

ON THE NUMERICAL SOLUTION OF
NONLINEAR PROBLEMS OF CONTINUA

By



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ABSTRACT

A state of the art survey of formulation aspects of geometric and material nonlinearity problem is given. The survey covers the formulation methods, the solution of nonlinear equilibrium equations, the incompressibility constraint, and, finally, the software aspects. A consistent Lagrangian, updated Lagrangian, and Eulerian formulation are derived from the energy balance equation transformed to the proper reference configuration. Difficulties opposed the pre-existence of a consistent Eulerian formulation in the literature are critically discussed. The proposed Eulerian formulation includes specific approximations which simplify the numerical treatment, and may restrict the applicability of the formulation to specific classes of problems, but do not alter the nature of the formulation. Differences between the presented Lagrangian and updated Lagrangian formulations and similar ones in the literature are found to be in specific geometric nonlinear terms in the final incremental equation as well as in the definition of the load increment vector. These differences are assessed within the framework of the basic equations of continuum mechanics. Specific forms of constitutive equations for elastic and elastoplastic materials are presented. For elastoplastic applications, it is shown that the use of proper frame indifferent stress rate leads to a constitutive equation which is a function of the incremental displacements and not the corresponding incremental strains.

A concise discussion of software aspects is provided which leads

to a suggestion of switching from the program package to the programming system concept. A programming system to account for material nonlinear behaviour is developed and tested.

In the application, a plasticity theory for porous metals is proposed. A simple model of porous material is analysed by the finite element method. An assessment is made of an existing yield criterion for porous metals. A modified yield criterion and plastic potential function, and consequently, different plasticity equations are given. Reasonable agreement is obtained between the present numerical results and previous experimental and analytical results in the literature.

TO THE MEMORY OF MY FATHER

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LIST OF SYMBOLS

The following is a list of frequently used symbols. Others will be defined when they are used. For a brief comment on the adopted system of notations, the reader is referred to page 77 of the context.

a	principal stretch	p	hydrostatic pressure
\bar{a}	acceleration vector	\bar{R}	generalized force vector
A	area of the loading surface	\bar{S}	second Piola-Kirchhoff stress tensor
\bar{b}	body force vector	t	time
C	configuration of a body	\bar{T}	first Piola-Kirchhoff stress tensor
$\frac{4}{C}$	fourth order tensor relating stresses to strains	\bar{u}	displacement vector
D	search direction	V	volume
\bar{D}	rate of deformation tensor	\bar{v}	velocity vector
$\frac{4}{D}$	fourth order tensor relating stresses to strains	w	strain energy density, or measure of work done
E	Young's modulus	X	material particle
\bar{e}	unit base vector	\bar{x}	current or Eulerian position vector
\bar{E}	Green-Lagrange strain tensor	\bar{X}	reference or Lagrangian position vector
F	force unbalance or yield function	Y	yield point
\bar{F}	deformation gradient tensor	\bar{I}	identity tensor
g	auxiliary field variable or plastic potential function	α	over-relaxation factor or coefficient function of relative density
H	hydrostatic pressure stiffness or plastic strain hardening modulus	β	coefficient function of relative density
I	identity matrix	γ	coefficient function of relative density
K	element stiffness, global stiffness or kinetic energy		

LIST OF SYMBOLS (continued)

δ variational operation	SUBSCRIPTS
Δ prefix indicating a finite increment	*Right Subscripts
$\bar{\epsilon}$ Euler-Almansi strain tensor	i direction
$d\lambda$ non-negative constant	j direction
ν Poisson's ratio	m incremental index
ρ mass density or relative density	n iteration index
$\bar{\sigma}$ Cauchy stress tensor	*Left Subscripts
ϕ pressure interpolation function	0,1,2 configuration to which the quantity is referred
ψ displacement interpolation function	SUPERSCRIPTS
	*Right Superscripts
	T transpose of a matrix, or of a tensor
	-1 inverse of a matrix, or of a tensor
	*Left Superscripts
	0,1,2 configuration at which the quantity is measured

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INTRODUCTION

1. PRELIMINARY REMARKS

(i) Preamble

In modern structural theory one observes an increasing interest in the solution of geometric and/or material nonlinearity problems. Geometric nonlinearity arises from nonlinear strain-displacement relations whereas material nonlinearity arises from material nonlinear behaviour as in plasticity and creep problems. The main aspects in formulating such problems are: the construction of proper constitutive equations for materials experiencing large strains, the introduction of an incompressibility constraint, the choice of proper formulation methods for specific applications, and the use of proper numerical techniques for solving the nonlinear equilibrium equations. Other aspects to be considered are numerous, e.g. convergence characteristics of solution techniques, proper treatment of boundary conditions, accuracy determination, feasibility of results, proper modelling of physical problems besides the basic requirements of software package that may handle problems of this nature. In this introduction, the main aspects in formulating general geometric and material nonlinearity problems are briefly discussed, with an emphasis on the state of recent progress in each particular point.

(ii) Formulation Methods

Four types of kinematic description of motion are in common use in continuum mechanics. According to Truesdell's definition, these are [1,2]:

- Material description; independent variables are the material particle X and the time t .
- Referential or Lagrangian description; independent variables are the reference or Lagrangian position \bar{X} of a particle X and the time t .
- Spatial or Eulerian description; independent variables are the current position \bar{x} of a particle and the time t .
- Relative description; independent variables are the current position \bar{x} of a particle and a variable time t .

In finite element formulations, only three descriptions are in common use. These are the Lagrangian, Eulerian and relative descriptions. To overcome some disadvantages of Lagrangian formulation, a modified version usually called "updated Lagrangian" formulation was developed. According to Truesdell's definitions, however, the updated Lagrangian formulation is a special case of the relative formulation.

It is noticed that most of the papers published on the formulation aspect dealt with Lagrangian formulation and few of them dealt with updated Lagrangian version whereas there is very little effort concerned with a detailed Eulerian formulation. Moreover, inconsistent transformation of the energy balance equation from the deformed current configuration to the undeformed Lagrangian

configuration led to creating artificial arguments in linearizing the equilibrium equations as in reference [16] and also led to incorrect definition of load and stiffness terms as in references [9-12,16]. It is also noticed that there is no critical survey published on this topic and most updated Lagrangian formulations are mistakenly specified as Eulerian formulations. Moreover, most of the existing formulations in literature arbitrarily omit specific nonlinear terms for the intended linearization process within the increment, when they appear with other incremental quantities [6]. Other formulations introduce an implicit, final form of nonlinear equilibrium equations where all the nonlinear stiffness contributions are combined in one stiffness term [9-12].

(iii) Solution of Nonlinear Equilibrium Equations

The main solution techniques may be classified as exact equilibrium, approximate equilibrium, and self-correcting methods. The main solution procedures under the exact equilibrium techniques are the successive approximation method, Newton-Raphson method, and direct minimization of potential energy methods. Under the approximate equilibrium techniques there are the incremental loading procedures and the static perturbation method. Finally, under the self-correcting techniques there are the first and second order self-correcting methods.

Even though there are many survey papers on this topic, little emphasis has been given to the applicability of these methods to particular nonlinear problems and to recent developments and modifications of each method to suit a particular nonlinear field.

(iv) Incompressibility Constraint

The usual energy formulation of the finite element method is not applicable for incompressible materials because the admissible deformation fields have to be incompressible. It was noticed that finite element solutions of elastoplastic solids usually exhibit too stiff a response in the fully plastic range, exceed the limit load and in some cases have no limit load. This was attributed to the lack of incompressibility constraint near the limit load [156]. For these reasons, alternate finite element formulations have been proposed. The introduction of an incompressibility constraint was carried out mainly through a Lagrange multiplier which led to a variational principle with two field variables; the displacement and the hydrostatic pressure. It is noticed that most of the work done in this field was applied to elastic materials and fluids, and there is no critical literature survey published on this topic.

2. SCOPE OF WORK

Although the main aspect of the thesis is concerned with both the formulation and the critical survey, we may classify the work into the following topics:

(i) State of the Art Survey

*Survey of Formulation Methods

Consistent classification of different methods of formulations and their pertinent characteristics is given. A discussion of the

reasons for differences in existing formulations in the literature is provided. These reasons cover the simplifying assumptions of the kinematic relations within the incremental formulation, the interpretation of the energy balance equation in the undeformed Lagrangian configuration and the use of different choices of stress measures and their conjugate strain definitions. Most of the mistakenly named Eulerian formulations in the literature are discussed and reclassified in a proper way.

*Survey of the Solution of Nonlinear Equilibrium Equations

Although there are some survey papers published on this topic, the work presented in this thesis emphasizes the applicability of numerical methods of solution to particular nonlinear problems. Also, a discussion of recent modifications and developments of particular numerical methods to suit specific nonlinear applications is provided.

*Survey of the Incompressibility Constraint

The introduction of the incompressibility constraint for elastic materials is critically discussed and different methods of formulation existing in the literature are assessed. The extension of these methods to account for elastoplastic material behaviour is discussed. An emphasis is put on the physical effect of the constraint in solving specific nonlinear problems as well as the numerical complications arising from different formulations.

(ii) Software Aspects

The main requirements, advantages and disadvantages of a general purpose finite element program as opposed to the concept of a programming system are discussed. Within the structure of these requirements, a justification for the switch from program to programming system is considered. A programming system is developed to account for general material nonlinear behaviour.

(iii) Formulation Methods

*Lagrangian Formulation

A consistent Lagrangian formulation is derived from the energy balance equation transformed to a Lagrangian frame of reference, for perfect mechanical systems. Explicit expressions for the individual stiffness matrices, contributing to the total stiffness matrix of the element are given. Differences between the presented formulation and existing ones in the literature [9-12,16,17] are discussed.

*Updated Lagrangian Formulation

An updated Lagrangian formulation is derived on a similar basis to the development of the Lagrangian formulation. Stiffness matrices contributing to the total stiffness matrix are presented. Once again, differences between the derived formulation and existing ones in the literature [9-12] are discussed.

*Eulerian Formulation

The main points opposing the existence of a complete and detailed Eulerian formulation in the literature are discussed. A detailed Eulerian formulation is given. Specific approximations which lead to a simplified Eulerian formulation suitable for numerical calculations are considered.

*Constitutive Equations

Only generalized infinitesimal strain constitutive equations written in an invariant form with respect to rigid body motion are considered. Discussion of specific stress rates and its applicability to particular materials is given. No survey to account for the general formulation of constitutive equations is attempted.

(iv) Applications

Applications only include the material nonlinearity part of the formulation. A study of the plastic behaviour of porous materials is presented. Effect of strain hardening and void separation are discussed. The achieved results represent a modified theory of plasticity for porous metals.

CHAPTER I

SOLUTION OF NONLINEAR CONTINUUM MECHANICS PROBLEMS

A STATE OF THE ART SURVEY*

I.1 SURVEY OF FORMULATION METHODS

I.1.1 INTRODUCTION

In describing the motion of a body, there exist four methods of formulations; which Truesdell [1,2] called: the material, the referential, the spatial, and the relative formulation. All four methods of formulation are equivalent for any smooth motion of the body.

1. In the material description, the independent variables are the particle or the body-point X , and the time t . This description is conceptually the most natural one and is the one exclusively used in analytical dynamics. However, it is rarely used in continuum mechanics, or specifically in finite element applications.
2. In the referential description, the independent variables are the position \bar{X} of a particle X in an arbitrarily chosen reference configuration, and the time t . It is important to notice that the choice of the reference configuration is arbitrary and

* A modified version of the work presented in this chapter is the subject of two survey articles submitted for publication in Applied Mechanics Reviews [227,228].

essentially this choice would not affect the results. A particular referential description was introduced by Euler where the position \bar{X} of the body-point X at the particular time $t = 0$ is used to describe the motion. This particular description is often called the Lagrangian formulation in the literature. However, any other choice of the reference configuration at a specific time other than $t = 0$ would still be Lagrangian in nature, in the sense that the independent variable \bar{X} is considered at a fixed time instant.

3. In the spatial description, the independent variables are the current position \bar{x} of the particle X , and the time t . This description was introduced by Bernoulli and D'Alembert but in literature it is usually called the Eulerian description. In the spatial description, we are concerned with what is happening in a fixed region of space as a function of time, which seems to be perfectly suited for the study of fluids. However, for the same reason the spatial description is awkward for the expression of the principles of mechanics, since, in fact, the laws of dynamics refer to what is happening to the body, and not to the region of space which the body momentarily occupies. It is also important to emphasize the fact that the independent variable \bar{x} is a function of the Lagrangian position \bar{X} and the time t , i.e. $\bar{x} = \bar{x}(\bar{X}, t)$. Therefore, all material derivatives will be much more difficult to handle in spatial description.

4. In the relative description, the independent variables are the position \bar{x} in a current or present configuration and the time τ . In relative description the reference configuration depends upon time. The current or present configuration is taken as the reference configuration and the past and future configurations are described relative to the current or present one. The variable time τ is the time when the particle X occupied a position $\bar{\xi}$, where $\bar{\xi} = \bar{\xi}_t(\bar{x}, \tau)$. It is important to realize that the relative description is referential or Lagrangian in nature, in the sense that the reference position is now denoted by \bar{x} at time t rather than \bar{X} at time $t = 0$. This will justify the classification of updated Lagrangian formulation as a special case of the relative description as will be discussed later.

In finite element approach to continuum mechanics problems, only the last three descriptions are used. In what follows, we will try to assess different finite element formulations, existing in the literature, within the frame work of the definitions given above. We will also discuss the simplifying assumptions and point out reasons of differences among these formulations.

I.1.2 LAGRANGIAN (REFERENTIAL) FORMULATION

Lagrangian formulation is by far the most widely used formulation in finite element applications to nonlinear structural problems [3-17]. While most of the Lagrangian finite element formulations utilize incremental method with linear increments, the differences basically lie

in the simplifying assumptions of the kinematic relations within the increment and in the interpretation of the energy balance equation in the undeformed Lagrangian configuration. Within the same simplifying assumptions, different forms of the final nonlinear equilibrium equations are owing to different choices of stress measures and their conjugate strain definitions.

Incremental Lagrangian procedures have found wide use in finite element applications, especially, to problems involving geometrical nonlinearity and elastic stability. Many of the early incremental finite element formulations are incomplete in that the variations in loading and rotations of the element are not properly represented [13]. The most general approach to incremental Lagrangian formulation is to consider complete nonlinear kinematic relations within a linear increment [13-17]. Linear increments are related to the rates of variables used in rate type theories of continuum mechanics or in the thermodynamics of continuous media. The basic principles of the incremental theory may be formulated by using either the basic equations of thermodynamics, i.e. the energy balance equation, of a continuum, or its equivalent variational description employing variations in the variables. When fully nonlinear kinematic relations are used within a linear increment, the number of stiffness contributions to the nonlinear element stiffness matrix are found to be three matrices [13-17]. These are: the usual small displacement or the incremental stiffness matrix, the initial stress, or the geometric, or the tangent stiffness matrix and, finally, the initial displacement, or the initial rotation stiff-

ness matrix. A fourth stiffness matrix: the initial load stiffness matrix, or the load correction matrix will be discussed when considering different implementations of the energy balance equation. Most of the early literature on finite element applications to geometrically nonlinear problems used the initial stress stiffness matrix to correct the linear stiffness matrix at the end of each successive load increment. As indicated by Oden [13], unless a new material frame of reference is established in the deformed element at the end of each increment, as was done for example by Murray and Wilson [18], this procedure is incorrect. Also, a local coordinate system rotating with the element may not be sufficient for higher order elements, because finite rotations within the element may take place.

Returning to the original point of the simplifying kinematic assumptions, Wunderlich [6] has introduced a systematically consistent development of an incremental Lagrangian formulation on the basis of the equations of continuum mechanics. However, he omits specific nonlinear terms for the intended linearization process within the increment, when they appear with other incremental quantities. In references [3,4] a similar simplified approach is applied to plate and shell problems. In reference [3] the stiffness equations are derived from strain energy expressions in which terms are neglected arbitrarily, whereas in [4] Von Karman simplification of the strain-displacement expressions is introduced. This implies that all quadratic strain terms are neglected except for those corresponding to rotation of the mid-surface. Fully nonlinear kinematic relations are used by Oden [13], Hibbit et al. [16]

and followed by many other authors. Needleman [5] introduces a Lagrangian scheme which is derived from a variational principle due to Hill [19,20]. Bathe et al. [9-12] also have followed a general approach but they present an implicit final form of the nonlinear equilibrium equations where all the nonlinear stiffness contributions are combined into one stiffness term. We may conclude this particular discussion by saying that none of the stiffness matrices contributing to the total stiffness of the element should in general be omitted. In specific applications, however, certain simplifications of these stiffness expressions may be achieved.

Another point worthy of discussion is the energy balance equation used by different authors. Most of the work done in Lagrangian and other formulation methods start with the energy balance equation, or with an equivalent virtual work principle, in the deformed or current configuration. If we consider three different configurations for the body; the original configuration C_0 at time $t = 0$, a current configuration C_1 at time $t = \tau$ and a neighbourhood configuration C_2 at time $t = \tau + \Delta t$, then we may write the energy balance equation in configuration C_2 as

$$\frac{d}{dt} {}^2_K = \int_{{}^2_V} {}^2_\rho {}^2_b \cdot {}^2_V d {}^2_V + \int_{{}^2_A} d {}^2_A \cdot {}^2_\sigma \cdot {}^2_V - \int_{{}^2_V} {}^2_\sigma : {}^2_D d {}^2_V \quad (I-1)$$

where the left superscripts indicate the configuration at which the quantity is measured whereas the left subscripts indicate the configuration to which the quantity is referred, and 2_K is the kinetic energy, ${}^2_\rho$ is the material density, 2_b is the body force per unit of current

volume, 2V is the current volume, ${}^2\bar{v}$ is the velocity, 2A is the area of the current loading surface, ${}^2\bar{\sigma}$ is the Cauchy stress tensor, ${}^2\bar{D}$ is the rate of deformation tensor and, finally, ${}^2\bar{\sigma} : {}^2\bar{D}$ indicates the double dot product of the two second order tensors ${}^2\bar{\sigma}$ and ${}^2\bar{D}$, i.e. ${}^2\bar{\sigma} : {}^2\bar{D} = {}^2\sigma_{ij} {}^2D_{ij}$.

A consistent transformation of eqn. (I-1) from the current deformed configuration to the Lagrangian undeformed configuration may give [13-15] (also see Appendix A)

$$\frac{d}{dt} {}^2K = \int_{\circ V} \circ\rho {}^2\bar{b} \cdot {}^2\bar{v} d \circ V + \int_{\circ A} d \circ\bar{A} \cdot {}^2\bar{S} \cdot {}^2\bar{F}^T \cdot {}^2\bar{v} - \int_{\circ V} {}^2\bar{S} : \frac{d}{dt} {}^2\bar{E} d \circ V \quad (I-2)$$

where 2K is the kinetic energy in the undeformed configuration, $\circ\rho$ is the material density in the undeformed configuration, ${}^2\bar{S}$ is the second Piola-Kirchhoff stress tensor, ${}^2\bar{F}^T$ is the transpose of the deformation gradient tensor and ${}^2\bar{E}$ is the Lagrangian strain tensor.

Many authors, however, differ in considering the surface traction term in eqn. (I-2) by dropping ${}^2\bar{F}^T$ from its expression [9-12,16]. The expression of the surface traction term presented here leads to a consistent development of the load correction matrix or the initial load stiffness matrix [13-15]. Hibbit et al. [16] introduce a rather intuitive argument as $\bar{\Delta F} = \Delta_{load} \bar{F} + \Delta_{geom} \bar{F}$, where \bar{F} is the surface traction vector or the body force vector, to obtain the form of the load correction vector. The term obtained by Hibbit et al., will have a zero value in some loading cases, while the term derived from the expression presented here will always possess some value in general. In [9-12],

Bathe et al. consider a similar treatment to that given by Hibbit et al. for the case of loads that are independent of the configuration of the body, however, for deformation-dependent loading they consider a similar approach to the one given here. It is our opinion that this term is obtained by consistent transformation of the energy balance equation with no reference to the type of surface traction, hence it should be applied to all types of surface tractions. The importance of including the load correction term in incremental formulations was pointed out by Oden [21] and Oden and Key [22]. We may conclude this discussion by asserting that the introduction of the load correction matrix is important both in large strain, and in infinitesimal strain but finite rotation analyses. This term should also be included for all types of surface tractions. The effect of including the correction matrix on the accuracy of the solution will depend on the nature of the application. Some authors connect this term with stability and buckling analysis. It is, however, not uncommon to have large rotations of the elements of the body prior to the onset of buckling, and thus it is generally necessary to add the correction term to the nonlinear equilibrium equation [13,163,223].

The final point of discussion under the scheme of Lagrangian formulation is the choice of compatible stress and strain measures. To provide a consistent discussion, we will first recall the following definitions

$$\bar{T} = |\bar{F}| \bar{F}^{-1} \cdot \bar{\sigma} \quad (I-3)$$

where $\bar{\bar{F}}^{-1}$ is the inverse of $\bar{\bar{F}}$, and $|\bar{\bar{F}}| = \bar{\bar{F}} \times \bar{\bar{F}} : \bar{\bar{F}}/3!$, and $\bar{\bar{T}}$ is the unsymmetric first Piola-Kirchhoff stress tensor or the nominal stress tensor, or the Piola-Lagrange stress tensor, whereas $\bar{\bar{F}} = \bar{x}\partial/\partial\bar{X}$ is the deformation gradient, $\bar{\bar{F}} \times \bar{\bar{F}}$ and $:$ represent double cross and double dot products of tensors, respectively.

$$\bar{\bar{S}} = |\bar{\bar{F}}| \bar{\bar{F}}^{-1} \cdot \bar{\bar{\sigma}} \cdot (\bar{\bar{F}}^{-1})^T = \bar{\bar{T}} \cdot (\bar{\bar{F}}^{-1})^T \quad (\text{I-4})$$

is the symmetric second Piola-Kirchhoff stress tensor in reference configuration, also called the Kirchhoff-Trefftz stress tensor.

$$\bar{\bar{F}} = \bar{\bar{R}} \cdot \bar{\bar{U}} = \bar{\bar{V}} \cdot \bar{\bar{R}} \quad (\text{I-5})$$

is the deformation gradient tensor, where $\bar{\bar{R}}$ is the rotation tensor, $\bar{\bar{U}}$ and $\bar{\bar{V}}$ are the right and left stretch tensors respectively.

$$\bar{\bar{E}} = \frac{1}{2} [\bar{\bar{F}}^T \cdot \bar{\bar{F}} - \bar{\bar{1}}] \quad (\text{I-6})$$

is the Euler-Lagrange, or the Lagrange, or the Green strain tensor,

$$\bar{\bar{\epsilon}} = \frac{1}{2} [\bar{\bar{1}} - (\bar{\bar{F}}^{-1})^T \cdot \bar{\bar{F}}^{-1}] \quad (\text{I-7})$$

is the Euler, or the Almansi strain tensor.

$$\bar{\bar{L}} = \frac{\bar{\bar{v}} \cdot \partial}{\partial \bar{x}} = \bar{\bar{D}} + \bar{\bar{W}} \quad (\text{I-8})$$

is the velocity gradient tensor, whereas $\bar{\bar{D}}$ is the rate of deformation tensor and $\bar{\bar{W}}$ is the spin, or vorticity tensor.

Based on the definition of the rate of work per unit final volume expressed as $\bar{\sigma} : \bar{D}$, Hill [23] defines particular conjugate pairs of stress and strain variables:

$$\{\bar{\sigma}, \bar{L}\}, \{\bar{S}, \dot{\bar{E}}\}, \{|\bar{F}| \bar{F}^T \cdot \bar{\sigma} \cdot \bar{F}, \dot{\bar{\epsilon}}\}, \{\bar{T} \cdot \bar{R}, \dot{\bar{U}}\} \text{ and } \{\bar{T}, \dot{\bar{F}}\},$$

where $\dot{(\)} \equiv d(\)/dt$. In finite element formulations, however, the most widely used stress measures are the second and the first Piola-Kirchhoff stress tensors \bar{S} and \bar{T} . Hence, an emphasis will be put on the two conjugate pairs $\{\bar{S}, \dot{\bar{E}}\}$ and $\{\bar{T}, \dot{\bar{F}}\}$.

The first conjugate pair $\{\bar{S}, \dot{\bar{E}}\}$ is used by many analysts [9-12,14-17] especially in the case of both geometric and material nonlinear problems. The transformation of the energy balance equation from the current deformed configuration to the Lagrangian undeformed configuration lends itself to an expression in terms of \bar{S} and $\dot{\bar{E}}$ [15,24]. Based on the symmetry of the second Piola-Kirchhoff stress tensor \bar{S} , the formulation identically satisfies the moment of momentum or the rotational equilibrium equations.

The use of the second conjugate pair $\{\bar{T}, \dot{\bar{F}}\}$ usually arises from formulating the problem based on a complementary energy principle [8, 24-30]. Professor G. AE. Oravas has given a comprehensive account of the history of the principle of minimum complementary energy in the linear theory of elasticity in his introduction and biographical portrait section to the English translation of Castigliano's pioneering book [31]. If the finite element formulation is based on a complementary energy principle, then unlike the linear theory of elasticity, it is not possible to derive a single field complementary

energy principle for finite elasticity involving S alone as a field variable [32]. Also, it is not a straightforward task to satisfy the linear momentum balance equation when it is expressed in undeformed coordinates. This, in fact, has motivated the study of constructing alternate complementary energy principles for finite elasticity. An early work on this subject is given by Libove [25]. Libove presents a complementary energy principle which is suitable for linear as well as nonlinear elastic materials undergoing finite deformations. The treatment is given in terms of a vectorial definition for the complementary energy and is most suitable for truss applications. Levinson [26] performs a similar treatment to the one given by Libove [25] but considers a continuum rather than a truss structure. Levinson's work, however, does not discuss explicitly the Euler equations of the given functional and hence it is not indicated which types of boundary conditions or specific field equations should be satisfied a priori [33]. Also, Levinson [26] does not explicitly discuss the moment equilibrium equation as related to the unsymmetry of the first Piola-Kirchhoff stress tensor. In a similar treatment to the one given in [25,26], Zubov [27] expresses the complementary energy functional in terms of the first Piola-Kirchhoff stress only. The stationary values of Zubov's functional actually characterizes possible solutions of the elasticity problem. Zubov's analysis is criticized by Koiter [28] but no additional features are given. The most careful treatment of the subject is given by the late Professor B.F. de Veubeke [24]. Veubeke discusses the linearizing assumptions which justify the

use of the first Piola-Kirchhoff stress tensor in a variational principle so that the condition of rotational symmetry or the rotational equilibrium equations should be satisfied. He points out that even for infinitesimal strains, finite rotations will not satisfy these assumptions and the rotational equilibrium equation has to be retained in the analysis. Based on polar decomposition, Veubeke [24], introduces a canonical variational principle, which is reduced to a principle of complementary energy function of the three field variables \bar{T} , \bar{u} and r_{ii} where $\bar{r} = 1/2 (\bar{T} \cdot \bar{R} + \bar{R}^T \cdot \bar{T}^T)$ is a stress tensor, and \bar{u} is the displacement vector. The canonical variational principle and its corresponding complementary energy principle has the rotational equilibrium equation as one of its Euler equations. On the basis of the work discussed above, Atluri [29], Murakawa and Atluri [8,30] discuss different hybrid finite element models in linear and nonlinear solid mechanics based on a complementary energy principle.

