF CONTENTS

INTRODUCTION

This thesis is a study of certain aspects of Universal Algebra modelled in a topos rather than in the category Ens of sets. It is intended to provide a deeper understanding of the real features of the algebraic notions considered here and to show that much of classical set-based Universal Algebra can be, as we believe it should be, studied in a topos-theoretic setting.

The notions, from classical Universal Algebra, which have been chosen for study in this thesis are: injectivity, residual smallness, essential boundedness, purity, equational compactness, bimorphisms and tensor products; the reason for this choice being that these have been extensively investigated during the last decade for the case of algebras in Ens (e.g., [2], [7], [8], [9], [21]).

Some of these concepts, in particular injectivity, have previously been considered by Howlett [14] for algebras in topoi other than Ens, but there the study was restricted to very special Grothendieck topoi, namely, those with enough points. Here, in Chapter (1), we deal with an arbitrary Grothendieck topos $\mathbf{E}$ and substantially improve on some of the work in [14] by proving, among other things, that injectivity is properly behaved in $\mathcal{M}od(\mathcal{H},\mathbf{E})$ iff it is in $\mathcal{M}od\,\mathcal{H}$, for any set $\mathcal{H}$ of quasi-equations. Note that [14] only considers sets of equations in this context, apart from the already mentioned restriction to Grothendieck topoi with enough points.

Chapter (2) deals with the notion of purity and equational compactness in $\mathcal{M}od(\mathcal{H}, \mathrm{Sh}\,\mathcal{L})$, proving a number of results corresponding to those for algebras in Ens. In particular, we show, for a <u>noetherian</u> locale, that pure homomorphisms, equational compactness, and the existence of equationally compact hulls are characterized here in the same way as in Ens. As a consequence, we have that equationally compact = pure injectivity in any $\mathcal{M}od(\mathcal{H}, \mathrm{Sh}\,\mathcal{L})$, providing a partial solution to an open problem of Howlett [page 154;14]. What is interesting about this is that it is not the Axiom of Choice in $\mathrm{Sh}\,\mathcal{L}$ which matters in this context (as speculated in [14]) but the properties of direct limits in $\mathrm{Sh}\,\mathcal{L}$.

Finally, in Chapter (3), we consider the notion of bimorphisms for algebras in a topos in analogy with the recent work by B. Banaschewski and E. Nelson [9] for algebras in Ens. Among other things, we prove the counterpart of the results for algebras in Ens regarding the equivalence of tensor products and Universal bimorphisms for suitable categories of algebras which have a functional internal hom-functor. We also show that the existence of Universal bimorphisms and functional internal hom-functors survives the passage from suitable categories of algebras in Ens to the corresponding categories of algebras in a Grothendieck topos.

Also, in this chapter, we consider an internal notion of tensor products and Universal bimorphisms, and show that these come out to be same as the usual ones, in the case of Sh$\mathcal{L}$.

Chapters (1-3) are independent of each other, and thus can be read in any order. Chapter (0), of course, contains a summary of the background material needed here, especially from Sheaf Theory and Universal Algebra.

Throughout the thesis, a single numbering is used for definitions, lemmas, propositions, theorems, remarks, etc.; the number n.s.t denotes the t-th numbered reference in the s-th section of Chapter n. However, if a result consists of different parts, then n.s.t (ii) means ii-nd part of the numbered result n.s.t.

All the references to the bibliography are enclosed in square brackets.

CHAPTER 0

PRELIMINARIES

In this chapter we intend to give some of the concepts and results of Sheaf Theory and Universal Algebra, which shall be used in this work, and to set out some standard, as well as some new, notations. Regarding general facts in Category Theory, Universal Algebra and Sheaf Theory, the reader is referred to one of the standard texts on these subjects, for instance [17] and [19] in category theory, [1], [13] and [10] in Universal Algebra and [12] and [20] in Sheaf Theory; also [15] for Topos Theory.

In Section (1), we briefly discuss the theory of set-valued presheaves and sheaves on a small category $\mathbb{C}$ and, in particular, the case of $\mathbb{C} = \mathscr{L}$ a locale.

In Section (2), we deal with the notion of algebras in a finitely complete category $\mathbf{E}$ and define quasi-equations and quasi-equational classes. In particular, we consider the case of a Grothendieck topos $\mathbf{E}$, and show that the two points of view, algebras in $\mathbf{E}$ on the one hand and algebra-valued sheaves on the other, give rise to the same category.

Finally, in Section (3), we define the $\mathbf{E}$-valued hom-functor:

$$[\,,\,] : (\mathcal{A}lg(\tau)\,\mathbb{E})^{*} \times \mathcal{A}lg(\tau)\mathbb{E} \to \mathbb{E},$$

and prove, in Proposition (0.3.5), that

$$\mathbb{E}(\mathbf{1},[A,B]) \simeq \mathcal{A}lg(\tau)\mathbb{E}\,(A,B),$$

for A and B in $\mathcal{A}lg(\tau)\mathbb{E}$, and hence conclude that, in the case of $\mathbb{E} = Sh\mathcal{L}$, this gives the usual $Sh\mathcal{L}$ -valued hom-functor.

0.1 SHEAF THEORY

In this section, we intend to give a brief review of the theory of sheaves: for more details the reader is referred to the standard texts introduced earlier.

0.1.1 If $\mathbb{C}$ is a small category, that is a category whose morphisms form a set, a *presheaf* on $\mathbb{C}$ is defined to be a contravariant functor from $\mathbb{C}$ to Ens, the category of sets and functions. Thus a presheaf P of sets on $\mathbb{C}$ is specified by two pieces of informations:

(i) a set PU, for each $U \in \mathbb{C}$, and

(ii) a "restriction map" $Pt : PU \to PV$,

for each morphism $t : V \to U$ in $\mathbb{C}$, subject to the usual compatibility conditions.

A morphism of presheaves is just a natural transformation of functors. The functor category $Ens^{\mathbb{C}^*}$ ($\mathbb{C}^*$ the opposite of the category $\mathbb{C}$) of all presheaves (of sets) on $\mathbb{C}$ and natural transformations is usually denoted by $\hat{\mathbb{C}}$.

0.1.2 In this thesis, a *locale* $\mathcal{L}$ is a complete lattice satisfying the distribution law

$$U \wedge \bigvee U_i = \bigvee U \wedge U_i$$

for binary meet "$\wedge$" and arbitrary join "$\bigvee$". For example, for a topological space X, the lattice $\mathcal{O}X$ of open subsets of X, with $\wedge$ and $\bigvee$ as intersection and union, respectively, is a locale. Further, any complete Boolean algebra is a locale.

0.1.3. Special Types of locales. The following two special types of locales will, in particular, be taken up in Chapter (2). Recall that an element $U \in \mathcal{L}$ is said to be *compact* iff, for any cover $U \leq \bigvee_{i \in I} U_i$ of U in $\mathcal{L}$, there exists a finite subset $J \subseteq I$ with $U \leq \bigvee_{i \in J} U_i$. In particular, a locale $\mathcal{L}$ is called *compactly generated* or *algebraic* when every element of $\mathcal{L}$ is a join of compact elements of $\mathcal{L}$. The most obvious examples of algebraic locales are the ideal lattices of distributive lattices (with zero and unit), and these turn out to be the same as the lattices of open sets of the spectra of commutative rings with unit. We further define a locale $\mathcal{L}$ to be *noetherian* when every element of $\mathcal{L}$ is compact. The reason for calling them noetherian is that the stated condition is equivalent to the "Ascending Chain Condition" (ACC) for $\mathcal{L}$. For; if all the $U \in \mathcal{L}$ are compact and $U_1 \leq U_2 \leq \ldots$ is an ascending chain in $\mathcal{L}$, then, by the compactness of $U = \bigvee U_n$, $U = U_k$ for some k, which says that the above sequence terminates after a finite number of steps. Conversely, let (ACC) hold in $\mathcal{L}$ and $U \leq \bigvee_{i \in I} U_i$ be any cover of an arbitrary element U in $\mathcal{L}$. Consider the ideal $\mathfrak{a}$ of $\mathcal{L}$ generated by all U_i: then $\mathfrak{a}$ must be principal for otherwise we can pick a strictly increasing infinite sequence in $\mathfrak{a}$, and hence $U \leq U_{i_1} v \ldots v U_{i_n}$ where the latter is the generating element of $\mathfrak{a}$. This shows that U is compact. Note that the noetherian locales are special "coherent locales" which are defined to be algebraic locales such that any finite meet of compact elements is again compact, and which are exactly the ideal lattices of distributive lattices (with zero and unit): on the other hand, the

the lattice of open sets of the spectrum of a commutative <u>noetherian</u> ring with unit is always a noetherian locale.

<u>0.1.4</u> Since a locale is a partially ordered set, it can be regarded as a small category in the usual way, and thus one has the notion of a presheaf on $\mathcal{L}$ (or on X if $\mathcal{L} = \mathcal{O}X$, for a topological space X). Here, for $P \in \text{PreSh}\mathcal{L}$, a presheaf (of sets) on $\mathcal{L}$, and each pair $V \leq U$ in $\mathcal{L}$, there is only one restriction map $\rho^U_V : PU \to PV$, and we write $\rho^U_V s = s|V$, for $s \in PU$.

<u>0.1.5 Sheaves on $\mathcal{L}$</u>. Of particular interest are those presheaves P on $\mathcal{L}$ which satisfy one or both of the following "exactness conditions".

(S) <u>Separation Axiom</u>: *For any cover* $U = \bigvee_{i \in I} U_i$ *in* $\mathcal{L}$ *and any two elements* s *and* t *of* PU, *if* $s|U_i = t|U_i$, *for all* $i \in I$, *then* $s = t$.

(P) <u>Patching Axiom</u>: *If* $U = \bigvee_{i \in I} U_i$ *is a cover in* $\mathcal{L}$ *and* $(s_i)_{i \in I}$ *is a family such that* $s_i \in PU_i$ *for all* $i \in I$ *and* $s_i|U_i \wedge U_j = s_j|U_i \wedge U_j$ *for all* $i, j \in I$, *then there exists an element* $s \in PU$ *with* $s|U_i = s_i$ *for all* $i \in I$.

A presheaf on $\mathcal{L}$ which satisfies (S) or both, (S) and (P), is called *separated* or a *sheaf*, respectively.

Note that a presheaf P on $\mathcal{L}$ is a sheaf iff, for any cover $U = \bigvee_{i \in I} U_i$ in $\mathcal{L}$, the following is an equalizer diagram:

$$PU \xrightarrow{h} \prod_{i \in I} PU_i \overset{f}{\underset{g}{\rightrightarrows}} \prod_{(i,j) \in I \times I} P(U_i \wedge U_j)$$

where the maps are determined by the restriction maps in the obvious way.

The full subcategory of PreSh$\mathcal{L}$ whose objects are the sheaves on $\mathcal{L}$ is denoted by Sh$\mathcal{L}$, and by ShX if $\mathcal{L} = \mathcal{O}X$ for a topological space X.

For any $U \in \mathcal{L}$, the representable presheaf $h_U = (-,U)$, defined by

$$h_U V = \begin{cases} 1 = \{0\}, & \text{if } V \leq U \\ \phi & \text{, if otherwise} \end{cases}$$

with the obvious restriction maps is a sheaf. These sheaves are in fact all the subsheaves of $h_{\mathbb{1}} = \mathbb{1}$, the terminal object of Sh$\mathcal{L}$; moreover, they form a generating set for the category Sh$\mathcal{L}$.

<u>0.1.6</u>. <u>The Sheaf Reflection.</u> The inclusion functor $i : \text{Sh}\mathcal{L} \rightarrowtail \text{Pre Sh}\mathcal{L}$ has a left adjoint $R : \text{Pre Sh}\mathcal{L} \to \text{Sh}\mathcal{L}$, which is left exact, that is preserves finite limits, called the *reflection functor*

or *the associated sheaf functor*. R is constructed as follows: For $U \in \mathcal{L}$, let $\mathcal{R}_U$ be the collection of all covers $U = \bigvee U_i$ in $\mathcal{L}$, directed by refinement. Then, for any $P \in \mathrm{PreSh}\,\mathcal{L}$ and any $C = \{U_i\}$ in $\mathcal{R}_U$, let P_C be the following equalizer:

$$P_C \rightarrowtail \prod PU_i \underset{g}{\overset{f}{\rightrightarrows}} \prod P(U_i \wedge U_j)$$

where f and g are determined by the restriction maps in the obvious way. The reflection $\tilde{P} = RP$ of P is now given by

$$\tilde{P}U = \varinjlim_{C \in \mathcal{R}_U} P_C$$

for each $U \in \mathcal{L}$. The effect $f \rightsquigarrow \tilde{f} = Rf$ on maps $f : P \to Q$ is obvious. Since $\{U\} \in \mathcal{R}_U$, the reflection map $P \xrightarrow{r} \tilde{P}$ is also clear. That $\tilde{P}$ is a sheaf and R is a left exact left adjoint to the inclusion functor then follows from the properties of limits and direct limits in Ens.

For a topological space X, the associated sheaf functor may also be constructed by using the "stalks" which we do not intend to give it in this work. However, for a presheaf P on X and $x \in X$, the *stalk* of P *at x* is defined to be the direct limit

$$P_x = \varinjlim_{U \ni x} PU ;$$

and if $s \in PU$ for some $U \ni x$, we write s_x for the image of s in P_x.

<u>0.1.6</u> For any two locales $\mathcal{L}$ and $\mathcal{M}$, a homomorphism is a map $\phi : \mathcal{L} \to \mathcal{M}$ which preserves finite meets and arbitrary joins. Any such ϕ determines a pair of functors

$$\mathrm{Sh}\mathcal{L} \underset{\phi_*}{\overset{\phi^*}{\rightleftarrows}} \mathrm{Sh}\mathcal{M}$$

such that ϕ^* is left adjoint to ϕ_* and is also left exact, where $(\phi_* S)\, U = S\phi(U)$, $(U \in \mathcal{L})$, and $\phi^* S$ is the sheaf reflection of the presheaf $U \rightsquigarrow \varinjlim_{\phi V \geq U} SV$, $(U \in \mathcal{M})$.

In particular, if $\phi = i : {\downarrow}U \rightarrowtail \mathcal{L}$, for $U \in \mathcal{L}$ and ${\downarrow}U = \{V \in \mathcal{L} : V \leq U\}$, is the inclusion map, then $\mathrm{Sh}({\downarrow}U)$ will simply be denoted by $\mathrm{Sh}U$ and $\phi_* = -|U : S \rightsquigarrow S|U$ will be called the restriction functor; moreover, by the construction of $\tilde{P}$, for $P \in \mathrm{PreSh}\mathcal{L}$, the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{PreSh}\mathcal{L} & \xrightarrow{-|U} & \mathrm{PreShU} \\
{\scriptstyle\sim}\downarrow & & {\scriptstyle\sim}\downarrow \\
\mathrm{Sh}\mathcal{L} & \xrightarrow{-|U} & \mathrm{ShU}
\end{array}$$

That is, $\widetilde{P|U} = \tilde{P}|U$.

0.1.7 Lemma. *If* $\mathbb{1} = \bigvee_{i \in I} U_i$ *is a cover in* $\mathcal{L}$, *then the restriction functors*

$$Sh\mathcal{L} \xrightarrow{-|U_i} ShU_i$$

are jointly faithful.

Proof. Let $F \overset{f}{\underset{g}{\rightrightarrows}} G$ be two morphisms in $Sh\mathcal{L}$ such that $f|U_i = g|U_i$ for all $i \in I$. We have to show that $f = g$. Let $U \in \mathcal{L}$ be any element of $\mathcal{L}$ and $a \in FU$, we have that

$$U = U \wedge \mathbb{1} = U \wedge \bigvee_{i \in I} U_i = \bigvee_{i \in I} U \wedge U_i \text{ ,}$$

the latter because $\mathcal{L}$ is a locale, is a cover of U in $\mathcal{L}$. Now,

$$f_U(a)|U \wedge U_i = f_{U \wedge U_i}(a|U \wedge U_i) = (f|U_i)_{U \wedge U_i}(a|U \wedge U_i)$$

$$= (g|U_i)_{U \wedge U_i}(a|U \wedge U_i) = g_{U \wedge U_i}(a|U \wedge U_i) = g_U(a)|U \wedge U_i$$

for all $i \in I$; since G is a sheaf and $a \in FU$ is arbitrary, we get $f_U = g_U$ for all $U \in \mathcal{L}$, and hence $f = g$. ///

<u>0.1.8</u> <u>Corollary</u>. *A diagram in Sh* $\mathcal{L}$ *is commutative iff there is a cover* $\mathbb{U} = \bigvee U_i$ *in* $\mathcal{L}$ *such that the diagram is commutative at each* U_i. ///

<u>0.1.9</u> A category $\mathbb{E}$ is called a Grothendieck topos iff it is equivalent to a reflective subcategory of some $\hat{\mathbb{C}}$ such that the reflection functor is left exact. In particular, then, the category Sh $\mathcal{L}$ for any locale $\mathcal{L}$ is a Grothendieck topos.

<u>0.1.10</u> Finally, the exponentiation F^G, for F and G in Sh $\mathcal{L}$, is a sheaf and is defined by

$$F^G U = \mathrm{Sh}U(G|U, F|U),$$

and for morphisms $G' \xrightarrow{f} G$ and $F \xrightarrow{g} F'$, the

$$g^f_U : \mathrm{Sh}U(G|U,F|U) \to \mathrm{Sh}U(G'|U,F'|U) \text{ given by } \alpha \rightsquigarrow g|U \circ \alpha \circ f|U,$$

for all $U \in \mathcal{L}$, are the components of $g^f : F^G \to F'^{G'}$.

0.2 ALGEBRAS IN A CATEGORY

As in the previous section, we wish to give a brief discussion of the notion of an algebra in a finitely complete category; for more detail, the reader is referred to one of the texts introduced in the beginning of this chapter or any other standard texts.

0.2.1 Definition. *Let $\mathbb{E}$ be a finitely complete category (in particular, it has a terminal object $\mathbb{1}$). Given a family $\tau = (n_\lambda)_{\lambda\in\Omega}$ of finite cardinal numbers n_λ, indexed by a set Ω, an* algebra *in $\mathbb{E}$ is an entity $A = (E, (\varepsilon_\lambda)_{\lambda\in\Omega})$, where E is an object of $\mathbb{E}$ and, for each $\lambda \in \Omega$, $\varepsilon_\lambda : E^{n_\lambda} \to E$ is a morphism in $\mathbb{E}$.*

For such an A, E is called the *underlying object* of A, $\varepsilon = (\varepsilon_\lambda)_{\lambda\in\Omega}$ the algebra structure of A and ε_λ the λ-th operation of A. We shall also write $|A|$ for the underlying object and λ_A for the λ-th operation, of an algebra A. The family $\tau = (n_\lambda)_{\lambda\in\Omega}$ is called *the type* of A and n_λ the *arity* of λ_A .

In the following, all algebras will be of a given fixed, but otherwise arbitrary type $\tau = (n_\lambda)_{\lambda\in\Omega}$ with all n_λ finite.

<u>0.2.2</u> <u>Definition</u>. *A <u>homomorphism</u> from an algebra A to an algebra B is a morphism $h : |A| \to |B|$ such that the diagram*

$$\begin{array}{ccc} |A|^{n_\lambda} & \xrightarrow{h^{n_\lambda}} & |B|^{n_\lambda} \\ \lambda_A \downarrow & & \downarrow \lambda_B \\ |A| & \xrightarrow{h} & |B| \end{array}$$

commutes, that is, $\lambda_B \circ h^{n_\lambda} = h \circ \lambda_A$, *for each* $\lambda \in \Omega$.

The set of all homomorphisms from A to B is normally denoted by (A,B).

For an algebra A, the identity morphism of $|A|$ in $\mathbb{E}$ is clearly a homomorphism $1_A : A \to A$, and for composable maps $|A| \xrightarrow{g} |B| \xrightarrow{f} |C|$ in $\mathbb{E}$ we have $(fg)^n = f^n \circ g^n$ for any finite cardinal number n, thus if f and g are underlying maps of homomorphisms, then so is $f \circ g$.

As a result, one has the category of all algebras of the type τ in $\mathbb{E}$ and all homomorphisms between them. This category will be denoted by $\mathcal{A}lg(\tau)\,\mathbb{E}$. We also have a faithful functor $|-| : \mathcal{A}lg(\tau)\,\mathbb{E} \to \mathbb{E}$ given by $|A|$ to be the underlying object of A and for a homomorphism $h : A \to B$, $|h| : |A| \to |B|$ the underlying map of h.

If $\mathbf{E}$ = Ens, the category of sets and functions between them, then $\mathcal{A}lg(\tau)\mathbf{E}$ will simply be denoted by $\mathcal{A}lg(\tau)$.

<u>0.2.3</u> Let F be the absolutely free algebra of the type τ on a set $X = \{z_1,\ldots z_n\}$ of n elements. For any algebra A in $\mathcal{A}lg(\tau)\mathbf{E}$, $\mathbf{E}(|A|^n, |A|)$ can easily be made into an algebra of the type τ in Ens. Indeed, the operations are defined by $\lambda(\phi_1,\ldots,\phi_{n_\lambda}) = \lambda_A \circ (\phi_1 \sqcap \cdots \sqcap \phi_{n_\lambda})$

for $\phi_i : |A|^n \to |A|$, $(i = 1,\ldots,n_\lambda)$, that is,

$$\lambda : (\phi_1,\ldots,\phi_{n_\lambda}) \rightsquigarrow |A|^n \xrightarrow{\prod_{i=1}^{n_\lambda} \phi_i} |A|^{n_\lambda} \xrightarrow{\lambda_A} |A| .$$

Extend the map $X \to \mathbf{E}(|A|^n,|A|)$, given by $z_i \rightsquigarrow p_{r_i}$ (p_{r_i} the i-th projections), freely to $\phi : F \to \mathbf{E}(|A|^n,|A|)$ and denote $\phi(p)$ by p_A for any $p \in F$. In fact, each $p \in F$ defines a natural transformation $\overline{p} : |-|^n \longrightarrow |-|$ with components p_A, $A \in \mathcal{A}lg(\tau)\mathbf{E}$.

