CONTRIBUTIONS TO THE THEORY OF CONTINUOUS MODULES

By

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To Abbu

who was my greatest source of inspiration as a mathematician

as well as a human being

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ABSTRACT

A module M is called continuous if (i) every submodule of M is essential in a summand of M, and (ii) if a submodule A is isomorphic to a summand of M, then A is itself a summand of M.

Injective and quasi-injective modules play an important role in module theory and continuous modules are a generalization of these concepts. Many of the important properties that hold for (quasi-) injective modules, still hold for continuous modules, and it is often more convenient to work with the above two conditions rather than the notion of (quasi-) injectivity.

This thesis deals with several important aspects of the theory of continuous modules. We give a decomposition theorem for continuous modules and, as a corollary, obtain a partial generalization of a result of Matlis and Papp. We also answer the open question: When is a finite direct sum of indecomposable continuous modules continuous? The continuity of infinite direct sums of indecomposable continuous modules is also examined.

The main chapter deals with the concept of continuous hulls. We give an appropriate definition, explicitly describe the continuous hulls for the classes of uniform cyclic modules, and of non-singular cyclic modules over commutative rings, and exhibit them by concrete examples. A necessary and sufficient condition for the existence of continuous hulls for arbitrary cyclic modules over a commutative ring

is also given. In our opinion, these results constitute an important development, since these first steps towards establishing the existence of continuous hulls should stimulate further research, and since the knowledge of their existence should prove valuable in related investigations.

Finally, we study in detail commutative rings for which every continuous module is quasi-injective. It is shown that this property holds true for large classes of rings such as noetherian ones, and semi-primary ones whose Jacobson radical has square zero. We characterize several other classes of rings with this property. Many examples are provided throughout the thesis which show the existence of continuous modules (and hulls) which are not (quasi-) injective.

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INTRODUCTION

The origin of the concept of continuous modules lies in Von Neumann's continuous geometries [34], where the continuous regular rings are the coordinate rings of these geometries.

Y. Utumi (1960) studied continuous regular rings extensively, and provided many of their characterizations in lattice theoretic terms [29].

Recall that a lattice L is called *upper continuous* if it is complete (that is, the least upper bound and the greatest lower bound of every subset of L exist), and if for every chain $\{a_\alpha\}$ and every element b, $(\vee a_\alpha) \wedge b = \vee(a_\alpha \wedge b)$ holds.
Utumi called a regular ring R, *right (left) continuous*, if the lattice of the principal right (left) ideals of R is upper continuous. We remark here, that, since the lattice of principal right ideals is anti-isomorphic to the lattice of principal left ideals, a left and right continuous regular ring (in Utumi's sense) is the same as a continuous regular ring (in Von Neumann's sense).

Brainerd and Lambek ([7] Corollary 4) showed that every complete Boolean ring is self-injective. Since every complete Boolean ring is continuous and regular, a motivation for Utumi's study, was the question whether every continuous regular ring is self-injective? He answered this in the negative, giving an example (Example 2.11, in our Chapter II) of a continuous regular ring which is not self-injective.

In 1961 [30], Utumi characterized continuous regular rings in ring theoretic terms and proved that a regular ring R is right continuous, if and only if every right ideal of R is essential in a direct summand of R.

A general definition of continuity for arbitrary rings was given, again by Utumi in 1965 [32]: A ring R is called *right continuous* if (i) every right ideal of R is essential in a summand of R, and (ii) if a right ideal A is isomorphic to a summand of R, then A is itself a summand of R. Note that every regular ring automatically satisfies condition (ii).

L. Jeremy 1974 [16], and S. Mohamed and T. Bouhy 1976 [21], independently applied Utumi's definition to modules, and obtained the following straightforward generalization:

Definition: A module M is called *continuous* if it has the following two properties:

I) Every submodule A of M is essential in a summand of M.

II) If a submodule A of M is isomorphic to a summand of M, then A is itself a summand of M.