I.1.3 UPDATED LAGRANGIAN OR RELATIVE FORMULATION

As discussed above, the referential description employs some assigned reference configuration. If the reference configuration is taken at $t = 0$, it will be a particular case of referential description which is called Lagrangian. If the reference configuration is taken at a variable time t , then we have a relative description, which is referential in nature. In a general relative description we describe the past and future relative to the present. Hence, we may be marching backward or forward with respect to the current, or present configura-

tion. In updated Lagrangian formulation, we describe the future with respect to the present, i.e. we march backward, which is a special case of the relative description. In updated Lagrangian or relative formulation, the independent variables are \bar{x} and τ , where \bar{x} is the position occupied by the material point X at time t . This indicates that \bar{x} is independent of the variable time τ . A feature which justifies the referential nature of the relative description and its distinction from spatial or Eulerian description in which the two variables \bar{x} and t used to describe the motion are dependent as $\bar{x} = \bar{x}(\bar{X}, t)$. For the same reason, material derivatives and other relations which are obvious and easy to derive in material or referential description seem to be more difficult to handle in spatial description. However, spatial or Eulerian description seems to be perfectly suited to the study of fluids, where we may observe the condition of flow in a fixed region of space. In such study, it is not easy to determine the paths of the particle, and at any time t we may not know what was the reference position \bar{X} occupied at time $t = 0$ by the fluid particle now at \bar{x} . In this case it is convenient to describe variables as functions of the position \bar{x} and the variable time t .

It is unfortunate that points raised in the above discussion are usually ignored in finite element literature and many analysts describe relative or updated Lagrangian formulation under the name of Eulerian formulation. Moreover, little effort has been made in the development of a conceptually consistent and detailed spatial, or Eulerian description [34-37].

As in Lagrangian formulation, reasons for the existing differences in updated Lagrangian formulation basically lie in the simplifying assumptions of the kinematic relations, and in the interpretation of the energy balance equation. Also, different choices of stress measures and their conjugate strain definitions will affect the final form of the nonlinear equilibrium equations. As shown in [15,38], utilizing the fully nonlinear kinematic relations within a linear increment gives the same number of stiffness contributions to the nonlinear element stiffness matrix as the Lagrangian formulation. However, certain differences in the final expressions do exist. One of the early tries of updated Lagrangian formulation is given by Murray and Wilson [18]. They employ the assumption of linearly elastic material behaviour and of displacement gradients that are small compared to the metric tensor. To justify the assumptions of small displacement gradients, they utilize a displaced local coordinate system with refined subdivisions of the structure. This approach essentially tackles the nonlinearity in the strain displacement relationships through a geometric transformation. As mentioned above, it should be noticed that local coordinate systems rotating with the element may not be sufficiently accurate for higher order elements, because finite rotations within the element may take place [13]. Yaghai and Popov [39] discuss another updated Lagrangian formulation utilizing an incremental variational principle based on a moving reference configuration. The incremental virtual work expression is obtained by taking the difference between the virtual work equations of two

consecutive current configurations. Lagrangian strain tensor and the second Piola-Kirchhoff stress tensor are used as the conjugate pair of variables in their analysis. Sharifi and Popov [40] extend the method of [39] to elastoplastic analysis involving infinitesimal strains with finite rotations. Gunasekera and Alexander [41] follow a similar formulation scheme by indicating that, under a proper updating of the coordinates of the body, the initial displacement stiffness matrix can be conveniently discarded. They do not, however, introduce a clear definition for their stress and strain measures. Their derivation for stiffness and finite element strain-displacement matrices seems to be computed for the Green strain tensor which requires the stress measure to be the second Piola-Kirchhoff stress tensor. However, this makes their constitutive laws to have an inadequate form of the Prandtl-Reuss equations which they assume. Another ambiguous formulation is given by Argyris and Chan [42] who, also, do not give a clear definition of their stress and strain terms. It is not clear that their stress measure is a member of the conjugate family and their Prandtl-Reuss equations are written with its ordinary time derivative as the stress rate. Atluri [43] describes another updated Lagrangian formulation utilizing the principle of virtual work written with respect to a current configuration. Expressions of the virtual work equation and the strain tensor indicate his utilization of the Green-Lagrange strain tensor and the second Piola-Kirchhoff stress tensor as a conjugate pair. Following Hofmeister et al. [44], Atluri [43] considers a correction term for the check of equilibrium of the initial stress state in the current

configuration as the reference configuration. Atluri's analysis is carried out using nonlinear kinematic relations, but in the numerical calculations he introduces linear incremental strain-displacement relations to develop the final finite element equations. Argyris and co-workers [45-47] introduce a distinct relative formulation utilizing a development given by Kleiber [48]. In [48] Kleiber discusses the kinematics of deformation processes for materials subjected to finite elastic and plastic strains. He introduces the concept of an intermediate stress-free configuration and uses it as a reference configuration. This configuration is obtained by the removal of stresses without further plastic deformation. Kleiber *opt. cit.*, shows that the usually accepted assumptions of the multiplicative decomposition of the deformation gradient lends itself in a natural way to the concept of the additive decomposition of strain measures. He introduces suitable definitions of the stress tensors that have to be taken into account to justify his development. Kleiber's work is essentially based on the previous work given by Lee [49] which was further elaborated by Freund [50]. Lee presents explicit expressions for elastic and plastic work in the case of polar decomposition but does not present a concise constitutive law for combined elastoplastic deformations. Such a constitutive law is discussed in more detail in the work of Argyris and Kleiber. As mentioned above, on the basis of Kleiber's development [48], Argyris et al. [45-47] introduce a distinct updated Lagrangian formulation utilizing the intermediate stress-free state as a reference configuration. However, they do not draw any

distinction between updated Lagrangian and Eulerian formulations. The work of Argyris et al. is based on what he calls the natural approach for defining strain measures and is applied to specific types of elements. Even though the treatment is referential in nature, they do utilize both the Euler-Almansi and the Green-Lagrange strain tensor in a parallel development. In [45,46] they assume infinitesimal increments of all independent and dependent variables. Hence they neglect nonlinear terms in all incremental strain - incremental displacement relations. In [47] they incorporate the nonlinear strain-displacement relations in an iterative manner. However, the use of the natural approach as applied to specific types of elements in Argyris's work makes it difficult to compare the details of their final equations with the corresponding ones in the literature.

McMeeking and Rice [51] introduce another relative formulation based on variational principle for incremental deformation given by Hill [20]. Even though they do present no details of their final equations, it could be noticed that for material time derivatives, they ~~take~~ no account of the spatial coordinate \bar{x} being a function of time. This indicates the relative and referential nature of their analysis. Also, they discuss some previous relative formulations under the name of Eulerian formulation. They close their analysis with a discussion of a general formulation of finite element equations for large deformations with an arbitrary choice of the conjugate stress and strain measures. Wunderlich [6] introduces a consistent development of an updated Lagrangian formulation on the basis of the equations of continuum

mechanics. However, as in his Lagrangian formulation, he omits specific nonlinear terms in the intended linearization process within the increment. Bathe et al. [9-12] follow a similar approach, but they introduce an implicit final form of nonlinear equilibrium equations where all the nonlinear stiffness contributions are combined into one stiffness term. Yamada [225] and Yamada and Sakurai [226] introduce an updated Lagrangian formulation utilizing rate quantities. Yamada's formulation leads to a development of a load correction factor for cases in which coupling exists between the stresses and the boundary velocities. Yamada's expression for the load correction matrix [225] is based on specifying the load vector associated with the updated surface area. The expression of the load vector in Yamada's formulation depends on the state of deformation. This dependency arises from the expression of the change in the surface area. The expression developed is, however, dependent on the type of the stress rate utilized in the constitutive equation, and, moreover, it is also noticed that Yamada suggests the use of an infinitesimal form of the strain rate-velocity relationship without the inclusion of second order terms. This procedure eliminates the initial displacement stiffness matrix from the final equilibrium equations. As explicitly shown in the development of Gadala, Dokainish and Oravas [15], we may conclude this particular discussion by saying that the most general approach for updated Lagrangian formulation is to consider fully nonlinear kinematic relations within a linear increment. Also, none of the stiffness contributions should in general be omitted. In specific applications,

however, certain simplifications of stiffness expressions may be achieved [224].

Considering the point of the interpretation of the energy balance equation in the current ~~reference~~ configuration, it will be shown that following the same ~~procedures~~ of Lagrangian formulation, we may have a surface traction term which is dependent on the relative deformation gradient between the current configuration C_1 at time $t = \tau$ and a neighbourhood configuration C_2 at time $t = \tau + \Delta t$. However, the relative deformation gradient in updated Lagrangian case is dependent on the incremental displacements from configuration C_1 to C_2 , and not on the total displacements as in Lagrangian formulation. This fact, together with the assumption of linear increment, gives differences between Lagrangian and updated Lagrangian formulations in the initial load stiffness matrix and the load increment. However, the same number of stiffness contributions are found and essentially similar concluding remarks to those given in the Lagrangian case may be stated here [15,38].

1.4 EULERIAN OR SPATIAL FORMULATION

As discussed above, the key difference between Eulerian formulation and other formulations is the fact that in Eulerian formulation the independent variables are the current position \bar{x} and the time t where \bar{x} is implicitly dependent on t . This fact complicates material derivatives and other relations when approached strictly by Eulerian, or spatial formulation. Also, while there are infinitely many referential

descriptions of a given motion, there is only one spatial description.

Nemat-Nasser and Shatoff [36] introduce an early accurate discussion of the differences between referential and spatial formulations. In [36], Nemat-Nasser and Shatoff utilize an absolute minimum principle for small deformations superimposed on a finitely deformed, stable configuration of an elastic solid, to develop their numerical solution. An incremental solution of large deformation problems including material as well as geometrical nonlinearities is presented. Nemat-Nasser and Shatoff opt. cit., give a short account of the basic concepts and equations which are necessary for the treatment of large deformation problems. They introduce an accurate discussion of the constitutive equation, and the stress and strain increments in both the Lagrangian and the Eulerian formulations. However, Nemat-Nasser and Shatoff opt. cit., seem to overlook the difficulties in obtaining material time derivatives in spatial formulation. They introduce a finite element displacement solution and final equations which are only applicable for the case of Lagrangian formulation.

Another attempt at the Eulerian formulation is made by Hartzman and Hutchinson [35]. In [35], Hartzman and Hutchinson analyze the nonlinear dynamic response of deformable solids subjected to time and space dependent thermal and mechanical loads. They also consider both the material and the geometrical nonlinearity in the analysis. The governing partial differential equations of the system are reduced to a set of simultaneous nonlinear ordinary differential equations which are solved by applying a step-by-step numerical technique in conjunction

with the constitutive relations. No explicit stiffness expressions are given and the procedure is meant to solve dynamic rather than static and/or dynamic structural behaviour. Hartzman and Hutchinson *opt. cit.*, introduce independent assumptions for the displacements and the incremental displacements in the form

$$u_i(\bar{x}, t) = Q_i(t) + C_{ij}(t) x_j \quad (I-9)$$

$$\Delta u_i(\bar{x}, t) = \Delta Q_i^t(t) + C_{ij}^t(t) x_j^t \quad (I-10)$$

where the superscript t indicates that the quantity is evaluated at time t and C_{ij} are the coefficient functions of the nodal coordinates and the nodal displacements. Owing to the special decomposition of the kinematics into products of functions of time and coordinates as given by eqns. (I-9, I-10), it seems that these assumptions do restrict the class of problems such formulation may handle. Expressions (I-9, I-10), in effect, account for linear coordinate translation through the term $Q(t)$. Even though rigid body rotations may be accounted for in the second term, finite rotations within the element are ignored. Moreover, the type of decomposition given by eqns. (I-9, I-10) seems to impose a referential nature on the formulation. As in [36] Hartzman and Hutchinson [35] introduce an accurate discussion of the treatment of the constitutive equations and stress and strain increments in spatial formulation. Apart from the assumptions (I-9, I-10) and the discussion given above, they utilize a proper time derivative for the acceleration term, point out its complications and provide a lumped mass approximation to overcome the difficulties.

Another treatment of the subject is given by Key [34]. Similar to Hartzman and Hutchinson [35], Key [34] discusses the solution of large deformation dynamic problems. Following Truesdell and Noll [52], Key [34] gives a brief account of the kinematics of the problem. He discusses basic differences between different formulations, and points out that if the mechanics is carried out properly and the numerical procedures are sound, the differences in the results should be minor when translated to a common frame of reference. Key's work accounts for both material and geometrical nonlinearities but, similar to the treatment of Hartzman and Hutchinson, Key [34] introduces the final equations in a form suitable only for dynamic analysis. The analysis is carried out in the current configuration described by a fixed spatial coordinate system and using the Cauchy stress. A careful treatment of the constitutive models and the time integration schemes is given. To develop the final equilibrium equations, Key introduces a displacement assumption in the form

$$u_i(\bar{x}, t) = \psi_{i\alpha}(\bar{x}) \underline{u}_\alpha \quad (\text{I-11})$$

where $\psi_{i\alpha}(\bar{x})$ represent the basic interpolation functions and \underline{u}_α represent the nodal displacements. If the interpolation functions are fixed in the coordinates of the reference configuration so that $\psi_{i\alpha} = \psi_{i\alpha}(\bar{X})$, which is the case in the referential and the relative formulation, then the material time derivatives are straightforward. In spatial formulation, however, $\psi_{i\alpha} = \psi_{i\alpha}^s(\bar{x})$, where \bar{x} is an implicit function of t , the material time derivatives become awkward. To

overcome this basic difficulty, Key [34] restricts his analysis to isoparametric elements for which the coordinates effectively become material coordinates and play the same role as the coordinates of the reference configuration. This approach essentially eliminates the basic nature of the spatial formulation and makes the treatment similar to the referential or the relative formulation.

As discussed by Gadala, Oravas and Dokainish [37], a general approach for developing a spatial formulation may depend on the energy balance equation in the current deformed configuration. Differences in spatial formulations may be owing to different simplifying kinematic assumptions. Within the framework of linear increments with fully nonlinear kinematic relations, specific simplifying assumptions have to be imposed so that we may obtain the final incremental equilibrium equations in a feasible form for numerical applications. Such simplifying assumptions, however, should not eliminate the basic nature of the spatial formulation. The final incremental equilibrium equations may be presented in a form suitable for both static and/or dynamic structural analysis. This would require detailed expressions for stiffness matrices, load vector as well as for the mass matrix. When such analysis is carried out [37], it is found that there are only two stiffness matrices contributing to the total nonlinear stiffness matrix of the element. Owing to the nature of each formulation, strict comparison between referential and relative formulations on one side and spatial formulation on the other, may not be logical. However, the two stiffness matrices in spatial formulation correspond to the usual small

displacement stiffness matrix and to the initial displacement or initial rotation stiffness matrix. In referential and relative formulations, the most used stress and strain measures are the second Piola-Kirchhoff stress $\bar{\bar{S}}$ and the Euler-Lagrange strain tensor $\bar{\bar{E}}$. The corresponding stress and strain measures in spatial formulation are the Cauchy stress $\bar{\bar{\sigma}}$ and the rate of deformation tensor $\bar{\bar{D}}$. The use of $\bar{\bar{\sigma}}$ and $\bar{\bar{D}}$ in spatial formulation eliminates a correspondence with the initial stress or the geometric stiffness matrix as could be seen from the analysis. Contrary to the referential and the relative formulations, the load vector in spatial formulation is independent of the deformation gradient. This fact eliminates the initial load or the load correction matrix in spatial formulation. Basically any conjugate pair for stress and strain measures may be utilized to develop an Eulerian or spatial formulation.

I.2 NUMERICAL SOLUTION OF THE NONLINEAR EQUILIBRIUM EQUATIONS

I.2.1 INTRODUCTION

In what follows, an attempt is made to assess different techniques currently employed for numerical solution of the nonlinear equilibrium equations. The development in this section is largely dependent on the work of Tillerson et al. [53], however, an emphasis is put on the recent modifications and improvements of the basic techniques. To allow a unified discussion, we will utilize the classification given by Tillerson, Stricklin & Haisler [53]. In [53], different solution techniques are classified according to the manner in which the

force unbalance is treated. The force unbalance may be defined as

$$\{F\} = - [K] \{u\} + \{Q\} + \{R^*\} \quad (I-12)$$

$$= - [K] \{u\} + R\{r\} + \{R^*\} \quad (I-13)$$

where $[K]$ is the linear global stiffness matrix, $\{u\}$ is the nodal displacement vector, $\{Q\}$ is the generalized force vector, $\{R^*\} = \{R^*(u)\}$ is a pseudo-force vector due to nonlinearities, R is a load parameter, and r is a normalized load vector. Equation (I-13) allows definitions of the derivatives of the force unbalance $\{F\}$ with respect to the load parameter R .

The first class of solution procedures seeks an exact equilibrium state, hence requires $\{F\} = \{0\}$. The second class of solution seeks to minimize the force unbalance vector by requiring $\{\dot{F}\} = \{0\}$, where the indicated differentiation is with respect to the load parameter R . The main disadvantage of the second class of solution is the tendency of the solution to deviate or drift from the exact equilibrium path. This drifting tendency resulted in the development of the so-called self-correcting solution procedures, which may be considered as a final class of solution techniques.

I.2.2 EXACT EQUILIBRIUM TECHNIQUES

Successive Approximations

The successive approximation method is used efficiently by many analysts [54-63]. The method is characterized by the iterative relation

$$[K] \{u\}_{n+1} = \{Q\} + \{R^*\}_n \quad n = 1, 2, \dots, N \quad (I-14)$$

where n denote the n th iteration and N is the total number of iterations

permitted for convergence. Depending on the behaviour of the nonlinear force-displacement relation, the procedure may or may not converge. In most of the literature on plasticity analysis, this procedure is termed the initial strain method. References [64,65] discuss different schemes of the initial strain method, depending on the way the total load vector and the pseudo-load vector are treated. However according to the above problem classification, some of these techniques, namely the direct incremental initial strain method, will fall under the category of the incremental loading procedures which will be discussed later.

Improvements in this method consists of either modifying the ~~initial~~ estimate of the displacement vector or accelerating the convergence. Wachspress [66,67] described a variational acceleration technique which involves periodic interruption of the iterative procedure with direct solution of a related coarse grid finite element problem. The solution is used to update the last fine grid iteration before continuing the iteration into the next cycle. Bathe and Wilson [68] discussed the numerical sensitivity of finite element systems in general and included the successive approximation method. Wilson [69] proposed a method of expressing the equilibrium equations in terms of absolute and relative, rather than only absolute, degrees of freedom. This was shown to give more flexibility in tackling the problem and a higher convergence rate. Argyris and Scharpf [70] combined the Newton-Raphson method with the successive approximation technique to give a modified initial strain procedure, which possesses better convergence characteristics when applied to material nonlinearity problems.

In [60,61], Young provides an extensive discussion about some important aspects in iterative solutions for linear and nonlinear systems. In reference [60] Young discusses different iterative schemes in relation with the convergence rate. In [61] two points are discussed. First, is the choice of the iteration parameters and its effect on the convergence rate. Second, is the proper timing and criteria for terminating the iteration procedure. A detailed discussion of convergence criteria is, however, beyond the scope of this section.

Only limited numbers of geometrical nonlinearity problems, with average degree of nonlinearity, have been analysed by the successive approximation method, whereas few successful applications of the method in the area of material nonlinearity are achieved. The problems of choosing a proper initial estimate, convergence criteria as well as the lack of physical interpretation of the intermediate steps in most of the iteration techniques lead to the limited applicability of the method to specific types of problems and to the feature that makes the method mostly acceptable by the experts rather than the beginners. The trend of the recent research might, however, lead to more successful applications of the method in both geometrical and material nonlinearity problems.

Newton-Raphson Method

The Newton-Raphson method seeks a Taylor's expansion of the force unbalance about the point $\{u\}$, retains only the first order increments of $\{u\}$, and assumes the force unbalance at point $\{u+\Delta u\}$ to be zero. The

method is characterized by the recurrence relations [13,53,71]

$$([K] + [K^*]_n) \{\Delta u\}_{n+1} = \{R\}_n \quad n = 1, 2, \dots, N \quad (I-15)$$

and $\{u\}_{n+1} = \{u\}_n + \{\Delta u\}_{n+1} \quad (I-16)$

$$= \{u\}_n + \alpha \{D\} \quad (I-17)$$

where $[K^*]$ is the nonlinear stiffness due to estimated displacement $\{u\}_n$, $[K^T] = [K] + [K^*]$ is the usual tangent stiffness matrix, $\{D\}$ is a current search direction or the direction of the displacement increment vector and α resembles an over-relaxation factor which governs the correction term or the displacement increment.

The method found good acceptance in the area of geometrically nonlinear problems, particularly for large displacement and stability analysis [71-74]. This is probably owing to the ability of the method to converge for highly nonlinear problems. The main drawback associated with the method is the large amount of computational effort required to invert the matrix $[K^T]$ at each cycle. This led to a modified Newton-Raphson procedure [74] wherein the coefficient matrix is held constant for several iterations and is updated only when the rate of convergence begins to deteriorate. It should be noted that the conventional Newton-Raphson method is equivalent to a search direction $\{D\}$ equal to $[K^T]^{-1} \{F\}$ and an over-relaxation factor α equal to unity, whereas the modified procedure would have a search direction equal to $[K]^{-1} \{F\}$ and an over-relaxation factor of unity. Irons and Elsayaf [71] and Oden [13] discussed other possible considerations of the factors $\{D\}$ and α . In material nonlinearity problems, the application of the Newton-Raphson technique has been rather limited to problems in

which no unloading occurs. Hofmeister et al. [44] employed an incremental formulation with the Newton-Raphson technique being retained as an equilibrium check and applied this method to large strain elasto-plastic problems.

Direct Minimization of Potential Energy

A distinct approach to nonlinear structural analysis is the direct minimization of the total potential energy [75-78]. Most of the minimization techniques applied in structural analysis have been sequential rather than random in nature. Furthermore the gradient methods in contrast to the direct search methods have proven most applicable [13]. Such gradient techniques are characterized by a recurrence relation of the same form of eqn. (I-17). The multiplicity of gradient techniques, such as univariant methods, the method of steepest descent, conjugate gradient methods and variable-metric method [13], differ in the determination of the corrective terms, α and $\{D\}$ in eqn. (I-17).

The main advantage of the method is probably its potential to save computer storage, as in most cases there is no need to form the conventional structural stiffness matrix. The main drawback of the method is the problem of convergence and the possible tendency of convergence to a local rather than a global minimum. Applications of the method has been successfully employed in geometric nonlinearity problems [77], whereas applications in material nonlinearity problems has been rather limited [78].

I.2.3 INCREMENTAL LOADING PROCEDURES

Incremental Stiffness Procedure

The incremental stiffness procedure is by far the most widely used technique in finite element applications to problems of nonlinear structural analysis. The technique seeks a minimum unbalance force vector $\{F\}$ by insuring $\{\dot{F}\} = \{0\}$. The basic incremental stiffness solution procedure considers the pseudo-force vector as a nonlinear stiffness term and hence it is characterized by the recurrence relation

$$[K^T]_m \{\Delta u\}_{m+1} = \{\Delta Q\} \quad m = 0, 1, 2, \dots, M \quad (I-18)$$

and

$$\{u\}_{m+1} = \{u\}_m + \{\Delta u\}_{m+1} \quad (I-19)$$

where $\{\Delta u\}_{m+1}$ is the incremental displacement at the $(m+1)$ th increment and M is the total number of load increments. An alternative formulation is to consider the pseudo force vector rather than the nonlinear stiffness term, and it is usually referred to as the incremental initial strain procedure [64]. This formulation would be characterized by a recurrence relation of the form

$$[K] \{\Delta u\}_{m+1} = \{\Delta Q\} + \{\Delta P^*\} \quad (I-20)$$

together with eqn. (19). This alternative formulation possesses the obvious advantage of formulating and inverting the stiffness matrix $[K]$ only once. Regarding the step increment size, Yamada et al. [79] suggested a varying load increment sufficient just to cause yielding in the successive elements and used this for elastoplastic problems. Cormeau [80] and Zienkiewicz and Cormeau [81] give an expression for the step size in accordance with the accuracy and stability of the solution

for viscoplasticity and creep applications.

The primary disadvantage associated with the incremental solution is the lack of accuracy determination at any load level. This leads to a drifting or deviation tendency from the nonlinear load behaviour path and creates the need for corrective techniques by using an equilibrium check after each increment [44,82]. A modification to the first procedure, in which a stiffness term for the nonlinear behaviour is utilized, is the midpoint Runge-Kutta technique [64]. In this technique, a stiffness matrix corresponding to the midpoint displacement of an increment is used for the whole increment to provide better accuracy than simply halving the increments. Another modification is the initial stress method proposed by Zienkiewicz et al. [57]. In this modification an internal iteration within each increment is performed without changing the stiffness matrix of the increment. The initial stress method was shown to overcome the convergence problems of the successive approximation and initial strain methods when applied to material nonlinearity, especially perfect plasticity, problems [64,70]. Argyris and Scharpf [70] combined the Newton-Raphson technique with the initial stress technique to give a modified initial stress procedure of improved convergence characteristics in cases when the original initial stress method was converging too slowly. Owing to the drifting phenomena, the application of this method has been rather limited in geometrically nonlinear problems, but widely used in materially nonlinear applications [82-87].

Static Perturbation Method

The static perturbation method is another solution procedure under the class seeking the minimum force unbalance vector [88-94]. Choosing the load increment as a path parameter between two displacement states u and $(u+\Delta u)$, the method seeks to determine the displacement response at a load level $(Q+\Delta Q)$ from that at a load level Q through a Taylor's expansion. The method is characterized by the recurrence formula of the form [93]

$$[K^T] \{\dot{u}\} = \{p\} \quad (I-21)$$

with similar equations for higher order terms.

In [93], Gallagher presents an extensive coverage of the perturbation procedures in nonlinear finite element analysis. Gallagher's work points out the correlation between simple incremental techniques, incremental-iterative techniques, and a corresponding approximation levels in the perturbation technique depending on the number of terms retained in the analysis through higher order equations similar to eqn. (I-21). Utilizing the perturbation technique, Gallagher, *opt. cit.*, discusses possible procedures for differentiating between limit point loads and bifurcation loads and indicates that the perturbation method is particularly effective in calculating the slope of the immediate post-bifurcation regime. This is probably owing to the ability of the method to retain a significant number of terms in the series expansion. The main drawback of the method is, however, the substantial increase in effort and computer time required to retain such higher order terms in the analysis.

I.2.4 SELF-CORRECTING SOLUTION PROCEDURES

One of the self-correcting solution techniques is the first order self-correcting procedure which utilizes a characteristic relation of the form

$$\{\dot{F}\} + z \{F\} = \{0\} \quad (I-22)$$

where z is a scalar amplification factor and the indicated differentiation is once again with respect to the load parameter R . Similar to the incremental technique, eqn. (I-22) leads to either considering a pseudo unbalance force vector or a nonlinear stiffness term in the final equilibrium equations. Substitution of eqn. (I-22) into eqn. (I-12) may be shown to give [53]

$$[K] \{\Delta u\}_{m+1} = \{\Delta Q\} + \{\Delta R^*\}_m + z \Delta R \{F\}_m \quad (I-23)$$

and

$$[K^T] \{\Delta u\}_{m+1} = \{\Delta Q\} + z \Delta R \{F\}_m \quad (I-24)$$

A logical argument of choosing the factor $z \Delta R$ equal to unity would obviously appear from eqns. (23) and (24) [ref. 44].

In reference [77], a self-correcting procedure is described in which the corrective load vector $z \Delta R \{F\}_m$ is combined in an iterative manner with the pseudo-force increment $\{\Delta R^*\}$. Bushnell [95,96] described a rather similar approach which he called the subincremental technique. In this technique there are two loops of calculations. The inner loop is an ordinary Newton-Raphson loop in which the material properties are held constant, while in the outer loop nonlinear strain increments are determined and the tangent stiffness properties are updated. Bushnell's technique seeks a response which is closer to both

the equilibrium path, through the use of Newton-Raphson loop, and the material property curve, through the iterative computation of the constitutive equation in the outer loop.

The first order self-correcting method was successfully applied to a wide range of both material and geometrical nonlinear problems as reported in references [95-99]. A logical extension to the first order self-correcting procedure is the second order self-correcting method. Similar arguments and extended forms of the recurrence relations, for the second order method, may be obtained in a manner similar to that discussed above [100].