<u>0.2.4</u> <u>Definition.</u> <u>*A law (identity or equation)*</u> *over Ω in the set X of variables is any pair $(p,q) \in F \times F$, or sometimes the equation $p = q$ formed from this pair.*

<u>0.2.5</u> <u>Definition.</u> *A quasi-identity (or quasi-equation) is any formula of the form*

$$\sigma : (p_1 = q_1) \wedge (p_2 = q_2) \wedge \ldots \wedge (p_k = q_k) \rightarrow (p = q).$$

We say that σ holds in an algebra $A \in \mathrm{Alg}(\tau)\mathbf{E}$ or that A satisfies σ, written as $A \models \sigma$, iff the pullback

$$\begin{tikzcd}
 & E_q(p_{1A},q_{1A}) \arrow[dr, tail] \arrow[d, dash, dashed] & & \\
P \arrow[ur] \arrow[r] \arrow[dr] & E_q(p_{iA},q_{iA}) \arrow[r, tail] \arrow[d, dash, dashed] & {|A|^n} \arrow[r, shift left, "p_{iA}"] \arrow[r, shift right, "q_{iA}"'] & {|A|} \\
 & E_q(p_{kA},q_{kA}) \arrow[ur, tail] & &
\end{tikzcd}$$

of the equalizers

$$E_q(p_{iA},q_{iA}) \rightarrowtail |A|^n \underset{q_{iA}}{\overset{p_{iA}}{\rightrightarrows}} |A|$$

(for $i = 1,\ldots k$), factors through the equalizer

$E_q(p_A,q_A) \rightarrowtail |A|^n \underset{q_A}{\overset{p_A}{\rightrightarrows}} |A|$. In particular, $A \models (p = q)$ iff $p_A = q_A$.

Let $\mathcal{H}$ be a set of quasi-equations, then we say that A satisfies $\mathcal{H}$, written as $A \models \mathcal{H}$, iff $A \models \sigma$ for all $\sigma \in \mathcal{H}$. The class of all algebras in $\mathbf{E}$ satisfying $\mathcal{H}$ will be denoted by $\mathcal{M}od(\mathcal{H},\mathbf{E})$ and is called a quasi-variety or a quasi-equational class. If $\mathcal{H}$ is a set of equations, then $\mathcal{M}od(\mathcal{H},\mathbf{E})$ is called an equational class or a variety. <u>From now on $\mathcal{H}$ always denote a set of quasi-equations, unless otherwise stated.</u>

<u>0.2.6</u> Let $k : \mathbf{E} \to \mathbb{F}$ be functor, preserving finite limits, then k induces another functor $\overline{k} : \mathcal{A}lg(\tau)\mathbf{E} \to \mathcal{A}lg(\tau)\,\mathbb{F}$ defined by $\overline{k}A = (k(|A|),(k\lambda_A)_{\lambda\in\Omega})$, and on homomorphisms $f : A \to B$, $|\overline{k}f| = k|f|$. Since k preserves finite limits, it then preserves pull-backs and equalizer diagrams; and hence if σ is a quasi-equation and $A \models \sigma$, for $A \in \mathcal{A}lg(\tau)\mathbf{E}$, then $\overline{k}\,A \models \sigma$. The converse is true if in addition k is faithful. In particular, if σ is an identity $p = q$, then $A \models (p=q)$ implies that $p_A = q_A$, and hence $kp_A = kq_A$ which implies that $p_{\overline{k}A} = q_{\overline{k}A}$; and thus $\overline{k}\,A \models (p = q)$.

By the above discussion, we get a functor

$$\overline{k}|\mathcal{M}od(\mathcal{H},\mathbf{E}) : \mathcal{M}od(\mathcal{H},\mathbf{E}) \to \mathcal{M}od(\mathcal{H},\mathbf{F})$$

for any set $\mathcal{H}$ of quasi-equations.

<u>0.2.7</u> <u>Lemma</u>. *Let $\mathbf{E}$ have a set $\mathcal{G}$ of generators. Then, for any $A \in \mathcal{A}lg(\tau)\mathbf{E}$ and any set $\mathcal{H}$ of quasi-equations, $A \in \mathcal{M}od(\mathcal{H},\mathbf{E})$ iff $\overline{h}_G A \in \mathcal{M}od\,\mathcal{H} = \mathcal{M}od(\mathcal{H},\mathcal{E}ns)$ for each $G \in \mathcal{G}$, where $h_G = \mathbf{E}(G,-)$.*

Proof: That $A \in \mathcal{M}od(\mathcal{H},\mathbf{E})$ implies $\overline{h}_G A \in \mathcal{M}od\,\mathcal{H}$ is clear, by what has been discussed in (0.2.6). Conversely, let $\sigma : \bigwedge_{i=1}^{k} (p_i,q_i) \to (p,q)$ be any quasi-equation and $A \in \mathcal{A}lg(\tau)\mathbf{E}$ such that $\overline{h}_G A \models \sigma$ for all $G \in \mathcal{G}$. Consider the diagram

$$\begin{tikzcd}
 & P \arrow[d, tail, "j"] & \\
E \arrow[r, tail, "i"] & {|A|^n} \arrow[r, shift left, "p_A"] \arrow[r, shift right, "q_A"'] \arrow[d, shift right, "p_{iA}"'] \arrow[d, shift left, "q_{iA}"] & {|A|} \\
 & {|A|} &
\end{tikzcd}$$

with P the pullback of the equalizers of the pairs (p_{iA}, q_{iA}) and E the equalizer of the pair (p_A, q_A). By the hypothesis on A, $h_G\ (p_A \circ j) = h_G\ (q_A \circ j)$, for all $G \in \mathcal{G}$, which this then implies that $p_A \circ j = q_A \circ j$, the latter because the set $\mathcal{G}$ of generators is collectively faithful, and hence j factors through i, by definition of equalizers; thus $A \models \sigma$. ///

<u>0.2.8</u> (a) The discussion in (0.2.6), in particular, shows that the category $\mathcal{A}lg(\tau)\ \hat{\mathbb{C}}$ is isomorphic to the category of all $\mathcal{A}lg(\tau)$-valued presheaves on $\mathbb{C}$; and for a Grothendieck topos $\mathbf{E} \subseteq \hat{\mathbb{C}}$, since the reflection functor $R : \hat{\mathbb{C}} \to \mathbf{E}$ preserves finite limits, it can be lifted to:

$$R : \mathcal{A}lg(\tau)\ \hat{\mathbb{C}} \to \mathcal{A}lg(\tau)\mathbf{E}$$

(denoted by the same letter R). In particular, the category $\mathcal{A}lg(\tau)\mathrm{Sh}\ \mathcal{L}$ is isomorphic to the category of all $\mathcal{A}lg(\tau)$-valued sheaves on $\mathcal{L}$; moreover, since $\hat{\mathbb{C}}$ has a set of generators, Lemma (0.2.7) implies that, for a set $\mathcal{H}$ of quasi-equations, $A \in \mathcal{M}od(\mathcal{H}, \hat{\mathbb{C}})$ iff $AU \in \mathcal{M}od\,\mathcal{H}$ for all $U \in \mathbb{C}$./

(b) For any two locales $\mathcal{L}$ and $\mathcal{M}$ and a homomorphism $\phi : \mathcal{L} \to \mathcal{M}$, the functors ϕ^* and ϕ_* given in (0.1.6) can be lifted to

$$\mathcal{M}od(\mathcal{H}, Sh\,\mathcal{L}) \underset{\phi_*}{\overset{\phi^*}{\rightleftarrows}} \mathcal{M}od(\mathcal{H}, Sh\,\mathcal{M})$$

(denoted by the same letters) such that ϕ^* is a left exact left adjoint to ϕ_*.

<u>0.2.9</u> Here, we give the free functor $\mathcal{F}$ on $\mathcal{M}od(\mathcal{H}, \mathbf{E})$, $\mathbf{E}$ a Grothendieck topos. The free functor

$$\# : \hat{\mathbb{C}} \longrightarrow \mathcal{M}od(\mathcal{H}, \hat{\mathbb{C}})$$

is given by $P \rightsquigarrow P^{\#}$ such that $P^{\#}U = F(PU)$ is the $\mathcal{M}od\,\mathcal{H}$ -free algebra on PU, and the effect on maps is provided by the definition of the free functor $F : Ens \to \mathcal{M}od\,\mathcal{H}$. The free functor $\mathcal{F} : \mathbf{E} \to \mathcal{M}od(\mathcal{H}, \mathbf{E})$ is now given to be the composite:

$$\mathbf{E} \xrightarrow{i} \hat{\mathbb{C}} \xrightarrow{\#} \mathcal{M}od(\mathcal{H}, \hat{\mathbb{C}}) \xrightarrow{R} \mathcal{M}od(\mathcal{H}, \mathbf{E})$$

where i is the inclusion functor and R is the lifting of the reflection functor; that is, for any $S \in \mathbf{E}$, $\mathcal{F}S = R(F \circ S)$.

Since $\mathcal{F}$ is left adjoint to the underlying functor $|-|$: $\mathcal{M}od(\mathcal{H},\mathbf{E}) \to \mathbf{E}$, it transfers the set $\mathcal{G} = \{\tilde{h}_U : U \in \mathbf{C}\}$ of generators of $\mathbf{E}$ to a set $\{\mathcal{F}_U = \mathcal{F}(\tilde{h}_U) : U \in \mathbf{C}\}$ of generators of $\mathcal{M}od(\mathcal{H},\mathbf{E})$.

0.2.10 We now turn to a discussion of the exponentiation functor.

Let $\mathbf{E}$ be a topos. For $G \in \mathbf{E}$ the exponentiation functor $(\ )^G$ from $\mathbf{E}$ into $\mathbf{E}$ is a right adjoint and so preserves limits and in particular it preserves finite products, pullbacks and equalizers. Hence if A, an object of $\mathbf{E}$, satisfies a quasi-equation σ then so does A^G. Indeed, if $\mathcal{H}$ is a set of quasi-equations and $A = (|A|,(\lambda_A)_{\lambda\in\Omega}) \in \mathcal{M}od(\mathcal{H},\mathbf{E})$ then $A^G = \left(|A|^G,(\lambda_A^G)_{\lambda\in\Omega}\right) \in \mathcal{M}od(\mathcal{H},\mathbf{E})$. Also, the exponential adjointness isomorphism:

$$\mathbf{E}(T,|A|^G) \xrightarrow{\sim} \mathbf{E}(T \times G,|A|)$$

is an isomorphism of set-valued algebras.

Moreover, for a morphism $f: F \to G$ in $\mathbf{E}$, the induced morphism $|A|^f : |A|^G \to |A|^F$ is the underlying map of a homomorphism A^f from the algebra A^G to the algebra A^F. To check this is to show that for any T in $\mathbf{E}$, the induced morphism

$$\mathbf{E}(T,|A|^G) \xrightarrow{\beta} \mathbf{E}(T,|A|^F)$$

(composition by $|A|^f$) is in fact a homomorphism of set-valued algebras. But, consider the diagram

$$\begin{tikzcd} \mathbf{E}(T \times G, |A|) \arrow[r, "\sim"] \arrow[d, "\alpha"'] & \mathbf{E}(T,|A|^G) \arrow[d, "\beta"'] \\ \mathbf{E}(T \times F, |A|) \arrow[r, "\sim"] & \mathbf{E}(T,|A|^F) \end{tikzcd}$$

where α is induced by the morphism $1_T \times f : T \times F \to T \times G$ and is then a homomorphism of set-valued algebras, the two horizontal morphisms are the exponential adjointness isomorphisms and then are isomorphisms of set-valued algebras. The diagram commutes, by adjointness, and hence β is a homomorphism of algebras./

0.3 $\mathbf{E}$-VALUED HOM-FUNCTOR ($\mathbb{E}$ a topos)

0.3.1 Construction. Let A and B be any two algebras in $\mathcal{A}lg(\tau)\mathbf{E}$. We define $[A,B]$ to be the largest subobject $[A,B] \xrightarrow{i_{AB}} |B|^{|A|}$ of $|B|^{|A|}$ such that, for each $\lambda \in \Omega$, the diagram:

$$\begin{tikzcd}
 & & {|B|^{|A|}\times|A|} \arrow[dr, "ev"] & \\
{[A,B]\times|A|^{n_\lambda}} \arrow[r, "i_{AB}\times 1"] & {|B|^{|A|}\times|A|^{n_\lambda}} \arrow[ur, "1\times\lambda_A"] \arrow[d, "\prod_{i=1}^{n_\lambda}(1\times p_{r_i})"] & & {|B|} \\
 & {(|B|^{|A|}\times|A|)^{n_\lambda}} \arrow[r, "(ev)^{n_\lambda}"] & {|B|^{n_\lambda}} \arrow[ur, "\lambda_B"] &
\end{tikzcd}$$

is an equalizer diagram, where "ev" is the evaluation map, i.e., the counit of the exponential adjunction; that is

$$\frac{|B|^{|A|} \xrightarrow{1} |B|^{|A|}}{|B|^{|A|} \times |A| \xrightarrow{ev} |B|}$$

Let $f : C \to A$ and $g : B \to D$ be any two homomorphisms in $\mathcal{A}lg(\tau)\mathbb{E}$. We shall define a morphism $[f,g] : [A,B] \to [C,D]$ which makes

$$[\,,\,] : (\mathcal{A}lg(\tau)\mathbb{E})^* \times \mathcal{A}lg(\tau)\,\mathbb{E} \to \mathbb{E}$$

into a functor (where $(\mathcal{A}lg(\tau)\mathbb{E})^*$ is the dual of the category $\mathcal{A}lg(\tau)\mathbb{E}$). To do so, consider the following diagram for each $\lambda \in \Omega$.

$$
\begin{tikzcd}[column sep=small]
{[A,B]\times|A|^{n_\lambda}} \arrow[r, tail, "i_{AB}\times 1"] & {|B|^{|A|}\times|A|^{n_\lambda}} \arrow[r, "\prod_{i=1}^{n_\lambda}(1\times pr_i)"] & {(|B|^{|A|}\times|A|)^{n_\lambda}} \arrow[rrr, "(ev)^{n_\lambda}"] \arrow[d, phantom, "(\mathrm{II})"] & & & {|B|^{n_\lambda}} \arrow[dll, "|g|^{n_\lambda}"'] \arrow[ddddd, "\lambda_B"] \\
 & & {} & {|D|^{n_\lambda}} \arrow[dr, "\lambda D"] & & \\
{[A,B]\times|C|^{n_\lambda}} \arrow[uu, "1\times|f|^{n_\lambda}"] \arrow[ddd, "1\times|f|^{n_\lambda}"] \arrow[r, tail, "i_{AB}\times 1"] & {|B|^{|A|}\times|C|^{n_\lambda}} \arrow[uu, "1\times|f|^{n_\lambda}"] \arrow[ddd, "1\times|f|^{n_\lambda}"] \arrow[r, "|g|^{|f|}\times 1"] \arrow[ddr, "1\times\lambda C"] & {|D|^{|C|}\times|C|^{n_\lambda}} \arrow[ur, "(ev)^{n_\lambda}\circ\prod_{i=1}^{n_\lambda}(1\times p_{ri})"] \arrow[dr, "1\times\lambda C"] & & {|D|} \arrow[dddr, "|g|"] & \\
 & & & {|D|^{|C|}\times|C|} \arrow[ur, "ev"] & & \\
 & & {|B|^{|A|}\times|C|} \arrow[ur, "|g|^{|f|}\times 1"] \arrow[d, "1\times|f|"] \arrow[rr, phantom, "(\mathrm{I})"] & & {} & \\
{[A,B]\times|A|^{n_\lambda}} \arrow[r, tail, "i_{AB}\times 1"] & {|B|^{|A|}\times|A|^{n_\lambda}} \arrow[r, "1\times\lambda_A"] & {|B|^{|A|}\times|A|} \arrow[rrr, "ev"] & & & {|B|}
\end{tikzcd}
$$

We shall show that the two inner most routes joining $[A,B] \times |C|^{n_\lambda}$ to $|D|$ are equal. Also, by the fact that epi-mono factorization is available in $\mathbf{E}$, we get the following factorization

$$\begin{tikzcd}
{[A,B] \times |C|^{n_\lambda}} \arrow[r, "i_{AB} \times 1"] \arrow[dr, "h \times 1"'] & {|B|^{|A|} \times |C|^{n_\lambda}} \arrow[r, "g^f \times 1"] & {|D|^{|C|} \times |C|^{n_\lambda}} \\
 & T \times |C|^{n_\lambda} \arrow[ur, tail, "k \times 1"'] &
\end{tikzcd}$$

and, then by the definition of $[C,D]$ we get a unique map from T into $[C,D]$, and hence a unique morphism.

$$[f,g] : [A,B] \longrightarrow [C,D]$$

which makes the diagram

$$\begin{tikzcd}
{[A,B]} \arrow[r, "{[f,g]}"] \arrow[d, "i_{AB}"] & {[C,D]} \arrow[d, "i_{CD}"] \\
{|B|^{|A|}} \arrow[r, "|g|^{|f|}"] & {|D|^{|C|}}
\end{tikzcd}$$

commutative; then one easily checks the functoriality of $[\,,]$ which is what we wished to show. We now show the commutativity of the main diagram in the following steps:

(i) we first show that the following diagram is commutative

$$\begin{tikzcd}
|B|^{|A|} \times |C| \arrow[r, "|B|^{|f|} \times 1"] \arrow[d, "1 \times |f|"'] \arrow[dr, phantom, "(1)"] & |B|^{|C|} \times |C| \arrow[d, "ev"] \\
|B|^{|A|} \times |A| \arrow[r, "ev"] & |B|
\end{tikzcd}$$

To prove this, apply the exponentiation functor $(\ )^{|C|}$ to the diagram (1) , then we get the following diagram:

$$\begin{tikzcd}
|B|^{|A|} \arrow[rrr, "|B|^{|f|}"] \arrow[dr, "n"] \arrow[ddd, "|B|^{|f|}"'] & & & |B|^{|C|} \arrow[ddd, "1"] \\
& (|B|^{|A|} \times |C|)^{|C|} \arrow[r, "(|B|^{|f|} \times 1)^{|C|}"] \arrow[d, "(1 \times |f|)^{|C|}"] & (|B|^{|C|} \times |C|)^{|C|} \arrow[ur, "(ev)^{|C|}"] \arrow[d, "(ev)^{|C|}"] & \\
& (|B|^{|A|} \times |A|)^{|C|} \arrow[r, "(ev)^{|C|}"] \arrow[dl, "ev^{|C|}"'] & |B|^{|C|} \arrow[dr, "1"] & \\
|B|^{|C|} \arrow[rrr, "1"] & & & |B|^{|C|}
\end{tikzcd}$$

where all the squares, except possibly the middle one, are commutative either by adjointness or for obvious reasons. Thus the two inner routes joining $|B|^{|A|}$ to $|B|^{|C|}$ are equal which then adjointness implies the commutativity of the diagram (1).

(ii) Now, by applying the functor $(-) \times |C|$ to the following commutative diagram:

$$\begin{tikzcd}
{|B|^{|A|}} \arrow[r, "{|B|^{|f|}}"] \arrow[dr, "{|g|^{|f|}}"'] & {|B|^{|C|}} \arrow[d, "{|g|^{|C|}}"] \\
& {|D|^{|C|}}
\end{tikzcd}$$

we obtain the following commutative diagram:

$$\begin{tikzcd}
{|B|^{|A|} \times |C|} \arrow[r, "{|B|^{|f|} \times 1}"] \arrow[d, "{|g|^{|f|} \times 1}"'] & {|B|^{|C|} \times |C|} \arrow[r, "{|g|^{|C|} \times 1}"] \arrow[d, phantom, "(2)"] & {|D|^{|C|} \times |C|} \arrow[d, "ev"] \\
{|D|^{|C|} \times |C|} \arrow[rr, "ev"] & & {|D|}
\end{tikzcd}$$

(iii) By combining the above diagrams (1) and (2), we obtain the subdiagram (I) in the main diagram, and hence it is commutative.

(iv) By a similar argument as above one can show the commutativity of the subdiagram (II) in the main diagram. Now, observe that all the other subdiagrams in the main diagram are commutative for obvious reasons. Thus, the two inner most routes connecting $[A,B] \times |C|^{n_\lambda}$ to $|D|$ are equal as it was claimed, and hence we are done.///

<u>0.3.2</u> <u>Definition</u>. *The functor*

$$[\,,\,] : (\mathcal{A}lg(\tau)\mathbb{E})^* \times \mathcal{A}lg(\tau)\mathbb{E} \to \mathbb{E}$$

is called the <u>E-valued</u> hom-functor for $\mathcal{A}lg(\tau)\mathbb{E}$.

<u>0.3.3</u> <u>Remark</u>. The evaluation map

$|B|^{|A|} \times |A| \xrightarrow{ev_{AB}} |B|$, for A and B in $\mathcal{A}lg(\tau)\mathbb{E}$ induces a morphism $[A,B] \times |A| \to |B|$ which will be denoted by the same letters ev_{AB} or simply by ev.

<u>0.3.4</u> <u>Lemma</u>. *A morphism* $f : |A| \to |B|$, *for A and B in* $\mathcal{A}lg(\tau)\mathbb{E}$, *is the underlying map of a homomorphism from A into B iff the exponential adjunction* $\overline{f}$,

$$\frac{\mathbb{1} \times |A| \xrightarrow{\sim} |A| \xrightarrow{f} |B|}{\mathbb{1} \xrightarrow{\overline{f}} |B|^{|A|}} \,, \quad \textit{factors through}$$

$$[A,B] \overset{i_{AB}}{\rightarrowtail} |B|^{|A|}.$$

Proof. To prove this we consider the following diagram, for each $\lambda \in \Omega$:

$$\begin{tikzcd}
 & & & {|A|} \arrow[d, "\overline{f}\times 1"'] \arrow[ddr, "f"] \arrow[dl, phantom, "(2)"] \arrow[dr, phantom, "(3)"] & \\
 & & & {|B|^{|A|} \times |A|} \arrow[dr, "ev"] & \\
{|A|^{n_\lambda}} \arrow[r, "\sim"] \arrow[uuurrr, "\lambda_A"] \arrow[drrr, "|f|^{n_\lambda}"'] & \mathbb{1}\times|A|^{n_\lambda} \arrow[r, tail, "\overline{f}\times 1"] \arrow[dr, phantom, "(1)"] & {|B|^{|A|}\times|A|^{n_\lambda}} \arrow[ur, "1\times\lambda_A"] \arrow[dr, "(ev)^{n_\lambda}\circ\prod_{i=1}^{n_\lambda}(1\times p_{ri})"] & & {|B|} \\
 & & & {|B|^{n_\lambda}} \arrow[ur, "\lambda_B"'] &
\end{tikzcd}$$

The subdiagrams (1) , (2) and (3) are commutative for obvious reasons. Thus the two routes connecting $\mathbb{1} \times |A|^{n\lambda}$ to $|B|$ are equal iff the outermost square commutes, which by the definition of [A,B] and the

definition of a homomorphism proves the assertion.///
This proves the following proposition:

<u>0.3.5</u> <u>Proposition</u>. *For A and B in* $\mathcal{A}lg(\tau)\mathbf{E}$*, the map* $\mathbf{E}(\mathbb{1}, [A,B]) \to \mathcal{A}lg(\tau)\mathbf{E}(A,B)$ *defined by* $f \rightsquigarrow f^{\#}$ *is an isomorphism, where* $f^{\#}$ *is given by:*

$$\frac{\dfrac{\mathbb{1} \xrightarrow{f} [A,B] \xrightarrow{i_{AB}} |B|^{|A|}}{\mathbb{1} \times |A| \xrightarrow{\bar{f}} |B|}}{f^{\#} : |A| \simeq \mathbb{1} \times |A| \xrightarrow{\bar{f}} |B|}$$

<u>0.3.6</u> Here, we give the exact description of the sheaf-valued hom-functor, that is the $\mathbf{E}$-valued hom-functor for the case $\mathbf{E} = \mathrm{Sh}\mathcal{L}$. Since the restriction functors : $\mathrm{Sh}\mathcal{L} \longrightarrow \mathrm{Sh}U, (U \in \mathcal{L})$, preserve products and equalizers, $[A,B]|U = [A|U, B|U]$, by definition of $[A,B]$. Thus, for any A and B in $\mathcal{M}od(\mathcal{H}, \mathrm{Sh}\mathcal{L})$, the sheaf $[A,B]$ is exactly defined by

$$\begin{aligned}[A,B]U &= ([A,B]|U)U = [A|U,B|U]U \\ &= \mathcal{A}lg(\tau)\mathrm{Sh}U(A|U, B|U);\end{aligned}$$

the last step is by an application of Proposition (0.3.5) to ShU.