Mohamed and Bouhy [21], studied continuous modules and showed that every (quasi-) injective module is continuous. The converse is not true. It was also shown that many properties of (quasi-) injective modules still hold for the continuous modules.

The present thesis focuses on some important aspects of the theory of continuous modules which have not been touched on as yet.

The first chapter provides the preliminaries and some background results to be used in subsequent chapters.

The second chapter is devoted to the decomposition of continuous modules into indecomposable parts. We give a decomposition theorem for continuous modules which generalizes a decomposition theorem for injective modules. As a consequence, we obtain a partial generalization of a characterization of right noetherian rings given by Matlis [20] and Papp [27].

The third chapter answers the open question: When is a finite direct sum of indecomposable continuous modules, continuous? We prove that such a direct sum is continuous if and only if each summand is continuous and is injective relative to the other summands. We also examine the continuity of infinite direct sums of indecomposable continuous modules.

The most important contribution of this thesis is Chapter IV, where we discuss possible definitions of continuous hulls and settle for an appropriate one. We then show the existence of continuous hulls for the classes of uniform cyclic modules, and of nonsingular cyclic modules, over commutative rings. In both the cases we explicitly describe these hulls. Examples are provided in both instances, to exhibit the existence of continuous hulls which are different from the (quasi-) injective hulls (and also different from the π-injective hulls in the nonsingular case). A necessary and sufficient condition is also provided, at the end of the chapter, for the existence of continuous hulls for arbitrary cyclic modules over commutative rings. We feel that these results constitute an important development, as they open a new dimension in the field and should provide impetus for further study and research.

In the fifth chapter, we undertake the study of rings for which every continuous module is quasi-injective. Our search is motivated by a general interest in identifying the rings for which continuity is a non-trivial concept, and in particular, by the fact that, only for such rings is the existence of continuous hulls problematic. We show that large classes of rings, such as commutative noetherian ones, and commutative semiprimary ones with Jacobson radical square zero, have the property that every continuous module over them is quasi-injective. We give a necessary and sufficient condition for every uniform continuous module over an arbitrary commutative ring. We also provide necessary and sufficient conditions for several special classes of rings, such as commutative perfect rings, commutative finitary rings, and commutative chain rings, to have the property that every continuous module is quasi-injective. The chapter concludes with an explicit example of a ring R, over which there are lots of continuous modules which are not quasi-injective, but very few quasi-injective modules.

All rings considered, have an identity element, and all modules are unital right modules. All items, such as theorems, propositions, lemmas, corollaries, examples and definitions, are numbered consecutively, m.n, where m indicates the chapter and n the place of the item within the chapter.

CHAPTER I

PRELIMINARIES

In this chapter, we provide some well-known facts for later use, concerning the concept of injectivity and some of its generalizations. Continuity and its generalizations are also dealt with, and we mention a number of results for continuous modules, which are analogous to familar results for (quasi-) injective modules. Finally, we define dual-continuous modules and list a few theorems for them, focussing on the similarity with those, which we shall prove for continuous modules in Chapters II and III.

§1. Injectivity, Quasi-Injectivity, and Relative Injectivity

Definition 1.1: A module $M \subset E$, is said to be *essential in* E (denoted by $M \subset' E$), if, for every submodule N of E, $M \cap N = 0$ implies $N = 0$. A module E is called *uniform* if every nonzero submodule of E is essential in E. A submodule M *is closed in* E if it has no proper essential extension in E.

Definition 1.2: A module E is said to be *injective*, if, for any two modules A and B such that $A \subset B$, any homomorphism $f: A \to E$ can be extended to $\hat{f}: B \to E$.

The concept of injective modules was initiated by Baer [5] and Nakayama [25]. It was shown by Eckmann and Schopf [9] that every module can be embedded in an injective module. In fact, they showed that, for any module M, there is a minimal injective overmodule E, unique upto isomorphism over M. It was also characterized as a maximal essential extension of M, and was called the *injective hull* of M. In Chapter IV, we give a detailed description of injective hulls and discuss the analogous concept of continuous hulls. We shall denote the injective hull of a module M by E(M).