I.3 INCOMPRESSIBILITY CONSTRAINT

I.3.1 ANALYTICAL TREATMENT

(1) Modifying the Strain Energy Density

One of the most common approaches to account analytically for incompressible material behaviour is to modify the strain energy function and the stress-strain relations such that the incompressibility condition may be included. For an isotropic elastic material behaviour, the strain energy density, W , may be considered as a function of the strain invariants I_1 , I_2 and I_3 . For an incompressible elastic material behaviour, however, the third strain invariant is unity and the strain

energy density may be considered as a function of only I_1 and I_2 . The modification of the stress-strain relations relies on the fact that, for an incompressible material, the addition of an arbitrary hydrostatic pressure will affect the stresses but it will not affect the strains. This indicates that the stresses are not completely determined by the deformations and, in fact, we may add an arbitrary hydrostatic pressure to the stress tensor in order to account for the incompressibility constraint.

In a series of paper [101-109], Rivlin utilizes the above approach in the analysis of large elastic deformations of isotropic incompressible and compressible materials. Truesdell [111] marks Rivlin's work [104] as the beginning of the modern theory of finite elastic strains. Rivlin, *opt. cit.*, utilizes a strain energy density function, for incompressible materials, which is first suggested by Mooney [112] and later represents the Mooney-Rivlin material. Mooney's work is based on semi-empirical grounds and on the assumptions that the material is isotropic, volume changes and hysteresis are negligible and, finally, the material obeys Hooke's law in simple shear when large elastic deformations are considered. Rivlin [106,107] points out that the simple form of the Mooney-Rivlin strain energy function

$$W = C_1 (I_1 - 3) + C_2 (I_2 - 3), \quad (I-25)$$

where C_1 and C_2 are physical constants, is the most general form that can be taken by the function if terms of third and higher orders in the principal extensions are neglected. In [106] Rivlin gives the explicit stress-strain relations for both compressible and incompressible

materials. In [101] a method for determining the unknown hydrostatic pressure is presented for a given stress state and it is pointed out that if only the strain state is specified, the hydrostatic pressure is indeterminate. Rivlin [103-106], and Rivlin and Thomas [110] show the application of this approach to different practical problems whereas Rivlin [107] uses the theory of large elastic deformations of incompressible materials to interpret the load deformation curves of certain types of deformations of incompressible vulcanized rubbers.

The work of Mooney and Rivlin, *opt. cit.*, is followed by the efforts of many authors to use similar, and sometimes modified, approaches for the solution of different practical problems. As an example, we mention the work of Green and Shield [113], Ericksen [114,115], Knowles [116], Truesdell in his continuum mechanics treatise [117], Oden [13], Ogden [118-121], and many others [122-127]. The work on the subject may be classified into two schemes. The first is concerned with a general formulation of the problem as a continuum mechanics problem [13,115,117] and an application of the approach to different field problems using the Mooney-Rivlin material model [113,114,116,122,123]. The second is concerned with a modification of the strain energy density function to account for different material types of incompressible rubbers and to achieve better correlation with experimental data [118-121, 124-127]. Oden [13] and Ogden [118-120] present a concise survey on the development of strain energy density functions for incompressible materials. Ogden [118-120] discusses the strain energy expressions for incompressible materials as a function

of the strain invariants I_1 , and I_2 as independent variables and points out that this complicates the associated mathematical analysis and creates the need for complicated forms of the strain energy function if better correlation with experiments is required. Ogden, *opt. cit.*, proposes a strain energy expression which is a function of the principal stretches a_1 , a_2 and a_3 as independent variables, where the principal stretches are subjected to the incompressibility constraint $a_1 a_2 a_3 = 1$. Ogden's strain energy function is a generalization of the strain energy function suggested by Verga [128]

$$W = \mu(a_1 + a_2 + a_3 - 3) \quad (\text{I-26})$$

where μ is a physical constant. Verga [128] specifically proposes his function as a first approximation to the behaviour of rubber like solids. This function, eqn. (I-26), is not expressible as a simple function of the strain invariants I_1 and I_2 . Ogden's approach is applied to problems of simple tension, equibiaxial tension and simple shear and is shown to retain simplicity throughout the analysis. In reference [129] similar development is applied to study the dynamics of a linearly inhomogeneous incompressible isotropic elastic half-space, where the equilibrium equations and the kinematic relations are developed for incompressible material under the condition that the third scalar invariant of the deformation gradient tensor is equal to unity.

(ii) Utilizing Mixed Variational Principles

Another method of approach which accounts for incompressible material behaviour is to formulate the problem as a variational problem

in which two field variables are varied: the displacements and the hydrostatic pressure. Much of this work stems from a paper by Reissner [130] in which he presents a variational principle that has both displacements and stresses as field variables. Reissner's work is based on the original work of Hellinger [131] and the resulting variational principle is usually referred to as Hellinger-Reissner principle. The Euler equations of the Hellinger-Reissner functional are the stress equilibrium equations, the stress-strain relations, and the stress boundary conditions as a natural boundary condition for the variational principle. The Hellinger-Reissner principle [130] is derived and applied to infinitesimal elastic strain problems. Reissner's work is extended by Reissner himself to cover problems of finite elastic deformations [132], by Biot [133] and Herrmann [134] to cover problems of thermoelasticity and heat conduction. The Euler equations of Biot's principle are the thermoelastic equilibrium equations and the heat conduction equations. Herrmann [134] extends Biot's work to the coupled processes of thermoelasticity and heat conduction in a three-dimensional anisotropic elastic body. Tonti [135] presents a thorough discussion and comparison of the above mentioned variational principles. The early extension of dual or multi-field variational principles to cover elastodynamical problems is due to Toupin [136]. Toupin's work is followed by similar work by Chen [137] and Gladwell et al. [138]. A very careful treatment of the subject is given by Veubeke [24, 139-142] and by Washizu [32]. By introducing Lagrange multipliers, Washizu [32] and Hu [143] generalize the principle of minimum potential energy to

give what is later known as the Hu-Washizu principle which has the strains, displacements, stresses and surface tractions as independent variables. Washizu points out that the original work of Hellinger [131] and the work of Reissner [130] may be interpreted as a special case of the Hu-Washizu principle. Following the original work of Friedrichs [144] for one dimensional problems, Veubeke [139-141] constructs a three-field variational principle from which most of the known variational principles including the Hellinger-Reissner one may be derived through "limitation principles" or simply specializing assumptions. Veubeke, *opt. cit.*, also discusses a canonical principle which when combined with his three-field variational principle leads to the two-field Hellinger-Reissner principle and points out that the latter is a special case of the Legendre transformation in which a unique inverse of the stress-strain relations should exist for the transformation to be valid (see also Washizu [32], and Courant and Hilbert [145]). An extensive survey on the historical development of energetical principles in elastomechanics is given by Oravas and McLean [148, 149].

The general Hellinger-Reissner principle, as well as the general dual or multi-field principles, usually introduces an excessive number of unknowns which make the utilization of such principles in approximate solutions and their application to practical problems rather difficult and uneconomical. One way of avoiding such problems is to restrict the dual principle in such a way that the only independent variation of the stress state be the hydrostatic pressure. Much of the work done on

incompressibility using the finite element method follows this particular simplification, and the discussion of this approach will be considered in section I.3.2.

(iii) Utilizing a Perturbation Technique

Initiated by a paper by Spencer [148], a number of researchers have examined analytically the problem of incompressible and nearly incompressible elastic deformations, by utilizing a perturbation technique. Spencer [148] presents a perturbation procedure in which the strain energy function is modified by adding to it a small perturbation. The stress-strain relations and the equations of equilibrium are then reformulated which, with suitable boundary conditions and a knowledge of the solution for the original strain energy, make possible the determination of the displacements and stress components required for the adjustment of the original solution. Later, Spencer [149] applies his approach to the solution of small compressibility problems with the deformations being nearly isochoric. Ogden [118-121] elaborates on Spencer's approach and points out that the isochoric part of the deformation in a nearly-incompressible material will differ by a term of order η from the corresponding exact isochoric deformation, where η is the ratio of the shear modulus, μ , over the bulk modulus, K . Ogden's development [121] is based on the assumption that the surface tractions are not allowed to reach the order of the shear modulus, μ . Based on the above result, and to a first order in $\mu\eta$, Ogden [121] presents expressions for the constitutive equations and the governing equations

for the solution of problems of compressible, nearly-incompressible and exact-incompressible material behaviour. Levinson [123] applies a similar perturbation scheme to the analysis of finite torsion of slightly incompressible rubber-like circular cylinders. Different strain energy functions representing different classes of incompressible materials have been considered. Through a comparison between the regular perturbation procedure and the Rayleigh-Ritz method, Levinson [122,123] points out that the perturbation procedure fails for certain constitutive relations whereas, in principle, the Rayleigh-Ritz method has general applicability. Levinson, *opt. cit.*, indicates that the assumed displacement field for the Rayleigh-Ritz procedure has to satisfy the continuity, the geometrical boundary conditions and, in case of incompressibility, it also has to satisfy the incompressibility constraints. Levinson provides neither account nor discussion on the accuracy of the solution when incompressibility is introduced directly on the displacement assumption (for details, see next section). Argyris et al. [150] present a similar approach in which the strain energy density is divided into two parts: the deviatoric or shear strain energy, and the volumetric or dilatational strain energy. Details of Argyris's numerical treatment will be considered in the following section.

I.3.2 FINITE ELEMENT APPLICATIONS

(1) Sources of Difficulties

In the words of Oden [13]: "Although in some exact solutions to problems in finite elasticity the assumption of an incompressible

material leads to certain simplifications in the analysis, such is not the case in finite element applications".

The numerical difficulties in treating incompressible isotropic elastic solids, i.e. solids for which Poisson's ratio, ν , is equal to one-half, are owing to the existence of a singularity in the displacement equations of equilibrium. The potential energy principle of such solids contains a coefficient which goes to infinity as Poisson's ratio approaches one-half. In finite element application the above point is reflected in a loss of accuracy in the approximate displacement field solution. This means that the approximation characteristics of the element can be entirely lost and the condition of the numerical calculation of stiffness matrices may deteriorate indefinitely. As a result, the finite element stress field will be of little, or even of no, value. Furthermore, for nearly-incompressible materials, i.e. materials for which Poisson's ratio only approaches one-half, the equivalent variational problem does not lead to a numerically well-behaved system of equations [151-155].

(ii) Direct Imposition of Incompressibility on Nodal Degrees of Freedom

Probably the most logical method for incorporating the incompressibility constraint is to directly restrict the displacement field to be an incompressible one. It is owing to studies by Nagtegaal et al. [156], Fried [153], Naylor [157], Argyris et al. [158], and others, that revealed that this apparently simple and logical approach is certainly deceptive and actually a difficult one. In the above work, it is shown

that finite element formulations, in which incompressibility is directly imposed on the displacement field, are especially sensitive to the type of element, element arrangement and orientation in the mesh, total degrees of freedom and its relation to the total number of constraints, type of the problem to be analyzed and its particular geometry, boundary conditions as well as other factors. In [156], Nagtegaal et al. present a simple analytical discussion of the idea of 'mesh locking' or 'locking of degrees of freedom' as a result of effectively reducing the nodal degrees of freedom through direct imposition of the incompressibility constraint. It is shown that such limitation will create an unrealistic and extrinsic constraint on the kinematics of the body and in some specific cases it enforces a uniform strain state all over the body regardless of the boundary conditions. Nagtegaal et al., opt. cit., show that the above result is closely related to the grid configuration and the type of element. In [156] the ratio of the degrees of freedom over the constraints is taken as a criterion for convergence and is calculated for typical element arrangements. When this ratio is less than unity, convergence will not in general occur. For planar elements, the ratio is shown to be greater than unity only for the 6-node linear strain triangles in plane strain whereas for three dimensional elements the ratio is greater than unity only for the 10-node linear strain tetrahedra arranged in cubes. In [159], Needleman and Shih present a similar approach which is applicable only for specific finite element meshes and specific boundary conditions utilizing plane strain quadrilateral elements with each element divided

into four triangles.

Nagtegaal et al. [156], propose an alternative of using a mixed variational principle of the Hellinger-Reissner type. They introduce a modified mixed principle which has the field variables as the displacements and only the dilational strain as an auxiliary field. The proposed principle is analytically transformed to an equivalent one which has the displacement as the only field variable.

It is, however, significant to notice that the treatment given by Nagtegaal et al. is carried out for specific element types with specific forms of interpolating function for the auxiliary field. It is also observed that the derived modified functional contains terms like Kg and Kg^2 where K is the bulk modulus and g is the auxiliary field. For exact incompressibility, where ν is equal to one-half and K is equal to infinity, Nagtegaal et al. propose the substitution $Kg = p$ where p is a new field variable. It is not, however, clear how this substitution is applied to the modified functional. Finally, the proof given by Nagtegaal et al. is only applicable for elastic materials whereas for elastoplastic materials, a variational principle given by Hill [20] is proposed as an alternative principle. However, the development of Hill's variational principle accounts only for stress rates which is suitable for incompressible materials. It is owing to the above complications that alternative finite element formulations for incompressible materials have been searched for.

(iii) Multi-Field Principles and Lagrange Multiplier Methods

Hellinger-Reissner Principle and its Modification

Many authors have used dual principles or Lagrange multiplier methods to account for incompressible material behaviour in finite element energy formulations [151, 152, 155 and 160-164]. Much of the work on the subject stems from Reissner's work [130] and the extensive treatment of Veubeke [139-142]. The immediate advantage of using Hellinger-Reissner principle, or other dual principles, in the numerical analysis of incompressible materials, is the elimination of the difficulties discussed above which appear when using the potential energy principle or other single-field variational principles. The two principles are, however, closely related through the canonical or Legendre transformation. Courant and Hilbert [145] discuss the derivation of Hellinger-Reissner principle from the potential energy principle through a Legendre transformation. The exact use of the Hellinger-Reissner principle in finite element applications is, however, uneconomical as it introduces an excessive number of unknowns. To decrease the total number of unknowns in the numerical treatment, a modified, or restricted Hellinger-Reissner principle is sought which has the displacements and only a single stress, the pressure, as field variables [139-142, 160-163]. Veubeke's derivation of Hellinger-Reissner principle [139-143] directly leads to the restricted principle which is suitable for the analysis of incompressible and nearly-incompressible material behaviour. One of the early accounts of the restricted Hellinger-Reissner principle is given by Herrmann [160]. Herrmann

states the equations of a boundary value problem in linear elasticity in terms of the displacements and a function of the mean pressure. In the limit, when Poisson's ratio takes the value of one-half, these equations become the governing equations for an incompressible elastic material [165]. The variational principle, presented by Herrmann [160], has the displacements and the mean pressure as field variables and its Euler equations are the governing equations for an incompressible elastic material. Later, Taylor et al. [151] extended Herrmann's approach to cover orthotropic thermoelastic problems. In the finite element application of Herrmann's approach, the pressure interpolation function is considered uniform over the element. The use of Herrmann's approach in finite element analysis leads to three different cases in considering the interpolation function for the hydrostatic pressure [162, 166]. These are

- a. The interpolation function of the pressure is assumed to be uniform over the element. This leads to the least number of additional unknowns, and to the satisfaction of incompressibility on the average which is usually called 'average incompressibility'.
- b. The interpolation function of the pressure is assumed to be of the same order as that for the displacement. This leads to what is usually called 'complete incompressibility'.
- c. The interpolation function of the pressure is assumed to be of higher order than that for the displacement. This leads to the highest number of additional unknowns, redundant equations

describing the constraints on the nodal displacements and, finally, results in a singular global stiffness matrix.

Thompson [162] and Thompson and Haque [166] point out that the use of complete incompressibility produces higher potential energy, whereas the use of average incompressibility may produce lower potential energy with respect to the exact solution. This is referred to the fact that, in the case of complete incompressibility, the incompressibility is satisfied everywhere and the resulting finite element displacement approximation is admissible with respect to the original energy formulation. In average incompressibility, however, the same condition is not satisfied and the resulting displacement field is not admissible with respect to the original energy formulation [166]. Thompson [162] presents a numerical comparison between the two approaches; average and complete incompressibility using plane strain linear stress-linear strain elements. The results obtained support the use of average incompressibility both from the convergence point of view as well as the computer time consideration. Oden [13] indicates that if element stresses may be used as a guide, the order of the local interpolation function of the pressure should be less than that of the displacement interpolation. The preference of using average incompressibility models, or models with pressure interpolation function of an order less than that of the displacement approximation, may be explained by the additional flexibility of these models provided by the relaxation of the incompressibility constraint. Considerations of convergence and accuracy of general dual models are, however, strongly related to the

use of reduced/selective integration techniques, which will be discussed in a later section.

Hybrid Methods

The use of hybrid techniques in the analysis of incompressible material behaviour, is a logical extension to the dual principle approach. As mentioned above, de Veubeke [141,142], Hu [143] and Washizu [32] present a general multi-field variational principle from which most of the finite element models may be deduced through specializing assumptions. In reference [164] Scharnhorst and Pian present a multi-field variational principle with the condition of incompressibility appearing as a constraint equation applied to the potential energy functional through the use of a Lagrange multiplier which has the physical meaning of a hydrostatic pressure. The Euler's equations of the variational principle are shown to be the governing equations of the elastic problem together with the incompressibility constraint. The field variables in their formulation are the displacements as well as both the distortional and hydrostatic stresses. Following the standard procedures of the hybrid element formulation, the distortional stresses may be eliminated on an element level giving final equilibrium equations which only contain the displacements as well as the hydrostatic pressures as field variables. Murakawa and Atluri [155] present another hybrid-stress finite element model based on a complementary energy principle. Their variational principle has, as independent variables, the first Piola-Kirchhoff stress tensor, a

point-wise rotation tensor, the hydrostatic pressure and an inter element boundary displacement field for the hybrid element formulation. The incompressibility constraint is introduced through a Lagrange multiplier applied to the strain energy function of the material.

Lagrange-Multiplier Method

Many authors use the Lagrange multiplier technique to introduce the incompressibility constraint into the variational formulation of the basic problem [155, 161-164]. The condition of incompressibility will appear as one of the Euler equations of the modified variational principle. The treatment, as many authors discussed [32, 141, 143, 145], is closely related to the multi-field variational principles. Courant and Hilbert [145] discuss the use of the Lagrange multiplier method to obtain several important transformations of the variational problems. More recently, de Veubeke [141], Hu [143] and Washizu [32], discuss similar treatment of the subject. Washizu uses the Lagrange multiplier method to introduce the conditions of compatibility and the geometrical boundary conditions into the basic potential energy principle. The resulting Hu-Washizu principle has the strains, the displacements and the Lagrange multipliers as independent field variables. The Euler equations of the variational principle indicate that the physical meaning of the Lagrange multipliers are to be the stresses and the surface tractions. As discussed above, Washizu [32] shows that the Hellinger-Reissner principle may be interpreted as a special case of the generalized Hu-Washizu principle. Hence, the

restricted Hellinger-Reissner principle, with the displacements and only the pressure as field variables, is similar to the use of Lagrange multiplier method with the physical meaning of the multiplier being interpreted as the hydrostatic pressure.

Discussion

As mentioned above, the advantage in using a multi-field variational principle is the elimination of the difficulties encountered when utilizing a single-field principle for the analysis of material incompressibility. This is, however, opposed by the additional computational effort required for the solution of a multi-field principle due to the excessive number of unknowns. Even for the restricted principle, where the field variables are the displacements and the hydrostatic pressure, the number of unknowns are expanded owing to the introduction of the nodal hydrostatic pressures, or the nodal generalized pressures, as additional unknowns. Furthermore, the expanded global stiffness matrix is not positive definite even if that for the original compressible material problem is, and the generalization to nonlinear problems is not always apparent [158, 159, 167-169]. To discuss the above numerical difficulties, we will consider the final equilibrium equations, for linear finite element application with incompressibility constraint, in the form

$$\begin{bmatrix} K & H \\ H^T & O \end{bmatrix} \begin{Bmatrix} u \\ p \end{Bmatrix} = \begin{Bmatrix} R \\ O \end{Bmatrix} \quad (I-27)$$

where K is the global stiffness matrix, H is the global hydrostatic pressure stiffness matrix, u is the displacement, p is the hydrostatic pressure and R is the generalized force vector.

Argyris et al. [158, 167] show that if the equation $[H]\{p\}=\{0\}$ has no non-zero solutions, there will be no self-equilibrating systems, the matrix on the left of eqn. (I-27) will be nonsingular, and the equations may be solved easily using any standard solution procedures. If, however, the equation $[H]\{p\}=\{0\}$ has a non-zero solution, the solution of eqn. (I-27) is not unique with the values of p being determined only up to scalar multiples of the linearly independent self-equilibrating solution of the equation $[H]\{p\}=\{0\}$. The prediction of linear dependency, or independency, of the constraints in a particular problem is, unfortunately, not easy, although some guiding principles may be available. This, in fact, is related to the condition of the matrix $[H]$, for if $[H]$ is not of full rank, the vector p is not uniquely determined. (If the matrix $[H]$ has n rows and m columns, then $[H]$ is not of full rank if: rank of $[H] < m$). To overcome the non-uniqueness of p , Argyris et al., opt. cit., suggest the addition of a new condition. They introduce a minimizing condition which minimizes the vector p with respect to some norm. The most logical choice being the dilatational strain energy associated with the hydrostatic pressure. Argyris et al. [167] give a comparison between the above approach and the approach of using a Poisson's ratio close to one-half. They show that it is advantageous to use the constraint matrix $[H]$, only when this matrix has a full rank.

(iv) Reduced-Selective Integration Technique

Original Work

A unique approach of treating the incompressibility constraint is the utilization of the reduced-selective integration technique. The method is motivated by the results of some numerical experiments published in the late 60's and early 70's. One of the first examples of a reduced integration elements is the plate and shell element presented by Zienkiewicz et al. [170]. In [170] two crucial points are discussed through the results of some numerical examples. First, it is shown that the stiffness of a numerically integrated element is reduced as the order of the numerical integration decreases. Second, it is indicated that the convergence of numerically integrated elements is guaranteed if the integration order allows sufficient accuracy for calculating the exact element volume in isoparametric formulation. The first point is utilized in improving specific plate and shell elements which behave poorly when reducing the shell thickness by giving an unrealistic large stiffness matrix. The method is termed selective-reduced integration technique and is applied specifically to the 8-node serendipity element. For this particular element, considerable improvement is obtained by using 2X2 Gauss quadrature in comparison to 3X3 Gauss points. The work of Zienkiewicz et al. [170] is pointed out earlier by Doherty, Wilson and Taylor [171] in the context of a plane quadrilateral element. Naylor [157] discusses a series of experiments in which he uses numerical integration in evaluating the stiffness of parabolic isoparametric elements for the analysis of incompressible material.

Naylor [157] finds that, in spite of the gross error in the stress values at the centre and the edges of each element as the compressibility is reduced, all stress components retain good accuracy at the reduced integration points using 2X2 Gauss quadrature. Exact integration using 3X3 Gauss points gives futile results, even at the integration points, when compared with the results of the reduced-selective integration scheme. In [157], Naylor^e makes an attempt to correlate the number of degrees of freedom to the number of integration points and their effect on the accuracy of the results. The basis of the above work is, however, the results of some numerical experiments and no effective trial has been given to investigate theoretically the observed phenomena. This point, led many authors to view the procedure more as a deception than a technique.

In [153], Fried presents what may be considered as the theoretical basis in understanding the behaviour of numerically integrated elements. Based on the equations of equilibrium and the stress-strain relations for both compressible and incompressible materials, Fried [153] derives an elastic energy term which represents the error in replacing the incompressible material by a compressible one. Based on the derived error term, a scheme is presented in which Poisson's ratio is increased gradually by increasing the order of the element, or refining the mesh. The proposed procedure balances the compressibility error with the best discretization accuracy. Fried, *opt. cit.*, points out that in order to make the error balance work, it is necessary to reduce the coupling between, or completely uncouple, the

value of the Poisson's ratio ν and the discretization error. He also indicates that the numerical integration scheme may be used to achieve the desired uncoupling.

Equivalence of Reduced Integration Techniques and Multi-Field Variational Principles

The work of Malkus and Hughes, with other co-workers [154, 172-176] marks the breakthrough effort in correlating the reduced-selective integration technique to the multi-field variational principle approach. In [154, 172] Malkus performs a series of numerical experiments in which he obtains similar results to the ones reported above by Naylor [157] and Fried [153]. Malkus, *opt. cit.*, concludes that the numerical integration scheme may be effectively utilized to achieve the worthwhile uncoupling between the value of the Poisson's ratio and the discretization error. He shows that the process does verily involve the number of degrees of freedom and the number of integration points as well as other factors. Hughes and co-workers [175] apply the concept to the case of plate bending elements. In an example of thin beams, they consider the shear term as a constraint which enforces the condition of negligible shear strains. Exact integration of this term is shown to lead to worthless numerical results whereas reduced-selective integration scheme gives superior results. This constraint condition is obviously analogous to the incompressibility constraint. Hughes et al. [175] explain the phenomena by indicating that the finite element mesh 'locks' when the number of

degrees of freedom is equal to the number of constraints. The point is further elaborated in an analytical manner by considering the individual stiffness contributions for shear and bending in two numerical integration cases, the one-point and the two-points Gaussian quadrature. In [176], Malkus and Hughes present more general equivalent results for multi-field variational principles and single-field ones employing the reduced-selective integration technique. The equivalence of two distinct cases is established. The two cases are:

- a. Mixed principle on a space (S, T) , $u \in S$, $g \in T$ where u is the displacement field and g is an auxiliary field, e.g. hydrostatic pressure field. It should be noted that S and T may have different nodes and different shape functions due to different discretization. The functional, F , of the mixed principle is evaluated utilizing a reduced-selective integration scheme based on the nodes of discretization of the space T , and the stationary points on (S, T) will correspond to the nodal values of (u, g) .
- b. A single-field principle which is obtained by a transformation of the mixed principle described above. Considering the Euler equation of the above mixed principle to have the form

$$\int L(u) - \alpha g = 0 \quad (I-28)$$

where L is an operator and $\alpha > 0$ is a constant. Equation (I-28) is obtained from the variation of a functional F in the form

$$F(u, g) = \int L(u) - \frac{\alpha}{2} g^2 \quad (I-29)$$

The transformation of the mixed principle in a, is obtained through the substitution $g = L(u)/\alpha$ into the functional F . The

problem, now, is carried out by evaluating the element matrices of the new functional with integration points that would have been the nodes of discretization of the space T . We obtain the stationary point on S and generate $g \in T$ by taking the integration points as nodal values for g . Usually the generated functions of g are, in general, discontinuous across the element boundaries.

The equivalence of the two cases a and b, discussed above, is proved in both linear and nonlinear analysis. Numerical examples are given and tables for constraint and accuracy conditions of typical elements are discussed by giving a 'constraint index'. This index is defined as the ratio of the total degrees of freedom minus the total constraints over the number of elements per side in the mesh, and is taken as an indication to the approximation capabilities of the element. It is shown that elements with constraint index less than or equal to zero usually lead to worthless numerical results.

The potential of Hughes and Malkus results lies in the essence that the convergence criteria, error bounds and other analytical and numerical aspects developed for mixed principles are readily applied to the reduced-selective integration displacement elements. It is also obvious that the technique combines the advantages of the simplicity of the single-field models with the theoretical desirability of using a mixed model in the analysis of constrained media.