CHAPTER 1

RESIDUAL SMALLNESS AND INJECTIVITY IN $\mathcal{M}od(\mathcal{H}, \mathbf{E})$

This chapter gives a study of injectivity and residual smallness in a quasi-equational class $\mathcal{M}od(\mathcal{H}, \mathbf{E})$ of algebras in an arbitrary Grothendieck topos $\mathbf{E}$.

The main results of this chapter are Propositions (1.2.10) and (1.3.4), describing the relationship between $\mathcal{M}od\,\mathcal{H}$ and $\mathcal{M}od(\mathcal{H}, \mathbf{E})$ regarding residual smallness and existence of enough injectives; we show that these notions hold in $\mathcal{M}od\,\mathcal{H}$ iff they hold in $\mathcal{M}od(\mathcal{H}, \mathbf{E})$. These results substantially improve the weaker results by Howlett [14].

WE also prove the counterparts of some results for equational classes of algebras in Ens, in the case of a quasi-equational class $\mathcal{M}od(\mathcal{H}, \mathbf{E})$; for instance, Proposition (1.4.6) which corresponds to Proposition (5) in [2].

Finally, we add a couple of remarks regarding characterizations of injectivity in $\mathcal{M}od(\mathcal{H}, \mathbf{E})$, which suggest problems for further study.

Regarding the general notions of injectivity and residual smallness for algebras in Ens, the reader is referred to [2], [3], [4], [11], [14], [16] and [21].

<u>1.1.</u> <u>The Adjointness of $\hat{\mathbf{C}}$.</u> Here, we construct a pair of adjoint functors:

$$\hat{\mathbf{C}} \underset{H}{\overset{G}{\rightleftarrows}} \mathrm{Ens}^{|\mathbf{C}|}$$

with G a left adjoint of H, where $|\mathbf{C}|$ denotes the discrete category whose set of objects is that of $\mathbf{C}$.

<u>1.1.1</u> Define $G : \hat{\mathbf{C}} \to \mathrm{Ens}^{|\mathbf{C}|}$ by $GP = (PU)_{U \in \mathbf{C}}$, for $P \in \hat{\mathbf{C}}$, and for any map $f : P \to Q$ in $\hat{\mathbf{C}}$, $Gf = (f_U)_{U \in \mathbf{C}}$. That G is actually a functor is easily checked.

<u>1.1.2</u> Define $H : Ens^{|\mathbb{C}|} \longrightarrow \hat{\mathbb{C}}$ in the following steps:

(I) For any $B = (B_V)_{V \in \mathbb{C}}$ in $Ens^{|\mathbb{C}|}$, we define a contravariant functor $\overline{B} \in \hat{\mathbb{C}}$ by:

$\overline{B}U = \prod_{V \in \mathbb{C}} B_V^{\mathbb{C}(V,U)}$, for each $U \in \mathbb{C}$, and to define

$\overline{B}s : \overline{B}W \longrightarrow \overline{B}U$, for each $s : U \longrightarrow W$, notice that s induces a morphism $s^* : h_U \to h_W$ with components s_V^* given by $s_V^* t = s \circ t$, for each V and $t : V \longrightarrow U$, in $\mathbb{C}$; then, s^* induces a morphism

$$s_V^B : B_V^{\mathbb{C}(V,W)} \longrightarrow B_V^{\mathbb{C}(V,U)}$$

given by $s_V^B \alpha = \alpha \circ s_V^*$, for each $V \in \mathbb{C}$ and $\alpha : \mathbb{C}(V,W) \to B_V$

Hence, define $\overline{B}s = \prod_{V \in \mathbb{C}} s_V^B$. This $\overline{B}$ is actually a functor. For, clearly $\overline{B}1_U = 1_{\overline{B}U}$ for each $U \in \mathbb{C}$, and for any composition of morphisms $U \xrightarrow{t} W_1 \xrightarrow{s} W_2$ in $\mathbb{C}$, $(st)^* = s^* t^*$, which this implies that

$$(st)_V^B \alpha = \alpha \circ (st)_V^* = \alpha \circ (s_V^* t_V^*) =$$

$$(\alpha \circ s_V^*) \circ t_V^* = t_V^B (\alpha \circ s_V^*) = t_V^B (s_V^B \alpha)$$

for any $\alpha : \mathbb{C}(V, W) \to B_V$, and hence

$$\overline{B}(st) = \prod_{V \in \mathbb{C}} (st)_V^B = \prod_{V \in \mathbb{C}} (t^B \circ s_V^B)$$

$$= \prod_{V \in \mathbb{C}} t_V^B \circ \prod_{V \in \mathbb{C}} s_V^B = \overline{B}t \circ \overline{B}s$$

(II) Let $f = (f_V)_{V \in \mathbb{C}}$ be any morphism from $B = (B_V)_{V \in \mathbb{C}}$ to $C = (C_V)_{V \in \mathbb{C}}$ in $Ens^{|\mathbb{C}|}$. For any pair V and U in $\mathbb{C}$, define

$$f_V^U : B_V^{\mathbb{C}(V,U)} \longrightarrow C_V^{\mathbb{C}(V,U)}$$

by $f_V^U \alpha = f_V \circ \alpha$, for $\alpha : \mathbb{C}(V, U) \to B_V$. Let $\overline{f} : \overline{B} \to \overline{C}$ be given by $\overline{f}_U = \prod_{V \in \mathbb{C}} f_V^U$, for each $U \in \mathbb{C}$. To check that $\overline{f}$ is actually a natural transformation is enough to show that the diagram:

$$\begin{array}{ccc} B_V^{\mathbb{C}(V,W)} & \xrightarrow{f_V^W} & C_V^{\mathbb{C}(V,W)} \\ {\scriptstyle s_V^B}\downarrow & & \downarrow{\scriptstyle s_V^C} \\ B_V^{\mathbb{C}(V,U)} & \xrightarrow{f_V^U} & C_V^{\mathbb{C}(V,U)} \end{array}$$

is commutative, for all V and any $s : U \to W$ in $\mathbb{C}$ which can easily be checked./

(III) We now define H by $HB = \overline{B}$ and $Hf = \overline{f}$, for $B = (B_V)_{V \in \mathbb{C}}$, $C = (C_V)_{V \in \mathbb{C}}$ and any $f : B \to C$, in $Ens^{|\mathbb{C}|}$. Clearly $H1_B = 1_{HB}$, and for any composite $B \xrightarrow{f} C \xrightarrow{g} D$ in $Ens^{|\mathbb{C}|}$, we have

$$(gf)_V^U\alpha = (gf)_V \circ\alpha = (g_V \circ f_V)\circ\alpha$$

$$= g_V \circ(f_V \circ\alpha) = g_V(f_V^U \alpha) = g_V^U(f_V^U\alpha)$$

for any pair V and U in $\mathbb{C}$ and $\alpha : \mathbb{C}(V,U) \to B_V$, and hence, for any $U \in \mathbb{C}$, we have

$$(Hfg)_U = (\overline{gf})_U = \prod_{V \in \mathbb{C}}(gf)_V^U = \prod_{V \in \mathbb{C}}(g_V^U f_V^U)$$

$$= \prod_{V \in \mathbb{C}} g_V^U \circ \prod_{V \in \mathbb{C}} f_V^U = \overline{g}_U \overline{f}_U = Hf_U \circ Hg_U ,$$

which shows that H is indeed a functor.///

<u>1.1.3</u> Here, we define two natural transformations

$$\eta : 1 \longrightarrow HG \quad \text{and} \quad \varepsilon : GH \longrightarrow 1$$

which shall serve as front and back adjunctions.

To define η, let $P \in \hat{\mathbb{C}}$ and $t : V \longrightarrow U$ be any map in $\mathbb{C}$. The map $Pt : PU \to PV$ induces a map $PU \longrightarrow PV^{\mathbb{C}(V,U)}$, by $a \rightsquigarrow \tilde{a}_V$ with $\tilde{a}_V s = (Ps)a$ for $s : V \to U$. Then, we get a map

$$\eta_{PU} : PU \longrightarrow \prod_{V \in \mathbb{C}} PV^{\mathbb{C}(V,U)} = (HGP)U,$$

by $a \rightsquigarrow (\tilde{a}_V)_{V \in \mathbb{C}}$. Define a natural transformation $\eta_P : P \longrightarrow HGP$ by $(\eta_P)_U = \eta_{PU}$. To check that η_P is natural is to show that the diagram

$$\begin{tikzcd} PW \arrow[r, "\eta_{PW}"] \arrow[d, "Ps"'] & \prod_{V \in \mathbb{C}} PV^{\mathbb{C}(V,W)} \arrow[d, "\overline{Ps}"] \\ PU \arrow[r, "\eta_{PU}"] & \prod_{V \in \mathbb{C}} PV^{\mathbb{C}(V,U)} \end{tikzcd}$$

commutes, for each $s : U \to W$ in $\mathbb{C}$. Let $a \in PW$, on the one hand,

$$\overline{P}s(\eta_{PW}\, a) = \overline{P}s(\tilde{a}_V)_{V\in\mathbb{C}}$$

$$= (s_V^P\, \tilde{a}_V)_{V\in\mathbb{C}} = (\tilde{a}_V \circ s_V^*)_{V\in\mathbb{C}}$$

and on the other,

$$\eta_{PU}(Ps)a = ((Psa)\tilde{}_V)_{V\in\mathbb{C}} \text{ ; but,}$$

for any $t : V \to U$,

$$(\tilde{a}_V \circ s_V^*)t = \tilde{a}_V(st) = P(st)a = (Pt \circ Ps)\, a$$

$$= Pt(Psa) = (Psa)\tilde{}_V\, t,$$

which shows that $\tilde{a}_V \circ s_V^* = (Psa)\tilde{}_V$, for all $V \in \mathbb{C}$ and hence the last diagram commutes.

Now, let $\eta = (\eta_P)_{P\in\hat{\mathbb{C}}}$. We still have to show that η is natural. For any $f : P \to Q$ in $\hat{\mathbb{C}}$, it is enough to establish commutativity of the diagram:

$$\begin{tikzcd} PU \arrow[r, "\eta_{PU}"] \arrow[d, "f_U"'] & \prod\limits_{V\in\mathbb{C}} PV^{\mathbb{C}(V,U)} \arrow[d, "\bar{f}_U"] \\ QU \arrow[r, "\eta_{QU}"] & \prod\limits_{V\in\mathbb{C}} QV^{\mathbb{C}(V,U)} \end{tikzcd}$$

Since f is natural, $f_V \circ Pt = Qt \circ f_U$, for any $t : V \to U$, and hence, by this and the definition of the given maps, one can then easily check the commutativity of the above diagram./

Define $\varepsilon : GH \to 1$ such that, for each $B = (B_V)_{V\in\mathbb{C}}$, ε_B has the composites

$$\prod_{V\in\mathbb{C}} B_V^{\mathbb{C}(V,U)} \xrightarrow{pr_U} B_U^{\mathbb{C}(U,U)} \xrightarrow{e} B_U$$

as its U-th component, for $U \in \mathbb{C}$, where pr_U is the U-th projection and $e(\alpha) = \alpha 1_U$ for $\alpha \in B_U^{\mathbb{C}(U,U)}$. To show the naturality of ε, we have to establish the commutativity of the diagram:

$$\begin{tikzcd}
\Big(\prod_{V\in\mathbb{C}} B_V^{\mathbb{C}(V,U)}\Big)_{U\in\mathbb{C}} \arrow[rr, "\varepsilon_B"] \arrow[d, "GHf"] & & B \arrow[d, "f"] \\
\Big(\prod_{V\in\mathbb{C}} C_V^{\mathbb{C}(V,U)}\Big)_{U\in\mathbb{C}} \arrow[rr, "\varepsilon_C"] & & C
\end{tikzcd}$$

for each $C = (C_V)_{V\in\mathbb{C}}$ and $f : B \to C$ in $Ens^{|\mathbb{C}|}$. To do so, let $\alpha = ((\alpha_{VU})_{V\in\mathbb{C}})_{U\in\mathbb{C}}$ with $\alpha_{VU} : \mathbb{C}(V,U) \to B$, for any V and U in $\mathbb{C}$,

$$\varepsilon_C(GHf)\alpha = \varepsilon_C(((f_V\circ\alpha_{VU})_{V\in\mathbb{C}})_{U\in\mathbb{C}}) = ((f_U\circ\alpha_{UU})1_U)_U$$

$$= (f(\alpha_{UU}1_U))_{U\in\mathbb{C}} = (f\varepsilon_B)\alpha.$$ Thus ε is also natural./

<u>1.1.4</u> <u>Proposition</u>. *G is a left adjoint to H, with η and ε as front and back adjunctions.*

Proof. It only remains to show that

$$1_G = G \xrightarrow{G*\eta} GHG \xrightarrow{\varepsilon * G} G$$

and $$1_H = H \xrightarrow{\eta * H} HGH \xrightarrow{H*\varepsilon} H$$

That is to show, for any $A \in \hat{\mathbb{C}}$ and $B = (B_V)_{V \in \mathbb{C}}$ in $Ens^{|\mathbb{C}|}$,

$$\varepsilon_{GA} \circ G\eta_A = 1_{GA} \text{ and } H\varepsilon_B \circ \eta_{HB} = 1_{HB}$$

to do so, let $U \in \mathbb{C}$ and $a \in AU$, we have

$$(GA)U \to (GHGA)U \to (GA)U$$

given by $a \rightsquigarrow (\tilde{a}_V)_{V \in \mathbb{C}} \rightsquigarrow e(\tilde{a}_U)$.

Now, $e(\tilde{a}_U) = \tilde{a}_U 1_U = (A1_U)a = a$, and hence $\varepsilon_{GA} \circ G\eta_A = 1_{GA}$

Similarly, one can show $H\varepsilon_B \circ \eta_{HB} = 1_{HB}$. ///

1.1.5 Lemma. *The front adjunction* η *is a monomorphism.*

Proof. It is enough to show that

$$\eta_{AU} : AU \to \prod_{V\in\mathbb{C}} AV^{\mathbb{C}(V,U)}$$

is a monomorphism, for any $A\in\hat{\mathbb{C}}$ and each $U\in\mathbb{C}$.
Let a and b be any two elements of AU with $\eta_{AU}a = \eta_{AU}b$, this implies that $\tilde{a}_V = \tilde{b}_V$, for all $V\in\mathbb{C}$. Then, for any $t : V \to U$, we have $(At)a = \tilde{a}_V t = \tilde{b}_V t = (At)b$; taking $t = 1_U$, this implies that $a = b$.///

<u>1.1.6</u> <u>Remark</u>. The functor G and H can be lifted to

$$\mathcal{M}od(\mathcal{H},\hat{\mathbb{C}}) \underset{H}{\overset{G}{\rightleftarrows}} \mathcal{M}od\,\mathcal{H}^{|\mathbb{C}|}$$

with G a left adjoint of H; the lifted functors have also been denoted by G and H. This is because both G and its right adjoint H preserve all inverse limits, hence carry algebras to algebras (and then remain adjoint.)

<u>1.2 RESIDUAL SMALLNESS</u>

<u>1.2.1</u> <u>Definition</u>. *A homomorphism $h : A \to B$ in a category $\mathbb{K}$ is called <u>essential</u> iff, for any homomorphism $g : B \to C$ in $\mathbb{K}$, whenever $g\circ h$ is a monomorphism, then so is g ($\mathbb{K}$ a category of algebras).*

Let $\mathbb{E}$ be a Grothendieck topos and $\mathcal{H}$ a set of quasi-equations.

<u>1.2.2.</u> <u>Lemma</u>. *In* $\mathcal{M}od(\mathcal{H},\mathbb{E})$,

(i) Any composite of essential monomorphisms is an essential monomorphism, and

(ii) any direct limit of essential monomorphisms is an essential monomorphism.

<u>Proof</u>. (i) is trivial. To prove (ii), let $f : A \to \underset{\alpha\in I}{\varinjlim} B_\alpha$ be a direct limit, in $\mathcal{M}od(\mathcal{H},\mathbb{E})$, of essential monomorphisms $f_\alpha : A \to B_\alpha$, with colimit maps $g_\alpha : B_\alpha \longrightarrow \underset{\alpha\in I}{\varinjlim} B\alpha$ and diagram maps $g_{\alpha\beta} : B_\alpha \to B_\beta$, for $\beta \geq \alpha$. Since each f_α is an essential monomorphism and $g_{\alpha\beta} \circ f_\alpha = f_\beta$ is a monomorphism, all $g_{\alpha\beta}$ are monomorphisms, and hence f is a monomorphis. The latter holds because it is true in Ens and hence in $\hat{\mathbb{C}}$, and a colimit in $\mathbb{E}$ is formed by first forming it in $\hat{\mathbb{C}}$ and then reflecting it to $\mathbb{E}$; since the reflection functor $R : \hat{\mathbb{C}} \to \mathbb{E}$ is left exact, and thus preserves monomorphisms, we are done. To show that f is essential, let $A \xrightarrow{f} \underset{\alpha\in I}{\varinjlim} B\alpha \xrightarrow{h} D$ be a

monomorphism, with $D \in \mathcal{M}od(\mathcal{H}, \mathbf{E})$. Then essentialness of f_α implies that all the $B_\alpha \xrightarrow{g_\alpha} \varinjlim_{\alpha \in I} B\alpha \xrightarrow{h} D$ are monomorphisms, and hence h is a monomorphism, the latter again because of the way direct limits are formed in $\mathcal{M}od(\mathcal{H}, \mathbf{E})$. Thus $f : A \to \varinjlim_{\alpha \in I} B\alpha$ is an essential monomorphism.///

__1.2.3__ __Lemma__. *In $\mathcal{M}od(\mathcal{H}, \mathbf{E})$, for any monomorphism $h : A \to B$ there exists a homomorphism $g : B \to C$ with $g \circ h$ an essential monomorphism.*

Proof. Take all the $\mathcal{M}od(\mathcal{H}, \mathbf{E})$-congruences θ on B with $A \xrightarrow{h} B \xrightarrow{\nu} B/\theta$ a monomorphism, where ν is the quotient map. Then, by the exactness discussed in the proof of the last lemma, any join of a chain of such congruences is again such a congruence, and hence there exists a maximal such congruence θ. Maximality of θ then implies that $A \to B/\theta$ is essential.///

__1.2.4__ __Corollary__. *In $\mathcal{M}od(\mathcal{H}, \mathbf{E})$, an* *algebra A is an absolute retract iff it has no proper essential extension.*

Proof. ($\Rightarrow$) If $f : A \rightarrowtail B$ is an essential monomorphism and $h : B \to A$ is a retraction, for A and B in $\mathcal{M}od(\mathcal{H}, \mathbb{E})$, then by essentialness of f, h is a monomorphism, and hence $A \simeq B$.

($\Leftarrow$) Given any monomorphism $f : A \rightarrowtail B$ in $\mathcal{M}od(\mathcal{H}, \mathbb{E})$, continue it to an essential monomorphism $A \xrightarrow{f} B \xrightarrow{g} C$, by the last lemma. By hypothesis on A, gf is an isomorphism and then $(gf)^{-1} \circ g : B \to A$ is the desired retraction.///

1.2.5 Definition. *A category $\mathbb{K}$ is called residually small, iff, it has a set of cogenerators.*

1.2.6 Definition. *A category $\mathbb{K}$ is called essentially bounded iff, each $A \in \mathbb{K}$, has up to isomorphism, only a set of essential extensions in $\mathbb{K}$.*

<u>1.2.7</u> <u>Lemma.</u> *For any well powered category with products and a set $\mathscr{G}$ of generators, residual smallness implies essential boundedness.*

<u>Proof</u>. Let $h : A \rightarrowtail B$ be an essential monomorphism in $\mathbb{K}$, and embed $B \overset{e}{\rightarrowtail} \prod_{\alpha \in I} C_\alpha$, for suitable cogenerators C_α. Then, for any generator G and a pair of distinct maps $G \underset{t}{\overset{s}{\rightrightarrows}} A$, we have $e \circ h \circ s \neq e \circ h \circ t$, and hence there exists some $\alpha \in I$ with $p_\alpha \circ e \circ h \circ s \neq p_\alpha \circ e \circ h \circ t$, with $p_\alpha : \prod_{\alpha \in I} C\alpha \to C_\alpha$ the α-th projection. Pick α_{st} as one such, then $A \xrightarrow{h} B \to \prod_{\beta \in J} C_\beta$ is a monomorphism, where $J = \{\alpha_{st} : s \neq t : G \to A\}$ and $\operatorname{card} J \leq \operatorname{card} \bigcup_{G \in \mathscr{G}} \mathbb{K}(G,A)^2$. Essentialness of h implies that $B \to \prod_{\beta \in J} C_\beta$ is a monomorphism, and since there exists only a set of products $\prod_{\beta \in J} C_\beta$, we are done.///

<u>1.2.8</u> <u>Lemma.</u> *For $\mathscr{M}od(\mathscr{H}, \mathbb{E})$, essential boundedness implies residual smallness.*

<u>Proof</u>. For any $A \in \mathscr{M}od(\mathscr{H}, \mathbb{E})$, take all $B_\alpha \leq A$,

the subalgebra of A generated by some $\alpha : U \coprod U \to |A|$, for $U \in \mathbb{C}$, and then continue them to essential extensions $B_\alpha \xrightarrow{i_\alpha} A \xrightarrow{f_\alpha} C_\alpha$, by Lemma (1.2.3). The homomorphism $\prod f_\alpha : A \to \prod C_\alpha$ is a monomorphism, for; if not, there exists some α with $B_\alpha \to A \to \prod C_\alpha$ not a monomorphism, which contradicts the fact that all $B_\alpha \to C_\alpha$ are monomorphisms. Since there exists, up to isomorphism, only a set of $B \in \mathcal{M}od(\mathcal{H}, \mathbf{E})$ generated by some $U \coprod U \to |B|$, and only a set of essential extensions of these B, by hypothesis, we are done.///

<u>1.2.9</u> <u>Corollary</u>. *For $\mathcal{M}od(\mathcal{H}, \mathbf{E})$, essential boundedness is equivalent to residual smallness.*

<u>Proof</u>. One way this is true by the last lemma, and since $\mathcal{M}od(\mathcal{H}, \mathbf{E})$ has a set of generators, namely the $\mathcal{M}od(\mathcal{H}, \mathbf{E})$-free algebras on the reflection of the representable presheaves h_U, for $U \in \mathbb{C}$, Lemma (1.2.7) implies the converse.///

<u>1.2.10</u> <u>Proposition</u>. *$\mathcal{M}od(\mathcal{H}, \mathbf{E})$ is*

residually small iff $\mathcal{M}od\,\mathcal{H}$ is residually small.