The concept of injectivity was generalized to that of relative injectivity by Azumaya [3], and Azumaya, Mbuntum and Varadarajan [4].

<u>Definition 1.3</u>: A module M is said to be *N-injective* (or *injective relative* to the module N), if for every submodule K of N, every homomorphism $f: K \to M$ can be extended to $\bar{f}: N \to M$.

It is obvious that a module M is N-injective for every module N, if and only if M is injective. Azumaya [3], proved the following:

<u>Theorem 1.4</u>: *Let* $\{M_\alpha ; \alpha \in I\}$ *be a family of R-modules and let* $M = \bigoplus_{\alpha \in I} M_\alpha$ *be their direct sum. If an R-module* Q *is* M_α*-injective for each* $\alpha \in I$, *then* Q *is M-injective.*

The next useful characterization of relative injectivity was also given by Azumaya [3], and generalizes a result by Johnson and Wong [17] for quasi-injective modules. We shall invoke this theorem frequently, in subsequent chapters; and since [3] is an unpublished paper, we provide a proof. Our proof is different from the one given by Azumaya, and is a generalization of the one given by Johnson and Wong for quasi-injective modules.

<u>Theorem 1.5</u>: *A module M is N-injective if and only if, for every homomorphism $f: N \to E(M)$, $f(N) \subset M$ holds.*

<u>Proof</u>: If every homomorphism $f: N \to E(M)$, maps N into M, then, trivially M is N-injective.

Conversely, let M be N-injective, and let $f: N \to E(M)$ be a homomorphism. Define $A = \{a \in N | f(a) \in M\}$, a submodule of N. Then $f|_A: A \to M$, is a well defined homomorphism. Since M is N-injective, $f|_A$ can be extended to a homomorphism $g: N \to M$, such that $g(a) = f(a)$ holds for all $a \in A$. Again, g can be extended to the injective hulls, as $\hat{g}: E(N) \to E(M)$, such that $\hat{g}(n) = g(n)$ for all $n \in N$.

Next, if $(\hat{g} - f)N \neq 0$, then $(\hat{g} - f)N \cap M \neq 0$, by essentiality of M in $E(M)$, and hence, there exists $m \in M$ such that $(\hat{g} - f)(n) = m \neq 0$ for some $n \in N$. Consequently, $\hat{g}(n) - f(n) = g(n) - f(n) = m$, implies that $f(n) = g(n) - m \in M$, and therefore, $n \in A$. This yields that $(\hat{g} - f)(n) = (g - f)(n) = 0$ holds, which contradicts that $(\hat{g} - f)N \neq 0$. Therefore, one obtains that $\hat{g} = f$ on N; and $f(N) = \hat{g}(N) = g(N) \subset M$ holds. □

As a corollary to Theorem 1.5, one obtains the result proved by Johnson and Wong [17], namely: A module M is quasi-injective if and only if it is invariant under every endomorphism of its injective hull; and that every module has a quasi-injective hull.

We remark here, that, if R_R is R-injective (that is, quasi-injective), then it is actually injective (by the well-known Baer's Criterion). Such a ring R is called *right-self-injective*.

Definition 1.6 [28]: A ring R is called *quasi-Frobenius* if it is right- and/or left-self-injective and right- and/or left-artinian.

§2. Continuity, Quasi-Continuity and π-Injectivity

Recall the definition of a continuous module given in the introduction:

Definition 1.7: A module M is called *continuous* if it satisfies:

I) Every submodule A of M is essential in a summand of M .

II) If a submodule A is isomorphic to a summand of M , then A is itself a summand of M .

Remark 1.8: It is obvious that condition (II) above is equivalent to: Condition (II'): Every monomorphism $M' \xrightarrow{f} M$, where M' is a summand of M , splits.

Most of the results on continuous modules in this section are by Mohamed and Bouhy [21].