(v) Other Methods

Other methods for incorporating incompressible material behaviour are intended mainly to overcome some of the difficulties in the above techniques, especially the numerical obstacles which arise when utilizing mixed principles. Some of these methods represent slight modifications of the ones discussed above. It is, however, noticed that these new techniques are not in general widely accepted. This is probably owing to some inherent difficulties pertinent to each particular method. In what follows, we will discuss two of these methods as an example. Techniques which are mainly applied to fluid mechanics problems, but may be extended to cover other areas, will be considered in a separate section.

a. Artificial Compressibility

Artificial compressibility simply means introducing some artificial factor which is a function of the Poisson's ratio and considering this factor as a compressibility measure. A series of problems may be analysed for decreasing values of the compressibility and the exact incompressible limit is obtained through an extrapolation. In [177], this approach is used with the compressibility being introduced in a nonlinear manner so that the power hardening characteristics of the utilized constitutive equations may be retained. The stress boundary value problem is formulated in terms of a single-field variational principle. The formulation is applied successfully to obtain the strain intensity factors for fully plastic crack

problems. It is, however, pointed out in [177] that the method is not fully adequate. The numerical experiments, carried out to check the accuracy and convergence of the overall method, are based on linear elastic problems. Valid extension of these results to nonlinear analysis is not guaranteed and progressive difficulties have been reported in the convergence characteristics and the extrapolation procedure when the method is applied to incompressible nonlinear problems with increasing degree of nonlinearity.

b. Splitting the Strain Energy Density

In [150] Argyris et al. present a method for incompressibility analysis which is closely related to the use of mixed variational principles. In the treatment of [150], the strain energy, U , is supposed to be divided into two parts; a deviatoric part U_d and a dilatational part U_v . The usual finite element discretization of the continuum is, then, assumed to be associated with the deviatoric part of the strain energy, and the behaviour of this continuum is entirely defined by the nodal displacements as a single field variable. The dilatational strain energy is, however, assumed to be associated with another discretized continuum the behaviour of which is defined by the nodal displacements as well as another unknown vector, p . The additional unknown vector, p , represents the lack of compatibility between the two continua and may be considered as an auxiliary field in comparison with mixed principles. The final equilibrium equa-

tions take the form of eqn. (I-27) with full diagonal elements. It is obvious to point out the similarity between this approach and the mixed variational principle approach and the discussion given above for mixed principles may be considered here.

(vi) A Note on Fluid Mechanics Applications

In this section, we will try to give a concise and brief view on the introduction of the incompressibility constraint in fluid mechanics problems. The main object of this discussion is to point out the applicability of the above methods to fluid mechanics problems, rather than to give a comprehensive survey of this particular topic. The interested reader is, however, referred to the versatile collection of papers edited by Gallagher, Oden, Taylor and Zienkiewicz [187-189].

The pioneering finite element applications to fluid mechanics are associated with a variational formulation of the continuum problem which, in general, lead to studies of potential flow problem as done, for example, by Zienkiewicz et al. [178]. A significant departure from this approach is pointed out by Oden [13, 182, 184] and Oden et al. [179,183]. Oden, opt. cit., underscores the fact that well defined variational principles are not essential to the formulation of fluid mechanics problems. Oden [182], Oden and Wellford [183] develop a typical finite element equations by constructing a Galerkin integral of Cauchy's first law of motion and by utilizing the velocity interpolation functions as weighting functions in the integral. This approach identically satisfies the linear momentum balance equation in an average sense

over the element. The method is equivalent to the energy balance method and, when incompressibility is considered, it provides final equilibrium equations similar to those obtained by utilizing a Lagrange multiplier method. Utilizing the same approach, Oden [184] develops general finite element models of compressible and incompressible flow and heat conduction in fluids. The incompressibility constraint is incorporated into the analysis by utilizing a modified stress tensor, together with both velocity and pressure approximations in the energy balance equation, after applying the Galerkin method of weighted residuals.

Other residual techniques, e.g. least square method and collocation method, are studied by different authors [187-188]. It is verified that the method of weighted residuals is equivalent to specific forms of variational methods [187], and also equivalent to the energy balance method [184]. The main advantages of the residual techniques are, however, the absence of any linear constraints in the derivation of the method. This leads to the establishment of a general theory of finite elements in the solution of nonlinear problems and specific nonlinear solutions of the Navier-Stokes equation. The principal disadvantages of the residual methods are the higher order differentiability requirements on the finite element interpolation functions and the associated computational difficulties.

A unique method of analysing incompressible and compressible flow problems is to formulate the problem in terms of a stream function which, in case of incompressible flow, identically satisfies the incompressibility constraint. Atkinson and co-workers [180-181]

describe this approach as applied to creeping flow problems and low Reynold's number developing flows. Atkinson et al. include the incompressibility constraint in the variational principle which, in turn, is transformed to a single field functional of the stream function. The most serious drawback of this approach is similar to that of the residual methods; the functional should be minimized with respect to the stream function as well as its derivatives.

Apart from the above specific examples, most of the methods described for solids may also be applied for fluid mechanics problems. The treatment is similar except for the specific constitutive laws that describe particular fluid behaviour [178-189].

I.4 SOFTWARE ASPECTS

I.4.1 INTRODUCTION

The formulation of nonlinear continuum mechanics problems usually leads to elegant matrix equations describing the behaviour of the system with different types of nonlinearities. However, there is a striking discrepancy between the matrix equations of the theory on one hand and the high degree of software complexity on the other hand [190].

One way of avoiding this software complexity is to develop software packages for the solution of particular structural problems. When the problem structure changes, the programs are changed accordingly. There are thousands of such finite element programs which may be considered as special purpose research tools but which can never

be qualified as reliable production software packages that can be used safely by others than the author.

An early alternative solution of the same problem of software complexity is the development of general purpose finite element programs or sometimes termed the concept of "black box software package", to which, the user submits a problem description and after some time of enigmatic calculations he is entitled to expect some kind of solution to his problem. To be able to assess this concept, we have to discuss some of its major characteristics.

The main requirements of a general purpose program or a black box software package can be interpreted as follows [190-194]:

- The capability of implementing different finite element formulations. This would lead to the ability of analyzing problems of elasticity, plasticity, elastoviscoplasticity, geometric non-linearity, dynamics, heat transfer, fluid mechanics and other field variable problems.
- The capability of incorporating different material constitutive equations. This requirement is imposed owing to the intensive research activities in this particular topic.
- State of development of data processing techniques. The program should be adjustable to the size of the problem and it should also provide sufficient facilities for checking all sections of the input data. Other requirements of the data structure are the consolidation and the flexibility of access. Intensive data management should be focused in such a way to minimize changes if

data transfer or adjustment is required. Accessing the data should not be restricted to a particular method of input and should provide maximum flexibility in problem description.

- Error control and detection by numerical analysis. At different stages of the solution, the program should check the results in comparison with the input commands, provide sufficient warnings of deviations and proper suggestions of altering the formulation.
- Integrability of the system. The program should be structured systematically in order to provide proper connections with other problem solution requirements, such as optimization, dimensioning and production planning.
- Flexibility in the application of the system. The user should be able to access dummy subroutines at appropriate locations in the program without getting involved into the large overall system.
- Rapid communication between users and program developers.
- Simplicity in changing from one computer system to another.

A survey of current capabilities of commercial programs shows that no program as yet possesses all the features discussed above [190-193]. Furthermore, black box software packages are subjected to a number of difficulties and disadvantages that are inherent to all bulky programs; the most important ones are [190/194]:

- Dependency: Large software packages are critically dependent on the continuous support, maintenance and development that has to be provided by the developers.

- Interference of functions: Large programs have hundreds of functions that can be explicitly or implicitly implemented by the user. Severe conflicts are often unavoidable in such cases.
- Loss of identity: Large programs are continuously updated by different implementors. A loss of identity will unavoidably occur in such cases.
- Obscuration: Owing to continuous development of large programs by various implementors, the program will very often change from a logical program the internal mechanism of which is fully understood by some people, to a chaotic program the internal mechanism of which is known to no one [190].
- Cost and time factors: Large programs are usually rented to users on the basis of the full capabilities of the package which, very often will not be fully exploited by the user. Moreover, it usually takes quite a long time to train someone to use and understand large software packages.
- Lack of detailed options. The broad fields of application covered by a general purpose program are usually accomplished by sacrificing comprehensive study of any particular option. In some cases, this forces the analyst to switch from one program to another or to add a sizeable part of the analysis on his own. Most of the available commercial programs require good effort to achieve such a task.

From the above discussion, it is suggested that an alternative solution, to the problem of software complexity, be considered. This is

thought to be a switch to the concept of programming systems rather than program packages [190,194]. A programming system means a software package for finite element analysis that provides various levels of analysis and formulation, each representing a closed domain of problem orientation. The package has to provide only the basic structure in the simplest possible flow sequence and has to possess high degree of dynamic flexibility. It should be always the user's job to modify and tailor the package to suit specific problem requirements.

I.4.2 REQUIREMENTS OF FINITE ELEMENT PROGRAMMING SYSTEMS

As mentioned above, a programming system package has to provide only the basic structure of different fields of finite element theory in the simplest flow with the maximum flexibility. This requires a functional decomposition of the solution procedure into closed domain operations. These operations require different operands (some kind of data) to produce physical output corresponding to different levels of finite element abstraction. Schrem [190] specifies the closed domain operations as 'processors' and interprets the operands as 'tablets'. A programming system has to satisfy some basic requirements which may be postulated through the basic functions of processors and tablets together with general organization requirements [190,194] as in the following:

Tablet Function Requirements [190]

- Every tablet should correspond to a closed domain data object in finite element theory and vice versa. No other sources of data object should be permitted in the system.
- Any set of tablets may be combined with every processor whenever a meaningful interpretation in finite element theory is possible.
- No tablet should possess any internal assumptions concerning the way its elements are processed by a particular processor.

Processor Function Requirements

- Every processor corresponds to a single operation in finite element theory. No finite element operations should be overlapped and hence no processor should call another processor.
- No processor may have built-in assumptions about the origin of its input tablets or about further post-processing of its output tablets.

General Organization Requirements

- Control information, i.e. decisions about generation of objects or choice of specific solution path, may not be implemented in processors. Similarly, data that determine the generation of objects or allow the choice of a solution path may not be implemented in tablets.
- The user may be able to create his own tablets and processors and should be able to attach them properly and easily to the system.
- Control units, which controls the flow and interaction between tablets and processors should be accessible to the user.

In spite of its immediate evidence, the system of requirements discussed above is violated in all contemporary finite element programs.

I.4.3 PROGRAMMING SYSTEM ORGANIZATION

(i) Data Handling Package

The requirements and structure of data handling packages are discussed by many analysts [195-197]. The main features, however, are outlined in the tablet and processor function requirements discussed above. A data handling package may be composed of input data tablets, input data pre-processors and result post-processors. The primary function of the data pre-processors is to generate internal data files defining the characteristics of the discrete system as prescribed by the data tablets. Data pre-processors also perform all different types of data check up. Result post-processors essentially reduce the output to the specific requirements in a manageable size. Realizing the wide range of finite element applications and the different requirements of each particular problem, this would be a formidable task within the framework of general purpose programs and it justifies the need for a programming system.

(ii) Finite Element Library

The finite element library may consist of processors which develop characteristics of the discrete system. The task may include numerical evaluation of discrete element matrices and evaluation of the solution field and secondary fields within each element. Even the most

versatile programs, may not have specific elements for particular applications. Once again this indicates a need for the switch from program to programming system. The compatible connections of a user's processor, in the finite element library or in any other section, is a user's job. This, however, should be an easy task within the frame work of the programming system requirements.

(iii) Material Model Library

The intensive research in developing constitutive equations describing material behaviour with different types of nonlinearities justifies the need for a material model section in a programming system. Once again it is an awesome task to incorporate in any program, all different types of constitutive equations, all physical parameters defining them, and all stress-rate definitions used to develop such equations. Hence, the programming system should provide processors for basic material behaviour, while material behaviour in specific applications should be amendable in a user's processor.

(iv) Sparse Matrix Processors

Sparse matrix processors may have the following basic tasks:

- Assembly of the overall or master structural matrices.
- Application of different types of boundary conditions.
- Direct matrix manipulation, which incorporates solution of linear or nonlinear algebraic equations, eigenvalue extraction and time integration schemes.

Reference [194] describes the basic structure and the input data tablets for a finite element programming system (NONFE programming system). The system does not meet all the requirements discussed above. The data structure of the 'NONFE' system is similar to the corresponding sections in MARC-CDC general purpose finite element program [198].

CHAPTER II
FORMULATION METHODS

II.1 INTRODUCTION

Formulation Methods

As discussed above, in finite element applications to continuum mechanics problems, only three major descriptions of motion are used. In what follows, we will discuss the basic equations and the general simplifying assumptions for each of the three formulations and we will develop consistent formulations leading to explicit final incremental equilibrium equations for each case. We will also discuss the basic differences between the presented formulations and similar ones in the literature. The development given in this chapter is based on the analysis given by Gadala, Oravas and Dokainish in reference [200].

Comment on Notations

In all formulations to follow we adopt the direct tensor method in which the tensor can be referred to its basis. In the base description, indicial notations with the summation convention is used. Bold face letters are used to indicate tensor quantities, lower case letters for vectors and upper case letters for higher order tensors (in type-written manuscript, the order of the tensors is identified by a number of superposed bars which equals the order of the tensor). All repeated indices are to be summed over their admissible range unless they are

enclosed by parentheses. Right subscripts in upper case refer to nodal points, whereas lower case right Greek scripts refer to generalized displacements. Left superscripts (numbers) indicate the configuration at which the quantity is measured whereas left subscripts (numbers) indicate the relative configuration to which the quantity is referred. All quantities associated with nodal points are underscored.

II.2 ASSUMPTIONS AND BASIC EQUATIONS

II.2.1 General Assumptions

A general body which occupies a finite region of Euclidean space is considered. Subjected to prescribed body forces and surface tractions the body undergoes the motion $\bar{x} = \bar{x}(\bar{X}, t)$. The particles of the body are identified by X . The relation between the particle X and its position vector \bar{X} in the reference configuration C_0 does not change with time. Three configurations of the body are considered. These are configuration C_0 at time $t = 0$, configuration C_1 at time $t = \tau$ and, finally, a neighbourhood configuration C_2 at time $t = \tau + \Delta t$ (see Figure II.1). For simplicity, without any loss of generality, a fixed rectangular frame with cartesian coordinate system in three dimensional space is established to describe the motion. In what follows, only perfect mechanical systems which take no account of thermodynamical effects will be considered. Complete nonlinear kinematic relations within a linear increment will be used throughout the formulation. Second order deformation gradients will be neglected. Further simplifying assumptions will be introduced where required.

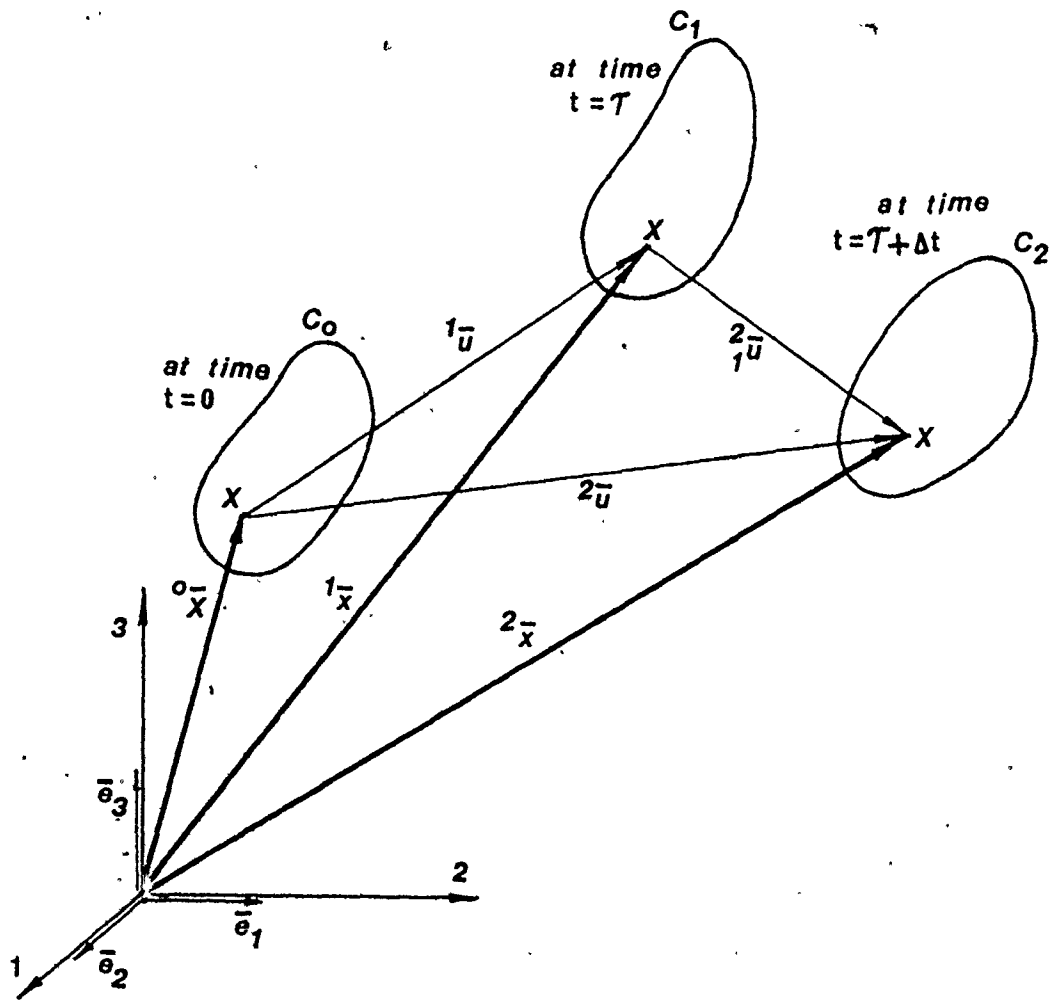


Figure (II.1) Geometry of motion from configuration C_0 to C_1 to C_2

II.2.2 Deformation Gradients

Referring to Figure (II.1), we may write the position vectors \bar{x} at configuration C_1 and C_2 as related to the displacement vectors \bar{u} in the form

$${}^1\bar{x} = {}^0\bar{X} + {}^1\bar{u} \quad (\text{II.1})$$

$${}^2\bar{x} = {}^0\bar{X} + {}^2\bar{u} = {}^1\bar{x} + {}^2_1\bar{u} \quad (\text{II.2})$$

The deformation gradient at configuration C_1 related to configuration C_0 may be given by

$${}^1_F = \frac{{}^1\bar{x} \partial}{{}^0\bar{X}} = \bar{1} + \frac{{}^1\bar{u} \partial}{{}^0\bar{X}} \quad (\text{II.3})$$

In a similar manner, other deformation gradients may be formed

$${}^2_F = \frac{{}^2\bar{x} \partial}{{}^0\bar{X}} = \bar{1} + \frac{{}^1\bar{u} \partial}{{}^0\bar{X}} + \frac{{}^2_1\bar{u} \partial}{{}^0\bar{X}} = {}^1_F + \frac{{}^2_1\bar{u} \partial}{{}^0\bar{X}} \quad (\text{II.4})$$

and

$${}^2_F = \frac{{}^2\bar{x} \partial}{{}^1\bar{x}} = \bar{1} + \frac{{}^2_1\bar{u} \partial}{{}^1\bar{x}} \quad (\text{II.5})$$

II.2.3 Kinematics of the Motion

Spatial Velocity

In spatial formulation, the material time derivative of the displacement ${}^2\bar{u}({}^2\bar{x})$ is given by

$$\frac{d}{dt} {}^2\bar{u} = {}^2\bar{v} = \frac{\partial}{\partial t} {}^2\bar{u} + {}^2\bar{v} \cdot \frac{\partial}{\partial {}^2\bar{x}} ({}^2\bar{u}) \quad (\text{II.6})$$

but from eqn. (II.2) we have

$$\frac{\partial \bar{u}}{\partial \bar{x}} = \frac{\partial}{\partial \bar{x}} (\bar{x} - \bar{o}) = \bar{1} - \frac{\partial \bar{o}}{\partial \bar{x}}$$

which after substitution into eqn. (II.6) and rearrangement, becomes

$$\bar{v} = \frac{\partial \bar{u}}{\partial t} \cdot \bar{2}_F^T \quad (\text{II.7})$$

The distinct form of eqn. (II.7) is a basic characteristic of the Eulerian formulation. In Lagrangian and updated Lagrangian formulations, the velocity is independent of the deformation gradient. The dependence of \bar{v} on \bar{F} in Eulerian formulation complicates the kinetic energy term in the energy balance equation and, for feasible numerical treatment, it imposes a lumped mass approximation as well as neglecting the second order deformation gradients as will be shown later. Equation (II.7) is, however, applicable for any displacement interpolation function and is subject to no due restrictions on the type of elements, the rigid body rotations or the finite rotations within the element.

Spatial Acceleration

Similar to the development of the velocity expression, we may write

$$\frac{d \bar{v}}{dt} = \frac{d^2 \bar{u}}{dt^2} \equiv \bar{2}_a = \frac{\partial}{\partial t} \left[\frac{\partial \bar{u}}{\partial t} \cdot \bar{2}_F^T \right] + \bar{v} \cdot \frac{\partial}{\partial \bar{x}} \left[\frac{\partial \bar{u}}{\partial t} \cdot \bar{2}_F^T \right] \quad (\text{II.8})$$

Utilizing the properties of the dot product of vectors and tensors

$$\frac{\partial \bar{u}}{\partial t} \cdot \bar{2}_F^T = \bar{2}_F \cdot \frac{\partial \bar{u}}{\partial t}$$

together with the substitution

$$\frac{\partial \frac{\partial \bar{F}}{\partial t}}{\partial t} = \frac{\partial}{\partial t} \left(\frac{\partial \bar{x}}{\partial \bar{x}} \frac{\partial}{\partial \bar{x}} \right) = \frac{\partial \frac{\partial \bar{x}}{\partial t}}{\partial \bar{x}} \frac{\partial}{\partial \bar{x}} = \frac{\partial \frac{\partial \bar{u}}{\partial t}}{\partial \bar{x}} \frac{\partial}{\partial \bar{x}}$$

eqn. (II.8) gives

$$\begin{aligned} \frac{\partial \bar{a}}{\partial t} = & \left[\frac{\partial \frac{\partial \bar{u}}{\partial t}}{\partial t} \cdot \frac{\partial \bar{F}}{\partial \bar{x}} + \frac{\partial \bar{u}}{\partial t} \cdot \frac{\partial \frac{\partial \bar{u}}{\partial t}}{\partial \bar{x}} \frac{\partial}{\partial \bar{x}} \right] + \frac{\partial \bar{v}}{\partial t} \cdot \left[\frac{\partial \bar{F}}{\partial \bar{x}} \left(\frac{\partial \bar{u}}{\partial t} \right) \cdot \frac{\partial \bar{F}}{\partial \bar{x}} \right. \\ & \left. + \frac{\partial \frac{\partial \bar{F}}{\partial t}}{\partial \bar{x}} \cdot \frac{\partial \bar{u}}{\partial t} \right] \end{aligned} \quad (II.9)$$

Equation (II.9) represents a very complicated and highly nonlinear coupled equation. The solution of such equations becomes highly uneconomical for all practical purposes. A considerable reduction in computation time and effort is therefore achieved when expression (II.9) is replaced by an equivalent expression associated with a lumped mass approximation. The lumped mass approximation does not, however, contradict the basic characteristics of a spatial formulation.

Velocity Gradient

Starting with eqn. (II.7), we have

$$\frac{\partial \frac{\partial \bar{v}}{\partial t}}{\partial \bar{x}} = \frac{\partial}{\partial \bar{x}} \left(\frac{\partial \bar{u}}{\partial t} \right) \cdot \frac{\partial \bar{F}}{\partial \bar{x}} + \frac{\partial \frac{\partial \bar{F}}{\partial t}}{\partial \bar{x}} \cdot \frac{\partial \bar{u}}{\partial t} \quad (II.10)$$

As mentioned in the general assumptions, the second term in (II.10) may be neglected as it represents a second order deformation gradient term. As will be shown later in the analysis, including such a term will complicate the final incremental equilibrium equations and make it

infeasible for numerical analysis. Such an approximation, however, restricts the class of problems the analysis is capable of handling but it does not contradict the basic nature of the spatial formulation. The class of problems suitable to treatment by such analysis may be assessed numerically.

II.2.4 Strain Measures

In finite deformations, many strain measures are useful, the majority of which can be computed from the deformation gradients. In Lagrangian and updated Lagrangian formulations, however, we will be mainly dealing with Green-Lagrange strain tensor, which at different configurations may be given by

$${}^2_{\circ}\bar{E} = \frac{1}{2} [{}^2_{\circ}\bar{F}^T \cdot {}^2_{\circ}\bar{F} - \bar{1}] = \frac{1}{2} \left[\frac{\partial {}^2_{\circ}\bar{u}}{\partial {}^{\circ}\bar{X}} + \frac{{}^2_{\circ}\bar{u}}{\partial {}^{\circ}\bar{X}} \frac{\partial}{\partial {}^{\circ}\bar{X}} + \frac{\partial {}^2_{\circ}\bar{u}}{\partial {}^{\circ}\bar{X}} \cdot \frac{{}^2_{\circ}\bar{u}}{\partial {}^{\circ}\bar{X}} \right] \quad (\text{II.11})$$

Substituting ${}^2_{\circ}\bar{u} = {}^1_{\circ}\bar{u} + {}^2_{\circ}\bar{u}$, eqn. (II.11) may be written in the form

$${}^2_{\circ}\bar{E} = {}^1_{\circ}\bar{E} + (\Delta {}^1_{\circ}\bar{E}^{\text{linear}} + \Delta {}^1_{\circ}\bar{E}^{\text{nonlinear}}) \quad (\text{II.12})$$

where

$${}^1_{\circ}\bar{E} = \frac{1}{2} \left[\frac{\partial {}^1_{\circ}\bar{u}}{\partial {}^{\circ}\bar{X}} + \frac{{}^1_{\circ}\bar{u}}{\partial {}^{\circ}\bar{X}} \frac{\partial}{\partial {}^{\circ}\bar{X}} + \frac{\partial {}^1_{\circ}\bar{u}}{\partial {}^{\circ}\bar{X}} \cdot \frac{{}^1_{\circ}\bar{u}}{\partial {}^{\circ}\bar{X}} \right] \quad (\text{II.13})$$

$$\Delta ({}^1_{\circ}\bar{E}^{\text{linear}}) = \frac{1}{2} \left[\frac{\partial {}^2_{\circ}\bar{u}}{\partial {}^{\circ}\bar{X}} + \frac{{}^2_{\circ}\bar{u}}{\partial {}^{\circ}\bar{X}} \frac{\partial}{\partial {}^{\circ}\bar{X}} + \frac{\partial {}^1_{\circ}\bar{u}}{\partial {}^{\circ}\bar{X}} \cdot \frac{{}^2_{\circ}\bar{u}}{\partial {}^{\circ}\bar{X}} + \frac{\partial {}^2_{\circ}\bar{u}}{\partial {}^{\circ}\bar{X}} \cdot \frac{{}^1_{\circ}\bar{u}}{\partial {}^{\circ}\bar{X}} \right] \quad (\text{II.14})$$

and

$$\Delta ({}^1_{\circ}\bar{E}^{\text{nonlinear}}) = \frac{1}{2} \left[\frac{\partial {}^2_{\circ}\bar{u}}{\partial {}^{\circ}\bar{X}} \cdot \frac{{}^2_{\circ}\bar{u}}{\partial {}^{\circ}\bar{X}} \right] \quad (\text{II.15})$$

We also have

$${}^1\bar{\epsilon} = \frac{1}{2} \left[\frac{\partial {}^1\bar{u}}{\partial {}^1\bar{x}} + \frac{{}^1\bar{u}}{\partial {}^1\bar{x}} \frac{\partial}{\partial {}^1\bar{x}} + \frac{\partial {}^1\bar{u}}{\partial {}^1\bar{x}} \cdot \frac{{}^1\bar{u}}{\partial {}^1\bar{x}} \frac{\partial}{\partial {}^1\bar{x}} \right] \quad (\text{II.16})$$

$$\Delta ({}^1\bar{\epsilon}) = \frac{1}{2} \left[\frac{\partial (\Delta {}^1\bar{u})}{\partial {}^1\bar{x}} + \frac{(\Delta {}^1\bar{u})}{\partial {}^1\bar{x}} \frac{\partial}{\partial {}^1\bar{x}} + \left(\frac{\partial (\Delta {}^1\bar{u})}{\partial {}^1\bar{x}} \cdot \frac{{}^1\bar{u}}{\partial {}^1\bar{x}} + \frac{\partial {}^1\bar{u}}{\partial {}^1\bar{x}} \cdot \frac{(\Delta {}^1\bar{u})}{\partial {}^1\bar{x}} \right) \frac{\partial}{\partial {}^1\bar{x}} \right] \quad (\text{II.17})$$

and

$${}^2\bar{\epsilon} = {}^1\bar{\epsilon} + \Delta ({}^1\bar{\epsilon}) \quad (\text{II.18})$$

In spatial formulation, the finite deformation strain measure is the Eulerian, or Almansi strain tensor

$${}^2\bar{\epsilon} = \frac{1}{2} \left[\frac{\partial {}^2\bar{u}}{\partial {}^2\bar{x}} + \frac{{}^2\bar{u}}{\partial {}^2\bar{x}} \frac{\partial}{\partial {}^2\bar{x}} - \frac{\partial {}^2\bar{u}}{\partial {}^2\bar{x}} \cdot \frac{{}^2\bar{u}}{\partial {}^2\bar{x}} \frac{\partial}{\partial {}^2\bar{x}} \right] \quad (\text{II.19})$$

and the incremental Eulerian strain tensor may be given by

$$\Delta {}^2\bar{\epsilon} = \frac{1}{2} \left[\frac{\partial (\Delta {}^2\bar{u})}{\partial {}^2\bar{x}} + \frac{(\Delta {}^2\bar{u})}{\partial {}^2\bar{x}} \frac{\partial}{\partial {}^2\bar{x}} - \left(\frac{\partial {}^2\bar{u}}{\partial {}^2\bar{x}} \cdot \frac{(\Delta {}^2\bar{u})}{\partial {}^2\bar{x}} + \frac{\partial (\Delta {}^2\bar{u})}{\partial {}^2\bar{x}} \cdot \frac{{}^2\bar{u}}{\partial {}^2\bar{x}} \right) \frac{\partial}{\partial {}^2\bar{x}} \right] \quad (\text{II.20})$$

II.2.5 Displacement Assumption

For Lagrangian (or total Lagrangian) formulation, we assume the displacement shape function in the form

$${}^2u_i = \psi_{i\alpha} ({}^0\bar{x}) {}^1u_\alpha \quad (\text{II.21})$$

and

$${}^1u_i = \psi_{i\alpha} ({}^0\bar{x}) {}^1u_\alpha \quad (\text{II.22})$$

where, in general, a displacement vector $\bar{u} = u_i \bar{e}_i$ and \bar{e}_i , $i = 1, 2, 3$, is the Cartesian basis. For the updated Lagrangian formulation, we have

$${}^2_1 u_i = \psi_{i\alpha} ({}^1\bar{x}) {}^2_1 u_\alpha \quad (\text{II.23})$$

and

$${}^1_1 u_i = \psi_{i\alpha} ({}^1\bar{x}) {}^1_1 u_\alpha \quad (\text{II.24})$$

and, finally, in case of Eulerian formulation, we assume the displacement shape function in the form

$${}^2_2 u_i = \psi_{i\alpha} ({}^2\bar{x}) {}^2_2 u_\alpha \quad (\text{II.25})$$

where \underline{u}_α is the vector of nodal displacements and $\psi_{i\alpha}$ is the usual finite element shape function. Later on, the implicit dependence of ψ on either \bar{X} or \bar{x} will be understood.