Proof. ($\Rightarrow$) Consider the following pair of adjoint functors:

$$\mathcal{M}od(\mathcal{H},\mathbb{E}) \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} \mathcal{M}od\,\mathcal{H}$$

where $\Gamma = (\mathbb{1},-)$ and Δ left exact, left adjoint to Γ; in fact, Δ is the composite

$$\mathcal{M}od\,\mathcal{H} \xrightarrow{\Delta_\circ} \mathcal{M}od\,\mathcal{H}^{|\mathbb{C}|} \xrightarrow{H} \mathcal{M}od(\mathcal{H},\hat{\mathbb{C}}) \xrightarrow{R} \mathcal{M}od(\mathcal{H},\mathbb{E}),$$

where $\Delta_\circ$ takes $\mathcal{M}od\,\mathcal{H}$ to the constant families indexed by $|\mathbb{C}|$. One can then easily check that the functor Γ transfers the set of cogenerators of $\mathcal{M}od(\mathcal{H},\mathbb{E})$ to a set of cogenerators. ($\Leftarrow$) If $\mathcal{M}od\,\mathcal{H}$ is residually small, then so is any $(\mathcal{M}od\,\mathcal{H})^I =$ $\mathcal{M}od(\mathcal{H},\mathrm{Ens}^I)$. Now, consider the pair of adjoint functors:

$$\mathcal{M}od(\mathcal{H},\hat{\mathbb{C}}) \underset{H}{\overset{G}{\rightleftarrows}} \mathcal{M}od(\mathcal{H},\mathrm{Ens}^{|\mathbb{C}|})$$

constructed earlier. The set of cogenerators of $\mathcal{M}od(\mathcal{H},\mathrm{Ens}^{|\mathbb{C}|})$ get transferred to a set of cogenerators in $\mathcal{M}od(\mathcal{H},\hat{\mathbb{C}})$, and hence $\mathcal{M}od(\mathcal{H},\hat{\mathbb{C}})$ is essentially bounded, by Lemma (1.2.7). Since monomorphisms in

$\mathcal{M}od(\mathcal{H},\mathbb{E})$ are also monomorphisms in $\mathcal{M}od(\mathcal{H}^{\sim},\hat{\mathbb{C}})$ and the reflection functor is left exact, essential monomorphisms in $\mathcal{M}od(\mathcal{H},\mathbb{E})$ are also essential in $\mathcal{M}od(\mathcal{H},\hat{\mathbb{C}})$, and hence $\mathcal{M}od(\mathcal{H},\mathbb{E})$ is also essentially bounded. Thus $\mathcal{M}od(\mathcal{H},\mathbb{E})$ is residually small, by Lemma (1.2.8).///

<u>1.3</u> <u>INJECTIVE ALGEBRAS IN $\mathcal{M}od(\mathcal{H},\mathbb{E})$</u>

<u>1.3.1</u> <u>Definition</u>. *In $\mathcal{M}od(\mathcal{H},\mathbb{E})$, <u>pushouts transfer monomorphisms</u> iff for any pushout diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

whenever f is a monomorphism, then g is also a monomorphism.

<u>1.3.2</u> <u>Proposition</u>. *Pushouts transfer monomorphisms in $\mathcal{M}od(\mathcal{H},\mathbb{E})$ iff they do in $\mathcal{M}od\,\mathcal{H}$.*

Proof. ($\Rightarrow$) Let the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

be a pushout diagram with f a monomorphism, in $\mathcal{M}od\mathcal{H}$.
Using the pair of adjoint functors $\Delta \dashv \Gamma$ given
in the proof of Proposition (1.2.10), we get that
the diagram

$$\begin{tikzcd} \Delta A \arrow[r, "\Delta f"] \arrow[d, "\Delta u"'] & \Delta B \arrow[d, "\Delta v"] \\ \Delta C \arrow[r, "\Delta g"] & \Delta D \end{tikzcd}$$

is a pushout in $\mathcal{M}od(\mathcal{H}, \mathbf{E})$ with Δf a monomorphism, and
then, by hypothesis on $\mathcal{M}od(\mathcal{H}, \mathbf{E})$, Δg is a monomorphism.
Since Δ is faithful, g is a monomorphism.

($\Leftarrow$) If the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

is a pushout in $\mathcal{M}od(\mathcal{H}, \mathbb{E})$ with f a monomorphism, then, by the construction of pushouts in $\mathcal{M}od(\mathcal{H}, \mathbb{E})$, D is the reflection of some $P \in \mathcal{M}od(\mathcal{H}, \hat{\mathbb{C}})$ with

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "\overline{g}"] & P \end{tikzcd}$$

a pushout in $\mathcal{M}od(\mathcal{H}, \hat{\mathbb{C}})$, $R\overline{g} = g$, and f is a monomorphism. Now, for each $U \in \mathbb{C}$,

$$\begin{tikzcd} AU \arrow[r, "f_U"] \arrow[d] & BU \arrow[d] \\ CU \arrow[r, "\overline{g}_U"] & PU \end{tikzcd}$$

is a pushout in $\mathcal{M}od\mathcal{H}$, and then, by the hypothesis on $\mathcal{M}od\mathcal{H}$, $\overline{g}_U$ is a monomorphism, for each $U \in \mathbb{C}$, and hence $\overline{g}$ is also a monomorphism. Now $R\overline{g} = g$, and since R is left exact, g is a monomorphism.///

<u>1.3.3</u> <u>Lemma</u>. *The category* $\mathcal{M}od(\mathcal{H}, E)$ *has enough injectives iff it is residually small and pushouts transfer monomorphisms.*

<u>*Proof*</u>. ($\Rightarrow$) To show that pushouts transfer monomorphisms, let

$$\begin{tikzcd} A \arrow[r, tail, "f"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

be a pushout in $\mathcal{M}od(\mathcal{H}, \mathbf{E})$ with f a monomorphism. Now, let $h : C \rightarrowtail E$ be a monomorphism to an injective algebra $E \in \mathcal{M}od(\mathcal{H}, \mathbf{E})$. Then, because f is a monomorphism, there exists a homomorphism $k : B \to E$ with $k \circ f = h \circ u$, and hence there exists a homomorphism $\ell : D \to E$ with $\ell \circ g = h$, the latter because the above diagram is a pushout. Now, since h is a monomorphism and $\ell \circ g = k$, g isa monomorphism, thus pushouts transfer monomorphisms. To show that $\mathcal{M}od(\mathcal{H},\mathbf{E})$ is residually small is to show it is essentially bounded, by Corollary (1.2.9). Now, let $h : A \to B$ be an essential monomorphism. Embedding A into any injective E, one readily sees that B can be embedded in E, and hence, up to isomorphism, there is only a set of such B. This shows that $\mathcal{M}od(\mathcal{H},\mathbf{E})$ is essentially bounded.

($\Leftarrow$) For $A \in \mathcal{M}od(\mathcal{H},\mathbf{E})$, take a maximal essential extension $f : A \rightarrowtail E$ of A in $\mathcal{M}od(\mathcal{H},\mathbf{E})$, which exists by Lemma 1.2.2(ii). We claim that E is an injective. To prove this, let $g : A \rightarrowtail B$ be any monomorphism and then form the following pushout diagram in $\mathcal{M}od(\mathcal{H},\mathbf{E})$:

$$\begin{tikzcd} A \arrow[r, tail, "g"] \arrow[d, tail] & B \arrow[d] \\ E \arrow[r, "h"] & C \end{tikzcd}$$

By hypothesis, h is a monomorphism, and hence retractable, by Colollary (1.2.4). Thus, E is injective, and this shows that $\mathcal{M}od(\mathcal{H},\mathbf{E})$ has enough injectives.///

1.3.4 Proposition. *The category $\mathcal{M}od(\mathcal{H}, \mathbf{E})$ has enough injectives iff $\mathcal{M}od\,\mathcal{H}$ has enough injectives.*

Proof. This follows from Propositions (1.2.10) and (1.3.2) and Lemma (1.3.3).///

This result substantially improves a similar result by Howlett [14]. Here, we deal with quasi-equational classes of algebras rather than equational classes as [14] does, but more importantly, our proof does not use the points of the topos whereas [14] only proves this result for a Grothendieck topos with enough points.

Moreover, Proposition (1.2.10) provides a positive answer to Howlett's question [14] page 108 whether essential boundedness of $\mathcal{M}od\,\mathcal{H}$ directly implies that of $\mathcal{M}od(\mathcal{H}, \mathbf{E})$.///

1.4 BEHAVIOR OF INJECTIVITY IN $\mathcal{M}od(\mathcal{H}, \mathbf{E})$

1.4.1 B. Banaschewski in [2] calls the notion of injectivity in a category $\mathbf{K}$ *properly behaved* if the following three propositions hold which describe the relationship between essential boundedness, residual smallness and the existence of injective hulls in $\mathbb{K}$. Actually [2] deals with injectivity with respect to a more general type of morphisms, but of course, here we only consider injectivity with respect to monomorphisms.

<u>1.4.2</u> (I) *For any $A \in \mathbb{K}$, the following conditions are equivalent:*

(I1) A is injective.

(I2) A is an absolute retract.

(I3) A has no proper essential extension.

<u>1.4.3</u> <u>(E)</u> *Every $A \in \mathbb{K}$ has an injective hull, unique up to isomorphism.*

<u>1.4.4</u> (H) *For any monomorphism $f : A \rightarrowtail B$, the following conditions are equivalent:*

(H1) $f : A \to B$ is an injective hull of A.

(H2) $f : A \to B$ is a maximal essential monomorphism.

(H3) $f : A \to B$ is a minimal injective extension.

<u>1.4.5</u> Also [2] gives sufficient conditions for the proper behavior of injectivity in $\mathbb{K}$ as follows:

(E3) *For any monomorphism $f : A \to B$ there exists a homomorphism $g : B \to C$ with $g \circ f$ an essential monomorphism.*

(E4) *Any diagram* $\begin{array}{ccc} A & \xrightarrow{f} & B \\ g\downarrow & & \\ C & & \end{array}$ *with f a monomorphism can be completed to a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g\downarrow & & \downarrow v \\ C & \xrightarrow[u]{} & D \end{array}$$

such that u is a monomorphism.

(E5) *Any direct limit of monomorphisms is a monomorphism.*

(E6) *The category $\mathbb{K}$ is essentially bounded.*

For $\mathbb{K} = \mathcal{M}od(\mathcal{H}, \mathbf{E})$ we now have the following counterpart of Proposition (5) in [2] for equational classes of algebras in Ens:

1.4.6 Proposition. *For $\mathcal{M}od(\mathcal{H},\mathbb{E})$, the following are equivalent:*

(i) *Injectivity is properly behaved.*

(ii) *$\mathcal{M}od(\mathcal{H},\mathbb{E})$ has enough injectives.*

(iii) *$\mathcal{M}od(\mathcal{H},\mathbb{E})$ is residually small and pushouts transfer monomorphisms.*

(iv) *(E4) and (E6) are satisfied.*

Proof. (i) ⇒ (ii): By (E) in the definition of proper behavior of injectivity.

(ii) ⇒ (iii): By Lemma (1.3.3).

(iii) ⇒ (iv): (E4) is trivial, by completing any diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \\ C & & \end{array}$$

, in $\mathcal{M}od(\mathcal{H},\mathbb{E})$ to a pushout. (E6) holds by Corollary (1.2.9).

(iv) ⇒ (i): It remains to show that $\mathcal{M}od(\mathcal{H},\mathbb{E})$ satisfies (E3) and (E5). But Lemma (1.2.3) proves (E3), and (E5) is discussed in the proof of Lemma 1.2.2(ii).///

In particular, one has, by Proposition (1.3.4): Injectivity is properly behaved in $\mathcal{M}od(\mathcal{H}, \mathbb{E})$ iff it is properly behaved in $\mathcal{M}od\,\mathcal{H}$ .///

We conclude this section with a couple of comments on injectivity in $\mathcal{M}od(\mathcal{H}, \mathbb{E})$.

Since the functor $\mathcal{M}od(\mathcal{H}, \mathbb{E}) \xrightarrow{(U,-)} \mathcal{M}od\,\mathcal{H}$ has a left adjoint which preserves monomorphisms and it is well known that such a functor preserves injectives, if $A \in \mathcal{M}od(\mathcal{H}, \mathbb{E})$ is injective, then so is AU, for each $U \in \mathbb{C}$. However, the converse of this is not true; for counter examples, in the case of abelian groups, the reader is referred to B. Banaschewski [3]./

For certain $\mathcal{H}$, one has characterizations of the injective $A \in \mathcal{M}od\,\mathcal{H}$ by properties of A in terms of its elements and subsets, for example divisibility for abelian groups, completeness for Boolean algebras, and completeness and Booleaness for distributive lattices. An obvious question to ask is to what extent, that is for what $\mathbb{E}$, such characterizations remain valid in $\mathcal{M}od(\mathcal{H}, \mathbb{E})$. The only case where anything is known about this is that of abelian groups: divisibility = injectivity for abelian groups in $Sh\,\mathcal{L}$ iff the locale $\mathcal{L}$ is Boolean (B. Banaschewski [4]).///

CHAPTER 2

PURITY AND EQUATIONAL COMPACTNESS IN $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$

In this chapter, we study the notions of purity and equational compactness in a quasi-equational class $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$ of sheaves of algebras on a locale $\mathcal{L}$, and prove the counterparts of some of the results for equational classes of algebras in Ens.

In section (1), we define a notion of finiteness of sheaves and then prove, in Proposition (2.1.7), that for a noetherian locale $\mathcal{L}$, $Sh\mathcal{L}$ is algebraic. This is used later to prove some of the results in section (2) and (3).

In section (2), we study pure homomorphisms in $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$, giving, among other results, the counterparts of a characterization of pure homomorphisms, and the Pure Representation Theorem.

Finally, section (3) gives a study of equationally compact algebras in $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$. We show, in Proposition (2.3.2), that these algebras are characterized here in the same way as in Ens, and in Proposition (2.3.8), we prove that the same conditions as in Ens are equivalent to the existence of equationally compact hulls in $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$.

Regarding the general definitions and results for algebras in Ens, the reader is referred to [5], [6], [7], [8], [15], [18], [22], [23].

2.1 Finite Sheaves

2.1.1 Definition. *For any collection* $\mathcal{F}$ *of sheaves on a locale* $\mathcal{L}$, *a sheaf* $F \in Sh\mathcal{L}$ *is said to be* *$\mathcal{F}$-finite iff, there exists an epimorphism* $\coprod_{i=1}^{n} F_i \twoheadrightarrow F$ *(n finite) with* $F_i \in \mathcal{F}$ $(i = 1,2,\ldots,n)$.

The collection of all $\mathcal{F}$-finite sheaves in $Sh\mathcal{L}$ will be denoted by $\bar{\mathcal{F}}$.

2.1.2 Proposition *(i)* $\bar{\mathcal{F}}$ *is closed under finite coproducts and epimorphic images.*

(ii) If $\mathcal{F}$ *is closed under finite non-empty products, then so is* $\bar{\mathcal{F}}$, *and hence if* $1 \in \mathcal{F}$, *then* $\bar{\mathcal{F}}$ *is closed under all finite products.*

(iii) If $\mathcal{F}$ *is closed under sub-objects,*

so is $\bar{\mathcal{F}}$.

Proof. (i) is trivial.

(ii) follows from the facts that, in any topos finite limits commute with colimits and the products of epimorphisms are epimorphisms, as follows: Suppose that each F_i, $i = 1,2,\ldots,m$, is $\mathcal{F}$-finite and let $\coprod_{i=1}^{n_j} S_{ij} \twoheadrightarrow F_j$ represents F_j, for each $j = 1,2,\ldots,m$.

We then obtain an epimorphism $\prod_{j=1}^{m} \coprod_{i=1}^{n_j} S_{ij} \twoheadrightarrow \prod_{j=1}^{m} F_j$.

Since finite products commute with coproducts, we have $\prod_{j=1}^{m} \coprod_{i=1}^{n_j} S_{ij} = \coprod \prod_{j=1}^{m} S_{i_j j}$, where the coproduct runs over all sequences $(i_1,\ldots,i_m)$ in $n_1 \times \ldots \times n_m$. By the hypothesis $\prod S_{i_j j} \in \mathcal{F}$, and hence $\prod_{j=1}^{m} F_j \in \bar{\mathcal{F}}$.

To prove (iii), suppose that $F \in \bar{\mathcal{F}}$ and the epimorphism $f : \coprod_{i=1}^{n} F_i \twoheadrightarrow F$ represents it. Let $g : G \rightarrowtail F$ be any monomorphism. By pulling back g along f we obtain:

$$\begin{tikzcd}
(\coprod_{i=1}^{n} F_i) \times_F G \arrow[r, two heads] \arrow[d, tail] & G \arrow[d, tail, "g"] \\
\coprod_{i=1}^{n} F_i \arrow[r, two heads, "f"] & F
\end{tikzcd}$$

But, from the properties of topoi, the top arrow is an epimorphism and pulling back preserves colimits, that is $(\coprod_{i=1}^{n} F_i) \times_F G = \coprod_{i=1}^{n} (F_i \times_F G)$, and each $F_i \times_F G$ is a subsheaf of F_i, and hence is in $\mathcal{F}$, for $i = 1,\ldots,n$.

Thus $G \in \bar{\mathcal{F}}$. ///

2.1.3 Definition. *A sheaf $F \in Sh\mathcal{L}$ is said to be* *finite* *iff every map $f : F \to \varinjlim G_\alpha$, from it to a direct limit of sheaves, factors through some component G_α.*

That is, there exists a morphism $\bar{f} : F \to G_\alpha$ for some α such that the following diagram commutes:

$$\begin{tikzcd}
F \arrow[rr, "f"] \arrow[dr, dashed, "\bar{f}"'] & & \underrightarrow{\ell im}\, G\alpha \\
& G_\alpha \arrow[ur, "g_\alpha"'] &
\end{tikzcd}$$

where g_α is a limit map.

The collection of all finite sheaves will be denoted by $FSh\mathcal{L}$.

<u>2.1.4</u> <u>Lemma</u>. *If U is any compact element of $\mathcal{L}$, then $(\underrightarrow{\ell im}\, F_\alpha)U = \underrightarrow{\ell im}\, F_\alpha U$, for any direct limit in $Sh\,\mathcal{L}$.*

<u>Proof</u>. It is enough to show that every $s \in (\underrightarrow{\ell im}\, F_\alpha)U$ is already in the image of some $F_\alpha U$ under the colimit map $g_{\alpha U}$. Let $s \in (\underrightarrow{\ell im}\, F_\alpha)U$ be arbitrary; since U is compact and by the construction of direct limits in

Sh$\mathcal{L}$, there exists a finite cover $U = \bigvee_{i=1}^{n} U_i$ of U in $\mathcal{L}$ such that $s|U_i \in \varinjlim F_\alpha U_i$, for $i = 1,\ldots n$, and hence $s|U_i \in$ image $F_{\alpha_i} U_i$ for some α_i. Now, since the colimit is direct, we can find some $\alpha \geq \alpha_i$, for all $i = 1,\ldots n$, such that $s|U_i \in$ image $F_\alpha U_i$, for all i, and F_α being a sheaf then implies that $s \in$ image $F_\alpha U$.///

<u>2.1.5</u> <u>Lemma</u>. *For any $U \in \mathcal{L}$, the sheaf $h_U = (-,U)$ is finite iff U is compact.*

<u>Proof</u>. ($\Rightarrow$) Let $U = \bigvee_I U_i$ be any cover of U in $\mathcal{L}$.

We write $U = \bigvee_{\text{all finite } J \subseteq I} \bigvee_{j \in J} U_j$, and hence

$h_U = \varinjlim_{\text{finite } J \subseteq I} h_{\bigvee_{j \in J} U_j}$ is a direct limit. Since h_U is finite, we get $h_U \leq h_{\bigvee_{j \in J} U_i}$ for some finite $J \subseteq I$.

Now, $1 = h_U U \leq h_{\bigvee_{j\in J} U_j} U$ implies that $h_{\bigvee_{j\in J} U_j} U = 1$,

which then implies that $U \leq \bigvee_{i\in J} U_i$; this shows that U

is a compact element. The converse is clear by the last lemma.

<u>2.1.6</u> <u>Lemma</u>. *For an algebraic locale $\mathcal{L}$, every finite sheaf is $\mathcal{L}$ -finite, for $\mathcal{L} = \{h_U : U \text{ compact}\}$ = set of all compact subsheaves of $\mathbb{1}$, the terminal object of Sh $\mathcal{L}$; and the converse is true if $\mathcal{L}$ is noetherian.*

<u>Proof</u>. Let $\mathcal{L}$ be algebraic and F be any finite sheaf on $\mathcal{L}$. Consider all the subsheaves T of F which are $\mathcal{L}$ -finite. Then, for a compact element $V \in \mathcal{L}$, $FV = \bigcup_{\text{all } T} TV$ by (2.1.4); and for a non-compact element $U \in \mathcal{L}$, let $U = \bigvee U_i$ $(i \in I)$ be a cover of U in $\mathcal{L}$ by compact elements (which exist because $\mathcal{L}$ is algebraic). Now, for any $s \in FU$, $s|U_i \in FU_i = \bigcup_{\text{all } T} TU_i$; for all $i \in I$, the latter equality becuase U_i are compact for all $i \in I$. This implies that $s \in (\underrightarrow{\lim}_{\text{all } T} T)U$, and hence

$FU = (\varinjlim_{\text{all } T} T)\, U$, for any $U \in \mathcal{L}$, and thus $F = \varinjlim_{\text{all } T} T$.

(direct). Since F is finite, it follows that $F \leq T$, for some $\mathscr{L}$ -finite subsheaf T of F, and hence $F = T$. This shows that F is $\mathscr{L}$ -finite.