They prove that every quasi-injective module is continuous, and give the following example to show that the converse is not true.

Example 1.9: Let R be a ring, which has only one proper nonzero right ideal, but is not left artinian. It can be easily shown that R_R

is continuous. However, R_R is not quasi-injective; since, otherwise, R_R is injective, and consequently, R becomes quasi-Frobenius. This contradicts that R is not left artinian.

In subsequent chapters (Chapters II, IV, V), we provide explicit examples of continuous modules which are not (quasi-) injective.

It is a well-known fact that every indecomposable (quasi-) injective module is uniform and has a local endomorphism ring. In the following we list some analogous properties of continuous modules [21].

Theorem 1.10:

1) *An indecomposable continuous module is uniform.*

2) *A closed submodule of a continuous module M is a direct summand.*

3) *Any direct summand of a continuous module is continuous.*

4) *If $M = M_1 \oplus M_2$ is continuous, then, M_1 is M_2-injective and vice-versa. Consequently, a module N is quasi-injective if and only if $N \oplus N$ is continuous.*

Faith and Utumi ([10], Theorem 3.1), and Osofsky ([26], Theorem 12), studied the endomorphism ring of quasi-injective modules and proved the following:

Theorem 1.11: *Let M be quasi-injective. Let $H = Hom_R(M,M)$, and let $J = J(H)$ be the Jacobson radical of H. Then*

1) $J = \{\alpha \in H \mid Ker(\alpha) \subset' M\}$.

2) *Idempotents modulo J can be lifted.*

3) *H/J is a right-self-injective Von Neumann regular ring.*

It was also shown by Faith and Utumi ([10], Corollary 2.4), that a ring R is semisimple artinian if and only if every R-module is quasi-injective.

Mohamed and Bouhy ([21] 4.1, 4.2) proved analogous statements for continuous modules.

Theorem 1.12: *Let M be a continuous module, and let J be the Jacobson radical of $H = Hom_R(M,M)$. Then*

1) $J = \{h \in H \mid Ker\, h \subset' M\}$.

2) Idempotents can be lifted modulo J.

3) H/J is a right continuous Von Neumann regular ring.

Corollary 1.13: *A continuous module M is indecomposable if and only if $H = Hom_R(M,M)$ is a local ring.*

Theorem 1.14 ([21], 3.7): *A ring R is semisimple artinian if and only if every R-module is continuous.*

Rings for which every cyclic R-module is quasi-injective were studied by Ahsan [1], and Koehler [18]. The latter characterized them completely in the following theorem.

Theorem 1.15 ([18]): *For a ring R, every cyclic R-module is quasi-injective if and only if $R = A \oplus B$, where A is semisimple artinian and B is a finite direct sum of rank 0, maximal valuation, duo rings.*

Jain and Mohamed [15] proved the following generalization of Theorem 1.15.

Theorem 1.16: ([15], Theorem 2.8 and Addendum). *For a ring* R*, every cyclic R-module is continuous if and only if* $R = A \oplus B$*, where* A *is semisimple artinian and* B *is a finite direct sum of right valuation, right duo rings with nil radical.*

L. Jeremy [16], defined quasi-continuous modules, as follows:

Definition 1.17: A module M is called *quasi-continuous* if it satisfies:

i) Every submodule A of M is essential in a summand of M.

ii) If M_1 and M_2 are summands of M such that $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a summand of M.

Jeremy [16], also observed that a module M is quasi-continuous if and only if M is invariant under all idempotents of the endomorphism ring of its injective hull. It can be easily checked that every continuous module is quasi-continuous.

Goel and Jain [11], independently defined and studied π-injective modules. Their definition is as follows:

Definition 1.18: A module M is called π-*injective* if, for any two submodules A_1 and A_2, with $A_1 \cap A_2 = 0$, the projections $\pi_i: A_1 \oplus A_2 \to A_i$ can be lifted to endomorphisms of M.

The next characterization of π-injective modules is an important tool in our study, and in the following chapters, we shall frequently refer to it.