Substituting (II.25) into (II.7) and rearranging, we may write the velocity vector in the form

$${}^2_2 \underline{v} = {}^2_2 v_i \bar{e}_i$$

where

$${}^2_2 v_i = \psi_{m\alpha} \frac{\partial {}^2 x_i}{\partial {}^2 X_m} {}^2_2 v_\alpha \quad (\text{II.26})$$

For a unit increment of time, eqn. (II.26) gives

$$\Delta {}^2_2 \underline{u} = \Delta {}^2_2 u_i \bar{e}_i$$

where

$$\Delta {}^2_2 u_i = \psi_{m\alpha} \frac{\partial {}^2 x_i}{\partial {}^2 X_m} \Delta {}^2_2 u_\alpha \quad (\text{II.27})$$

A corresponding expression for (Δv) should depend on eqn. (II.9). Owing to the complications discussed above, this expression will be only considered through a lumped mass approximation.

II.2.6 Incremental and Total Strain Rates

Substituting the displacement assumptions, eqns. (II.21-II.25), into the incremental Lagrangian strain tensor, eqns. (II.11-II.18), and utilizing the symmetry of the strain tensor, we may obtain

$$\Delta \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \bar{\mathbf{E}} \right) = \left\{ \left[\frac{\partial \psi_{i\alpha}}{\partial \circ X_j} + \frac{\partial \psi_{k\beta}}{\partial \circ X_i} \frac{\partial \psi_{k\alpha}}{\partial \circ X_j} \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \underline{u}_\beta + \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \underline{u}_\beta \right) \right] \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \underline{u}_\alpha \right\} \bar{e}_i \bar{e}_j$$

which for a linear increment will be

$$\Delta \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \bar{\mathbf{E}} \right) = \left\{ \left[\frac{\partial \psi_{i\alpha}}{\partial \circ X_j} + \frac{\partial \psi_{k\beta}}{\partial \circ X_i} \frac{\partial \psi_{k\alpha}}{\partial \circ X_j} \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \underline{u}_\beta \right) \right] \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \underline{u}_\alpha \right\} \bar{e}_i \bar{e}_j \quad (\text{II.28})$$

The material time derivative of eqn. (28) may be given by

$$\frac{d}{dt} \left(\Delta \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \bar{\mathbf{E}} \right) = \left\{ \left[\frac{\partial \psi_{i\alpha}}{\partial \circ X_j} + \frac{\partial \psi_{k\beta}}{\partial \circ X_i} \frac{\partial \psi_{k\alpha}}{\partial \circ X_j} \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \underline{u}_\beta \right) \right] \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \underline{v}_\alpha \right\} \bar{e}_i \bar{e}_j \quad (\text{II.29})$$

where $\begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \underline{v}_\alpha$ are the incremental nodal velocities from configuration C_1 to C_2 . It should be noted that in eqn. (II.29) the displacement $\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \underline{u}$ is a function of the relative time τ at configuration C_1 whereas the displacement $\begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \underline{u}$ is a function of the actual time t at configuration C_2 . This, as discussed above, indicates the referential nature of the analysis [also see Truesdell and Noll, reference [52] p. 49].

In a similar way, starting from eqn. (II.17) and utilizing the relation $\Delta \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \underline{u} = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \underline{u}$, then we may write

$$\Delta \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \bar{\mathbf{E}} \right) = \left\{ \left[\frac{\partial \psi_{i\alpha}}{\partial \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} X_j} + \frac{\partial \psi_{k\beta}}{\partial \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} X_i} \frac{\partial \psi_{k\alpha}}{\partial \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} X_j} \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \underline{u}_\beta \right) \right] \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \underline{u}_\alpha \right\} \bar{e}_i \bar{e}_j \quad (\text{II.30})$$

which yields

$$\frac{d}{dt} (\Delta \bar{1}\bar{E}) = \left\{ \left[\frac{\partial \psi_{i\alpha}}{\partial \bar{1}x_j} + \frac{\partial \psi_{k\beta}}{\partial \bar{1}x_i} \frac{\partial \psi_{k\alpha}}{\partial \bar{1}x_j} (\bar{1}u_\beta) \right] \bar{2}v_\alpha \right\} \bar{e}_i \bar{e}_j \quad (\text{II.31})$$

Once again, the referential nature of the analysis is implied when obtaining eqn. (II.31) from (II.30) by taking $\bar{1}\bar{x} = \bar{1}\bar{x}(\bar{0}\bar{X}, \tau)$. This indicates that $\bar{1}\bar{x}$ is independent of the actual time t , and that configuration C_1 is acting as a variable or updated configuration which is Lagrangian in nature. For the same reason, it is obvious that similar relations will be awkward in Eulerian formulation. Analogously, we may write

$$\frac{d}{dt} (\bar{2}\bar{E}) = \frac{d}{dt} (\bar{1}\bar{E} + \Delta \bar{0}\bar{E}) = \frac{d}{dt} (\Delta \bar{0}\bar{E}) \quad (\text{II.32})$$

$$\frac{d}{dt} (\bar{2}\bar{E}) = \frac{d}{dt} (\bar{1}\bar{E} + \Delta \bar{1}\bar{E}) = \frac{d}{dt} (\Delta \bar{1}\bar{E}) \quad (\text{II.33})$$

Utilizing the deformation gradients given by eqns. (II.3-II.5) together with the displacement assumptions, eqns. (II.21-II.25), and the general assumptions discussed above, we may write

$$\bar{1}F_{ij} = \frac{\partial \bar{1}x_i}{\partial \bar{0}X_j} = \delta_{ij} + \frac{\partial \psi_{i\alpha}}{\partial \bar{0}X_j} \bar{1}u_\alpha \quad (\text{II.34})$$

$$\bar{2}F_{ij} = \frac{\partial \bar{2}x_i}{\partial \bar{1}x_j} = \delta_{ij} + \frac{\partial \psi_{i\alpha}}{\partial \bar{1}x_j} \bar{2}u_\alpha \quad (\text{II.35})$$

II.2.7 Energy Balance Equation

Referring to Fig. (II.1), we may write the energy balance equation in configuration C_2 and its transformation to the Lagrangian reference configuration as given in Chapter (I), eqn. (I.1-I.2). For quick reference eqn. (I.1) and (I.2) are reported here as

$$\frac{d}{dt} \int_{2V} \rho \bar{b} \cdot \bar{v} d^2V + \int_{2A} d^2\bar{A} \cdot \bar{\sigma} \cdot \bar{v} - \int_{2V} \bar{\sigma} : \bar{D} d^2V \quad (II.36)$$

and

$$\frac{d}{dt} \int_{0V} \rho \bar{b} \cdot \bar{v} d^0V + \int_{0A} d^0\bar{A} \cdot \bar{\sigma} \cdot \bar{v} - \int_{0V} \bar{\sigma} : \frac{d}{dt} d^0\bar{E} d^0V \quad (II.37)$$

II.3 LAGRANGIAN (REFERENTIAL) FORMULATION

II.3.1 Incremental Equilibrium Equations

We base the formulation on the mechanical energy balance equation in the deformed state, eqn. (II.36), written for a single element at some time t during the deformation. The transformed energy balance equation in the undeformed Lagrangian state, configuration C_0 , is given by eqn. (II.37). For the intended incrementation procedure, it is convenient to focus attention on a single element with the understanding that equations for an individual element are only valid when contributions from all elements sharing a particular node are added together. With this in mind, then considering eqn. (II.37) term by term and substituting the displacement assumption, eqns. (II.21-II.22), we

may have

Kinetic energy term:

$$\begin{aligned} J_1 &\Rightarrow \frac{d}{dt} \int_{\mathcal{O}_V} \frac{1}{2} \rho \left(\frac{d^2 \bar{u}}{dt} \cdot \frac{d^2 \bar{u}}{dt} \right) d \mathcal{O}_V \\ &= \int_{\mathcal{O}_V} \rho \frac{d^2 \bar{u}}{dt} \cdot \frac{d^2 \bar{u}}{dt^2} d \mathcal{O}_V \end{aligned}$$

Substituting the displacement assumption and considering arbitrary virtual velocities $\frac{2}{1} \underline{v}_\alpha$, we have

$$J_1 = \int_{\mathcal{O}_V} \left[\rho (\psi_{i\alpha} \psi_{i\beta}) d \mathcal{O}_V \right] \frac{d^2}{dt^2} (\underline{u}_\beta + \underline{1} \underline{u}_\beta),$$

which for incremental formulation gives

$$\Delta J_1 = \int_{\mathcal{O}_V} \left[\rho (\psi_{i\alpha} \psi_{i\beta}) d \mathcal{O}_V \right] \frac{d^2}{dt^2} (\Delta \underline{1} \underline{u}_\beta + \Delta \underline{2} \underline{u}_\beta)$$

where $\Delta \underline{2} \underline{u} = \Delta(\Delta \underline{1} \underline{u})$. Hence for a linear increment the above equation becomes.

$$\Delta J_1 = [{}^1 M_{\alpha\beta}] \{ \Delta \underline{1} \underline{u}_\beta \} \quad (\text{II.38})$$

where

$${}^1 M_{\alpha\beta} = \int_{\mathcal{O}_V} \rho (\psi_{i\alpha} \psi_{i\beta}) d \mathcal{O}_V \quad (\text{II.39})$$

is the consistent mass matrix.

Load vector term:

$$J_2 \Rightarrow \int_{\circ_V} \circ_\rho \bar{b}_i \frac{d \bar{u}_i}{dt} d \circ_V + \int_{\circ_A} d \circ_A \cdot \bar{S}_i \cdot \bar{F}_i \cdot \frac{d \bar{u}_i}{dt} \Rightarrow \bar{R}_i$$

Substituting expressions for the displacement assumption, the deformation gradient and considering the following relations

$$\frac{d \bar{u}_i}{dt} = \frac{d}{dt} (\bar{u}_i + \bar{u}_i) = \bar{v}_i,$$

$$d \circ_A \cdot \bar{S}_i = \bar{S}_i d \circ_A,$$

then for arbitrary virtual velocities \bar{v}_i , we may have

$$J_2 = \int_{\circ_V} \circ_\rho \bar{b}_i \psi_{i\alpha} d \circ_V + \int_{\circ_A} \bar{S}_i n_i (\psi_{i\alpha} + \frac{\partial \psi_{i\beta}}{\partial X_j} \psi_{j\alpha} \bar{u}_\beta) d \circ_A$$

Increments of the above equation give

$$\begin{aligned} \Delta J_2 &= \Delta \bar{R}_i \\ &= \Delta \bar{R}_i + [K_{\alpha\beta}^{(4)}] \{\Delta \bar{u}_\beta\} \end{aligned} \quad (\text{II.40})$$

where

$$\Delta \bar{R}_i = \left\{ \int_{\circ_V} \Delta (\circ_\rho \bar{b}_i) \psi_{i\alpha} d \circ_V + \int_{\circ_A} (\Delta \bar{S}_i n_i) \left[\psi_{i\alpha} + \frac{\partial \psi_{i\beta}}{\partial X_j} \psi_{j\alpha} \bar{u}_\beta \right] d \circ_A \right\} \quad (\text{II.41})$$

is the load increment vector, and

$$K_{\alpha\beta}^{(4)} = \int_{\circ_A} \bar{S}_i n_i \frac{\partial \psi_{i\beta}}{\partial X_j} \psi_{j\alpha} d \circ_A \quad (\text{II.42})$$

is the initial load or the load correction matrix.

Strain energy rate term:

$$J_3 \Rightarrow \int_{\mathcal{O}_V} \dot{\bar{S}} : \frac{d}{dt} \dot{\bar{E}} d \mathcal{O}_V$$

Substituting eqns. (II.28-II.33) and considering arbitrary virtual velocities $\dot{\bar{v}}_\alpha$, we obtain

$$J_3 = \int_{\mathcal{O}_V} \dot{\bar{S}}_{ij} \left[\frac{\partial \psi_{i\alpha}}{\partial \dot{x}_j} + \frac{\partial \psi_{k\beta}}{\partial \dot{x}_i} \frac{\partial \psi_{k\alpha}}{\partial \dot{x}_j} \dot{\bar{u}}_\beta \right] d \mathcal{O}_V$$

Increments of the above equation give

$$\begin{aligned} \Delta J_3 = & \left[\int_{\mathcal{O}_V} \dot{\bar{S}}_{ij} \left(\frac{\partial \psi_{k\beta}}{\partial \dot{x}_i} \frac{\partial \psi_{k\alpha}}{\partial \dot{x}_j} \right) d \mathcal{O}_V \right] \{ \Delta \dot{\bar{u}}_\beta \} \\ & + \left\{ \int_{\mathcal{O}_V} \left(\frac{\partial \psi_{i\alpha}}{\partial \dot{x}_j} + \frac{\partial \psi_{k\beta}}{\partial \dot{x}_i} \frac{\partial \psi_{k\alpha}}{\partial \dot{x}_j} \dot{\bar{u}}_\beta \right) (\Delta \dot{\bar{S}}_{ij}) d \mathcal{O}_V \right\} \end{aligned}$$

Introducing the assumptions

$$\begin{aligned} \Delta (\dot{\bar{S}}_{ij}) &= \Delta (\dot{\bar{S}}_{ij} + \Delta \dot{\bar{S}}_{ij}) = \Delta \dot{\bar{S}}_{ij} + O(\Delta(\Delta \dot{\bar{S}}_{ij})) \\ (\Delta \dot{\bar{S}}_{ij}) &= {}^1D_{ijkl} (\Delta \dot{\bar{E}}_{kl}) \end{aligned} \quad (\text{II.43})$$

where possible forms of D_{ijkl} for specific applications may be discussed independently, then the strain energy term may be put in the form

$$\Delta J_3 = [{}^1K_{\alpha\beta}^{(1)} + {}^1K_{\alpha\beta}^{(2)} + {}^1K_{\alpha\beta}^{(3)}] \{ \Delta \dot{\bar{u}}_\beta \} \quad (\text{II.44})$$

where

$${}^1K_{\alpha\beta}^{(1)} = \int_{\mathcal{O}_V} \frac{\partial \psi_{i\alpha}}{\partial \dot{x}_j} {}^1D_{ijkl} \frac{\partial \psi_{k\beta}}{\partial \dot{x}_l} d \mathcal{O}_V \quad (\text{II.45})$$

is the usual small displacement or the incremental stiffness matrix;

$${}^1_{\circ}K_{\alpha\beta}^{(2)} = \int_{\circ V} {}^1S_{ij} \frac{\partial \psi_{k\beta}}{\partial \circ X_i} \frac{\partial \psi_{k\alpha}}{\partial \circ X_j} d \circ V \quad (\text{II.46})$$

is the initial stress or the geometric or the tangent stiffness matrix;

$$\begin{aligned} {}^1_{\circ}K_{\alpha\beta}^{(3)} = \int_{\circ V} {}^1D_{ijkl} \{ & (\frac{\partial \psi_{r\gamma}}{\partial \circ X_i} \frac{\partial \psi_{r\alpha}}{\partial \circ X_j} \frac{\partial \psi_{k\beta}}{\partial \circ X_l} + \frac{\partial \psi_{l\alpha}}{\partial \circ X_j} \frac{\partial \psi_{w\gamma}}{\partial \circ X_k} \frac{\partial \psi_{w\beta}}{\partial \circ X_l}) {}^1_{\underline{u}}{}_{-\gamma} \\ & + (\frac{\partial \psi_{r\gamma}}{\partial \circ X_i} \frac{\partial \psi_{r\alpha}}{\partial \circ X_j} \frac{\partial \psi_{w\eta}}{\partial \circ X_k} \frac{\partial \psi_{w\beta}}{\partial \circ X_l}) {}^1_{\underline{u}}{}_{-\gamma} {}^1_{\underline{u}}{}_{-\eta} \} d \circ V \end{aligned} \quad (\text{II.47})$$

is the initial displacement or the initial rotation stiffness matrix.

Collecting results of eqns. (II.38-II.47), the incremental equilibrium equations take the form

$$\Delta {}^1_{\circ}R_{\alpha} = [{}^1_{\circ}K_{\alpha\beta}^{(1)} + {}^1_{\circ}K_{\alpha\beta}^{(2)} + {}^1_{\circ}K_{\alpha\beta}^{(3)} - {}^1_{\circ}K_{\alpha\beta}^{(4)}] \{\Delta {}^1_{\underline{u}}{}_{\beta}\} + [{}^1_{\circ}M_{\alpha\beta}] \{\Delta {}^1_{\underline{u}}{}_{\beta}\} \quad (\text{II.48})$$

where all individual terms have been defined above.

It should be noticed that the final incremental equilibrium equations possess two types of nonlinearities: the geometric nonlinearity arising from the generalized strain-displacement relations, eqns. (II.11-II.18), and the material nonlinearity since we cannot in general assume a linear relation between \bar{S} and \bar{E} .

So far, the formulation developed here differs from similar ones in the literature [9-12 and 16-17] in the development of the load correction matrix, eqn. (II.42), and the definition of the load increment, eqn. (II.41). As mentioned above, in [16] Hibbit et al. introduce a rather intuitive argument in $\bar{\Delta F} = \Delta_{\text{load}} \bar{F} + \Delta_{\text{geom}} \bar{F}$, where \bar{F} is the surface traction vector or the body force vector, to obtain the

form of the load correction vector. The term obtained by Hibbit et al. will have a zero value in some loading cases, while the term presented here will always possess some value in general. In [9-12] Bathe et al. introduce similar arguments to the ones presented here but for deformation dependent loadings. As stated above, this term exists owing to consistent transformation of the energy balance equation with no reference to loading condition and hence it can be applied to any type of surface tractions. The formulation presented here coincides with the procedure given by Oden [13].

II.3.2 Incompressibility Constraint

The incompressibility condition may be expressed as

$$J = \int_{2V} {}^2p ({}^2D : \bar{1}) \, d {}^2V = 0 \quad (\text{II.49})$$

It should be noted that eqn. (II.49) implies incompressibility for both elastic and plastic strains and provides the relationship ${}^2D_{ij} = {}^2D'_{ij}$, where ${}^2D'_{ij}$ is the deviatoric rate of the deformation tensor. The pressure 2p acts as a Lagrange multiplier for the constraint equation. Incompressibility on the average may be satisfied if 2p is assumed constant throughout the element, whereas complete incompressibility may be obtained by assuming an independent shape function for 2p .

Equation (II.49) may be transformed to the reference configuration as

$$J = \int_{\Omega_V} 2p \, 2D_{ii} \, d\Omega_V = \int_{\Omega_V} 2p(\delta_{ij} 2D_{ij}) \frac{d\Omega_V}{d\Omega_V} \, d\Omega_V = \int_{\Omega_V} 2p(\bar{\Gamma}:2\bar{D}) |J^2_{\bar{F}}| \, d\Omega_V$$

but $|J^2_{\bar{F}}| = 1$ for incompressibility, hence

$$J = \int_{\Omega_V} 2p \left[\bar{\Gamma} : \frac{1}{2} \left(\frac{\partial \bar{X}}{\partial \bar{x}} \cdot \frac{\partial 2\bar{v}}{\partial \bar{x}} + \frac{2\bar{v}}{\partial \bar{x}} \cdot \frac{\partial \bar{X}}{\partial \bar{x}} \right) \right] d\Omega_V = 0 \quad (\text{II.50})$$

Assuming an independent interpolation function for the pressure

$$2p(\bar{X}) = \phi_N(\bar{X}) \, 2p_N \quad (\text{II.51})$$

where N is the number of nodes per element, then adding the constraint eqn. (II.50) to the energy balance eqn. (II.37) and following essentially the same procedure as that used to derive eqn. (II.48) with the use of the interpolation function (II.51), we obtain, instead of eqn. (II.48),

$$\begin{aligned} \Delta \, {}^1_{\alpha} \bar{R} &= [{}^1_{\alpha\beta} K^{(1)} + {}^1_{\alpha\beta} K^{(2)} + {}^1_{\alpha\beta} K^{(3)} - {}^1_{\alpha\beta} K^{(4)}] \{ \Delta \, {}^1_{\beta} \underline{u} \} \\ &+ [{}^1_{\alpha N} H] \{ \Delta \, {}^1_{N} p \} + [{}^1_{\alpha\beta} M] \{ \Delta \, {}^1_{\beta} \underline{u} \} \end{aligned} \quad (\text{II.52})$$

and

$$[{}^1_{\beta N} H]^T \{ \Delta \, {}^1_{\beta} \underline{u} \} = \{ 0 \} \quad (\text{II.53})$$

where

$${}^1_{\alpha N} H = \int_{\Omega_V} \left(\phi_N \frac{\partial \bar{X}_r}{\partial \bar{x}_i} \frac{\partial \psi_{i\alpha}}{\partial \bar{x}_r} \right) d\Omega_V \quad (\text{II.54})$$

is the hydrostatic pressure stiffness matrix, and other terms have been defined above. Although the form of eqns. (II.52-II.54) is similar to

corresponding forms in the literature, but the procedure and the matrix expressions are different.

II.4 UPDATED LAGRANGIAN (RELATIVE) FORMULATION

II.4.1 Incremental Equilibrium Equations

As discussed above, in updated Lagrangian formulation we describe the future in terms of the current configuration. Referring to Figure (II.1), the independent variables will be the position ${}^2\bar{x}$ and the variable time τ , where ${}^2\bar{x} = {}^2\bar{x}({}^0\bar{x}, t)$, i.e. ${}^2\bar{x}$ is independent of τ which indicates the referential nature of the analysis (compare to independent variables of Lagrangian formulation; ${}^0\bar{x}$ and t which have the same character). In a much similar way to the development given in Appendix A, the energy balance equation referred to the updated or variable Lagrangian reference configuration C_1 may be written in the form

$$\frac{d}{dt} {}^2K = \int_V {}^1\rho {}^2\bar{b} \cdot \frac{d}{}^2\bar{u}}{dt} dV + \int_A {}^1\bar{A} \cdot {}^2\bar{S} \cdot {}^2\bar{F}^T \cdot \frac{d}{}^2\bar{u}}{dt} - \int_V {}^2\bar{S} \cdot \frac{d}{}^2\bar{E}}{dt} dV \quad (II.55)$$

Considering eqn. (II.55) term by term, we have

Kinetic energy term:

$$J_1 \Rightarrow \frac{d}{dt} {}^2K = \frac{d}{dt} \int_V \frac{1}{2} {}^1\rho \left(\frac{d}{}^2\bar{u}}{dt} \cdot \frac{d}{}^2\bar{u}}{dt} \right) dV$$

following the same procedure of developing eqns. (II.38-II.39), we may write

$$\Delta J_1 = [{}^1M_{\alpha\beta}] \{\Delta \overset{1}{u}_{-\beta}\} \quad [\text{II.38}] \quad (\text{II.56})$$

where

$${}^1M_{\alpha\beta} = \int_V \rho (\psi_{i\alpha} \psi_{i\beta}) dV \quad [\text{II.39}] \quad (\text{II.57})$$

is the consistent mass matrix, and for the sake of comparison, the numbers of the corresponding Lagrangian expressions are indicated in square brackets.