Conversely, let now $\mathcal{L}$ be noetherian and F a $\mathscr{L}$ -finite (which is now same as $\mathcal{L}$ -finite) sheaf on $\mathcal{L}$.

Let F be represented by $f\colon \coprod_{i=1}^{n} h_{U_i} \twoheadrightarrow F$ and $\sigma : F \to \varinjlim G_\alpha$ be any map from F to a direct limit, in $Sh\mathcal{L}$. Since each U_i is compact, each h_{U_i} is finite, by (2.1.5), for $i = 1,\ldots,n$ and so is their coproduct, the latter is easily checked. This provides us with the following factorization:

$$\begin{tikzcd}
\coprod_{i=1}^{n} h_{U_i} \arrow[r, two heads, "f"] \arrow[dr, dashed, "\bar{f}"'] & F \arrow[r, "\sigma"] & \varinjlim G_\alpha \\
& G_\alpha \arrow[ur, "g_\alpha"'] &
\end{tikzcd}$$

through some colimit map g_α. Now, let $K \leq (\coprod_{i=1}^{n} h_{U_i})^2$ be the kernel of f, and then by the properties of topoi

we have $K = K \cap (\coprod_{i=1}^{n} h_{U_i})^2 = \coprod K \cap (h_{U_i} \times h_{U_j}) = \coprod h_{V_{ij}}$

for some $V_{ij} \leq U_i \wedge U_j$ $(i,j = 1,\ldots,n)$. But, $\mathcal{L}$ being noetherian, all V_{ij} are also compact, and hence finite, which implies that K is finite. Next, we have that the kernel of g_α is the union of the $\mathrm{Ker}\, g_{\alpha\beta}$ for all $\beta \geq \alpha$ because every $U \in \mathcal{L}$ is compact and hence $(\varinjlim G_\alpha)U = \varinjlim G_\alpha U$ for all $U \in \mathcal{L}$, by (2.1.4). Since K is finite and is mapped to $\bigcup_{\beta \geq \alpha} \mathrm{Ker}\, g_{\alpha\beta}$ by $\overline{f} \times \overline{f}$, we get a factorization of $\overline{f}^2 | K$ through some $\mathrm{Ker}\, g_{\alpha\beta}$. Hence, $g_{\alpha\beta} \circ \overline{f}$ composes equally with the two morphisms of the kernel pair of f, and since f is the coequalizer of its kernel pair, $g_{\alpha\beta} \circ \overline{f}$ factors through f. Thus, we get the following diagram:

$$\begin{tikzcd}
\coprod_{i=1}^{n} h_{U_i} \arrow[r, two heads, "f"] \arrow[dr, "g_{\alpha\beta} \circ \overline{f}"'] & F \arrow[r, "\sigma"] \arrow[d, dashed, "\overline{\sigma}"] & \varinjlim G_\alpha \\
& G_\beta \arrow[ur, "g_\alpha"'] &
\end{tikzcd}$$

with $\overline{\sigma} \circ f = g_{\alpha\beta} \circ \overline{f}$. Then, we have $g_\beta \circ \overline{\sigma} \circ f = g_\beta \circ g_{\alpha\beta} \circ \overline{f} = g_\alpha \circ \overline{f} = \sigma \circ f$, and since f is an epimorphism also $g_\beta \circ \overline{\sigma} = \sigma$, which is the required factorization of σ through some colimit map g_β, showing that F is finite.///

2.1.7 Proposition. *For a noetherian locale $\mathcal{L}$, $Sh\mathcal{L}$ is algebraic; that is each $F \in Sh\mathcal{L}$ is the join of its finite subsheaves.*

Proof. By the first part of the proof of the last lemma, we have $F = \bigvee T$ over all $\mathcal{L}$-finite subsheaves T of F which are finite by the second part of that lemma.///

2.2 PURE HOMOMORPHISMS

2.2.1 For an algebra $A \in \mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$ and a sheaf S on $\mathcal{L}$, we have *the algebra A[S] of polynomials with variables in S and with coefficients in A,* defined to be the coproduct $A[S] = A \coprod \mathcal{F}s$

in $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$, where $\mathcal{F}$ is the free functor. *A sheaf of equations E in variables S with coefficients in A* is a subsheaf of $|A[S]|^2$. If C is any extension of A and $h : A \to B$ is any homomorphism, in $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$, then a homomorphism $f : C \to B$ whose restriction $f|A$ to A is h will be called *over h*. Here, if A is a subalgebra of B and h is the inclusion map, then f is said to be *over A*. Also, if furthermore $B = A$, then f is called a *retraction* of C to A, and A is called a *retract* of C iff such a retraction exists.

2.2.2 Remark. To obtain the desired results concerning pure homomorphisms, we shall need that the conclusion of Lemma (2.1.4) be true for all $U \in \mathcal{L}$ and also that the conclusion of Porposition (2.1.7) holds. Thus, for the remainder of this chapter, we let $\mathcal{L}$ be a noetherian locale. Some intermediate results do not require this condition, and we shall point these out at the end of the chapter.

2.2.3 Definition. *A homomorphism $h : A \to B$ in $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$ is said to be pure iff, for every finite subsheaf F of the kernel of any homomorphism $f : A[S] \to B$ over h, there also exists a retraction*

$g : A[S] \to A$ whose kernel contains F.

<u>2.2.4.</u> <u>Remark</u>. Note that every pure homomorphism is indeed a monomorphism. For; let $h : A \to B$ be a pure homomorphism, then any finite subsheaf F of the kernel of h is contained in the kernel of any homomorphism $f : A[S] \to B$ over h. By purity of h, F is contained in the kernel of some retraction $g : A[S] \to A$, and so $F \le \Delta \le A \times A$ which then implies that $\bigvee_{\text{all } F} F \le \Delta$. By Proposition (2.1.7) $\operatorname{Ker} h = \bigvee_{\text{all } F} F$, and hence $\operatorname{Ker} h \le \Delta$. This shows that h is a monomorphism.

<u>2.2.5</u> <u>Lemma</u> *(i) Any composite of pure homomorphisms is pure.*

(ii) If $f \circ g$ is pure, so is g.

(iii) Any direct limit of pure homomorphisms is pure.

<u>Proof</u>. (i) is trivial.

(ii) Let $g : A \to B$ and $f : B \to C$ be any two homomorphisms with $f \circ g$ pure. Consider the diagram

$$\begin{tikzcd} & A[S] \arrow[d, "h"] & \\ A \arrow[ur, tail, "i"] \arrow[r, "g"] & B \arrow[r, "f"] & C \end{tikzcd}$$

with i the natural monomorphism and h any homomorphism over g. Let F be any finite subsheaf of the kernel of h. Then, $F \leq \operatorname{Ker} f \circ h$ and hence, since $f \circ g$ is pure, there exists a retraction $\alpha : A[S] \to A$ whose kernel contains F. This shows that g is a pure homomorphism.

(iii) This is a crucial reuslt to the rest of this chapter, and here is where the condition of $\mathcal{L}$ being noetherian is needed most. To prove this, let $h_\alpha : A \to B_\alpha (\alpha \in I)$ be a direct family of pure homomorpisms and $h : A \to B$ be its colimit with the colimit maps $g_\alpha : B_\alpha \longrightarrow B$ (i.e. $h_\alpha = g_\alpha \circ h$ for all $\alpha \in I$). To show that h is pure, let $f : A[S] \to B$ be any homomorphism over h. Then, by (2.1.7), $S = \bigvee S_\alpha$ with $S_\alpha \leq S$ finite, and since we only deal with finite subsheaves of $A[S]$, S can be taken to be finite without loss of generality. Now, let F be any finite subsheaf of the kernel of f. By finiteness of S, we get a factorization:

$$\begin{tikzcd}
A[S] \arrow[rr, dashed, "\overline{f}"] \arrow[d, "f"'] & & B_\alpha \arrow[dll, "g_\alpha"] \\
B & &
\end{tikzcd}$$

through some g_α. Thus $\overline{f}^2(F) \leq \operatorname{Ker} g_\alpha = \bigcup_{\beta \geq \alpha} \operatorname{Ker} g_{\alpha\beta}$,

the latter is a consequence of Lemma (2.1.4), and so we have a morphism from F to the direct union $\bigcup_{\beta \geq \alpha} \operatorname{Ker} g_{\alpha\beta}$, where $G_\alpha \xrightarrow{g_{\alpha\beta}} G_\beta$ for $\beta \geq \alpha$, and hence, by finiteness of F, this factors through some $\operatorname{Ker} g_{\alpha\beta}\,(\beta \geq \alpha)$, and then $F \leq \operatorname{Ker}(g_{\alpha\beta} \circ \overline{f})$. Now $g_{\alpha\beta} \circ \overline{f}|A = g_{\alpha\beta} \circ h_\alpha = h_\beta$, showing that $g_{\alpha\beta} \circ \overline{f}$ is over h_β. Since $h_\beta : A \to B_\beta$ is a pure homomorphism, we get that $F \leq \operatorname{Ker} g$ for some retraction $g : A[S] \to A$. This shows that h is a pure homomorphism. ///

<u>2.2.6</u> <u>Proposition</u>. *The pure homomorphisms in a quasi-equational class $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$ are exactly the direct limits of retractable ones.*

Proof. By (iii) of the last lemma and the fact that retractable homomorphisms are pure, the proposition is true in one direction.

Conversely, let $f : A \to B$ be a pure homomorphism and $h : A[S] \to B$ any homomorphism over f inducing an isomorphism $A[S]/\mathrm{Ker}\ h \xrightarrow{\sim} B$. For any finite subsheaf F of the kernel of h, there exists a retraction $g_F : A[S] \to A$ whose kernel contains F (this is because f is pure). Let θ_F be the $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$-congruence generated by F. Since $\theta_F = \mathrm{Ker}\ \nu$ (where $\nu : A[S] \twoheadrightarrow A[S]/\theta_F$ is the quotient map) is contained in the kernel of g_F and the quotient maps are epimorphisms, we have a homomorphism $t_F : A[S]/\theta_F \to A$ with $t_F \circ \nu = g_F$, by the homomorphism decomposition theorem. Hence, all the homomorphisms $A \to A[S]/\theta_F$ are retractable. Now, by (2.1.7),

$\mathrm{Ker}\ h = \bigvee_{\mathrm{all}\ F} \theta_F$, and the fact that $B \simeq A[S]/\bigvee \theta_F =$

$\underset{\mathrm{all}\ F}{\underrightarrow{\lim}}\ A[S]/\theta_F$ in $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$ implies the assertion. ///

<u>2.2.7</u> <u>Corollary</u>. *Any left adjoint functor between quasi-equational classes of sheaves of algebras on noetherian locales preserves pure homomorphisms.*

Proof. This follows from the fact that a left adjoint functor preserves any kind of colimits, and that it preserves retractable homomorphisms is easily checked.///

<u>2.2.8</u> <u>Corollary</u>. *In any quasi-equational class $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$, if*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ C & \xrightarrow{g} & D \end{array}$$

is a pushout diagram with f a pure homomorphism, then g is also pure.

Proof. This follows from a well known categorical fact that h produces a pair of adjoint functors:

$$(A \downarrow \mathcal{M}od(\mathcal{H}, Sh\mathcal{L})) \underset{G}{\overset{F}{\rightleftarrows}} (C \downarrow \mathcal{M}od(\mathcal{H}, Sh\mathcal{L}))$$

where F is defined by pushing out along h and is left adjoint to G which is "preceeding by h". Hence, F preserves colimits of any kind, and that it preserves retractable homomorphisms is easy to check. Thus F preserves pure homomorphisms.///.

2.2.9 Corollary. *If $f : A \to B$ is a pure homomorphism, then so are all $f_U : AU \to BU$, for $U \in \mathcal{L}$.*

Proof. This follows from Lemma (2.1.4) and the fact that, for any $g : C \to D$ in $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$ which is retractable, all $g_U : CU \to DU$ are retractable, for $U \in \mathcal{L}$ .///

2.2.10 Corollary. *If $f : A \twoheadrightarrow B$ is a pure homomorphism in $\mathcal{M}od(\mathcal{H}, ShX)$, for a noetherian topological space X, then so are all the stalk maps $f_x : A_x \to B_x$, $x \in X$*

Proof. This follows from Corollary (2.2.7) and the fact that the stalk functors $\mathcal{M}od(\mathcal{H}, ShX) \to \mathcal{M}od\,\mathcal{H}$, are left adjoint. The latter is a consequence of 0.2.8(b), taking $\mathcal{L} = \mathcal{O}X$, $\mathcal{M} = \mathcal{O}Y$ for a singleton set Y and, for each $x \in X$, $\phi_x : \mathcal{L} \to \mathcal{M}$ is given by,

$\phi_x U = 1$ if $x \in U$ and empty otherwise.

In fact, one can prove this proposition directly as follows: We know, by Proposition (2.2.6), that $f = \varinjlim f_\alpha$ (direct), for retractable homomorphisms $f_\alpha : A \to B_\alpha$, and $f_x = \varinjlim f_{\alpha x}$ (direct), and that all the components $f_{\alpha x}$ are retractable is easily checked. Since, in $\mathcal{M}od\,\mathcal{H}$, pure homomorphisms are exactly the direct limits of retractable ones [5], we are done.///

<u>2.2.11</u> . Example. The converse of the above corollary is not true. Here is a counterexample:

Let $\mathcal{M}od\,\mathcal{H} = \mathcal{A}b$, the category of abelian groups, and let $X = \{0, 1,2\}$ with open sets $\mathcal{O}X = \{\phi, \{0\}, \{0,1\}, \{0,2\}, X\}$ be a topological space. Notice that, any presheaf $F \in \mathcal{A}b\,\mathrm{PreSh}X$ is a sheaf iff the following diagram is a pullback diagram:

$$\begin{tikzcd}
 & FX \arrow[dl] \arrow[dr] & \\
F\{0,1\} \arrow[dr] & & F\{0,2\} \arrow[dl] \\
 & F\{0\} &
\end{tikzcd}$$

where the maps are the restrictions. The stalks are easily checked to be $F_0 = F\{0\}$, $F_1 = F\{0,1\}$ and $F_2 = F\{0,2\}$. We now define two sheaves A and B in $\mathcal{A}b\mathrm{Sh}X$ as follows: Let A be represented by the pullback

$$\begin{tikzcd}
Z_2 \arrow[r] \arrow[d, tail, "i"'] & 0 \arrow[d] \\
Z_4 \arrow[r, "\alpha"] & Z_2
\end{tikzcd}$$

where i is the natural inclusion and α is multiplication by 2 the other maps are trivial. Let B be given by the pullback

$$\begin{tikzcd}
Z_4 \arrow[r] \arrow[d, tail, "j"'] & 0 \arrow[d] \\
Z_4 \oplus Z_2 \arrow[r, "\alpha \times 1_{Z_2}"] & Z_2
\end{tikzcd}$$

with the natural homomorphisms. Let $f : A \rightarrowtail B$ be the

...ly by 1

of purity in $\mathcal{A}b$, that all $f_x : A_x \to B_x$ are pure homomorphisms, for $x \in X$; but $f_X : AX \to BX$, i.e. $Z_2 \xrightarrow{f} Z_4$, is not pure, because Z_4 does not have any non-trivial direct summand, in particular Z_2 is not a summand of Z_4. By Corollary (2.2.9), $f : A \to B$ is not pure.///

<u>2.2.12</u> <u>Definition</u>. *A pure homomorphism $h : A \to B$ is said to be <u>pure-essential</u> iff, for any homomorphism $g : B \to C$, if $g \circ h$ is pure, then g is a monomorphism.*

<u>2.2.13</u> <u>Remark</u>. Note that, this means that if $A \xrightarrow{h} B \xrightarrow{\nu} B/\theta$, with ν the quotient map, is a pure homomorphism for any congruence θ on B, then θ must be trivial. For, if $h : A \to B$ is pure-essential and $\nu \circ h$ is pure, then by essentialness, ν is an embedding, and this shows that $\theta = \Delta$. Conversely, let $A \xrightarrow{f} B \xrightarrow{g} C$ be a pure homomorphism and consider the factorization

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, two heads, "\nu"'] & C \\ & B/\mathrm{Ker}\ g \arrow[ur, tail, "h"'] & \end{tikzcd}$$

Since $h \circ \nu \circ f$ is pure, $\nu \circ f$ is also pure, by Lemma 2.2.5(ii) and then by hypothesis Ker g is trivial, that is, g is a monomorphism. Hence f is essential.///

<u>2.2.14</u> <u>Lemma</u>. *In* $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$, *for any pure homomorphism* $h : A \to B$, *there exists a homomorphism* $f : B \to C$ *such that* $f \circ h$ *is pure-essential.*

Proof. Consider the set of all $\mathcal{M}od(\mathcal{H}, \mathrm{Sh}\mathcal{L})$-congruences θ on B with $\nu \circ h : A \to B/\theta$ a pure homomorphism. This family is not empty, Δ is such a congruence, and union of a chian of such congruences has the same property,. the latter is by Lemma 2.2.5(iii). Hence, there exists a maximal such congruence $\theta_\circ$; which $A \to B/\theta_\circ$ is indeed pure-essential, the latter by maximality of $\theta_\circ$.///

<u>2.2.15</u> <u>Corollary</u>. *If* $\{B_\alpha\}_{\alpha \in I}$ *is an*

updirected family of subalgebras B_α *of* B *in* $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$ *with* $B = \bigcup B_\alpha$ *and each* B_α *is a pure-essential extension of the subalgebra* A *of* B, *then* B *is also a pure-essential extension of* A.

Proof. Since B is a direct limit of pure homomorphisms, it is pure. It remains to show that it is essential. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a pure homomorphism, then $A \xrightarrow{j_\alpha} B_\alpha \xrightarrow{i_\alpha} B \xrightarrow{g} C$ is pure for each $\alpha \in I$, and since the j_α are pure, this implies that the $g \circ i_\alpha$ are pure which then implies that g is a monomorphism. Thus f is essential.///

2.2.16. Definition. *An algebra* A *in* $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$ *is said to be* *pure-irreducible* *iff, for any pure homomorphism* $A \xrightarrow{f} \prod_{\alpha\in I} B_\alpha$ *in* $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$, *there exists some* $\alpha \in I$ *such that* $p_\alpha \circ f$ *is a monomorphism, where* $p_\alpha : \prod_{\alpha\in I} B_\alpha \to B_\alpha$ *is the* α-*th projection.*

is pure-irreducible iff there exists a finite sheaf $F \leq |A[S]|^2$ such that F is not in the kernel of any retraction $A[S] \to A$ but for any non-trivial $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$-congruence θ on A, there exists a homomorphism $h_\theta : A[S] \to A/\theta$ over $\nu_\theta : A \twoheadrightarrow A/\theta$ whose kernel contains F.

This means that F is not solvable in A but it is solvable in any proper quotient of A.

Proof. Let F be any finite subsheaf of $|A[S]|^2$ satisfying the stated condition and $f: A \rightarrowtail \prod_{\alpha\in I} B_\alpha$ a pure homomorphism. Suppose that $\theta_\alpha = \mathrm{Ker}\ p_\alpha \circ f \neq \Delta$ for all $\alpha \in I$. Then, by hypothesis, one has a homomorphism $h_{\theta_\alpha} : A[S] \to A/\theta_\alpha$ over $\nu_{\theta_\alpha} : A \twoheadrightarrow A/\theta_\alpha$ with $F \leq \mathrm{Ker}\ h_{\theta_\alpha}$, for each $\alpha \in I$. But $\mathrm{Ker}\ h_{\theta_\alpha} = \mathrm{Ker}(A[S] \xrightarrow{h_{\theta_\alpha}} A/\theta_\alpha \to B_\alpha)$, and hence, $F \leq \mathrm{Ker}\ h$ for some homomorphism $h : A[S] \to \prod_{\alpha\in I} B_\alpha$ over f. Purity of f implies that there exists a retraction $g : A[S] \to A$ whose kernel contains F, which is a contradiction. Hence, at least one θ_α is trivial which shows that A is pure-irreducible.

Conversely, let A be pure-irreducible, and suppose there does not exist any finite subsheaf F of $|A[S]|^2$ satisfying the stated condition. This implies that, for any finite subsheaf F of $|A[S]|^2$ not contained in the kernel of any retraction $g : A[S] \to A$, there also exists a non-trivial $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$-congruence θ_F such that F is not contained in the kernel of any homomorphism $A[S] \to A/\theta_F$ over A. Now, consider the homomorphism $h : A \to \prod_{\text{all } F} A/\theta_F$ determined by all $\nu_F : A \to A/\theta_F$. We claim that h is a pure homomorphism. To prove this claim, let G be any finite subsheaf of $|A[S]|^2$ which is contained in the kernel of some homomorphism $g : A[S] \to \prod_{\text{all } F} A/\theta_F$ over h. Then G is contained in the kernel of $p_F \circ g$, p_F is the projection, for all such F as above. Hence, G is not one of these F, and thus there exists a retraction $\overline{g} : A[S] \to A$ whose kernel contains G. This implies that h is pure as was claimed. Since A is pure-irreducible, at least one of these congruences must be trivial which then is a contradiction.///

2.2.18 Theorem. (Pure Representation Theorem).

Every algebra $A \in \mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$ has a pure embedding into a product of pure-irreducible algebras.

Proof. If A is pure-irreducible there is nothing to prove; if not, let F be a finite subsheaf of $|A[S]|^2$ which is not contained in the kernel of any retraction $A[S] \to A$. Then, for any $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$-congruence θ on A, F is also not contained in the kernel of any homomorphism $A[S] \to A/\theta$ over A. Union of any chain of such congruences has the same property. Thus, there exists a maximal such congruence, say θ_F. Then, clearly $A \to \prod A/\theta_F$ is pure-subdirect representation of A, via pure-irreducible algebras.///

2.3 EQUATIONAL COMPACTNESS

2.3.1 Definition. *In $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$, an algebra A is said to be* ***equationally compact*** *iff, for any subsheaf T of $|A[S]|^2$, T is in the kernel of some retraction $A[S] \to A$ whenever this holds for all finite subsheaves of T.*

<u>2.3.2</u> <u>Proposition</u>. *In $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$, for an algebra A, the following are equivalent.*

(i) A is equationally compact.

(ii) Every pure homomorphism $A \to B$ is retractable.

(iii) A is pure-injective.

(iv) A has no proper pure-essential extension.

Proof. (i) $\Rightarrow$ (ii) Let $h : A \to B$ be a pure homomorphism and $f : A[S] \twoheadrightarrow B$ any epimorphism over h. Note that, for any finite subsheaf F of the kernel of f, there exists a retraction $A[S] \to A$ whose kernel contains F, this is because h is pure. Since A is equationally compact, this implies that Ker f is contained in the kernel of some retraction $g : A[S] \to A$. This, by the homomorphism decomposition theorem, provides us with a factorization.

$$\begin{tikzcd} A[S] \arrow[r, two heads, "f"] \arrow[d, "g"'] & B \arrow[dl, dashed, "k"] \\ A & \end{tikzcd}$$

that is, $k \circ f = g$. Now, $k \circ h = k \circ f | A = g | A = 1_A$, showing that h is retractable.