Theorem 1.19: ([11], Theorem 1.1). *The following are equivalent:*

i) *M is π-injective.*

ii) *$e(M) \subset M$, for all $e^2 = e \in Endo_R(E(M))$.*

iii) *If $E(M) = \sum_{i=1}^{n} \oplus E_i$, then $M = \sum_{i=1}^{n} \oplus (M \cap E_i)$.*

iv) *If $E(M) = \sum_{i \in I} \oplus E_i$, then $M = \sum_{i \in I} \oplus (M \cap E_i)$.*

We point out here that the equivalence of (i) and (ii) obviously implies that π-injectivity and quasi-continuity are equivalent concepts. Again, it is easy to see that every continuous module is π-injective (by [21], Proposition 3.4), and the converse is not true since, for example, Z_Z is π-injective but is not continuous. We shall provide more examples in Chapter IV. We remark here that π-injective hulls exist; they are also dealt with in detail, and extensively used, in Chapter IV.

§3. Dual-Continuous Modules

Mohamed and Singh [24], defined the concept of dual-continuous modules as an exact dual of that of continuous modules, as follows:

Definition 1.20: A module M is called *dual-continuous* if it satisfies:

i) For every submodule A of M, M can be decomposed as $M = M_1 \oplus M_2$, such that $M_1 \subset A$ and $A \cap M_2$ is small in M_2.

ii) Every epimorphism $M \to M'$, onto a summand M' of M, splits.

In the following, we give a few known results on dual-continuous modules because of their similarity to the results proved in Chapters II and III.

A decomposition theorem for dual-continuous module was obtained by Mohamed and Singh [24], and later refined and improved by Mohamed and Müller [22], who proved:

Theorem 1.21: *Let M be a dual-continuous module. Then M can be decomposed as $M = \sum_{i \in I} \oplus A_i \oplus N$, uniquely upto isomorphism, where each A_i is a local module, and $N = Rad\, N$.*

Mohamed and Müller also investigated dual-continuity of finite direct sums of indecomposable dual continuity and showed the following:

Theorem 1.22: *Let $M = \sum_{i=1}^{n} \oplus A_i$, then M is dual-continuous if and only if each A_i is dual-continuous and A_j-projective for all $j \neq i$.*

We conclude this chapter by the remark that recently [23], the above result has been extended to infinite direct sums by the same authors, who show:

Theorem 1.23: *Let M be a module with small radical. Then M is dual-continuous if and only if $M = \sum_{i \in I} \oplus A_i$, where A_i is local dual-continuous and A_j-projective for all $j \neq i$.*

CHAPTER II

THE DECOMPOSITION OF CONTINUOUS MODULES

§1. Introduction

The decomposition theory for any algebraic structure has always been a useful tool in the study of its properties and structure theory.

Warfield [35], gave the following result for injective hulls, which can be used to derive a decomposition theorem for injective modules.

Theorem 2.1: ([35], Corollary 4.1). *Any two representations of an injective module as the injective hull of a direct sum of injective submodules, have isomorphic refinements.*

Next, we obtain a decomposition theorem for injective modules from this result.

Theorem 2.2: *Let* E *be an injective module. Then* E *can be decomposed as* $E = E_1 \oplus E_2$, *where* E_1 *is essential over a direct sum* $\sum_{i \in I} \oplus A_i$ *of indecomposable summands of* E, *and* E_2 *has no uniform submodule. Moreover, if* $E = E_1' \oplus E_2'$ *is another decomposition (of* E*) such that* E_1' *is essential over a direct sum* $\sum_{j \in J} \oplus A_j'$ *of indecomposable summands of* E, *and* E_2' *has no uniform submodule; then* $E_2 \cong E_2'$ *and there exists a bijection* $P: I \to J$, *such that* $A_i \cong A_{P(i)}$.