Load vector term:

$$J_2 \Rightarrow \int_V \rho \overset{2}{b}_i \cdot \frac{d \overset{2}{u}_i}{dt} dV + \int_A \overset{1}{A} \cdot \overset{2}{S}_i \cdot \overset{2}{F}_i \cdot \frac{d \overset{2}{u}_i}{dt} \Rightarrow \overset{2}{R}_i$$

Substituting expressions for the displacement assumption, the deformation gradient and considering similar relations to those given for the Lagrangian case in the corresponding term and, finally, for arbitrary virtual velocities $\overset{2}{v}_{1-\alpha}$ we may have

$$J_2 = \int_V \rho \overset{2}{b}_i \psi_{i\alpha} dV + \int_A \overset{2}{S}_{n_i} \left(\psi_{i\alpha} + \frac{\partial \psi_{i\beta}}{\partial x_j} \psi_{j\alpha} \overset{2}{u}_{-\beta} \right) dA$$

Increments of the above equation give

$$\Delta J_2 = \left[\int_V \Delta (\rho \overset{1}{b}_i) \psi_{i\alpha} dV \right] + \left[\int_A (\Delta \overset{2}{S}_{n_i}) \left(\psi_{i\alpha} + \frac{\partial \psi_{i\beta}}{\partial x_j} \psi_{j\alpha} \overset{2}{u}_{-\beta} \right) dA \right]$$

$$+ \left[\int_A \overset{2}{S}_{n_i} \left\{ \frac{\partial \psi_{i\alpha}}{\partial x_j} \psi_{j\alpha} (\Delta \overset{2}{u}_{-\beta}) \right\} dA \right]$$

substituting the assumptions

$$\Delta \overset{2}{S}_{n_i} = \Delta (\overset{1}{S}_{n_i} + \Delta \overset{1}{S}_{n_i}) = \Delta \overset{1}{\sigma}_{n_i} + O[\Delta (\Delta \overset{1}{S}_{n_i})]$$

where $\bar{1}S = \bar{1}\sigma$, and

$$\Delta \bar{1}u_B = \Delta (\Delta \bar{1}u_B) = O[\Delta(\Delta \bar{1}u_B)]$$

we obtain

$$\Delta J_2 = \Delta \bar{1}R = \Delta \bar{1}R + [{}_{1}K_{\alpha\beta}^{(4)}] \{\Delta \bar{1}u_B\} \quad \text{[II.40] (II.58)}$$

where

$$\Delta \bar{1}R = \left\{ \int_{1V} \Delta ({}^1\rho {}^1b_i) \psi_{i\alpha} d {}^1V + \int_{1A} (\Delta {}^1\sigma_{ni}) \psi_{i\alpha} d {}^1A \right\} \quad \text{[II.41] (II.59)}$$

is the load increment vector, and

$${}_{1}K_{\alpha\beta}^{(4)} = \int_{1A} \Delta {}^1\sigma_{ni} \frac{\partial \psi_{i\beta}}{\partial {}^1x_j} \psi_{j\alpha} d {}^1A \quad \text{[II.42] (II.60)}$$

is the initial load or the load correction matrix.

Strain energy rate term:

$$J_3 \Rightarrow \int_{1V} \bar{1}S : \frac{d}{dt} \bar{1}E d {}^1V$$

Once again, following the same procedures of developing eqns.

(II.44-II.47), we may obtain

$$\Delta J_3 = [{}_{1}K_{\alpha\beta}^{(1)} + {}_{1}K_{\alpha\beta}^{(2)} + {}_{1}K_{\alpha\beta}^{(3)}] \{\Delta \bar{1}u_B\} \quad \text{[II.44] (II.61)}$$

where

$${}_{1}K_{\alpha\beta}^{(1)} = \int_{1V} \frac{\partial \psi_{i\alpha}}{\partial {}^1x_j} {}^1D_{ijkl} \frac{\partial \psi_{k\beta}}{\partial {}^1x_l} d {}^1V \quad \text{[II.45] (II.62)}$$

is the usual small displacement or the incremental stiffness matrix;

$${}^1K_{\alpha\beta}^{(2)} = \int_{V} {}^1\sigma_{ij} \frac{\partial \psi_{k\beta}}{\partial x_i} \frac{\partial \psi_{k\alpha}}{\partial x_j} dV \quad [\text{II.46}] \quad (\text{II.63})$$

is the initial stress or the geometric or the tangent stiffness matrix;

$$\begin{aligned} {}^1K_{\alpha\beta}^{(3)} = \int_{V} {}^1D_{ijkl} \{ & \left(\frac{\partial \psi_{r\gamma}}{\partial x_i} \frac{\partial \psi_{r\alpha}}{\partial x_j} \frac{\partial \psi_{k\beta}}{\partial x_l} + \frac{\partial \psi_{i\alpha}}{\partial x_j} \frac{\partial \psi_{w\gamma}}{\partial x_k} \frac{\partial \psi_{w\beta}}{\partial x_l} \right) {}^1\underline{u}_\gamma \\ & + \left(\frac{\partial \psi_{r\gamma}}{\partial x_i} \frac{\partial \psi_{r\alpha}}{\partial x_j} \frac{\partial \psi_{w\eta}}{\partial x_k} \frac{\partial \psi_{w\beta}}{\partial x_l} {}^1\underline{u}_\gamma {}^1\underline{u}_\eta \right) \} dV \quad [\text{II.47}] \quad (\text{II.64}) \end{aligned}$$

is the initial displacement or the initial rotation stiffness matrix.

Collecting results of eqns. (II.55-II.64), the final incremental equilibrium equations may take the form

$$\Delta {}^1\bar{R}_\alpha = [{}^1K_{\alpha\beta}^{(1)} + {}^1K_{\alpha\beta}^{(2)} + {}^1K_{\alpha\beta}^{(3)} - {}^1K_{\alpha\beta}^{(4)}] \{\Delta {}^1\underline{u}_\beta\} + [{}^1M_{\alpha\beta}] \{\Delta {}^1\underline{u}_\beta''\} \quad [\text{II.48}] \quad (\text{II.65})$$

where all individual terms are defined above.

The formulation given here differs from similar ones in the literature in the development of the load correction matrix, the definition of the load increment, the expressions and the number of stiffness matrices contributing to the total incremental nonlinear stiffness matrix. In [9-11], Bathe et al. introduce the same load increments for both Lagrangian and updated Lagrangian formulations. In the formulation presented here, there is a distinct difference between the two expressions, eqns. (II.41, II.59). The dependence of the load increment vector on the deformation gradient in only Lagrangian formulation and not in the updated Lagrangian version may be visualized from the nature of the reference configuration in each case. For the

same reason, it would be expected that the load correction matrix in updated Lagrangian case is dependent on $\Delta \bar{\sigma}_n$, the increment of the surface traction vector from configuration C_1 to C_2 , with comparison to its dependence on $\bar{\sigma}_n$, the surface traction vector from configuration C_0 to C_2 in the Lagrangian case. In [6] Wunderlich introduces kinematic simplifications which eliminate the initial displacement stiffness matrix in the updated Lagrangian formulation. As stated above, the number of stiffness matrices contributing to the total nonlinear stiffness matrix of the element are the same in both Lagrangian and updated Lagrangian formulations and only simplifying kinematic assumptions would eliminate some of these matrices. It is this reason which lends similar nature to both the Lagrangian and the updated Lagrangian formulations and makes them distinct from the features of an Eulerian formulation. It should be noticed that the definition of the material property tensor, eqn. (II.43), is different in each formulation. However, proper transformation should make it possible to change one into the other, as will be discussed later.

II.4.2 Incompressibility Constraint

In a much similar way to the development given in section (II.3.2), the incremental equilibrium equations in updated Lagrangian Formulation after considering the incompressibility constraint may be written as:

$$\Delta \bar{R}_\alpha = [{}^1K_{\alpha\beta}^{(1)} + {}^1K_{\alpha\beta}^{(2)} + {}^1K_{\alpha\beta}^{(3)} - {}^1K_{\alpha\beta}^{(4)}] \{\Delta \bar{u}_\beta\} \\ + [{}^1H_{\alpha N}] \{\Delta \bar{p}_N\} + [{}^1M_{\alpha\beta}] \{\Delta \bar{u}_\beta\} \quad \text{[II.52] (II.66)}$$

and

$$[{}^1H_{\beta N}]^T \{\Delta \bar{u}_\beta\} = \{0\} \quad \text{[II.53] (II.67)}$$

where

$${}^1H_{\alpha N} = \int_{1V} (\theta_N \frac{\partial^1 x_r}{\partial^2 x_i} \frac{\partial \psi_{1\alpha}}{\partial^1 x_r}) d^1V \quad \text{[II.54] (II.68)}$$

is the hydrostatic pressure stiffness matrix, and other terms have been defined above.

II.5 EULERIAN (SPATIAL) FORMULATION

II.5.1 Incremental Equilibrium Equations

Similar to Lagrangian and updated Lagrangian formulations, we base the Eulerian formulation on the mechanical energy balance equation in the deformed state written for a single element e at some time t during the deformation. Hence, starting with eqn. (II.36), the kinetic energy term may be written in the form

$$\frac{d}{dt} \frac{2K}{2} = \frac{d}{dt} \left\{ \frac{1}{2} \int_{2V} \rho \cdot \bar{v} \cdot \bar{v} d^2V \right\}$$

which for mass conserving systems becomes

$$= \frac{1}{2} \int_{2V} \rho \left(\frac{d \bar{v}}{dt} \cdot \bar{v} \right) d^2V = \int_{2V} \rho \bar{a} \cdot \bar{v} d^2V$$

Substituting the above equation together with

$$\bar{D} = \frac{1}{2} \left(\frac{\partial \bar{v}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{x}} \right) \quad \text{and} \quad d \bar{A} \cdot \bar{\sigma} = \bar{\sigma}_n d \bar{A}$$

into (II.36), and then considering a finite unit increment of time, the variational form of the energy balance equation may be written as

$$\int_{\bar{V}} \rho \bar{a} \cdot \delta \bar{u} d \bar{V} = \int_{\bar{V}} \rho \bar{b} \cdot \delta \bar{u} + \int_{\bar{A}} \bar{\sigma}_n \cdot \delta \bar{u} d \bar{A} - \int_{\bar{V}} \bar{\sigma} : \delta \left[\frac{1}{2} \left(\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{u}}{\partial \bar{x}} \right) \right] d \bar{V} \quad (II.69)$$

Substituting the displacement assumption (II.25), into (II.69) we have

$$\left\{ \int_{\bar{V}} \rho \bar{a} \cdot (\psi_{i\alpha} \bar{e}_i) d \bar{V} \right\} \delta \bar{u}_\alpha = \left\{ \int_{\bar{V}} \rho \bar{b} \cdot (\psi_{i\alpha} \bar{e}_i) d \bar{V} + \int_{\bar{A}} \bar{\sigma}_n \cdot (\psi_{i\alpha} \bar{e}_i) d \bar{A} - \int_{\bar{V}} \bar{\sigma} : \left[\frac{1}{2} \left(\frac{\partial \psi_{i\alpha}}{\partial \bar{x}_j} + \frac{\partial \psi_{j\alpha}}{\partial \bar{x}_i} \right) \bar{e}_i \bar{e}_j \right] d \bar{V} \right\} \delta \bar{u}_\alpha$$

where \bar{e}_i is the unit base vector in the Cartesian frame of reference.

For a virtual displacements $\delta \bar{u}$, the above expression yields

$$\int_{\bar{V}} \rho \bar{a} \cdot (\psi_{i\alpha} \bar{e}_i) d \bar{V} = \int_{\bar{V}} \rho \bar{b} \cdot (\psi_{i\alpha} \bar{e}_i) d \bar{V} + \int_{\bar{A}} \bar{\sigma}_n \cdot (\psi_{i\alpha} \bar{e}_i) d \bar{A} - \int_{\bar{V}} \bar{\sigma} : \frac{1}{2} \left[\left(\frac{\partial \psi_{i\alpha}}{\partial \bar{x}_j} + \frac{\partial \psi_{j\alpha}}{\partial \bar{x}_i} \right) \bar{e}_i \bar{e}_j \right] d \bar{V} \quad (II.70)$$

Taking increments of the energy balance equation in the form of eqn. (II.70) yields

$$\int_{2V} \Delta({}^2\rho \ 2\bar{a}) \cdot (\psi_{i\alpha} \bar{e}_i) d 2V = \int_{2V} \Delta({}^2\rho \ 2\bar{b}) \cdot (\psi_{i\alpha} \bar{e}_i) d 2V + \int_{2A} (\Delta \ 2\bar{\sigma}_n) \cdot (\psi_{i\alpha} \bar{e}_i) d 2A - \int_{2V} (\Delta \ 2\bar{\sigma}) : \left[\frac{1}{2} \left(\frac{\partial \psi_{i\alpha}}{\partial x_j} + \frac{\partial \psi_{j\alpha}}{\partial x_i} \right) \bar{e}_i \bar{e}_j \right] d 2V \quad (\text{II.71})$$

Introducing a load vector definition in the form

$$2\bar{R}_\alpha = \int_{2V} ({}^2\rho \ 2\bar{b}) \cdot (\psi_{i\alpha} \bar{e}_i) d 2V + \int_{2A} (2\bar{\sigma}_n) \cdot (\psi_{i\alpha} \bar{e}_i) d 2A \quad (\text{II.72})$$

and hence, the incremental load vector will take the form

$$\Delta \ 2\bar{R}_\alpha = \int_{2V} \Delta({}^2\rho \ 2\bar{b}) \cdot (\psi_{i\alpha} \bar{e}_i) d 2V + \int_{2A} (\Delta \ 2\bar{\sigma}_n) \cdot (\psi_{i\alpha} \bar{e}_i) d 2A, \quad (\text{II.73})$$

then, eqn. (70) takes the form

$$\Delta \ 2\bar{R}_\alpha = \int_{2V} (\Delta \ 2\bar{\sigma}) : \left[\frac{1}{2} \left(\frac{\partial \psi_{i\alpha}}{\partial x_j} + \frac{\partial \psi_{j\alpha}}{\partial x_i} \right) \bar{e}_i \bar{e}_j \right] d 2V + \int_{2V} \Delta({}^2\rho \ 2\bar{a}) \cdot (\psi_{i\alpha} \bar{e}_i) d 2V \quad (\text{II.74})$$

It is important to observe that, contrary to the Lagrangian and the updated Lagrangian formulation, expression (II.72) for the load vector is independent of the deformation gradients. This form eliminates the need for an initial load or a load correction matrix as observed in the Lagrangian and the updated Lagrangian formulation. Also, a simpler expression for the incremental load vector, equation (II.73), is obtained. The incremental load vector in both the Eulerian and the updated Lagrangian formulation is independent of the displacements, whereas in Lagrangian formulation the same vector is dependent on the

current displacements for all types of loading.

To complete the formulation, we now introduce a constitutive equation in the form of

$$\Delta \overset{2}{\sigma} = \overset{4}{C} : \Delta \overset{2}{\epsilon} \quad (\text{II.75})$$

where $\overset{4}{C}$ is a fourth order tensor which may include the effect of frame indifference. Utilizing (II.7-II.10 and II.25-II.27) and neglecting second order deformation gradients as discussed above, we obtain the following relations

$$\frac{\partial \overset{2}{u}}{\partial \overset{2}{x}} = \left(\frac{\partial \psi_{j\gamma}}{\partial \overset{2}{x}_i} \overset{2}{u}_{\gamma} \right) \bar{e}_i \bar{e}_j \quad (\text{II.76})$$

$$\frac{\partial \overset{2}{v}}{\partial \overset{2}{x}} = \left(\frac{\partial \psi_{m\beta}}{\partial \overset{2}{x}_i} \frac{\partial \overset{2}{x}_j}{\partial \overset{0}{X}_m} \overset{2}{v}_{\beta} \right) \bar{e}_i \bar{e}_j \quad (\text{II.77})$$

$$\frac{\partial (\Delta \overset{2}{u})}{\partial \overset{2}{x}} = \left[\frac{\partial \psi_{m\beta}}{\partial \overset{2}{x}_i} \frac{\partial \overset{2}{x}_j}{\partial \overset{0}{X}_m} (\Delta \overset{2}{u}_{\beta}) \right] \bar{e}_i \bar{e}_j \quad (\text{II.78})$$

$$\frac{\partial \overset{2}{u}}{\partial \overset{2}{x}} \cdot \frac{(\Delta \overset{2}{u})}{\partial \overset{2}{x}} = \left[\left(\frac{\partial \psi_{k\gamma}}{\partial \overset{2}{x}_i} \frac{\partial \psi_{m\beta}}{\partial \overset{2}{x}_j} \frac{\partial \overset{2}{x}_k}{\partial \overset{0}{X}_m} \overset{2}{u}_{\gamma} \right) \Delta \overset{2}{u}_{\beta} \right] \bar{e}_i \bar{e}_j \quad (\text{II.79})$$

Substituting (II.76-II.79) into (II.20) we obtain

$$\Delta \overset{2}{\epsilon} = \left[\frac{1}{2} \left(\frac{\partial \psi_{m\beta}}{\partial \overset{2}{x}_i} \frac{\partial \overset{2}{x}_j}{\partial \overset{0}{X}_m} + \frac{\partial \psi_{m\beta}}{\partial \overset{2}{x}_j} \frac{\partial \overset{2}{x}_i}{\partial \overset{0}{X}_m} \right) \right]$$

$$- \frac{1}{2} \left(\frac{\partial \psi_{k\gamma}}{\partial x_i} \frac{\partial \psi_{m\beta}}{\partial x_j} \frac{\partial^2 x_k}{\partial X_m} + \frac{\partial \psi_{k\gamma}}{\partial x_j} \frac{\partial \psi_{m\beta}}{\partial x_i} \frac{\partial^2 x_k}{\partial X_m} \right) \delta_{\underline{u}_\gamma} \left(\Delta \delta_{\underline{u}_\beta} \right) \bar{e}_i \bar{e}_j \quad (\text{II.80})$$

Hence, eqn. (II.75) may take the form

$$\Delta \delta_{\underline{\sigma}} = \delta C_{ijkl} \left[\frac{1}{2} \left(\frac{\partial \psi_{m\beta}}{\partial x_k} \frac{\partial^2 x_l}{\partial X_m} + \frac{\partial \psi_{m\beta}}{\partial x_l} \frac{\partial^2 x_k}{\partial X_m} \right) \right. \\ \left. - \frac{1}{2} \left(\frac{\partial \psi_{p\gamma}}{\partial x_k} \frac{\partial \psi_{m\beta}}{\partial x_l} \frac{\partial^2 x_p}{\partial X_m} + \frac{\partial \psi_{p\gamma}}{\partial x_l} \frac{\partial \psi_{m\beta}}{\partial x_k} \frac{\partial^2 x_p}{\partial X_m} \right) \delta_{\underline{u}_\gamma} \right] \left(\Delta \delta_{\underline{u}_\beta} \right) \bar{e}_i \bar{e}_j \quad (\text{II.81})$$

To obtain the final incremental equilibrium equations, we substitute the stress-strain relations, equation (II.81), into the incremental form of the energy balance equation, equation (II.74),

$$\Delta \delta \bar{R}_\alpha = \int_{\delta V} \left\{ \frac{1}{4} \delta C_{ijkl} \left[\left(\frac{\partial \psi_{i\alpha}}{\partial x_j} + \frac{\partial \psi_{j\alpha}}{\partial x_i} \right) \left(\frac{\partial \psi_{m\beta}}{\partial x_k} \frac{\partial^2 x_l}{\partial X_m} + \frac{\partial \psi_{m\beta}}{\partial x_l} \frac{\partial^2 x_k}{\partial X_m} \right) \right. \right. \\ \left. \left. - \left(\frac{\partial \psi_{i\alpha}}{\partial x_j} + \frac{\partial \psi_{j\alpha}}{\partial x_i} \right) \left(\frac{\partial \psi_{p\gamma}}{\partial x_k} \frac{\partial \psi_{m\beta}}{\partial x_l} \frac{\partial^2 x_p}{\partial X_m} + \frac{\partial \psi_{p\gamma}}{\partial x_l} \frac{\partial \psi_{m\beta}}{\partial x_k} \frac{\partial^2 x_p}{\partial X_m} \right) \delta_{\underline{u}_\gamma} \right] \right\} \\ \delta \delta V \left(\Delta \delta_{\underline{u}_\beta} \right) + \int_{\delta V} \Delta \left(\delta \rho \delta \bar{a} \right) \cdot \left(\psi_{i\alpha} \bar{e}_i \right) \delta \delta V \quad (\text{II.82})$$

Introducing a lumped mass approximation for the dynamical term, the final incremental equilibrium equations may take the form

$$\Delta \delta \bar{R}_\alpha = \left[\delta K_{\alpha\beta}^{(1)} - \delta K_{\alpha\beta}^{(2)} \right] \left\{ \Delta \delta_{\underline{u}_\beta} \right\} + \left[\delta M_{\alpha\beta} \right] \left\{ \Delta \delta_{\underline{u}_\beta}'' \right\} \quad (\text{II.83})$$

where $\Delta \delta \bar{R}_\alpha$ is the incremental load vector given by eqn. (II.73),

$$2_{\alpha\beta}^{K(1)} = \int_{2_V} \frac{1}{4} \left(\frac{\partial \psi_{i\alpha}}{\partial 2_{x_j}} + \frac{\partial \psi_{j\alpha}}{\partial 2_{x_i}} \right) 2C_{ijkl} \left(\frac{\partial \psi_{m\beta}}{\partial 2_{x_k}} \frac{\partial 2_{x_l}}{\partial 0_{X_m}} + \frac{\partial \psi_{m\beta}}{\partial 2_{x_l}} \frac{\partial 2_{x_k}}{\partial 0_{X_m}} \right) d 2_V \quad (\text{II.84})$$

corresponds to the usual small displacement stiffness matrix,

$$2_{\alpha\beta}^{K(2)} = \int_{2_V} \frac{1}{4} 2C_{ijkl} \left(\frac{\partial \psi_{i\alpha}}{\partial 2_{x_j}} + \frac{\partial \psi_{j\alpha}}{\partial 2_{x_i}} \right) \left(\frac{\partial \psi_{p\gamma}}{\partial 2_{x_k}} \frac{\partial \psi_{m\beta}}{\partial 2_{x_l}} \frac{\partial 2_{x_p}}{\partial 0_{X_m}} + \frac{\partial \psi_{p\gamma}}{\partial 2_{x_l}} \frac{\partial \psi_{m\beta}}{\partial 2_{x_k}} \frac{\partial 2_{x_p}}{\partial 0_{X_m}} \right) 2_{u_\gamma} d 2_V \quad (\text{II.85})$$

corresponds to the initial displacement or the initial rotation stiffness matrix and, finally,

$$2_{\alpha\beta}^M = \frac{1}{N} I_{\alpha\beta} \int_{2_V} 2_\rho d 2_V \quad (\text{II.86})$$

is the lumped mass matrix, where N is the number of nodes per element and $[I]$ is the identity matrix.

As mentioned above, a strict comparison between spatial and relative or referential formulations may not be strictly logical owing to the specific nature of each. However, the two stiffness matrices in the Eulerian formulation correspond to the usual small displacement stiffness matrix and to the initial displacement or initial rotation stiffness matrix when the Eulerian formulation is compared with the Lagrangian and the updated Lagrangian formulation. Once again, the use of $\bar{\sigma}$ and \bar{D} as the conjugate pair of variables in the Eulerian formulation avoids the analogy with the initial stress or the geometric stiffness matrix. The analogy between different formulations is based on a comparison between the linear and the nonlinear parts in the expressions of stress and strain increments in each formulation.

II.5.2 Incompressibility Constraint

Once again, in a much similar way to the development of sections (II.3.2) and (II.4.2), the incremental equilibrium equations in Eulerian formulation after implementing the incompressibility constraint may be written as

$$\Delta \underline{\underline{R}}_{\alpha} = [\underline{\underline{K}}_{\alpha\beta}^{(1)} - \underline{\underline{K}}_{\alpha\beta}^{(2)}] \{ \Delta \underline{\underline{u}}_{\beta} \} + [\underline{\underline{H}}_{\alpha N}] \{ \Delta \underline{\underline{P}}_{-N} \} + [\underline{\underline{M}}_{\alpha\beta}] \{ \Delta \underline{\underline{u}}_{\beta} \} \quad (\text{II.87})$$

and

$$[\underline{\underline{H}}_{\beta N}]^T \{ \Delta \underline{\underline{u}}_{\beta} \} = \{ 0 \} \quad (\text{II.88})$$

where

$$\underline{\underline{H}}_{\alpha N} = \int_{\underline{\underline{V}}} (q_N \frac{\partial \psi_{i\alpha}}{\partial x_1}) d \underline{\underline{V}} \quad (\text{II.89})$$

is the hydrostatic pressure stiffness matrix, whereas other terms have been defined above.

II.6 CONSTITUTIVE EQUATIONS

II.6.1 Elastic Material

a. Lagrangian and Updated Lagrangian Formulations

For the classical hyperelastic material, one may introduce the strain energy density measured per unit initial volume so that $\bar{W} = W(\bar{E}) = w(\bar{F})$. This will lead to stress-strain relations in terms of the derivatives of the strain energy W with respect to E or F . Such approach is discussed extensively by Truesdell and Noll [52] and Oden [13]. For specific applications of the same approach, see for example references [36, 199 and 201].

On the other hand, one may use an incremental formulation based on the current configuration as a reference state, so that

$$\Delta \overset{1}{\sigma} = \overset{1}{C} : \Delta \overset{1}{\epsilon} \quad (\text{II.90})$$

where $\Delta \overset{1}{\sigma}$ and $\Delta \overset{1}{\epsilon}$ are the increments of Cauchy stress and strain tensors, and $\overset{1}{C}$ is the tensor of the appropriate elastic constants, then through transformation to the undeformed configuration with the use of Jaumann and Truesdell stress increments, Hibbit et al. [16] give a form of the constitutive equation as

$$\Delta \overset{1}{S}_{ij} = \left| \frac{\partial \overset{1}{x}}{\partial \overset{0}{X}} \right| \frac{\partial \overset{0}{X}_1}{\partial \overset{1}{x}_m} \frac{\partial \overset{0}{X}_j}{\partial \overset{1}{x}_n} \frac{\partial \overset{0}{X}_r}{\partial \overset{1}{x}_k} \{ C_{mnkl} \frac{\partial \overset{0}{X}_s}{\partial \overset{1}{x}_l} + \overset{1}{\sigma}_{mn} \frac{\partial \overset{0}{X}_s}{\partial \overset{1}{x}_k} - \overset{1}{\sigma}_{mk} \frac{\partial \overset{0}{X}_s}{\partial \overset{1}{x}_n} - \overset{1}{\sigma}_{nk} \frac{\partial \overset{0}{X}_s}{\partial \overset{1}{x}_m} \} \Delta \overset{1}{E}_{rs} \quad (\text{II.91})$$

Contrary to the development given by Hibbit et al. [16], and due to arguments given by Prager [202], Masur [203] and discussed in [14], eqn. (II.91) will be restricted to elastic materials. Owing to the same arguments, any possible frame indifferent rate of stress may be applied to derive an acceptable constitutive equation for elastic material. As an alternative example we will consider the use of a convected stress rate in the form

$$\left(\frac{\delta \bar{\sigma}}{\delta t}\right)^C = \left(\frac{\delta \bar{\sigma}}{\delta t}\right)^J + \bar{D} \cdot \bar{\sigma} + \bar{\sigma} \cdot \bar{D} \quad (II.92)$$

where $(\delta \bar{\sigma}/\delta t)^C$ is the convected stress rate, $(\delta \bar{\sigma}/\delta t)^J$ is the Jaumann stress rate, and $\bar{\sigma}$ is the Cauchy stress tensor. Equation (II.92) yields (see similar development in Appendix B)

$$\Delta \overset{1}{S}_{ij} = \left| \frac{\partial \overset{1}{x}}{\partial \overset{0}{X}} \right| \frac{\partial \overset{0}{X}_i}{\partial \overset{1}{x}_m} \frac{\partial \overset{0}{X}_j}{\partial \overset{1}{x}_n} \frac{\partial \overset{0}{X}_r}{\partial \overset{1}{x}_k} \{ C_{mnkl} \frac{\partial \overset{0}{X}_s}{\partial \overset{1}{x}_l} + \sigma_{mk} \frac{\partial \overset{0}{X}_s}{\partial \overset{1}{x}_n} + \sigma_{nk} \frac{\partial \overset{0}{X}_s}{\partial \overset{1}{x}_m} \} \Delta \overset{1}{E}_{rs} \quad (II.93)$$

Transformation of specific forms of constitutive equations from total Lagrangian to updated Lagrangian formulation may be achieved through the relations

$$\overset{1}{D}^4 = \left| \overset{1}{F} \right| \overset{1}{F}^{-1} \cdot \overset{1}{F}^{-1} \cdot \overset{1}{D}^4 \cdot (\overset{1}{F}^{-1})^T \cdot (\overset{1}{F}^{-1})^T, \quad (II.94)$$

$$\overset{1}{D}^4 = \frac{1}{\left| \overset{1}{F} \right|} \overset{1}{F} \cdot \overset{1}{F} \cdot \overset{1}{D}^4 \cdot \overset{1}{F}^T \cdot \overset{1}{F}^T \quad (II.95)$$

Equations (II.94 & II.95) have components

$$\overset{1}{D}_{ijkl} = \left| \frac{\partial \overset{1}{x}}{\partial \overset{0}{X}} \right| \frac{\partial \overset{0}{X}_i}{\partial \overset{1}{x}_m} \frac{\partial \overset{0}{X}_j}{\partial \overset{1}{x}_n} \overset{1}{D}_{mnpq} \frac{\partial \overset{0}{X}_k}{\partial \overset{1}{x}_p} \frac{\partial \overset{0}{X}_l}{\partial \overset{1}{x}_q} \quad (II.96)$$

and

$$\overset{1}{D}_{ijkl} = \left| \frac{\partial \overset{0}{X}}{\partial \overset{1}{x}} \right| \frac{\partial \overset{1}{x}_i}{\partial \overset{0}{X}_m} \frac{\partial \overset{1}{x}_j}{\partial \overset{0}{X}_n} \overset{1}{D}_{mnpq} \frac{\partial \overset{1}{x}_k}{\partial \overset{0}{X}_p} \frac{\partial \overset{1}{x}_l}{\partial \overset{0}{X}_q} \quad (II.97)$$

b. Eulerian Formulation

In Eulerian formulation, we may start with an incremental equation similar to eqn. (II.90) (equation (II.90) may, as well, be written with respect to a configuration C_2 instead of C_1). To this point, however, a careful interpretation of this equation is to be discussed. The stress increment in eqn. (II.90) is meant to represent the Cauchy stress increment. This increment is referred to and measured per unit area of the current configuration, which is the configuration just preceding the application of the load increment. By definition, this is no longer a Cauchy stress increment and in fact it represents the increment of the first Piola-Kirchhoff stress tensor referred to the current configuration, whereas the Cauchy stress increment should be referred to and measured per unit area in the new configuration which results after applying the load increment. The stress ${}^1\sigma$ and the incremental stress $\Delta {}^1\sigma$ are, however, referred to and measured per unit area of the same configuration. As a result, ${}^1\sigma$ and $\Delta {}^1\sigma$ may be added to give a stress tensor referred to and measured per unit area of the current configuration just preceding the application of the increment. The actual Cauchy stress tensor at configuration C_2 , just following the application of the load increment, may be obtained through the transformation

$${}^2\sigma = \frac{1}{|{}^2F|} {}^2F \cdot ({}^1\sigma + \Delta {}^1\sigma) \cdot {}^2F^T \quad (\text{II.98})$$

The treatment of $\Delta {}^1\epsilon$ is similar and we may write the corresponding

transformation in the form

$${}^2\bar{\epsilon} = ({}^2\bar{F}^{-1})^T \cdot ({}^1\bar{\epsilon} + \Delta {}^1\bar{\epsilon}) \cdot {}^2\bar{F}^{-1} \quad (\text{II.99})$$

Expressions similar to (II.91 and II.93) may be obtained for the Eulerian case. For a detailed example, see Appendix (B).