(ii) ⇒ (iii) is clear. (iii) ⇒ (iv): Let $h : A \to B$ be a pure-essential homomorphism. Then by (iii), there exists some retraction $f : B \to A$ which then, by essentialness of h, is a monomorphism. Hence, $B \simeq A$.

(iv) ⇒ (i): Given $T \leq |A[S]|^2$ with a retraction $h_F : A[S] \to A$ whose kernel contains F, for each finite subsheaf F of T. Then, the $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$-congruences θ_F on $A[S]$ generated by these F form an updirected set, and

$\theta = \bigvee_{\text{all } F} \theta_F$ is a $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$-congruence

containing T. Then, all $A \to A[S]/\theta_F$ are retractable and $A \xrightarrow{\nu|A} A[S]/\theta = \xrightarrow[\text{all } F]{\ell im} (A \to A[S]/\theta_F)$. Hence $\nu|A$

is a pure homomorphism, by Proposition (2.2.6). Continue $\nu|A$ to a pure-essential homomorphism

$$A \xrightarrow{\nu|A} A[S]/\theta \xrightarrow{g} B.$$ Then by (iv), $g \circ \nu|A$ is an isomorphism, and then $(g \circ \nu|A)^{-1} \circ g \circ \nu$ is a retraction whose kernel contains T. This shows that A is equationally compact.///

2.3.3 Corollary. *Absolute retracts in*

$\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$ are equationally compact.///

<u>2.3.4</u> <u>Corollary</u>. *Products and retracts of equationally compact algebras in $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$ are again equationally compact.///*

<u>2.3.5</u> <u>Corollary</u>. *In $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$, maximal pure-essential extensions are equationally compact.*

Proof. Let $i : A \rightarrowtail B$ be a maximal pure essential extension and $f : B \rightarrowtail C$ be any pure homomorphism. Continue $f \circ i$ to a pure-essential homomorphism $g \circ f \circ i : A \to D$. Then, maximality of B implies that $g \circ f$ is an isomorphism, which then implies that f is retractable. This shows that B is equationally compact.///

<u>2.3.6</u> <u>Corollary</u>. *Any right adjoint functor between quasi-equational classes of sheaves of algebras on noetherian locales preserves equational compactness.*

Proof. This is because equationally compact algebras are pure injectives, by Proposition 2.3.2 (iii), and that, any left adjoint functor preserves pure homomorphisms. Then by the properties of adjointness, the assertion easily follows.///

2.3.7 Definition. *An equationally compact, pure-essential extension of an algebra A, in $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$ is called an* *equational compact hull of A*.

2.3.8 Proposition. *In $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$, for any algebra A, the following are equivalent.*

(i) A has an equationally compact hull.

(ii) A has a pure embedding into an equationally compact algebra.

(iii) A has, up to isomorphism, only a set of pure-essential extensions.

Proof. (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii): Since an equationally compact algebra is pure-injective, then, by essentialness, every pure-essential extension of A is embedded in any equationally compact pure-extension of A. Thus, there is only a set of pure-essential extensions of A.

(iii) $\Rightarrow$ (i): Since the union of any chain of pure-essential extensions of A is again a pure-essential extension of A, (iii) implies that there is a maximal such in $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$, which then, by Corollary (2.3.5), is an equationally compact hull of A in $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$.///

<u>2.3.9</u> <u>Corollary</u>. *Equationally compact hulls in $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$, as far as they exist, are essentially unique.///*

<u>2.3.10</u> <u>Remark</u>. As we have promised earlier in this chapter, we now list those results which have been proved without using the assumption of $\mathcal{L}$ being a noetherian locale. For all the other results we do need $\mathcal{L}$ to be noetherian or, as one can see, at least the following consequences of $\mathcal{L}$ being noetherian: on the one hand, Proposition (2.1.7) and on the other that, for a direct limit $A = \varinjlim_{\alpha \in I} A\alpha$ in $\mathcal{M}od(\mathcal{H}, Sh\mathcal{L})$, with the limit maps $g_\alpha : A_\alpha \to A$ and diagram maps $g_{\alpha\beta} : A_\alpha \to A_\beta$ (for $\beta \geq \alpha$), $AU = (\varinjlim A_\alpha)U = \varinjlim A_\alpha U$, for all $U \in \mathcal{L}$, and especially the consequence of this that

$$\mathrm{Ker}\ g_\alpha = \bigcup_{\beta \geq \alpha} \mathrm{Ker}\ g_{\alpha\beta}, \text{ for all } \alpha \in I.$$

Here is the list of results we could do without $\mathcal{L}$ being noetherian.

(1) Remark (2.2.4); which actually could be avoided by letting h be a monomorphism in the very definition of a pure homomorphism.

(2) Lemma (2.2.5); in which the assertions (i) and (ii) are always true, but, as one can see, to prove (iii) we do need $\mathcal{L}$ to be noetherian.

(3) Remark (2.2.13), Lemma (2.2.17) and also in Proposition (2.3.2), the implications (i) $\Rightarrow$(ii) $\Rightarrow$(iii) $\Rightarrow$ (iv) are always true, but to prove (iv) $\Rightarrow$ (i) one does need $\mathcal{L}$ to be noetherian.

(4) Finally, in the proposition (2.3.8), the implication (i) $\Rightarrow$(ii) is clearly always true.///

CHAPTER 3

BIMORPHISMS AND TENSOR PRODUCTS

For a topos $\mathbb{E}$, let $\mathbb{K}$ be a full subcategory of $\mathcal{A}lg(\tau)\mathbb{E}$ which is hereditary and closed under exponentiations.

The aim of this chapter is to consider the notion of bimorphisms for algebras in a topos and, in particular, to study the relationship between tensor products, Universal bimorphisms and functional internal hom-functor, leading to Proposition (3.3.2), which is the counterpart of a similar result for categories of algebras in Ens.

In section (4), we introduce an internal notion of tensor products and Universal bimorphisms, and prove a similar result, Proposition (3.4.5), to Proposition (3.3.2). Later, in Section (5), we show for $\mathbb{E} = Sh\mathcal{L}$ that these internal notions are same as the usual ones.

Also, in Section (5), we deal with the existence of Universal bimorphisms, and prove, in Proposition (3.5.3), that for a Grothendieck topos $\mathbf{E}$, if $\mathcal{A} \subseteq \mathcal{A}lg(\tau)$ has a Universal bimorphism functor, then so does any reflective subcategory of $\mathcal{A}_{\mathbf{E}}$.

Finally, in Section (6), we show that if $\mathcal{A}$ has a functional hom-functor, then so does $\mathcal{A}Sh\,\mathcal{L}$.

Regarding the notion of bimorphisms and tensor products for the case of algebras in Ens, the reader is referred to [9].

3.1 BIMORPHISMS

3.1.1 Definition. *A morphism $f : |A| \times |B| \to |C|$ (for A, B and C in $\mathbb{K}$) is called a bimorphism iff for each $\lambda \in \Omega$, the diagram:*

$$\begin{array}{ccccccc}
|A|\times|B|^{n_\lambda} & \xrightarrow{\Delta\times 1} & |A|^{n_\lambda}\times|B|^{n_\lambda} & \xrightarrow{\sim} & (|A|\times|B|)^{n_\lambda} & \xrightarrow{f^{n_\lambda}} & |C|^{n_\lambda} \\
\downarrow{\scriptstyle 1\times\lambda_B} & & & & & & \downarrow{\scriptstyle \lambda_C} \\
|A|\times|B| & & & \xrightarrow{f} & & & |C|
\end{array}$$

is commutative, and, analogously, for A and B reversed.

3.1.2 **Proposition.** *For a morphism* $f : |A| \times |B| \to |C|$, *the following are equivalent:*

(i) f *is a bimorphism.*

(ii) *The exponential adjunction* $\bar{f}$ *of* f, *that is*

$$\frac{|A| \times |B| \xrightarrow{f} |C|}{|A| \xrightarrow{\bar{f}} |C|^{|B|}}$$

factors through $i_{BC} : [B,C] \to |C|^{|B|}$ *in* $\mathbf{E}$

and, analogously, for A and B reversed.

(iii) *The exponential adjunctions*

$$|A| \xrightarrow{\bar{f}} |C|^{|B|} \quad \text{and} \quad |B| \xrightarrow{\overline{f\sigma}} |C|^{|A|}$$

of f *and* $f\sigma$ *respectively, for*

$\sigma : |B| \times |A| \simeq |A| \times |B|$ *the natural isomorphism, are underlying maps of homomorphisms.*

(iv) $\bar{f} : |A| \to |C|^{|B|}$ *is the underlying map of a homomorphism and factors through*

$$i_{BC} : [B,C] \rightarrowtail |C|^{|B|}.$$

(v) Reversing A and B in (iv).

Proof. (i) $\Leftrightarrow$ (ii) Consider the diagram:

$$\begin{tikzcd}
 & & & & {|A|\times|B|} \arrow[d, "f\times 1"] \arrow[ddr, "f"] & \\
 & & & & {|C|^{|B|}\times|B|} \arrow[dr, "ev"] & \\
{|A|\times|B|^{n_\lambda}} \arrow[uurrrr, "1\times\lambda_B"] \arrow[r, "\bar{f}\times 1"] \arrow[ddrr, "\Delta\times 1"'] & {|C|^{|B|}\times|B|^{n_\lambda}} \arrow[urrr, "1\times\lambda B"] \arrow[drr, "\prod_{i=1}^{n_\lambda}(1\times pr_i)"] \arrow[d, phantom, "(1)"] & & & & {|C|} \\
 & {} & & {(|C|^{|B|}\times|B|)^{n_\lambda}} \arrow[r, "(ev)^{n_\lambda}"] & {|C|^{n_\lambda}} \arrow[ur, "\lambda_C"] & \\
 & & {|A|^{n_\lambda}\times|B|^{n_\lambda}} \arrow[rr, "\sim"] & & {(|A|\times|B|)^{n_\lambda}} \arrow[ul, "(\bar{f}\times 1)^{n_\lambda}"] \arrow[u, "f^{n_\lambda}"'] &
\end{tikzcd}$$

in which, the subdiagram (1) commutes, because if composing it with the projections

$$(|C|^{|B|} \times |B|)^{n_\lambda} \xrightarrow{pr_i} |C|^{|B|} \times |B| \quad (i = 1,\dots,n_\lambda)$$

produces equal maps. The other sub-diagrams, except possibly the inner most square, commute for obvious reasons. Thus the two innermost routes connecting $|A| \times |B|^{n_\lambda}$ to $|C|$ are equal iff the two outer routes joining them are equal. By reversing A and B, we get the same result as above. Further the equality of the two inner routes means, by the definition of [B,C] and the following epi-mono factorization,

$$\begin{tikzcd} |A| \times |B|^{n_\lambda} \arrow[rr, "\bar{f} \times 1"] \arrow[dr, two heads, "u \times 1"'] & & |C|^{|B|} \times |B|^{n} \\ & X \times |B| \arrow[ur, tail, "v \times 1"'] & \end{tikzcd}$$,

the factorization of $X \overset{v}{\rightarrowtail} |C|^{|B|}$ through

$i_{BC} : [BC] \rightarrowtail |C|^{|B|}$, and thus the factorization of $\bar{f}$ through i_{BC}. Thus, we have proved the assertion (i) $\Leftrightarrow$ (ii).

(i) $\Leftrightarrow$ (iii) To prove this claim, we consider the following diagram, for each $\lambda \in \Omega$:

$$
\begin{tikzcd}
{|A|^{n_\lambda}\times|B|} \arrow[r, "1\times\Delta"] \arrow[ddd, "\lambda_A\times 1"'] & {|A|^{n_\lambda}\times|B|^{n_\lambda}} \arrow[r, dash, "\sim"] & {(|A|\times|B|)^{n_\lambda}} \arrow[r, "f^{n_\lambda}"] & {|C|^{n_\lambda}} \arrow[ddd, "\lambda_C"] \\
 & {|A|^{n_\lambda}\times|B|} \arrow[ul, "1"'] \arrow[r, "\bar{f}^{n_\lambda}\times 1"] \arrow[d, "\lambda_A\times 1"] & {(|C|^{n_\lambda})^{|B|}\times|B|} \arrow[ur, "(ev)^{n_\lambda}"] \arrow[d, "\lambda_C^{|B|}\times 1"] & \\
 & {|A|\times|B|} \arrow[r, "\bar{f}\times 1"] \arrow[dl, "1"] & {|C|^{|B|}\times|B|} \arrow[dr, "ev"] & \\
{|A|\times|B|} \arrow[rrr, "f"] & & & {|C|}
\end{tikzcd}
$$

By a similar argument as above, we can show that all the subdiagrams in the above diagram are commutative, except possibly the middle one. Thus, we conclude that the two inner routes joining $|A|^{n_\lambda} \times |B|$ to $|C|$ are equal iff the two outer routes connecting them are equal.

Observe that the outer square is the one given in the definition of a bimorphism, and, by exponential adjointness; the equality of the two inner routes means the commutativity of the following diagram (which is the diagram required to prove $\overline{f}$ is a homomorphism):

$$\begin{array}{ccc} |A|^{n_\lambda} & \xrightarrow{\overline{f}^{n_\lambda}} & (|C|^{|B|})^{n_\lambda} \\ \Big\downarrow \lambda & & \Big\downarrow \lambda \\ |A| & \xrightarrow{\overline{f}} & |C|^{|B|} \end{array}$$

Also, the same result follows by reversing A and B. This proves the equivalence of (i) and (iii).

(i) ⇒ (iv): Let f be a bimorphism, by (i) ⟺ (iii), we get that $\overline{f} : |A| \to |C|^{|B|}$, the exponential adjunction to f, is the underlying map of a homomorphism, and, by (i) ⟺ (ii), we get the following required factorization

$$\begin{tikzcd} {|A|} \arrow[rr, "\overline{f}"] \arrow[dr, dashed, "f^{\#}"'] & & {|C|^{|B|}} \\ & {[B,C]} \arrow[ur, hook, "i_{BC}"'] & \end{tikzcd}$$

(iv) ⇒ (i): By the fact that $\overline{f}$ is the underlying map of a homomorphism, and arguing the same way as in the proof of (i) ⟺ (iii), we get one of the two commutative diagrams in the definition of a bimorphism. The factorization of $\overline{f}$ through i_{BC} and using the diagram given in the definition of [B,C] provides us with the other required commutative diagram, showing f is a bimorphism.

(i) ⟺ (v) This follows from (i) ⟺ (iv), by reversing A and B in the proof of (i) ⟺ (iv).///

3.1.3 The notion of bimorphism gives rise to the following two functors, one Ens-valued and the other $\mathbb{E}$-valued:

$$\text{(I)} \qquad \mathrm{BIM} : \mathbb{K}^* \times \mathbb{K}^* \times \mathbb{K} \to \mathrm{Ens}$$

$$\text{(II)} \qquad {}^{\ulcorner}[-,-,-] : \mathbb{K}^* \times \mathbb{K}^* \times \mathbb{K} \to \mathbb{E}$$

be defined as follows:

For A, B and C in $\mathbb{K}$, let BIM(A,B,C) be the set of all bimorphisms $f : |A| \times |B| \to |C|$, and for morphisms $A' \xrightarrow{u} A$, $B' \xrightarrow{v} B$ and $C \xrightarrow{w} C'$, let $f \rightsquigarrow |w| f(|u| \times |v|)$ be

$$\mathrm{BIM}(u,v,w) : \mathrm{BIM}(A,B,C) \to \mathrm{BIM}(A',B',C')$$

To see that BIM is actually a functor, we first show that, for a bimorphism $f : |A| \times |B| \to |C|$, the morphism

$$|w| f(|u| \times |v|) : |A'| \times |B'| \to |C'|$$

is a bimorphism. To do so, consider the following diagram, for each $\lambda \in \Omega$:

$$
\begin{tikzcd}
{|A'|\times|B'|^{n_\lambda}} \arrow[r, "\Delta\times 1"] \arrow[dr, "|u|\times|v|^{n_\lambda}"] \arrow[dd, "1\times\lambda"'] & {|A'|^{n_\lambda}\times|B'|^{n_\lambda}} \arrow[r, phantom, "\simeq"] \arrow[dr, "|u|^{n_\lambda}\times|v|^{n_\lambda}"] & {(|A'|\times|B'|)^{n_\lambda}} \arrow[rr, "f^{n_\lambda}\cdot(|u|\times|v|)"] \arrow[dr, "(|u|\times|v|)^{n_\lambda}"] & & {|C|^{n_\lambda}} \arrow[r, "|w|^{n_\lambda}"] \arrow[d, "1"] & {|C'|^{n_\lambda}} \arrow[d, "1"] \\
& {|A|\times|B|^{n_\lambda}} \arrow[r, "\Delta\times 1"] \arrow[d, "1\times\lambda"] & {|A|^{n_\lambda}\times|B|^{n_\lambda}} \arrow[r, phantom, "\simeq"] & {(|A|\times|B|)^{n_\lambda}} \arrow[r, "f^{n_\lambda}"] & {|C|^{n_\lambda}} \arrow[r, "|w|^{n_\lambda}"] \arrow[d, "\lambda_C"'] & {|C'|^{n_\lambda}} \arrow[d, "\lambda_C"'] \\
{|A'|\times|B'|} \arrow[r, "|u|\times|v|"] & {|A|\times|B|} \arrow[rrr, "f"] & & & {|C|} \arrow[r, "|w|"] & {|C'|}
\end{tikzcd}
$$

Again, all the subdiagrams in the above diagram are commutative for abvious reasons and by the fact that f is a bimorphism and u, v and w are homomorphisms of algebras. Thus, the outer diagram, which is one of the two diagrams needed in proving that the morphism $|w|f(|u|\times|v|)$ is a bimorphism, commutes. By reversing A and B we get the other diagram, and hence conclude that $|w|f(|u|\times|v|)$ is a bimorphism. The functoriality of BIM can easily be checked.

Next, we define [-,-,-] as follows:

Let A,B and C be in $\mathbb{K}$. We define [A,B,C] to be the largest subobject of $|C|^{|A|\times|B|}$ such that, for each $\lambda \in \Omega$, the following diagram is an equalizer diagram:

$$\begin{tikzcd}
 & & {|C|^{|A|\times|B|}\times|A|\times|B|} \arrow[drr, "ev"] & & \\
{[A,B,C]\times|A|^{n_\lambda}\times|B|} \arrow[r, tail, "i\times1\times1"] & {|C|^{|A|\times|B|}\times|A|^{n_\lambda}\times|B|} \arrow[ur, "1\times\lambda_A\times1"] \arrow[dr, "\prod(1\times pr_i\times1)"'] & & & {|C|} \\
 & & {(|C|^{|A|\times|B|}\times|A|\times|B|} \arrow[r, "n\ (ev)^{n_\lambda}"] & {|C|^{n_\lambda}} \arrow[ur, "\lambda_C"'] &
\end{tikzcd}$$

and,analogously, for A and B reversed. The effect on maps $A' \xrightarrow{u} A$, $B' \xrightarrow{v} B$ and $C \xrightarrow{w} C'$ is provided by the commutativity of the following diagram:

$$
\begin{tikzcd}
{[A,B,C]\times|A|^{n_\lambda}\times|B|} \arrow[r, "i\times1\times1"] & {|C|^{|A|\times|B|}\times|A|^{n_\lambda}\times|B|} \arrow[rrr] & & & {|C|^{n_\lambda}} \arrow[dl, "|w|^{n_\lambda}"'] \arrow[dddd] \\
& & & {|C|^{n_\lambda}} \arrow[d, "\lambda_{C'}"] & \\
{[A,B,C]\times|A'|^{n_\lambda}\times|B'|} \arrow[u, "1\times|u|\times|v|"] \arrow[dd, "1\times|u|\times|v|"'] \arrow[r, "i\times1\times1"] & {|C|^{|A|\times|B|}\times|A'|^{n_\lambda}\times|B|} \arrow[u, "1\times|u|\times|v|"'] \arrow[d, "1\times\lambda\times1"'] \arrow[dd, "1\times|u|\times|v|", shift left=6] \arrow[r, "|w|^{|u|\times|v|}\times1\times1"] & {|C|^{|A|\times|B|}\times|A|^{n_\lambda}\times|B|} \arrow[ur] \arrow[d] & {|C|} & \\
& {|C|^{|A|\times|B|}\times|A'|\times|B'|} \arrow[r, "|w|^{|u|\times|v|}\times1\times1"] & {|C|^{|A|\times|B|}\times|A|\times|B|} \arrow[ur, "ev"] & & \\
{[A,B,C]\times|A|^{n_\lambda}\times|B|} \arrow[r, "i\times1\times1"] & {|C|^{|A|\times|B|}\times|A|^{n_\lambda}\times|B|} \arrow[r, "1\times\lambda\times1"] & {|C|^{|A|\times|B|}\times|A|\times|B|} \arrow[rr, "ev"] & & {|C|} \arrow[uul, "\lambda_C"']
\end{tikzcd}
$$

The commutativity of the above diagram is shown by the same argument used to show the commutativity of a similar diagram in (0.3.1) used to define $[f,g] : [A,B] \to [C,D]$ and then, by the definition of $[A',B',C']$, we get a unique morphism

$$[u,v,w] : [A,B,C] \to [A',B',C'] .$$

It is not hard to check that $[-,-,-]$ is in fact a functor.///

<u>3.1.4</u> <u>Proposition</u>. *For A,B and C in* $\mathbb{K}$, *$[A,B,C]$ is isomorphic to the following pullback:*

$$\begin{tikzcd}
P \arrow[r, "s"] \arrow[d, "t"'] & {[A,C^{|B|}]} \arrow[d, "i"] \\
{[B,C]^{|A|}} \arrow[r, "(i_{BC})^{|A|}"'] & (|C|^{|B|})^{|A|}
\end{tikzcd}$$

Proof. We shall first find two morphisms $s : [A,B,C] \to [A,C^{|B|}]$ and $t : [A,B,C] \to [B,C]^{|A|}$ to replace P by $[A,B,C]$ in the above diagram. To define s, consider the following diagram:

$$
\begin{tikzcd}
 & & {|C|^{|B|^{|A|}} \times |A| \times |B|} \arrow[d, "ev"] & \\
{[A,B,C] \times |A|^{n_\lambda} \times |B|} \arrow[r, "i_{ABC}\times 1\times 1"] & {|C|^{|B|^{|A|}} \times |A|^{n_\lambda} \times |B|} \arrow[ur, "1\times\lambda_A\times 1"] \arrow[d, "(1\times p_{r_i})\times 1"] \arrow[ddl, "\prod(1\times p_{r_i}\times 1)"'] & {|C|^{|B|} \times |B|} \arrow[r, "ev"] & {|C|} \\
 & {(|C|^{|B|^{|A|}} \times |A|)^{n_\lambda} \times |B|} \arrow[r, "ev\times 1"] & {|C|^{n_\lambda |B|} \times |B|} \arrow[r, "ev"] \arrow[u, "\lambda_C^{n_\lambda}\times 1"] & {|C|^{n_\lambda}} \arrow[u, "\lambda_C"'] \\
{(|C|^{|B|^{|A|}} \times |A| \times |B|)^{n_\lambda}} \arrow[ur, "1\times p_r"] \arrow[rr, "(ev)^{n_\lambda}"] & & {(|C|^{|B|} \times |B|)^{n_\lambda}} \arrow[u, "1\times p_r"] \arrow[ur, "(ev)^{n_\lambda}"'] &
\end{tikzcd}
$$

Again, arguing the same way as we have been doing and using the definition of $[A,B,C]$, we get that the two inner routes joining $[A,B,C] \times |A|^{n_\lambda} \times |B|$ to $|C|$ are equal. Hence, as before, the definition of $[A,C^{|B|}]$

and the existence of epi-mono factorization in $\mathbb{E}$ provides us with a map

$$s : [A,B,C] \to [A,C^{|B|}].$$

To define the map t, we notice that $[B,C]^{|A|}$ is the largest subobject of $(|C|^{|B|})^{|A|}$ making the following diagram an equalizer diagram, for each $\lambda \in \Omega$:

$$\begin{tikzcd}
 & & & {|C|^{|B|}\times|B|} \arrow[dr, "ev"] & \\
{[B,C]^{|A|}\times|A|\times|B|} \arrow[r, "ev\times 1"] & {[B,C]\times|B|^{n_\lambda}} \arrow[r, "i_{BC}\times 1"] & {|C|^{|B|}\times|B|^{n_\lambda}} \arrow[ur, "1\times\lambda_B"] \arrow[dr] & & {|C|} \\
 & & & {|C|^{n_\lambda}} \arrow[ur, "\lambda_C"'] &
\end{tikzcd}$$

Now, consider the following diagram:

$$\begin{tikzcd}
 & & {(|C|^{|B|})^{|A|}\times|A|\times|B|} \arrow[d] & \\
 & (2) & {|C|^{|B|}\times|B|} \arrow[dr] & \\
{[A,B,C]\times|A|\times|B|^{n_\lambda}} \arrow[uurr, "i\times1\times\lambda"] \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & {|C|^{|B|}\times|B|^{n_\lambda}} \arrow[ur] \arrow[d] & & {|C|} \\
{((|C|^{|B|})^{|A|}\times|A|\times|B|)^{n_\lambda}} \arrow[r] & {(|C|^{|B|}\times|B|)^{n_\lambda}} \arrow[r] & {|C|^{n_\lambda}} \arrow[ur] &
\end{tikzcd}$$

(the maps are clear from the previous diagram.)