Proof: Let E be an injective module. The decomposition of E is obtained if we apply Zorn's Lemma on the family of sets $\{A_k \mid k \in K;\ A_k$ indecomposable summands of E such that $\sum_{k \in K} A_k$ is a direct sum$\}$. Then we get a maximal direct sum $\sum_{i \in I} \oplus A_i$ in E, for some $I \subseteq K$. Thus E contains the injective hull E_1 of the direct sum $\sum_{i \in I} \oplus A_i$, and hence $E = E_1 \oplus E_2$. E_2 does not have any uniform submodule, since, otherwise the maximality of $\sum_{i \in I} \oplus A_i$ is violated.

Now, if $E = E_1' \oplus E_2'$ is another decomposition as given in the hypothesis, then

$$E = E(\sum_{i \in I} \oplus A_i) \oplus E_2 = E(\sum_{i \in I} \oplus A_i \oplus E_2)\text{, and also}$$

$$E = E(\sum_{j \in J} \oplus A_j') \oplus E_2' = E(\sum_{j \in J} \oplus A_j' \oplus E_2').$$

Consequently, Theorem 2.1 yields the required uniqueness property.

Recall the definition (1.20) of a dual-continuous module given in the previous chapter: A module M is called *dual-continuous* if

i) for any submodule A of M, there exists a decomposition $M = M_1 \oplus M_2$, with $M_1 \subseteq A$ and $A \cap M_2$, small in M_2, and

ii) every exact sequence $M \to M' \to 0$, with M' a summand of M, splits.

Recall also, the decomposition theorem (1.21) for dual continuous modules, given by Mohamed and Müller [22], which states that a dual continuous module M can be decomposed as $M = \sum_{i \in I} \oplus A_i \oplus N$, uniquely up to isomorphism, where each A_i is a local module and $N = \mathrm{Rad}N$.

There is no decomposition theorem for continuous modules in the present literature. We provide such a theorem in this chapter, which actually generalizes Theorem 2.2 for injective modules. As a corollary, we obtain a partial generalization of a result of Matlis [20] and Papp [27]. We also provide an example, to exhibit that the decomposition is best possible in a certain sense.

§2. A Decomposition Theorem

We first state the following lemma [28], without proof, and then give the decomposition theorem.

Lemma 2.3: *Let* $A \oplus B = C_1 \oplus \cdots \oplus C_n$, *and that* $Hom_R(A,A)$ *be a local ring. Then there exists an* i, *such that* $C_i \cong C_i' \oplus C_i''$, *with* $A \cong C_i''$ *and* $B \cong C_i' \oplus (\sum_{j \neq i} \oplus C_j)$.

Theorem 2.4: *Let* M *be a continuous module. Then there exists a decomposition* $M = M_1 \oplus M_2$, *where* M_1 *is essential over a direct sum* $\sum_{i \in I} \oplus A_i$ *of indecomposable (hence uniform) summands of* M, *and* M_2 *has no uniform submodule. Moreover, if* $M = M_1' \oplus M_2'$ *is another decomposition such that,* M_1' *is essential over a direct sum* $\sum_{j \in J} \oplus A_j'$ *of indecomposable summands of* M *and* M_2' *has no uniform submodule, then* $E(M_2) \cong E(M_2')$; *and there is a bijection* $P: I \to J$, *such that* $A_i \cong A'_{P(i)}$, *and consequently, also* $E(M_1) \cong E(M_1')$.

Remark 2.5: The direct sum $\sum_{i \in I} \oplus A_i$ which appears above, is in fact a maximal direct sum, and every finite direct sum of the A_i is still a summand of M. The latter holds true since M is quasi-continuous and hence satisfies Definition 1.17. This also implies that each A_i is A_j-injective, for all $j \neq i$.

<u>Proof of the Theorem</u>: Let M be continuous. Let $\{A_k | k \in K\}$ be the family of all indecomposable summands. We call a subset J of K "direct" if the sum $\sum_{j \in J} A_j$ is a direct sum. Consider the family $\{J_\ell | \ell \in \mathcal{L}\}$ of all direct subsets of K, ordered by inclusion. This is nonempty, as the empty set belongs to the family. It can be easily seen that the union of a chain of members of the family is still a member of the family. Hence, by Zorn's Lemma, we get a maximal direct set $I \subset K$.