II.6.2 Elastoplastic Material

a. General Considerations

Intensive effort has been attempted to describe elastoplastic material behaviour at finite strains but little of this effort has become widely accepted, see for example Green and Naghdi [204-206]. If an infinitesimal theory is used to describe the material behaviour at finite strains with the stress replaced by the second Piola-Kirchhoff stress and the infinitesimal strain replaced by Green-Lagrange strain tensor, then a special case of the theory of Green and Naghdi is obtained. In this approach, the stresses may be converted to Cauchy stresses while the strains may be converted to Eulerian strains for a spatial formulation. The computational complications, arising from the accurate utilization of the theory of Green and Naghdi and other similar approaches, have so far made the use of the resulting constitutive equations unfeasible in programs for large scale analysis.

Another widely used approach, is to employ a generalized infinitesimal constitutive equation written in an invariant form with respect to rigid body motion [9-12 and 16-17]. In this approach, we may still assume a linear decomposition of strain increments into elastic

and plastic parts. This is not the case in general [49], but for static deformations of most metals, we can assume that the recoverable elastic strains are small so that the above decomposition is valid. Then, the use of tangent modulus method in finite element analysis leads to an incremental equation of the form of eqn. (II.90) given above, where C_{ijkl} are now, either the appropriate elastic constants for a purely elastic increment, or are derived from the elastic constants, the current state of yielding and the work hardening characteristics for an elastoplastic increment. The dependence of the C_{ijkl} on only the current state implies that both the material and the hardening rule should be isotropic. However anisotropic behaviour could be included as it will only complicate the formulation of the tensor $\frac{4}{C}$. Hence, with the restriction of isotropic hardening and small recoverable strains, the tensor $\frac{4}{C}$ is formed in the same way as in the small strain analysis.

b. Lagrangian and Updated Lagrangian Formulations

To insure that the constitutive equation would be frame-indifferent, we can only use Jaumann derivative as a frame indifferent rate in elastoplastic deformation [6,19,20]

$$\left(\frac{\delta \bar{\sigma}}{\delta t}\right)^J = \left(\frac{d \bar{\sigma}}{dt}\right) - \bar{W} \cdot \bar{\sigma} + \bar{\sigma} \cdot \bar{W} \quad (\text{II.100})$$

where \bar{W} is the skew part of the velocity gradient tensor, $\bar{L} = \bar{v}\partial/\partial\bar{x}$, and $\bar{L} = \bar{D} + \bar{W}$.

Equation (II.100) may be shown to give (see Appendix B)

$$\Delta \overset{1}{S}_{ij} = \overset{1}{D}_{ija} \Delta \overset{1}{u}_a \quad (\text{II.101})$$

where

$$\begin{aligned} \overset{1}{D}_{ija} = & \left| \frac{\partial \overset{1}{x}}{\partial \overset{0}{X}} \right| \frac{\partial \overset{0}{X}_i}{\partial \overset{1}{x}_m} \frac{\partial \overset{0}{X}_j}{\partial \overset{1}{x}_n} \left\{ \frac{\partial \overset{0}{X}_r}{\partial \overset{1}{x}_k} C_{mnkl} \frac{\partial \overset{0}{X}_s}{\partial \overset{1}{x}_l} \left(\frac{\partial \psi_{r\alpha}}{\partial \overset{0}{X}_s} + \frac{\partial \psi_{w\beta}}{\partial \overset{0}{X}_r} \frac{\partial \psi_{w\alpha}}{\partial \overset{0}{X}_s} \overset{1}{u}_\beta \right) \right. \\ & \left. - \frac{1}{2} \left(\frac{\partial \psi_{p\alpha}}{\partial \overset{0}{X}_v} \frac{\partial \overset{0}{X}_v}{\partial \overset{1}{x}_m} - \frac{\partial \psi_{m\alpha}}{\partial \overset{0}{X}_v} \frac{\partial \overset{0}{X}_v}{\partial \overset{1}{x}_p} \right) \overset{1}{\sigma}_{pn} - \frac{1}{2} \left(\frac{\partial \psi_{n\alpha}}{\partial \overset{0}{X}_v} \frac{\partial \overset{0}{X}_v}{\partial \overset{1}{x}_p} - \frac{\partial \psi_{p\alpha}}{\partial \overset{0}{X}_v} \frac{\partial \overset{0}{X}_v}{\partial \overset{1}{x}_n} \right) \overset{1}{\sigma}_{mp} \right\} \end{aligned} \quad (\text{II.102})$$

Once again, the transformation of specific constitutive equations from Lagrangian to updated Lagrangian formulation, and vice versa, may be achieved through eqns. (II.94-II.97).

c. Eulerian Formulation

The same comments on the treatment of incremental stresses and incremental strains, in case of Eulerian formulation of elastic constitutive equations, may be considered for the elastoplastic case. Owing to the same arguments given in references [14,202,203], we may only use Jaumann derivative, eqn. (II.100), as a frame indifferent rate in elastoplastic deformations. The use of eqn. (II.100) may be shown to give a constitutive equation in the form of (see appendix (B))

$$\Delta \overset{2}{\sigma}_{ij} = \overset{2}{c}_{ija} \Delta \overset{2}{u}_a \quad (\text{II.103})$$

where

$$\begin{aligned}
{}^2C_{ij\alpha} = & \left\{ \frac{1}{2} {}^2C_{ijkl} \left[\left(\frac{\partial \psi_{m\alpha}}{\partial x_k} \frac{\partial^2 x_l}{\partial X_m} + \frac{\partial \psi_{m\alpha}}{\partial x_l} \frac{\partial^2 x_k}{\partial X_m} \right) \right. \right. \\
& - \left. \left(\frac{\partial \psi_{r\gamma}}{\partial x_k} \frac{\partial \psi_{m\alpha}}{\partial x_l} \frac{\partial^2 x_r}{\partial X_m} + \frac{\partial \psi_{r\gamma}}{\partial x_l} \frac{\partial \psi_{m\alpha}}{\partial x_k} \frac{\partial^2 x_r}{\partial X_m} \right) {}^2u_{\gamma} \right] \\
& + \left. \frac{1}{2} \left(\frac{\partial \psi_{m\alpha}}{\partial x_p} \frac{\partial^2 x_i}{\partial X_m} - \frac{\partial \psi_{m\alpha}}{\partial x_i} \frac{\partial^2 x_p}{\partial X_m} \right) {}^2\sigma_{pj} + \frac{1}{2} \left(\frac{\partial \psi_{m\alpha}}{\partial x_p} \frac{\partial^2 x_j}{\partial X_m} - \frac{\partial \psi_{m\alpha}}{\partial x_j} \frac{\partial^2 x_p}{\partial X_m} \right) {}^2\sigma_{pi} \right\}
\end{aligned}$$

(II.104)

CHAPTER III

APPLICATION: A MODIFIED PLASTICITY THEORY FOR POROUS METALS

III.1 INTRODUCTION

In recent years, the behaviour of porous materials under mechanical loads has been considered of increasing importance. The problems to be studied include the production of powder metal products by die pressing and forging techniques and the behaviour of these products in use, the behaviour of rocks when considered to be structural components, the effect of large dose nuclear radiation on metals and ceramics, and the behaviour of solids weakened by numerous cracks or voids when subjected to high intensity stress waves.

One of the main aspects of this study is the development of a plasticity theory for porous materials. In conventional plasticity theory volume constancy is assumed for the material undergoing plastic deformation, and this assumption applies to pore-free materials very well. In the deformation of porous materials, however, the volume does not remain constant. Some progress has been made in the construction of plasticity theories for soils [207-210]. Such theories aim to account for some important characteristics of soil material behaviour, such as the dependence of the failure envelope on the hydrostatic stress state, the ability to either soften or harden according to some measure of density, and the ability of dilation or compaction when deviatorically

stressed at different densities. These soil plasticity theories generally involve macroscopic scale assumptions concerning the means by which the plastic work is dissipated. Theories which have been presented for porous metal plasticity [211-216], however, have generally concerned themselves with microstructural details and the treatment in this chapter will continue this approach in an attempt to refine these porous metal models. Also there is apparently no evidence for the existence of a critical state or a cap phenomena corresponding to that observed for soils, so it is difficult to compare the two materials. The main assumptions included in previous work [211-216] on porous metals may be restated here as: isotropy of the material, rigid perfectly plastic matrix material obeying Von Mises yield criterion and the corresponding flow rule, and isotropic void growth or decay.

All the previous work suggests that the yield criterion for a porous metal is a function of the first invariant of stress, J_1 , and the second invariant of deviatoric stress, J_2 . The particular form used by previous authors was

$$F = \{\alpha J_2' + \beta J_1^2\}^{1/2} \quad (\text{III.1})$$

where α and β are functions of the relative density. The relative density, ρ , is defined as the ratio of the density of matrix plus voids to the density of the void-free matrix material, where the matrix material is the actual material of the body without voids. The stress-strain relations for this yield criterion were of the form,

$$d\epsilon_i = d\lambda (\sigma_i - \phi \sigma_m) \quad i = 1, 2, 3 \quad (\text{III.2})$$

where ϕ is a function of α and β , $d\lambda$ is a non-negative constant and σ_m is the hydrostatic stress. The determination of α and β was done analytically by Green [211] through specific assumptions on the stress distribution in his model which consists of a uniform cubic array with spherical voids inside. Oyane et al. [212,214] and Kuhn et al. [213] determined the same parameters experimentally.

The form of equation (III.1) was assumed arbitrarily and the experiments with which it was compared considered only a few simple states of stress such as pure shear and die compression. This seems to be insufficient and the main object of this chapter is to examine the validity of this yield criterion and the consequent basic equations of plasticity with respect to different stress states.

III.2 MATERIAL MODEL

The material model used in this analysis is based on that used by Green [211] who considered a material containing spherical voids in a cubic array. To simplify numerical calculations, a modification to two dimensions is adopted, although this is, of course, not necessary. The material thus contains a regular array of cylindrical voids. Such an assumption may be reasonably accurate for metals affected by neutron radiation as this would probably leave roughly cylindrical damaged zones. However, powder metal products contain roughly spherical voids because of the surface tension effects during sintering. Plane strain conditions around cylindrical pores, as opposed to plane stress, probably results in a stress distribution which is closer to that which

would exist for spherical pores. Hence, all results in this chapter are for plane strain rather than plane stress. The results obtained in these two dimensional analyses are then extended to three dimensions by the assumption of isotropic behaviour. Since the model is intended to represent an isotropic material, any anisotropy of the model would have no physical significances although it could affect the accuracy of the model in reproducing the behaviour of the real material. The fairly good agreement obtained between the analysis and the experimental results, however, indicates that this source of error is probably not significant. Von Mises yield criterion and the associated flow rule are assumed to apply to the matrix material.

A finite element approach, utilizing the NONFE programming system [194], is used. Such finite element approach reduces the number of restricting assumptions regarding the stress distribution as done, for example, by Green [211]. In addition more accurate material properties, primarily the addition of elastic strains and of plastic strain hardening and more realistic geometries may be considered. Also, material anisotropy and the effect of void growth or decay may be handled in an easier manner.

Figure (III.1a) shows the material model. By symmetry conditions, only the section shown in Figure (III.1b) need be considered. The boundary of the pore centered on the origin is considered stress free. Displacements, u , along the edge $x = 0$ and displacements, v , along the edge $y = 0$ are constrained to be zero, where u and v are the displacements in x and y directions, respectively. The other two edges

of the square are given fixed normal displacements, constant throughout their length. This is a condition obtained from symmetry for all strain states having $\gamma_{xy} = 0$ which are the only cases considered. A state of pure shear can be produced by imposing $\epsilon_x/\epsilon_y = -1$ with $\gamma_{xy} = 1$, but the latter is difficult to impose because of the lack of a guiding symmetry condition to use in defining the boundary conditions for such case. The anisotropy caused by the particular pore distribution used would be an interesting point to examine in future studies, but, since the intention in this application is to model isotropic solids only, no further account was taken of this point and only strain states having principal strain directions parallel to the x and y axes were considered. The average strain as a local measure of deformation, i.e. end displacement divided by the total length, would be the one measured in a test on the material at a macroscopic scale when all similar units to that shown in Figure (III.1b) are assembled into the complete lattice as shown in Figure (III.1a). Similarly the stress calculated would be the average stress appearing along the boundaries of the model unit. These average stresses and strains are given later as stress-strain curves for the material.

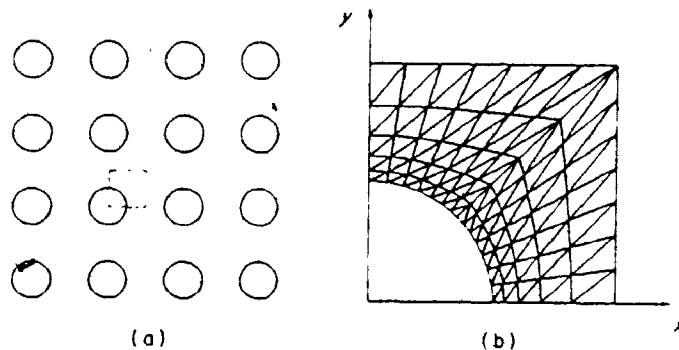


Figure (III.1) Material model and finite element grid for $\rho = 0.8$

The geometry used in this model is obviously a gross simplification of the distribution of pores in a real material. In order to examine the effect of changes of this distribution on the results, some analysis was carried out on the modified model shown in Figure (III.2). Again by consideration of symmetry only the element enclosed by dotted lines in the model need be considered. Similar arguments to those discussed for the original model, may be shown to apply for this new model concerning the boundary conditions and the average stress and strain. Some of the results of this modified model will be shown together with the corresponding results from the original one for comparison. Both grids are symmetrical about their pertinent diagonals. No attempt has been made to check the accuracy of the solution with respect to changes in the number of nodal points of the grid.

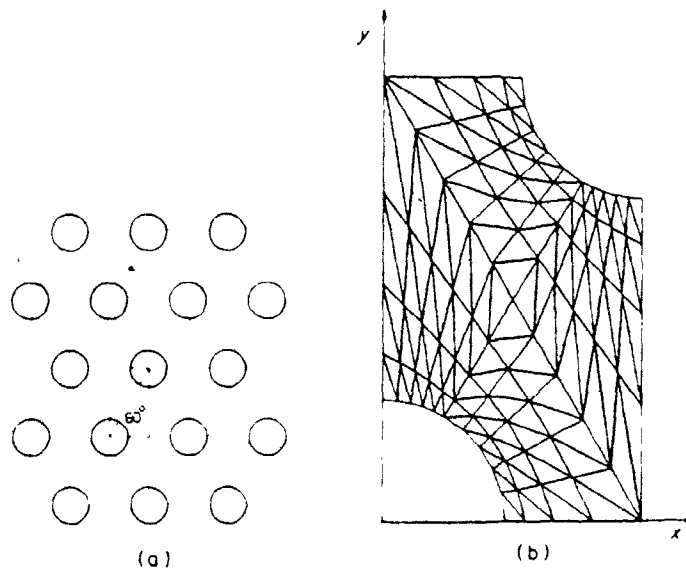


Figure (III.2) Modified material model and finite element grid for $\rho = 0.8$

The finite element technique used is an infinitesimal strain, elastoplastic analysis using constant strain triangular elements similar to that given by Yamada et al. [79] and Zienkiewicz [217]. Nonlinearity is only considered to arise from material behaviour, so the strains are kept infinitesimal. A large strain analysis with geometrical nonlinearities would be desirable if material behaviour during a forming operation were required. However, the small strain analysis including elastic behaviour of the material would be of greater help in studying the initial flow stress of the material and investigating a proper yield surface. In addition, the information obtained from the infinitesimal strain analysis would be more important to the analysis of structural components than those obtained from the large strain behaviour.

The material properties used were, in terms of Young's modulus, E :

$$\text{Yield point of bulk material, } Y = E/1800$$

$$\text{Elastic strain hardening modulus, } H = E/40$$

$$\text{Poisson's ratio, } \nu = 0.355$$

When the Young's modulus, E , is taken equal to 10800 kg/mm^2 then the previous properties correspond to solid copper ($\rho = 1$). This was chosen since some experimental results for copper are available [214] to which the analysis will be compared.

The strain states analyzed were, $\epsilon_y/\epsilon_x = -1.0, -0.75, -0.5, 0.0, +0.5$ and $+1.0$, for relative densities, ρ , of 0.6, 0.8 and 0.95 for the original model.

III.3 NUMERICAL RESULTS

Figures (III.3) shows stress-strain curves for the model material in two strain states. These are typical for all cases studied. For each strain state, the results of the original and the modified models are plotted together for comparison. The results of the two models are in good agreement. The discrepancy in the two results is greater at late stages of the analysis. However, we are not concerned about this stage in studying a yield criterion for the material, and the difference at the yield point is small for the stress states studied.

The yield stress of the material is defined in two ways, first, a two tangent method, and second at the step when a fully plastic section is developed in the model [218]. The results of the two definitions are again very close as shown in Figure (III.3). To check the numerical results against experiments, Figure (III.4) shows a plot of the experimental yield stresses versus the relative density for sintered copper in die compression [214], where they are compared to the results given by previous authors and to those from the present model. The results shown compare favourably with those previously determined. The finite element predictions lie between the two empirically derived formulae [213, 214].

In order to determine whether the yield criterion, equation (III.1), describes the numerical results we obtained, plots were made of J_2' versus J_1^2 . An example of these for $\rho = 0.6$ is shown in Figure (III.5).

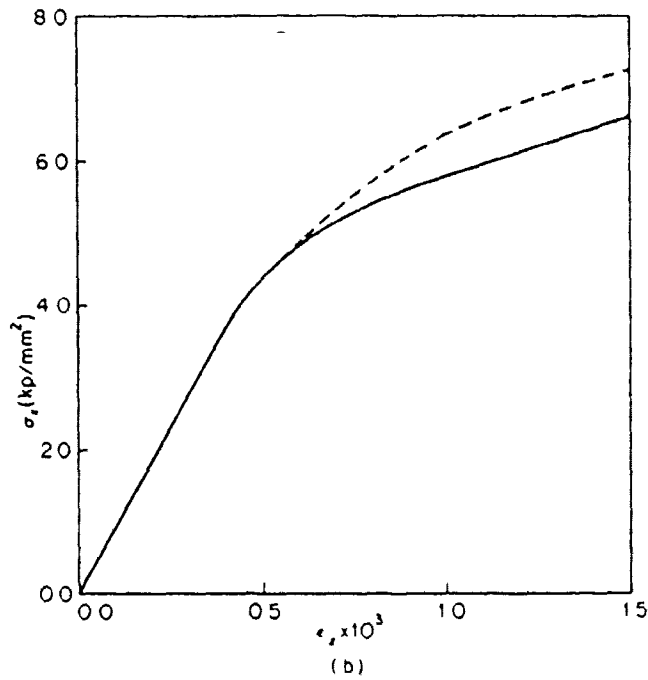
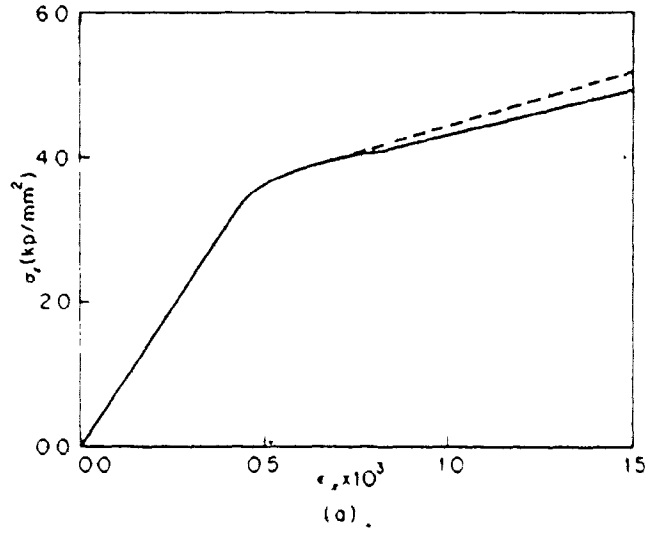


Figure (III.3) Stress-strain curves for original and modified models, $\rho = 0.8$: (a) $\epsilon_y/\epsilon_x = -0.5$, (b) $\epsilon_y/\epsilon_x = 0.0$

— original model
 - - - modified model

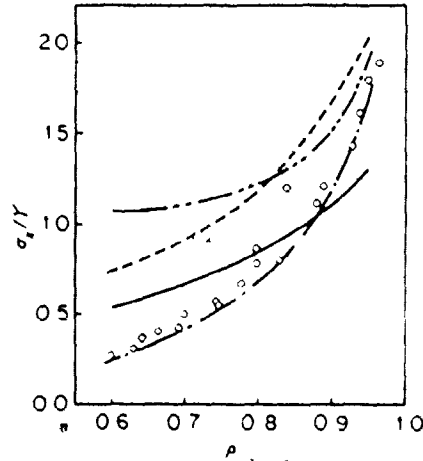


Figure (III.4) Flow stress vs. void fraction for die compression,
 $\epsilon_y/\epsilon_x = 0.0$

- Green [211]
- Shima and Oyane [214]
- · - · - Kuhn and Downey [213]
- present analysis
- o experimental data [214]

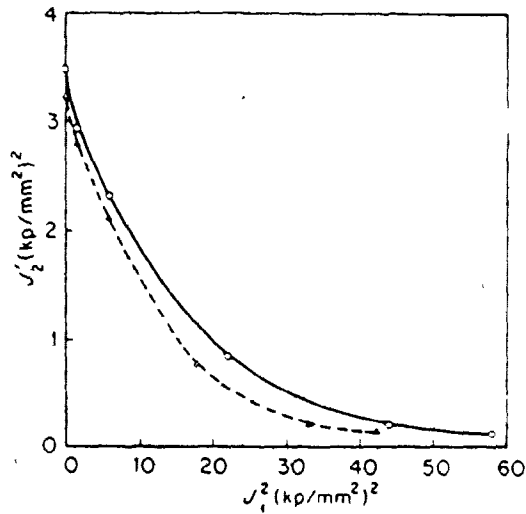


Figure (III.5) A plot of J_1^2 vs. J_2^2 for the numerical results
 obtained, $\rho = 0.6$

- o— yield defined by two tangents
- △- yield defined by fully plastic section

The relation is definitely not the linear one predicted by equation (III.1). Similar results were found for other cases, $\rho = 0.8$ and $\rho = 0.95$. The relation between the two invariants seems to be best fitted by a linear one in J_2' and J_1 only. However, when this was adopted, there was a large discrepancy in the strain increments when considering the yield function to be also a plastic potential function. This might indicate that, for this kind of material, the yield function and the plastic potential function may be different for a best fit to the data. Consequently, our material would be plastically unstable according to Drucker's definition [219]. Although porous materials show some kind of plastic instability in tension [211], Palmer et al. [220] have shown that convexity of the yield surface and normality of the strain vector carry over to unstable materials. So, we will postulate the normality of the strain vector to the yield surface.

According to Drucker's definition for a stable plastic material [221], a plastic potential function exists, and it is identical to the yield function, which must represent a convex surface in stress space. From this argument, we were directed to make a best fit for the yield points and the slopes at these points, given by the strain results, into one function representing both the yield criterion and the plastic potential function of the material. We postulate a general yield function of the form

$$F = \{\alpha J_2' + \beta J_1^2 + \gamma Y |J_1|\}^{1/2} \quad (\text{III.3})$$

where

J_2, J_1 are the second deviatoric and first stress invariants respectively,

α, β & γ are different functions of relative density, ρ , for a given material, and

Y is the yield stress of the matrix material.

The absolute value sign is included in order to make the yield function symmetrical about the stress origin. We have left out dependence on the third invariant of the stress tensor, J_3 . Inclusion of J_3 would seem an unwarranted refinement especially in light of the scant experimental observations available with which to determine the effect of J_3 on the yield surface. The yield surface represented by equation (III.3) is an ellipsoid whose major axis coincides with the σ_m axis, as shown diagrammatically in Figure (III.6). The factors α, β and γ should be such that when $\rho = 1$, equation (III.3) should reduce to Von Mises equation, so with $F = Y$, we should have $\beta = \gamma = 0$ and $\alpha = 3$. Figure (III.7) shows the numerical results obtained for the yield stress and the slope of the yield function at points of yielding. A least square fit for the yield points and the slopes was carried out and the results are plotted in Figure (III.7) for relative density $\rho = 0.95, 0.80$ and 0.60 .