The subdiagrams (1) and (2) are commutative for obvious reasons, and also by the definition of $[A,B,C]$, we get that the two outer routes connecting $[A,B,C]\times|A|\times|B|^{n_\lambda}$ to $|C|$ are equal. Hence, the two inner routes joining them are also equal. Thus, by the definition of an equalizer, we are provided with a map $t : [A,B,C] \to [B,C]^{|A|}$. Now, by how s and t were defined we can easily see that the following diagram is commutative:

$$\begin{tikzcd}
{[A,B,C]} \arrow[r, "s"] \arrow[d, "t"'] & {[A,C^{|B|}]} \arrow[d, tail, "i"] \\
{[B,C]^A} \arrow[r, tail, "i^{|A|}"'] & (|C|^{|B|})^{|A|}
\end{tikzcd}$$

To show that the above diagram is a pullback, let there be an object $T \in \mathbf{E}$ and maps α and β making the following diagram commutative:

$$\begin{tikzcd} T \arrow[r, "\alpha"] \arrow[d, "\beta"'] & {[A, C^{|B|}]} \arrow[d, tail] \\ {[B,C]^{|A|}} \arrow[r] & (|C|^{|B|})^{|A|} \end{tikzcd}$$

One can argue the same way as before and show that the two maps $T \xrightarrow{\alpha} [A, C^{|B|}] \to (|C|^{|B|})^{|A|}$ and $T \xrightarrow{\beta} [B,C]^{|A|} \longrightarrow (|C|^{|B|})^{|A|}$ satisfy the two conditions required in the definition of $[A,B,C]$ and hence they both factors through $[A,B,C] \rightarrowtail (|C|^{|B|})^{|A|}$, which is required in the definition of a pullback diagram. Thus the proposition is proved.///

Next, we show how the above two functors BIM and [-,-,-] are related to each other, which yields a similar result to the case of [,].

3.1.5 Lemma. *A morphism* $|A| \times |B| \xrightarrow{f} |C|$

(for A, B and C in $\mathbb{K}$) is a bimorphism iff the exponential adjunction $\overline{f}$, given by

$$\frac{\mathbb{1} \times |A| \times |B| \simeq |A| \times |B| \xrightarrow{f} |C|}{\mathbb{1} \xrightarrow{\overline{f}} |C|^{|A| \times |B|}},$$

factors through $[A,B,C] \xrightarrow{i_{ABC}} |C|^{|A| \times |B|}$.

<u>*Proof*</u>. The proof is similar to the proof of Lemma (0.3.4). Let $\lambda \in \Omega$ be arbitrary. Consider the following diagram:

$$\begin{tikzcd}
 & & {|A|^{n_\lambda} \times |B|^{n_\lambda}} \arrow[rr, phantom, "\simeq"] & & {(|A| \times |B|)^{n_\lambda}} \arrow[d] \\
 & & {(|C|^{|A|\times|B|} \times |A| \times |B|)^{n_\lambda}} \arrow[rr] \arrow[u, phantom, "(3)"] & & {|C|^{n_\lambda}} \arrow[d] \\
{|A|^{n_\lambda} \times |B|} \arrow[uurr] \arrow[r, "\simeq"] \arrow[ddrrrr] & {\mathbb{1} \times |A|^{n_\lambda} \times |B|} \arrow[r] \arrow[d, phantom, "(1)"] & {|C|^{|A|\times|B|} \times |A|^{n_\lambda} \times |B|} \arrow[u] \arrow[dr] & & {|C|} \\
 & {} & & {|C|^{|A|\times|B|} \times |A| \times |B|} \arrow[ur] \arrow[r, phantom, "(2)"] & {} \\
 & & & & {|A| \times |B|} \arrow[ul] \arrow[uu]
\end{tikzcd}$$

(the maps are clear from the previous diagrams.)

Again as before, one can easily show that the subdiagrams (1), (2) and (3) are commutative, and thus the two inner routes connecting $|A|^{n_\lambda} \times |B|$ to $|C|$ are equal, which is one of the diagrams required to define $[A,B,C]$, iff the two outer routes are equal, which is one of the two diagrams required to define a bimorphism. By reversing A and B we get the other two corresponding diagrams, which proves the lemma.///

This lemma proves the following proposition:

<u>3.1.6</u> <u>Proposition</u>. *For A,B and C in $\mathbb{K}$, the map $\mathbb{K}(\mathbb{I},[A,B,C]) \longrightarrow BIM(A,B,C)$ given by $g \rightsquigarrow g^{\#}$ is an isomorphism, where $g^{\#}$ is given by the following:*

$$\frac{\dfrac{\mathbb{I} \xrightarrow{g} [A,B,C] \xrightarrow{i_{ABC}} |C|^{|A| \times |B|}}{\mathbb{I} \times |A| \times |B| \xrightarrow{\bar{g}} |C|}}{g^{\#} : |A| \times |B| \xrightarrow{\sim} \mathbb{I} \times |A| \times |B| \xrightarrow{\bar{g}} |C|} \quad ///$$

<u>3.1.7</u> <u>Definition</u>. *A bimorphism $f : |A| \times |B| \to |C|$ (for A,B and C in $\mathbb{K}$) is called <u>universal</u> iff any other bimorphism $g : |A| \times |B| \to |D| (D \in \mathbb{K})$ factors through*

a unique $|h| : |C| \to |D|$, *for* $h : C \to D$ *in* $\mathbb{K}$ *(i.e.,* $g = |h|f$*).*

Notice that the notion of a universal bimorphism depends on the category $\mathbb{K}$.

If the universal bimorphisms exist for all pairs of objects of $\mathbb{K}$, they determine a functor,

$$M : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$$

called the *functor of universal bimorphism,* defined by taking $|M(A,B)|$ to be the codomain of the universal bimorphism $\beta_{AB} : |A| \times |B| \to |M(A,B)|$, for each pair A and B in $\mathbb{K}$, and the effect on maps $f : A \to A'$, and $g : B \to B'$ is denoted by $M(u,v) : M(A,B) \to M(A',B')$ and is defined to be the unique map obtained by the following factorization, provided by the definition of a universal bimorphism:

$$\begin{tikzcd} {|A| \times |B|} \arrow[r, "\beta_{AB}"] \arrow[d, "|f| \times |g|"] & {|M(A,B)|} \arrow[d, dashed, "|M(f,g)|"] \\ {|A'| \times |B'|} \arrow[r, "\beta_{A'B'}"] & {|M(A',B')|} \end{tikzcd}$$

Using the definition of a universal bimorphism, it is easily checked that M is in fact a functor. Moreover, by the commutativity of the above diagram, the correspondence $\beta : |-| \times |-| \to |M(-,-)|$ with components β_{AB} (for A and B in $\mathbb{K}$) is indeed a natural transformation between the two functors. Also, there is a natural equivalence $\mathbb{K}(M(A,B),C) \xrightarrow{\sim} BIM(A,B,C)$ given by $h \rightsquigarrow |h|\beta_{AB}$, which says that the functor $BIM(A,B,-)$ is representable. The universal bimorphisms are entirely determined by the above natural transformation; β_{AB} being the map corresponding to the identity morphism of $C = M(A,B)$./

3.2 FUNCTIONAL INTERNAL HOM-FUNCTOR

3.2.1 Definition. *A functor $H : \mathbb{K}^* \times \mathbb{K} \to \mathbb{K}$ will be called an internal hom-functor iff, $|H(A,B)| = [A,B]$ for all A and B in $\mathbb{K}$; that is to say the following diagram is commutative:*

$$\begin{tikzcd} \mathbb{K}^* \times \mathbb{K} \arrow[rr, "H"] \arrow[dr, "{[\,,\,]}"'] & & \mathbb{K} \arrow[dl, "|-|"] \\ & \mathbb{E} & \end{tikzcd}$$

<u>3.2.2</u> <u>Definition</u>. *An internal hom-functor H of $\mathbb{K}$ is called* <u>*functional*</u> *iff $H(A,B)$ is a subalgebra of the algebra $B^{|A|}$, that is, the inclusion morphism $i_{AB} : [A,B] \rightarrowtail |B|^{|A|}$ is the underlying map of a homomorphism*

$$i_{AB} : H(A,B) \rightarrowtail B^{|A|}$$

(denoted by the same letters).

This notion is analogous to that in [9].

<u>3.2.3</u> <u>Proposition</u>. *For an internal hom-functor $H : \mathbb{K}^* \times \mathbb{K} \to \mathbb{K}$, the following are equivalent:*

(i) *H is functional.*

(ii) *For any A, B and C in $\mathbb{K}$, there is a natural isomorphism:*

$$\mathbb{K}(A,H(B,C)) \xrightarrow{\Phi} \mathrm{BIM}(A,B,C)$$

given by $f \rightsquigarrow \overline{f}$, where $\overline{f}$ is defined, by exponential adjointness as follows:

$$\frac{|A| \xrightarrow{|f|} [B,C] \overset{i_{BC}}{\rightarrowtail} |C|^{|B|}}{|A| \times |B| \xrightarrow{\overline{f}} |C|}$$

(iii) *For any A and B in $\mathbb{K}$, there exists a homomorphism $A \xrightarrow{e_{AB}} H(H(A,B),B)$ such that the exponential adjunction $\bar{e}_{AB}$, given by*

$$\frac{|A| \xrightarrow{|e_{AB}|} [H(A,B),B] \rightarrowtail |B|^{[A,B]}}{|A| \times [A,B] \xrightarrow{\bar{e}_{AB}} |B|}$$

is the evaluation map.

(iv) *The evaluation map $ev : [A,B] \times |A| \to |B|$ is a bimorphism.*

Proof. (i)$\Longleftrightarrow$(ii) Let H be a functional internal hom-functor and A,B and C belong to $\mathbb{K}$. Since i_{BC} is the underlying map of a homomorphism, for any homomorphism $f : A \to H(B,C)$, the composite $i_{BC} \circ |f|$ is the underlying map of a homomorphism from A to $C^{|B|}$, and it obviously factors through i_{BC}. Thus, by Proposition 3.1.2(iv), $\bar{f} : |A| \times |B| \to |C|$, defined in (ii), is a bimorphism. Now, Φ is clearly a monomorphism, because if $\bar{f} = \bar{g}$, for $f,g : |A| \to [B,C]$, then, by the adjointness isomorphism, $i_{BC} \circ |f| = i_{BC} \circ |g|$, and since i_{BC} is a monomorphism, $|f| = |g|$. To show that Φ is onto, let

$f : |A| \times |B| \to |C|$ be any bimorphism. Then, by the Proposition 3.1.2(iv), the exponential adjunction $\overline{f}$ of f,

$$\frac{|A| \times |B| \xrightarrow{f} |C|}{|A| \xrightarrow{\overline{f}} |C|^{|B|}},$$

is the underlying map of a homomorphism and has the following factorization:

$$\begin{tikzcd} {|A|} \arrow[rr, "\overline{f}"] \arrow[dr, dashed, "f^{\#}"'] & & {|C|^{|B|}} \\ & {[B,C]} \arrow[ur, "i_{BC}"'] & \end{tikzcd}$$

where $f^{\#}$ is clearly the underlying map of a homomorphism, because both i_{BC} and $\overline{f}$ are underlying maps of homomorphisms, and also $\Phi\,(f^{\#}) = \overline{f}$. This shows that Φ is an isomorphism.

Conversely, let $\Phi : f \rightsquigarrow \overline{f}$ be a natural isomorphism. Take $A = H(B,C)$, then $\Phi\,(1_A) = \overline{1}_A$ given by,

$$\frac{[B,C] \xrightarrow{1_A} [B,C] \xrightarrow{i_{BC}} |C|^{|B|}}{[B,C] \times |B| \xrightarrow{\overline{1}_A} |C|}$$

is a bimorphism, by 3.1.2(ii). Thus, by 3.1.2(iii), the exponential adjunction i_{BC} of $\overline{I}_A$ is the underlying map of a homomorphism; that is H is functional.

(i) <=> (iv) Let H be functional and A and B be any two algebras in $\mathbb{K}$. Then, $i_{AB} : H(A,B) \to B^{|A|}$ is a homomorphism with the underlying map $i_{AB}:[A,B] \to |B|^{|A|}$, which clearly has a factorization through itself. Thus, by the Proposition 3.1.2(iv), the evaluation map $ev : [A,B] \times |A| \to |B|$ is a bimorphism.

Conversely, if $ev : [A,B] \times |A| \to |B|$ is a bimorphism, then the corresponding adjunction map $i_{AB} : [A,B] \rightarrowtail |B|^{|A|}$ to ev is the underlying map of a homomorphism, by 3.1.2(iii). This shows that H(A,B) is a subalgebra of $B^{|A|}$, for any A and B in $\mathbb{K}$.

(iv) => (iii) Let $ev : [A,B] \times |A| \to |B|$ be a bimorphism. Then, the corresponding adjunction $\overline{ev} : |A| \to |B|^{[A,B]}$ is the underlying map of a homomorphism and factors through $[H(A,B),B] \xrightarrow{i} |B|^{[A,B]}$, by the Proposition 3.1.2(iv). This shows that there exists a map $s : |A| \to [H(A,B),B]$ such that the following diagrma is commutative:

$$\begin{tikzcd} {|A|} \arrow[rr, "\overline{ev}"] \arrow[dr, dashed, "s"'] & & {|B|^{[A,B]}} \\ & {[H(A,B),B]} \arrow[ur, "i_{B(A,B)B}"'] & \end{tikzcd}$$

But, the functionláity of H implies that $i_{H(A,B)B}$ is the underlying map of a homomorphism, and, as proved above, $\overline{ev}$ is the underlying map of a homomorphism, too. Hence, s is the underlying map of a homomorphism $e_{AB} : A \to H(H(A,B),B)$.

(iii) $\Rightarrow$ (iv) This is clear, by 3.1.2(ii).///

<u>3.2.4</u> . <u>Remark.</u> Observe that, for a functional internal hom-functor H, the object [A,B,C] being a pullback of two subalgebras, by Proposition (3.1.4), is itself a subalgebra of $|C|^{|A|\times|B|} \simeq (|C|^{|B|})^{|A|}$ which will be denoted by <u>$\mathbb{B}(A,B,C)$</u>, for A,B and C in $\mathbb{K}$./

<u>3.2.5</u> <u>Proposition</u>. *For a functional internal hom-functor $H : \mathbb{K}^* \times \mathbb{K} \to \mathbb{K}$, we have*

$$H(A,H(B,C)) \simeq B(A,B,C) ,$$

for any A, B and C in $\mathbb{K}$.

Conversely, an internal hom-functor H is functional if

$$[A,H(B,C)] \simeq [A,B,C]$$

for any A, B and C in $\mathbb{K}$.

<u>Proof</u>. ($\Rightarrow$) By the last remark and the Proposition (3.1.4), it is enough to show that, for any A,B and C in $\mathbb{K}$, the following diagram is a pullback:

$$\begin{tikzcd}
{[A,H(B,C)]} \arrow[r, "{[1_A, i_{BC}]}"] \arrow[d, tail, "i"'] \arrow[dr, phantom, "(\mathrm{I})"] & {[A,C^{|B|}]} \arrow[d, tail, "i"] \\
{[B,C]^{|A|}} \arrow[r, "i_{BC}^{|A|}"] & {(|C|^{|B|})^{|A|}}
\end{tikzcd}$$

To show the commutativity of the above diagram, apply the functor ( ) × |A| to the diagram and then form the following diagram:

$$
\begin{tikzcd}
{[B,C]} \arrow[rrr, "i_{BC}"] \arrow[ddd, "1"'] & & & {|C|^{|B|}} \arrow[ddd, "1"] \\
& {[A,H(B,C)]\times|A|} \arrow[ul] \arrow[r] \arrow[d] & {[A,C^{|B|}]\times|A|} \arrow[ur] \arrow[d] & \\
& {([B,C]^{|A|}\times|A|)} \arrow[r] \arrow[dl, "ev"'] & {(|C|^{|B|})^{|A|}\times|A|} \arrow[dr, "ev"] & \\
{[B,C]} \arrow[rrr, "i_{BC}"] & & & {|C|^{|B|}}
\end{tikzcd}
$$

All the subdiagrams commute except possibly the middle square. Thus, the two inner routes connecting $[A,H(B,C)]\times|A|$ to $|C|^{|B|}$ are equal, and hence, by the exponential adjointness isomorphism, we get that the diagram (I) is commutative. Now, let the object $T \in \mathbf{E}$ together with two morphisms α and β make the following diagram commutative:

$$\begin{tikzcd}
T \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\mathrm{II}"] & {[A, C^{|B|}]} \arrow[d] \\
{[B,C]^{|A|}} \arrow[r] & (|C|^{|B|})^{|A|}
\end{tikzcd}$$

We now claim that, for each $\lambda \in \Omega$, the following diagram commutes:

$$\begin{tikzcd}
 & & {[B,C]^{|A|} \times |A|} \arrow[dr, "ev"] & \\
T \times |A|^{n_\lambda} \arrow[r, "\beta\times 1"] & {[B,C]^{|A|} \times |A|^{n_\lambda}} \arrow[ur, "1\times\lambda_A"] \arrow[d, "\prod(1\times P_{r_i})"] & & {[B,C]} \\
 & ({[B,C]^{|A|}} \times |A|)^{n_\lambda} \arrow[r, "(ev)^{n_\lambda}"] & {[B,C]^{n_\lambda}} \arrow[ur, "\lambda_{H(B,C)}"'] &
\end{tikzcd}$$

To prove this, we know that by the definition of

$[A, C^{|B|}]$ the following diagram is commutative:

$$\begin{tikzcd}
 & & & (|C|^{|B|})^{|A|}\times|A| \arrow[dr, "ev"] & \\
T\times|A|^{n_\lambda} \arrow[r, "\alpha\times 1"] & {[A, C^{|B|}]\times|A|^{n_\lambda}} \arrow[r, tail] & (|C|^{|B|})^{|A|}\times|A|^{n_\lambda} \arrow[ur, "1\times\lambda"] \arrow[drr, "ev\circ\prod(1\times p_{r_i})"'] & & |C|^{|B|} \\
 & & & & (|C|^{n_\lambda})^{|B|} \arrow[u, "\lambda"']
\end{tikzcd}$$

This, together with the commutativity of the diagram (II) implies that the following diagram commutes:

$$\begin{tikzcd}
 & & & (|C|^{|B|})^{|A|}\times|A| \arrow[dr, "ev"] & \\
T\times|A|^{n_\lambda} \arrow[r, "\beta\times 1"] & {[B, C]_{\mathfrak{D}}^{|A|}\times|A|^{n_\lambda}} \arrow[r, tail, "i^{|A|}\times 1"] & (|C|^{|B|})^{|A|}\times|A|^{n_\lambda} \arrow[ur, "1\times\lambda"] \arrow[dr, "ev\circ\prod(1\times p_r)"'] & & |C|^{|B|} \\
 & & & (|C|^{n_\lambda})^{|B|} \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

Now, we consider the following diagram:

$$
\begin{tikzcd}
 & & & & {[B,C]^{|A|}\times|A|} \arrow[d] \arrow[ddr] & \\
 & & (1) & & {(|C|^{|B|})^{|A|}\times|A|} \arrow[d] & (3) \\
T\times|A|^{n_\lambda} \arrow[r, "\beta\times 1"] & {[B,C]^{|A|}\times|A|} \arrow[r, "\lambda"] \arrow[uurrr] \arrow[ddr] & (|C|^{|B|})^{|A|}\times|A| \arrow[urr] \arrow[dr] & & |C|^{|B|} & {[B,C]} \arrow[l] \\
 & & (2) & ((|C|^{|B|})^{|A|}\times|A|)^{n_\lambda} & (|C|^{n_\lambda})^{|B|} \arrow[u] & (5) \\
 & & ({[B,C]^{|A|}}\times|A|)^{n_\lambda} \arrow[ur] \arrow[rr] & (4) & {[B,C]^{n_\lambda}} \arrow[u] \arrow[uur] &
\end{tikzcd}
$$

(the maps are clear from the previous diagrams) in which the square (5) commutes, because H is functional, that is i_{BC} is a homomorphism, and all the other subdiagrams (1), (2), (3) and (4) are again commutative for obvious reasons. Thus, the two outer routes joining $T \times |A|^{n_\lambda}$ to $[B,C]$ are equal which is what claimed to be.

Using the above, by the definition of $[B,C]^{|A|}$, we get the following unique factorization:

$$\begin{tikzcd}
T \arrow[r, "\beta"] \arrow[dr, dashed, "\beta^{\#}"'] & {[B,C]^{|A|}} \\
& {[A,H(B,C)]} \arrow[u]
\end{tikzcd}$$

The morphism $\beta^{\#}$ makes the following diagram commutative, too:

$$\begin{tikzcd}
T \arrow[r, "\alpha"] \arrow[dr, "\beta^{\#}"'] & {[A,C^{|B|}]} \\
& {[A,H(B,C)]} \arrow[u]
\end{tikzcd}$$

This is because,

$$T \xrightarrow{\beta^{\#}} [A,H(B,C)] \rightarrowtail [A,C^{|B|}] \rightarrowtail (|C|^{|B|})^{|A|}$$

$$= \quad T \xrightarrow{\beta^{\#}} [A,H(B,C)] \rightarrowtail [B,C]^{|A|} \rightarrowtail (|C|^{|B|})^{|A|}$$

$$= \quad T \xrightarrow{\beta} [B,C]^{|A|} \rightarrowtail (|C|^{|B|})^{|A|}$$

$$= \quad T \xrightarrow{\alpha} [A,C^{|B|}] \rightarrowtail (|C|^{|B|})^{|A|}$$

and since $[A,C^{|B|}] \rightarrowtail (|C|^{|B|})^{|A|}$ is a monomorphism, we get that

$$T \xrightarrow{\beta^{\#}} [A,H(B,C)] \longrightarrow [A,C^{|B|}] = T \xrightarrow{\alpha} [A,C^{|B|}].$$

Thus, the diagram (I) is a pullback, and hence the conclusion.

Conversely, by applying the functor $\mathbb{K}(\mathbb{1},-)$ to diagram (I) and to the similar pullback diagram for $[A,B,C]$, we get the following pullback diagram:

$$\begin{tikzcd}
BIM(A,B,C) \arrow[dr, "\wr\Phi"] \arrow[drr, "s"] \arrow[ddr, "t"'] & & \\
& \mathbb{K}(A,H(B,C)) \arrow[r, tail, "-\circ i_{BC}"] \arrow[d, "|-|"] & \mathbb{K}(A,C^{|B|}) \arrow[d, tail, "|-|"] \\
& \mathbb{E}(|A|,[B,C]) \arrow[r, tail, "-\circ i_{BC}"] & \mathbb{E}(|A|,|C|^{|B|})
\end{tikzcd}$$

with $\Phi : BIM(A,B,C) \simeq \mathbb{K}(A,H(B,C))$ an isomorphism. It remains to show that the isomorphism Φ is the same as that in the Proposition 3.2.3(ii). That is to show that, for a bimorphism $f : |A| \times |B| \to |C|$, Φf is given by the following factorization:

$$\frac{|A| \times |B| \xrightarrow{f} |C|}{|A| \xrightarrow{\overline{f}} |C|^{|B|}}$$

$$\begin{tikzcd}
|A| \arrow[rr, "\overline{f}"] \arrow[dr, dashed, "|\Phi f|"'] & & |C|^{|B|} \\
& |H(B,C)| \arrow[ur, tail] &
\end{tikzcd}$$

But, this is readily checked, by the definition of s in the above commutative diagram.///

We clearly have that $[A,B,C] \simeq [B,A,C]$, and, for a functional internal hom-functor, $B(A,B,C) \simeq B(B,A,C)$. This proves the following corollary:

<u>3.2.6</u> <u>Corollary</u>. *For A,B and C in $\mathbb{K}$ and H a functional internal hom-functor, we have*

$$H(A,H(B,C)) \simeq H(B,H(A,C)).///$$

3.3 TENSOR PRODUCTS

<u>3.3.1</u> <u>Definition</u>. *A <u>tensor multiplication</u> for a functor $H : \mathbb{K}^* \times \mathbb{K} \to \mathbb{K}$ is a functor $T : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ essentially unique if it exists, such that there is a natural equivalence*

$$\mathbb{K}(T(A,B),C) \xrightarrow{\sim} \mathbb{K}(A,H(B,C))$$

for A,B and C in $\mathbb{K}$.

This notion is similar to that in [9].

3.3.2 Proposition. *For a functional internal hom-functor $H : \mathbb{K}^* \times \mathbb{K} \to \mathbb{K}$, a functor $S : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ is a tensor multiplication for H iff S is a functor of universal bimorphisms.*

Proof. This is an immediate consequence of what have been shown before about these conditions in terms of natural isomorphisms, namely, for A,B and C in $\mathbb{K}$,

(T) S is a tensor multiplication for H iff

$$\mathbb{K}(S(A,B),C) \simeq \mathbb{K}(A,H(B,C))$$

(F) H is functional internal hom-functor iff

$$\mathbb{K}(A,H(B,C)) \simeq BIM(A,B,C)$$

(U) S is a universal bimorphism functor iff

$$BIM(A,B,C) \simeq \mathbb{K}(S(A,B),C).///$$

3.4 INTERNAL TENSOR PRODUCTS

3.4.1 Definition. *An internal tensor multiplication for a functor $G : \mathbb{K}^* \times \mathbb{K} \to \mathbb{K}$ is a* *functor IT : $\mathbb{K} \times \mathbb{K} \to \mathbb{K}$, essentially unique if it exists, such that there is a natural equivalence:*

$$[IT(A,B),C] \longrightarrow [A,G(B,C)].$$

<u>3.4.2</u> <u>Remark</u>. If IT is an internal tensor product of G, then

$$\mathbb{E}(\mathbb{1},[IT(A,B),C]) \simeq \mathbb{E}(\mathbb{1},[A,G(B,C)]$$

and hence, by the Proposition (0.3.5), we have

$$\mathbb{K}(IT(A,B),C) \simeq \mathbb{K}(A,G(B,C));$$

This shows that IT is also a tensor product of G.///

<u>3.4.3</u> <u>Definition</u>. *An <u>internal universal</u> <u>bimorphism</u> is a functor IM : $\mathbb{K} \times \mathbb{K} \to \mathbb{K}$, such that there is a natural equivalence*

$$[IM(A,B),C] \longrightarrow [A,B,C]$$

for A,B and C in $\mathbb{K}$.

<u>3.4.4</u> <u>Remark</u>. If IM is an internal universal bimorphism functor, then one has

$$\mathbf{E}(\mathbb{1},[\mathrm{IM}(A,B),C]) \simeq \mathbf{E}(\mathbb{1},[A,B,C]),$$

and hence, by Propositions (0.3.5) and (3.1.6), we have

$$\mathbb{K}(\mathrm{IM}(A,B),C) \simeq \mathrm{BIM}(A,B,C).$$

This shows that IM is also the functor of universal bimorphisms.///

We do not know whether the converse of these two remarks is true in general, but we do know that they are true in the special case of $\mathbf{E} = \mathrm{Sh}\mathcal{L}$, which will be discussed later.

<u>3.4.5</u> <u>Proposition</u>. *For a functional internal hom-functor $H : \mathbb{K}^* \times \mathbb{K} \to \mathbb{K}$, a functor $S : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ is an internal tensor multiplication for H iff S is an internal universal bimorphism functor.*

Proof. This follows from Proposition (3.2.5).///

3.5 EXISTENCE OF UNIVERSAL BIMORPHISMS

Recall that, any hereditary, productive subcategory $\mathcal{A} \subseteq \mathcal{A}lg(\tau)$ has a universal bimorphism functor. In the following, $\mathcal{A}$ will always be such a subcategory of $\mathcal{A}lg(\tau)$.

3.5.1 Lemma. *If $\mathbb{K} \subseteq \mathcal{A}lg(\tau)^{\mathbb{E}}$ has a functor of universal bimorphisms, then so does any reflective subcategory $\mathbb{L} \subseteq \mathbb{K}$.*

Proof. Let M be the functor of universal bimorphisms on $\mathbb{K}$ and $R : \mathbb{K} \to \mathbb{L}$ be the reflection functor, then for A and B in $\mathbb{L}$, $(A,B) \rightsquigarrow R \circ M(A,B)$ defines the functor of Universal bimorphisms on $\mathbb{L}$. ///

3.5.2 Lemma. *If M is the functor of universal bimorphisms of $\mathcal{A} \subseteq \mathcal{A}lg(\tau)$, then $\mathcal{A}^{\mathbb{E}}$ has a functor of universal bimorphisms given by $M(A,B)U = M(AU,BU)$.*

Proof. Since bimorphisms are defined by commutative diagrams, any limit preserving functor preserves them;

and hence a morphism $f : |A| \times |B| \to |C|$ is a bimorphism iff $f_U : |AU| \times |BU| \to |CU|$ is a bimorphism for all $U \in \mathbb{C}$. Using this, the assertion is then easily checked.///

3.5.3 Proposition. *For any Grothendieck topos $\mathbb{E}$, if $\mathcal{A} \subseteq \mathcal{A}lg(\tau)$ has a universal bimorphisms functor, then so does any reflective subcategory $\mathbb{K}$ of $\mathcal{A}\mathbb{E}$.*

Proof. Since $\mathbb{E}$ is a Grothendieck category, $\mathbb{K}$ is then a reflective subcategory of $\mathcal{A}\hat{\mathbb{C}}$ and hence Lemma (3.5.1) and Lemma (3.5.2) proves the conclusion.///

Case $\mathbb{E} = Sh\mathcal{L}$

3.5.4 Definition. *A morphism $f : |A| \times |B| \to |C|$, for A,B and C in $\mathcal{A}lg(\tau)Sh\mathcal{L}$, is said to be a local bimorphism iff there exists a cover $\mathbb{1} = \bigvee_{i \in I} U_i$ of $\mathbb{1}$ in $\mathcal{L}$ such that*

$$f|U_i : |A|U_i| \times |B|U_i| \to |C|U_i|$$

is a bimorphism, for all $i \in I$.

3.5.5 Remark. In $\mathcal{A}lg(\tau)Sh\mathcal{L}$, for any $f : |A| \times |B| \to |C|$, the following are equivalent:

(i) f is a bimorphism.

(ii) The maps $f_U : |AU| \times |BU| \to |CU|$ are bimorphisms, for $U \in \mathcal{L}$.

(iii) f is a local bimorphism.

Here, (i) $\Leftrightarrow$ (ii) follows from the fact that limit preserving functors preserve bimorphisms, and (i) $\Leftrightarrow$ (iii) by Corollary (0.1.8). ///

3.5.6 Lemma. *For any A, B and C in $\mathcal{A}lg(\tau)Sh\mathcal{L}$ and any $U \in \mathcal{L}$,*

$$[A,B,C]U = BIM(A|U, B|U, C|U).$$

Proof. Since the restriction functor preserves products and equalizers, $[A,B,C]|U = [A|U, B|U, C|U]$, by definition of $[A,B,C]$, and thus

$$[A,B,C]U = ([A,B,C]|U)U = [A|U,B|U,C|U]U$$
$$= BIM(A|U,B|U,C|U),$$

the last step by applying the Proposition (3.1.6) in ShU.///

<u>3.5.7</u> <u>Lemma</u>. *If $\mathcal{A}Sh\mathcal{L}$ and $\mathcal{A}ShU$, for $U \in \mathcal{L}$, have functors of universal bimorphisms, M_1 and M_2 respectively, then, for A and B in $\mathcal{A}Sh\mathcal{L}$,*

$$M_1(A,B)|U = M_2(A|U,B|U).$$

Proof. This follows by the remark in (0.1.6) that $\widetilde{P|U} = \tilde{P}|U$, for $P \in \mathcal{A}PreSh\mathcal{L}$, and the construction of universal bimorphisms in $\mathcal{A}Sh\mathcal{L}$, implicit in the proof of the Proposition (3.5.3), as the sheaf reflection of the presheaf $U \rightsquigarrow M(AU,BU)$, M the functor of universal bimorphisms in $\mathcal{A}$.///

<u>3.5.8</u> <u>Proposition</u>. *For any category $\mathcal{A}Sh\mathcal{L}$, M is the functor of universal bimorphisms iff*

$$[M(A,B),C] \simeq [A,B,C].$$

Proof. One way is true, by Remark (3.4.4). Conversely, let M be the functor of Universal bimorphisms, if $U \in \mathcal{L}$ and M_1 and M_2 are as in the preceding lemma, then

$$[M(A,B),C]U = (M_1(A,B)|U, C|U)$$

$$= (M_2(A|U, B|U), C|U) = BIM(A|U, B|U, C|U)$$

$$= [A,B,C]U. ///$$

3.6 We conclude this chapter with a discussion of the internal hom-functor.

3.6.1 Proposition. *If $\mathcal{A}$ has a functional internal hom-functor, then so does $\mathcal{A}Sh\mathcal{L}$.*

Proof. Let H be the functional internal hom-functor of $\mathcal{A}$. For A and B in $\mathcal{A}Sh\mathcal{L}$ and each $U \in \mathcal{L}$, the λ-th operation, for any $\lambda \in \Omega$ of the algebra.

$\prod_{V\leq U} H(AV,BV)$ is given by

$$\lambda((h_{1V})_{V\leq U},\ldots,(h_{n_\lambda V})_{V\leq U})=(\lambda_{H(AV,BV)}(h_{1V},\ldots,h_{n_\lambda V}))_{V\leq U}$$

for any homomorphisms $h_{iV} : AV \to BV$, $i = 1,\ldots,n_\lambda$ and all $V \leq U$. The set $[A,B]U = \mathcal{A}ShU(A|U,B|U)$ is a subset of the underlying set of the algebra $\prod_{V\leq U} H(AV,BV)$, consisting of all the sequences

$h = (h_V)_{V\leq U}$, $h_V : AV \to BV$, such that the diagram,

$$\begin{tikzcd} AV \arrow[r, "h_V"] \arrow[d] \arrow[dr, phantom, "(1)"] & BV \arrow[d] \\ AW \arrow[r, "W"] & BW \end{tikzcd}$$

commutes, for each pair $W \leq V$ in $\downarrow U$; where the vertical arrows are the restriction maps. Now, if $h_1,\ldots,h_{n_\lambda}$ belong to $[A,B]U$, then, for any pair $W \leq V$ in $\downarrow U$ and $a \in AV$, we have,

$$\lambda(h_1,\ldots,h_{n_\lambda})_V(a)|W = \lambda_{H(AV,BV)}(h_{1V},\ldots,h_{n_\lambda V})(a)|W$$

$$\left(\begin{array}{l}\text{because H is functional and}\\ \text{thus the operations are pointwise}\end{array}\right) = \lambda_{BV}(h_{1V}a,\ldots,h_{n_\lambda V}a)|W$$

$$= \lambda_{BW}(h_{1V}(a)|W,\ldots,h_{n_\lambda V}(a)|W) \quad \left(\begin{array}{l}\text{because restriction maps}\\ \text{are homomorphisms.}\end{array}\right)$$

$$= \lambda_{BW}(h_{1W}(a|W),\ldots,h_{n_\lambda W}(a|W)) \quad \left(\begin{array}{l}\text{because all } h_i \in [A,B]U \text{ and}\\ \text{thus satisfy diagram (1).}\end{array}\right)$$

$$= \lambda_{H(AW,BW)}(h_{1W},\ldots,h_{n_\lambda W})(a|W) \quad \left(\begin{array}{l}\text{because the operations}\\ \text{are pointwise.}\end{array}\right)$$

$$= \lambda(h_1,\ldots,h_{n_\lambda})_W\ (a|W).$$

Thus, $\lambda(h_1,\ldots,h_{n_\lambda})$ satisfies the diagram (1), and hence belongs to $[A,B]U$; this implies that $[A,B]U$ is closed under the algebra operations of $\prod_{V\leq U} H(AV,BV)$, and hence $[A,B]U$ together with the operations $\lambda_{[A,B]U} = \lambda|[A,B]U$, for all $\lambda \in \Omega$, is a subalgebra of $\prod_{V\leq U} H(AV,BV)$, and hence belongs to $\mathcal{A}$, by the hypothesis on $\mathcal{A}$. Now, for each $\lambda \in \Omega$, define an n_λ-ary operation $\lambda_{[A,B]}$ on $[A,B]$ with components $\lambda_{[A,B]U}$, for each $U \in \mathcal{L}$. To check that $\lambda_{[A,B]}$ is actually a natural transformation, let $V \leq U$ and $h_1,\ldots,h_{n_\lambda}$ belong to $[A,B]U$, then we have,

$$\lambda_{[A,B]U}(h_1,\ldots,h_{n_\lambda})|V$$

$$= \quad \lambda_{H(AW,BW)}(h_{1W},\ldots,h_{n_\lambda W})_{W\le U}|V$$

$$= \quad \lambda_{H(AW,BW)}(h_{1W},\ldots,h_{n_\lambda W})_{W\le V}$$

$$= \quad \lambda_{H(AW,BW)}((h_1|V)_W,\ldots,(h_{n_\lambda}|V)_W)_{W\le V}$$

$$= \quad \lambda_{[A,B]V}(h_1|V,\ldots,h_{n_\lambda}|V).$$

This proves the naturality of $\lambda_{[A,B]}$. Thus, for any A and B in $\mathcal{A}Sh\mathcal{L}$, the sheaf $[A,B]$ together with these λ-th operations, $\lambda \in \Omega$, is an algebra and belongs to $\mathcal{A}Sh\mathcal{L}$. We now define an internal hom-functor

$$G : \mathcal{A}Sh\mathcal{L}^{*} \times \mathcal{A}Sh\mathcal{L} \to \mathcal{A}Sh\mathcal{L}$$

by $G(A,B) = ([A,B], \lambda_{[A,B]})$, and for homomorphisms

$A' \xrightarrow{f} A$ and $B \xrightarrow{g} B'$, $|G(f,g)| = [f,g]$. To check that $G(f,g)$ is actually a homomorphism, let $U \in \mathcal{L}$ and, $\lambda \in \Omega$, $h_1,\ldots,h_{n_\lambda}$ be arbitrary elements of $[A,B]U$, we have $g_V \circ \lambda(h_{1V},\ldots,h_{n_\lambda V}) \circ f_V(a') = g_V \circ \lambda(h_{1V},\ldots,h_{n_\lambda V})(f_V a')$

(because operations are pointwise) $= g_V\ \lambda(h_{1V} f_V a',\ldots,h_{n_\lambda V} f_V a')$

(because g_V is a homomorphism) $= \lambda(g_V \circ h_{1V} \circ f_V a',\ldots,g_V \circ h_{n_\lambda V} \circ f_V a')$

(pointwise operations)$=\ \lambda(g_V \circ h_{1V} \circ f_V,\ldots,g_V \circ h_{n_\lambda V} \circ f_V) a'$,

for any $V \leq U$ and $a' \in A'V$, and then we get

$$\begin{aligned}
[f,g]_U \lambda(h_1,\ldots,h_{n_\lambda}) &= g|U \circ \lambda(h_1,\ldots,h_{n_\lambda}) \circ f|U \\
&= (g_V \circ \lambda(h_{1V},\ldots,h_{n_\lambda V}) \circ f_V)_{V \leq U} \\
&= \lambda(g_V \circ h_{1V} \circ f_V,\ldots,g_V \circ h_{n_\lambda V} \circ f_V)_{V \leq U} \\
&= \lambda(g|U \circ h_1 \circ f|U,\ldots,g|U \circ h_{n_\lambda} \circ f|U) \\
&= \lambda[f,g]_U(h_1,\ldots,h_{n_\lambda}) .
\end{aligned}$$

Thus $\mathcal{G}$ is an internal hom-functor of $\mathcal{A}Sh\mathcal{L}$. Finally, it remains to show that G is functional. By the Proposition 3.2.3(iv), it is enough to show that, for

BIBLIOGRAPHY

[1] B. Banaschewski, An Introduction to Universal Algebra, Lecture Notes, I.I.T. Kanpur, Summer 1972.

[2] B. Banaschewski, Injectivity and Essential Extensions in Equational Classes of Algebras, Proceedings of the Conference on Universal Algebra (1969), Queen's Paper in Pure and Applied Mathematics, No. 25 (1970), 131-147.

[3] B. Banaschewski, Injective Sheaves of Abelian Groups: Counterexamples, Can. J. Math (to appear)

[4] B. Banaschewski, When Are Divisible Abelian Groups injective, Preprint, University of Cape Town, South Africa (1979)

[5] B. Banaschewski, Equational Compactness in Universal Algebra, McMaster University (1973).

[6] B. Banaschewski, On Equationally Compact Extensions of Algebras, Alg. Univ. Vol. 4 (1973), 20-35.

[7] B. Banaschewski and E. Nelson, Equational Compactness in Equational Classes of Algebras, Alg. Univ. Vol. 2 (1972), 152-165.

[8] B. Banaschewski, Equational Compactness of G-sets, Can. Math. Bull. Vol. 17(i), (1974), 11-18.

[9] B. Banaschewski and E. Nelson, Tensor Products and Bimorphisms, Can. Math. Bull. Vol. 19(4), (1976), 385-402.

[10] P.M. Cohn, Universal Algebra, Harper and Row, (New York, 1965).

[11] D.E. Dobbs, On Characterizing Injective Sheaves, Can. J. Math., 24 (1977), 1031-1039.

[12] R. Godement, Topologie Algébrique et Théorie des Faisceaux, Herman, (Paris, 1958).

[13] G. Gratzer, Universal Algebra, Van Nostrand, (Princeton, 1968).

[14] C.R. Howlett, Doctoral Dissertation, McMaster University, Hamilton, Ont., Canada (1973).

[15] P.T. Johnstone, Topos Theory, Academic Press, London, (1977).

[16] H. Kleisli and Y.C. Wu, On Injective Sheaves, Can. Math. Bull., Vol. 7(3), (1964), 415-423.

[17] S. MacLane, Categories for the Working Mathematician, Graduate Texts in Mathematics, No. 5, Springer-Verlag, (1971).

[18] J.M. Maranda, On Pure Subgroups of Abelian Groups, Arch. Math., 11 (1960), 1-13.

[19] B. Mitchell, Theory of Categories, Acadmeic Press, (1965).

[20] B.R. Tennison, Sheaf Theory, Cambridge University Press, (1975).

[21] W. Taylor, Residually Small Varieties, Alg. Univ. Vol. 2 (1972), 33-53.

[22] B. Weglorz, Equationally Compact Algebras (I), Fund. Math., 59 (1966), 289-298.

[23] B. Weglorz, Equationally Compact Algebras (III), Fund. Math. 60 (1967), 89-93.