The results obtained show that for equation (3) with $F = Y$, we have, for the material properties shown above

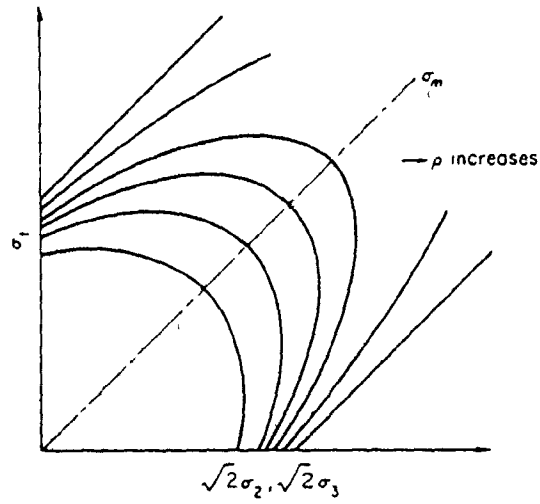


Figure (III.6) Schematic illustration of the yield surface

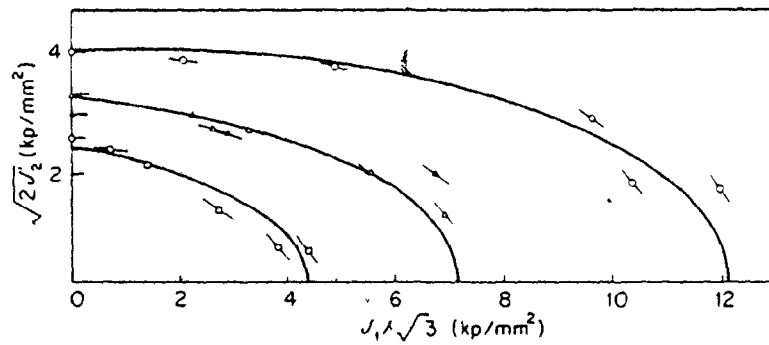


Figure (III.7) Yield surfaces of the present analysis

- $\rho = 0.95$
- $\rho = 0.8$
- $\rho = 0.6$
- $\rho = 0.8$
- (modified model)

For $\rho = 0.95$

$$\alpha = 4.114 \quad \beta = 0.114 \quad \gamma = -0.111$$

For $\rho = 0.80$

$$\alpha = 6.314 \quad \beta = 0.159 \quad \gamma = 0.155$$

For $\rho = 0.60$

$$\alpha = 11.834 \quad \beta = 0.260 \quad \gamma = 0.468$$

The relative values of β and γ show that for applicability over the whole range of J_1 values considered, neither may be neglected. A plot of these coefficients against relative density is shown in Figure (III.8).

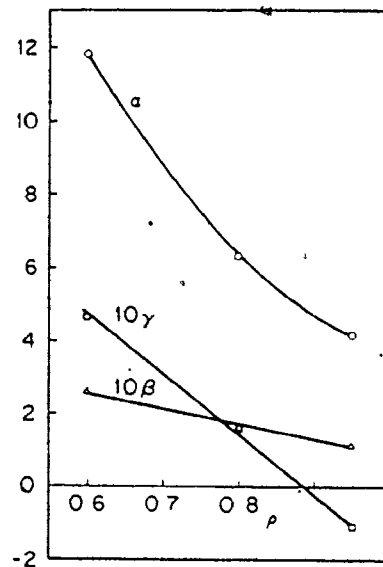


Figure (III.8) Coefficients of the yield function vs. porosity

- coefficient of J_1'
- △— coefficient of $J_1'' \times 10$
- coefficient of $J_1''' \times 10$

A further check on the normality criterion is carried out in the π -plane, where the projections of the yield surface should be concentric circles with diameters depending on porosity and hydrostatic pressure (or J_1). In Figure (III.9) the yield points are plotted together with the incremental strain vector directions as solid lines. The theoretical directions are shown by dotted lines. It should be noted that each line represents a vector directed away from the origin. A favourable agreement is obtained for most of the stress states.

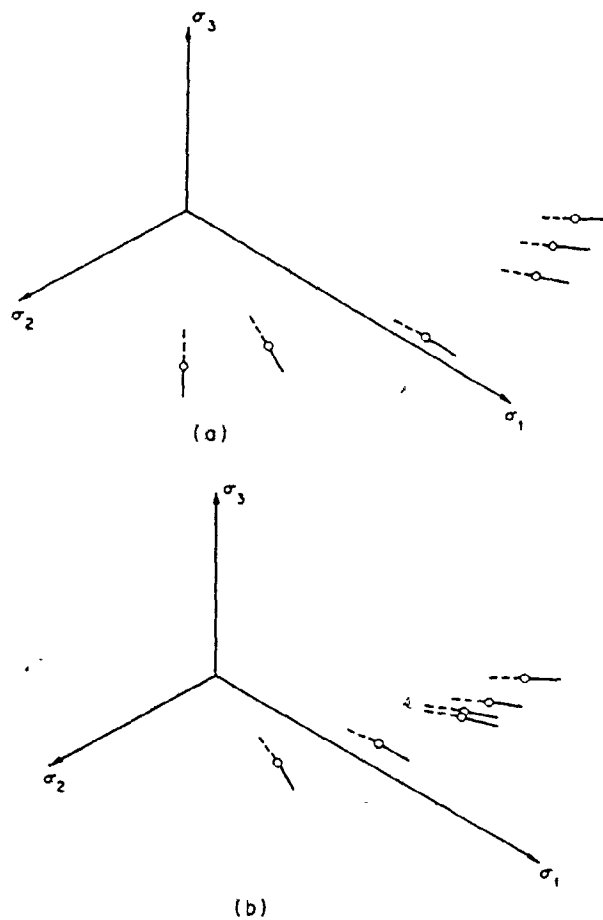


Figure (III.9) Normality of the strain vector to the yield surface in the π -plane: (a) $\rho = 0.95$; (b) $\rho = 0.8$
 ——— numerical results obtained
 - - - - - expected theoretical results

The three points which were obtained from analysis of the modified model of Figure (III.2) are plotted in Figure (III.7). It is clear that there is some small effect of the array geometry on the yield surface. In the future it might be interesting to examine this effect as it pertains to pore geometries in a real material, but since it appears to be of secondary importance, no further account is taken of it in this study.

It would be useful to see if the effect on α , β and γ of the Young's modulus and strain hardening properties could be neglected and therefore if the yield criterion with these coefficients as obtained above could also describe the behaviour of other sintered materials such as iron and aluminum. For this purpose, Figure (III.10) shows the variation of the yield stress in uniaxial tension of sintered iron with relative density [222]. Assuming that the density of pure iron is 7.87 g/cm^3 and that $Y = 10.0 \text{ kg/mm}^2$, the yield stress may be calculated from equation (III.3) with the same values of α , β and γ . The results obtained by Shima and Oyane [214] are plotted on the same curve. Our curve shows some difference between experimental and theoretical results, but this might not be significant for some purposes. Even better results would probably be obtained by using the properties for the specific material in concern to evaluate the coefficients α , β and γ .

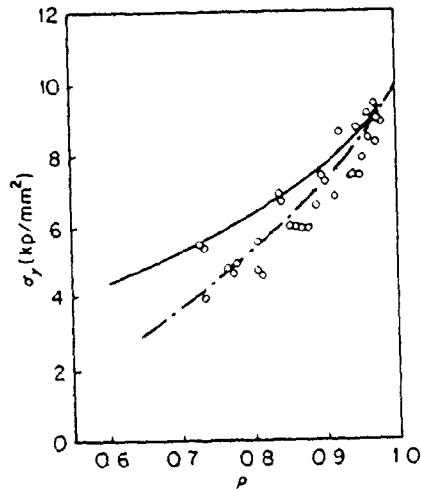


Figure (III.10) Variation of the yield stress vs. density for sintered iron

Shima and Oyane [214]
 experimental data [222]
 present analysis

III.4 BASIC EQUATIONS OF PLASTICITY

From the previous analysis, it is a straight forward procedure to derive the basic equations of plasticity as applied to porous materials. From equation (III.3), we may define a plastic potential function as

$$g = \{\alpha J_2' + \beta J_1^2 + \gamma Y |J_1|\}^{1/2} \quad (\text{III.4})$$

The principal strain increments can be derived from g if we consider the case of elastic-perfectly plastic material or in cases of small strain hardening so that the effect of changes in Y may be neglected

$$d\epsilon_i = d\lambda' \frac{\partial g}{\partial \sigma_i} = d\lambda \left[\sigma_i - \left(1 - \frac{6\beta}{\alpha}\right) \sigma_m + \frac{\gamma}{\alpha} Y \frac{J_1}{|J_1|} \right] \quad (\text{III.5})$$

where $i = 1, 2, 3$.

and further

$$\begin{aligned} d\epsilon_v &= d\epsilon_1 + d\epsilon_2 + d\epsilon_3 \\ &= d\lambda \left[\frac{18\beta}{\alpha} \sigma_m + \frac{3\gamma}{\alpha} Y \frac{J_1}{|J_1|} \right] \end{aligned} \quad (\text{III.6})$$

where $d\lambda$ is a proportionality constant, a function of both stress and strain state, and it will be determined later. Now, if we define an equivalent stress by

$$\sigma_{eq} = \left\{ \alpha J_2' + \beta J_1^2 + \gamma Y |J_1| \right\}^{1/2} \quad (\text{III.7})$$

and if an equivalent strain increment is denoted by $d\epsilon_{eq}$, then the plastic work done per unit volume of the porous body, dW_p , is expressed by

$$dW_p = \sigma_1 d\epsilon_1 + \sigma_2 d\epsilon_2 + \sigma_3 d\epsilon_3 = \sigma_{eq} d\epsilon_{eq} \quad (\text{III.8})$$

Substituting equation (III.5) into equation (III.8) and rearranging, we have

$$d\lambda = \frac{\alpha}{2} \frac{d\epsilon_{eq}}{\sigma_{eq} - \frac{3}{2} \gamma \frac{\sigma_m}{\sigma_{eq}} Y \frac{J_1}{|J_1|}} \quad (\text{III.9})$$

which reduces to the same $d\lambda$ of the ordinary plasticity theory when $\lambda = 0$ and $\alpha = 3$.

Eliminating σ_m from equations (III.5) and (III.6) and rearranging, σ_i can be expressed in terms of $d\epsilon_i$ and $d\epsilon_v$ ($i = 1, 2, 3$). Substituting these into equation (III.8), coupled with equation (III.7),

we may have an expression for $d\epsilon_{eq}$. This expression is algebraically complicated but easily determined numerically for a particular case.

CONCLUSIONS

A critical state of the art survey is provided for the major formulation aspects, namely formulation methods, solution of nonlinear equilibrium equations, the incompressibility constraint and, finally, software aspects. Consistent Lagrangian, updated Lagrangian and Eulerian formulation are derived from the energy balance equation transformed to the proper reference configuration. It is shown that many analysts describe relative or updated Lagrangian formulation under the name of Eulerian formulation. Hence, little effort has been devoted to the development of a consistent and detailed spatial or Eulerian description. Moreover, the existing trials of Eulerian formulation, which are aware of the major obstacles and differences, do include basic assumptions which impose a Lagrangian nature on the formulation.

In both the Lagrangian and the updated Lagrangian formulations, it is shown that the most general approach is to consider complete nonlinear kinematic relations within a linear increment. This leads to three stiffness matrices contributing to the total nonlinear stiffness matrix of the element. These are: the usual small displacement or the incremental stiffness matrix, the initial stress or the geometric or the tangent stiffness matrix, and, finally, the initial displacement or the initial rotation stiffness matrix. It is shown in principle that none of the above matrices should generally be omitted. In specific applications, however, certain simplifications of these stiffnesses may be

achieved.

A consistent transformation of the energy balance equation from the current deformed configuration to both Lagrangian and updated Lagrangian configurations is provided. Such transformation eliminates the need for intuitive arguments for the development of the load vector expression and provides a consistent definition of the load correction matrix. The introduction of the load correction matrix is important both in large strain, and in infinitesimal strain but finite rotation analysis. This term should also be included for all types of loading except when specific simplifying assumptions are imposed.

Certain differences in stiffness expressions between the Lagrangian and the updated Lagrangian formulation exist. The distinct ones are those in the initial load stiffness matrix and in the load increment. These differences are pertinent to the nature of the reference configuration in each formulation. However, the same number of stiffness contributions are found and essential similarities may be underscored.

In the Eulerian formulation, the expression for the load vector is independent of the deformation gradients. This is not the case in both the Lagrangian and the updated Lagrangian formulation. The form of the load vector expression, in the Eulerian formulation, eliminates the need for an initial load or a load correction matrix as observed in the Lagrangian and the updated Lagrangian formulation. Also, a simpler expression for the incremental load vector is obtained. The incremental load vector in both the Eulerian and the updated Lagrangian formulation

is independent of the displacements, whereas in the Lagrangian formulation the same vector is dependent on the current displacements for all types of loadings.

In the Eulerian formulation, only two stiffness matrices contribute to the element stiffness matrix. A strict comparison between the spatial and the relative or the referential formulation may not be logical owing to the specific nature of each. However, the two stiffness matrices in the Eulerian formulation correspond to the usual small displacement stiffness matrix and to the initial displacement or initial rotation stiffness matrix when compared with the Lagrangian and the updated Lagrangian formulation. The use of the Cauchy stress tensor and the rate of deformation tensor as the conjugate pair of variables in the Eulerian formulation avoids the analogy with the initial stress or the geometric stiffness matrix. The analogy between different formulations is based on a comparison between linear and nonlinear parts in the expressions of stress and strain increments in each formulation.

In the formulation of constitutive equations for elastoplastic materials, only Jaumann derivative may be used as a frame indifferent rate. The constitutive equation arising from this derivative is shown to be a function of the displacement increments and not the corresponding strain increments. Explicit expression is given for such constitutive equations in terms of the displacement function.

In software aspects, the considerations of the basic characteristics, advantages and disadvantages of both a program package, or black box package, and a programming system strongly favour the

latter. A finite element programming system should provide various levels of abstraction, each representing a closed domain of problem orientation. The package has to provide only the basic structure in the simplest possible flow sequence and has to possess high degree of dynamic flexibility. It should be always the user's job to modify the package to suit the specific problem requirements.

In the application a study of the plastic behaviour of porous materials is performed. A check on the previously assumed yield criterion, for such materials, shows the inadequacy of the assumption of a linear relationship between the second stress invariant and the square of the first stress invariant. A modified yield criterion is proposed. The modified criterion is a function of the second strain invariant, the square of the first stress invariant and the first stress invariant. The coefficients in the modified function are porosity dependent.

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APPENDIX (A)

TRANSFORMATION OF THE ENERGY BALANCE EQUATION

Starting with the energy balance equation in current deformed state

$$\frac{d}{dt} \int_{\mathcal{V}} \rho \bar{K} = \int_{\mathcal{V}} \rho \bar{b} \cdot \bar{v} \, d\mathcal{V} + \int_{\mathcal{A}} d\bar{A} \cdot \bar{\sigma} \cdot \bar{v} - \int_{\mathcal{V}} \bar{\sigma} : \bar{D} \, d\mathcal{V} \quad (\text{A.1})$$

where the individual terms have been defined above, then considering it term by term, we have

* Rate of strain energy term

$$\int_{\mathcal{V}} \bar{\sigma} : \bar{D} \, d\mathcal{V}$$

Substitute the transformation

$$\bar{\sigma} = \frac{1}{|\bar{F}|} \bar{\sigma}_F \cdot \bar{S} \cdot \bar{F}^T \quad \text{and} \quad d\mathcal{V} = |\bar{F}| \, d\mathcal{V}_0,$$

then

$$\int_{\mathcal{V}} \bar{\sigma} : \bar{D} \, d\mathcal{V} = \int_{\mathcal{V}_0} (\bar{\sigma}_F \cdot \bar{S} \cdot \bar{F}^T) : \bar{D} \, d\mathcal{V}_0 \quad (\text{i})$$

From the definition of the scalar product of two second order tensors, it can be shown that

$$\bar{P} : \bar{V} = (\bar{P}^T \cdot \bar{V}) : \bar{1} = (\bar{P} \cdot \bar{V}^T) : \bar{1} \quad (\text{ii})$$

where \bar{P} , \bar{V} are general second order tensors and $\bar{1}$ is the identity tensor. Substituting eqn. (i) into eqn. (ii) and utilizing the symmetry,

(where \bar{T} is the first Piola-Kirchhoff stress tensor), then

$$\int_{\mathcal{A}} d \bar{A} \cdot \bar{\sigma} \cdot \bar{v} = \int_{\mathcal{A}} d \bar{A} \cdot \bar{S} \cdot \bar{F}^T \cdot \bar{v} \quad (\text{A.3})$$

* Body force term

$$\int_{\mathcal{V}} \rho \bar{b} \cdot \bar{v} d \mathcal{V} = \int_{\mathcal{V}} \rho \bar{b} \cdot \bar{v} |J_F| d \mathcal{V} = \int_{\mathcal{V}} \rho \bar{b} \cdot \bar{v} d \mathcal{V} \quad (\text{A.4})$$

with similar transformation for the kinetic energy term.

Substituting eqns. (A.2-A.4) into eqn. (A.1), then the energy balance equation in the undeformed state may be written as

$$\frac{d}{dt} \int_{\mathcal{V}} \rho \bar{K} = \int_{\mathcal{V}} \rho \bar{b} \cdot \bar{v} d \mathcal{V} + \int_{\mathcal{A}} d \bar{A} \cdot \bar{S} \cdot \bar{F}^T \cdot \bar{v} - \int_{\mathcal{V}} \rho \bar{S} : \frac{d \bar{E}}{dt} d \mathcal{V} \quad (\text{A.5})$$

APPENDIX (B)

CONSTITUTIVE EQUATION USING JAUMANN STRESS RATE

(i) LAGRANGIAN CASE

Starting with the Jaumann stress rate definition

$$\left(\frac{\partial \bar{\sigma}}{\partial t}\right)^J = \left(\frac{d \bar{\sigma}}{dt}\right) - \bar{W} \cdot \bar{\sigma} + \bar{\sigma} \cdot \bar{W} \quad (\text{B.1})$$

then the Jaumann stress increment over the increment of unit time

$$\begin{aligned} (\Delta \bar{\sigma})^J &= (\Delta \bar{\sigma}) - \frac{1}{2} \left[\frac{(\Delta \bar{u})}{\partial \bar{x}} \frac{\partial}{\partial \bar{x}} - \frac{\partial (\Delta \bar{u})}{\partial \bar{x}} \right] \cdot \bar{\sigma} \\ &+ \bar{\sigma} \cdot \frac{1}{2} \left[\frac{(\Delta \bar{u})}{\partial \bar{x}} \frac{\partial}{\partial \bar{x}} - \frac{\partial (\Delta \bar{u})}{\partial \bar{x}} \right] \end{aligned} \quad (\text{B.2})$$

substitute

$$(\Delta \bar{\sigma})^J = \bar{C}^4 : \Delta \bar{\epsilon}, \quad \Delta \bar{\sigma} = \frac{1}{|\bar{F}|} \bar{F} \cdot \Delta \bar{S} \cdot \bar{F}^T$$

into eqn. (B.2), we obtain

$$\begin{aligned} \Delta \bar{S} &= |\bar{F}| \bar{F}^{-1} \cdot \left\{ \bar{C}^4 : \Delta \bar{\epsilon} + \frac{1}{2} \left[\frac{(\Delta \bar{u})}{\partial \bar{x}} \frac{\partial}{\partial \bar{x}} - \frac{\partial (\Delta \bar{u})}{\partial \bar{x}} \right] \cdot \bar{\sigma} \right. \\ &\left. - \bar{\sigma} \cdot \frac{1}{2} \left[\frac{(\Delta \bar{u})}{\partial \bar{x}} \frac{\partial}{\partial \bar{x}} - \frac{\partial (\Delta \bar{u})}{\partial \bar{x}} \right] \right\} \cdot (\bar{F}^T)^{-1} \end{aligned} \quad (\text{B.3})$$

Considering term by term

$$\bar{F} = \frac{\bar{x}}{\bar{X}} \frac{\partial}{\partial \bar{X}} = \frac{\partial \bar{x}_i}{\partial \bar{X}_j} \bar{e}_i \bar{e}_j, \quad \bar{F}^{-1} = \frac{\bar{X}}{\bar{x}} \frac{\partial}{\partial \bar{x}} = \frac{\partial \bar{X}_i}{\partial \bar{x}_j} \bar{e}_i \bar{e}_j,$$

$$\Delta \bar{\epsilon} = \frac{\partial \bar{X}_r}{\partial \bar{x}_m} \Delta \bar{E}_{rs} \frac{\partial \bar{X}_s}{\partial \bar{x}_n} \bar{e}_m \bar{e}_n,$$

$$\Delta \overset{1}{S} = (\Delta \overset{1}{S}_{ij}) \bar{e}_i \bar{e}_j \quad (\text{B.6})$$

where

$$\Delta \overset{1}{S}_{ij} = \overset{1}{D}_{ij\alpha} (\Delta \overset{1}{u}_\alpha) \quad (\text{B.7})$$

and

$$\begin{aligned} \overset{1}{D}_{ij\alpha} = & \left| \frac{\partial \overset{1}{x}}{\partial \overset{0}{X}} \right| \frac{\partial \overset{0}{X}_1}{\partial \overset{1}{x}_m} \frac{\partial \overset{0}{X}_j}{\partial \overset{1}{x}_n} \left\{ \frac{\partial \overset{0}{X}_r}{\partial \overset{1}{x}_k} \overset{1}{C}_{mnkl} \frac{\partial \overset{0}{X}_s}{\partial \overset{1}{x}_1} \left(\frac{\partial \psi_{r\alpha}}{\partial \overset{0}{X}_s} + \frac{\partial \psi_{w\beta}}{\partial \overset{0}{X}_r} \frac{\partial \psi_{w\alpha}}{\partial \overset{0}{X}_s} \overset{1}{u}_\beta \right) \right. \\ & \left. - \frac{1}{2} \left(\frac{\partial \psi_{p\alpha}}{\partial \overset{0}{X}_v} \frac{\partial \overset{0}{X}_v}{\partial \overset{1}{x}_m} - \frac{\partial \psi_{m\alpha}}{\partial \overset{0}{X}_v} \frac{\partial \overset{0}{X}_v}{\partial \overset{1}{x}_p} \right) \overset{1}{\sigma}_{pn} - \frac{1}{2} \left(\frac{\partial \psi_{n\alpha}}{\partial \overset{0}{X}_v} \frac{\partial \overset{0}{X}_v}{\partial \overset{1}{x}_p} - \frac{\partial \psi_{p\alpha}}{\partial \overset{0}{X}_v} \frac{\partial \overset{0}{X}_v}{\partial \overset{1}{x}_n} \right) \overset{1}{\sigma}_{mp} \right\} \end{aligned} \quad (\text{B.8})$$

(ii) EULERIAN CASE

In case of Eulerian formulation, the stress increment may be directly obtained from eqn. (B.2) written with respect to configuration C_2 so that

$$\Delta \overset{2}{\sigma} = \overset{2}{C} : \Delta \overset{2}{\epsilon} + \frac{1}{2} \left[\frac{\partial (\Delta \overset{2}{u})}{\partial \overset{2}{x}} - \frac{\partial (\Delta \overset{2}{u})}{\partial \overset{2}{x}} \right] \cdot \overset{2}{\sigma} - \overset{2}{\sigma} \cdot \frac{1}{2} \left[\frac{\partial (\Delta \overset{2}{u})}{\partial \overset{2}{x}} - \frac{\partial (\Delta \overset{2}{u})}{\partial \overset{2}{x}} \right] \quad (\text{B.9})$$

Considering eqn. (B.9) term by term, we have

$$* \overset{2}{C} : \Delta \overset{2}{\epsilon} = \overset{2}{C}_{ijkl} \Delta \overset{2}{\epsilon}_{kl} \bar{e}_i \bar{e}_j,$$

substituting $\Delta \overset{2}{\epsilon}_{kl}$ from eqn. (II.80) we obtain

$$\begin{aligned} \overset{2}{C} : \Delta \overset{2}{\epsilon} = & \frac{1}{2} \overset{2}{C}_{ijkl} \left\{ \left(\frac{\partial \psi_{m\alpha}}{\partial \overset{2}{x}_k} \frac{\partial \overset{2}{x}_1}{\partial \overset{0}{X}_m} + \frac{\partial \psi_{m\alpha}}{\partial \overset{2}{x}_1} \frac{\partial \overset{2}{x}_k}{\partial \overset{0}{X}_m} \right) \right. \\ & \left. - \left(\frac{\partial \psi_{r\gamma}}{\partial \overset{2}{x}_k} \frac{\partial \psi_{m\alpha}}{\partial \overset{2}{x}_1} \frac{\partial \overset{2}{x}_r}{\partial \overset{0}{X}_m} + \frac{\partial \psi_{r\gamma}}{\partial \overset{2}{x}_1} \frac{\partial \psi_{m\alpha}}{\partial \overset{2}{x}_k} \frac{\partial \overset{2}{x}_r}{\partial \overset{0}{X}_m} \right) \overset{2}{u}_\gamma \right\} (\Delta \overset{2}{u}_\alpha) \bar{e}_i \bar{e}_j \end{aligned} \quad (\text{B.10})$$

$$* \frac{1}{2} \left[\frac{\partial (\Delta \overset{2}{u})}{\partial \overset{2}{x}} - \frac{\partial (\Delta \overset{2}{u})}{\partial \overset{2}{x}} \right] \cdot \overset{2}{\sigma}$$

Substituting eqn. (II.78), we obtain for the above term

$$\begin{aligned} & \left\{ 2 \left[\frac{\partial \psi_{m\alpha}}{\partial^2 x_p} \frac{\partial^2 x_i}{\partial^0 X_m} - \frac{\partial \psi_{m\alpha}}{\partial^2 x_i} \frac{\partial^2 x_p}{\partial^0 X_m} \right] (\Delta^2 u_\alpha) \bar{e}_i \bar{e}_p \right\} \cdot \left\{ 2 \sigma_{rj} \bar{e}_r \bar{e}_j \right\} \\ & = \left[\frac{1}{2} \frac{\partial \psi_{m\alpha}}{\partial^2 x_p} \frac{\partial^2 x_i}{\partial^0 X_m} - \frac{\partial \psi_{m\alpha}}{\partial^2 x_i} \frac{\partial^2 x_p}{\partial^0 X_m} \right] 2 \sigma_{pj} (\Delta^2 u_\alpha) \bar{e}_i \bar{e}_j \end{aligned} \quad (B.11)$$

with the last term in eqn. (B.9) treated in a similar way to that used to develop (B.11).

Collecting results of eqns. (B.9) to (B.11), we may have

$$\Delta^2 \bar{\sigma} = (\Delta^2 \sigma_{ij}) \bar{e}_i \bar{e}_j \quad (B.12)$$

where

$$\Delta^2 \sigma_{ij} = 2 C_{ij\alpha} (\Delta^2 u_\alpha) \quad (B.13)$$

and

$$\begin{aligned} 2 C_{ij\alpha} = & \left\{ \frac{1}{2} 2 C_{ijkl} \left[\left(\frac{\partial \psi_{m\alpha}}{\partial^2 x_k} \frac{\partial^2 x_l}{\partial^0 X_m} + \frac{\partial \psi_{m\alpha}}{\partial^2 x_l} \frac{\partial^2 x_k}{\partial^0 X_m} \right) \right. \right. \\ & - \left. \left(\frac{\partial \psi_{r\gamma}}{\partial^2 x_k} \frac{\partial \psi_{m\alpha}}{\partial^2 x_l} \frac{\partial^2 x_r}{\partial^0 X_m} + \frac{\partial \psi_{r\gamma}}{\partial^2 x_l} \frac{\partial \psi_{m\alpha}}{\partial^2 x_k} \frac{\partial^2 x_r}{\partial^0 X_m} \right) 2 u_\gamma \right] \\ & + \left. \frac{1}{2} \left(\frac{\partial \psi_{m\alpha}}{\partial^2 x_p} \frac{\partial^2 x_i}{\partial^0 X_m} - \frac{\partial \psi_{m\alpha}}{\partial^2 x_i} \frac{\partial^2 x_p}{\partial^0 X_m} \right) 2 \sigma_{pj} + \frac{1}{2} \left(\frac{\partial \psi_{m\alpha}}{\partial^2 x_p} \frac{\partial^2 x_j}{\partial^0 X_m} - \frac{\partial \psi_{m\alpha}}{\partial^2 x_j} \frac{\partial^2 x_p}{\partial^0 X_m} \right) 2 \sigma_{pi} \right\} \end{aligned} \quad (B.14)$$