Since $\sum_{i \in I} \oplus A_i \subset M$ holds, continuity of M yields the existence of a summand M_1 of M, such that M_1 is essential over $\sum_{i \in I} \oplus A_i$, and $M = M_1 \oplus M_2$. We observe that M_2 does not have any uniform submodule U, since otherwise there exists a summand N_2 of M_2, such that $N_2 \,{}'\!\supset U$. But then N_2 itself becomes uniform (hence indecomposable) and the direct set I could be enlarged, contradicting its maximality.

Next, let M have another decomposition, $M = M_1' \oplus M_2'$, such that $M_1' \,{}'\!\supset \sum_{j \in J} \oplus A_j'$, the A_j' are indecomposable summands, and M_2' has no uniform submodule. Since $\sum_{i \in I} \oplus A_i$ is essential in M_1 as well as in $\sum_{i \in I} \oplus E(A_i)$, we get

$$E(M_1) \cong E(\sum_{i \in I} \oplus A_i) \cong E(\sum_{i \in I} \oplus E(A_i)) . \quad \text{Similarly,}$$

$$E(M_1') \cong E(\sum_{j \in J} \oplus A_j') \cong E(\sum_{j \in J} \oplus E(A_j')) \quad \text{holds.}$$

Thus $E(M) \cong E(M_1) \oplus E(M_2) \cong E(M_1') \oplus E(M_2')$, yields $E(\sum_{i \in I} \oplus E(A_i) \oplus E(M_2)) \cong E(\sum_{j \in J} \oplus E(A_j') \oplus E(M_2'))$, where each $E(A_i)$ (respectively $E(A_j')$) is injective, indecomposable for all $i \in I$ (respectively for all $j \in J$) , and $E(M_2)$ (respectively, $E(M_2')$) has no indecomposable summand.

Now, by Theorem 2.1, we obtain $E(M_2) \cong E(M_2')$, and $E(A_i) \cong E(A'_{P(i)})$, where $P: I \to J$ is a bijection. We can write $M = A_i \oplus S = A'_{P(i)} \oplus S'$, where $\text{Endo}_R(A'_{P(i)})$ is a local ring (as $A'_{P(i)}$ is indecomposable and continuous); S and S' are the complementary summands. Lemma 2.3, then applies and yields that $A'_{P(i)}$ is isomorphic to a summand either of A_i or of S. Therefore, either $A_i \cong A_{P(i)}$ or $S \cong A'_{P(i)} \oplus S_1$ holds. In the latter case, we obtain $M \cong A_i \oplus A'_{P(i)} \oplus S_1$. However, continuity of M then implies that $A_i \oplus A'_{P(i)}$ is continuous, and A_i and $A'_{P(i)}$ are injective relative to each other. This, together with the above established fact $E(A_i) \cong E(A_{P(i)})$, implies that $A_i \cong A_{P(i)}$ holds in this case too. □

<u>Proposition 2.6</u>: ([13], Proposition 1.4). *Every right module with right Krull dimension has finite uniform dimension.*

<u>Corollary 2.7</u>: *Let R be a ring with right Krull dimension. Then, every continuous R-module M is essential over a direct sum of indecomposable summands, which is unique up to isomorphism.*

<u>Proof</u>: As M is continuous, $M = M_1 \oplus M_2$ where $\sum_{i \in I} \oplus A_i \subset' M_1$ and M_2 has no nonzero uniform submodule. If $M_2 \neq 0$, then each nonzero

cyclic submodule of M_2, has right Krull dimension (being an epimorphic image of R), and therefore, has a nonzero uniform submodule by the previous proposition. This contradicts that M_2 has no nonzero uniform submodules. Consequently, $M_2 = 0$; and M is essential over $\sum_{i \in I} \oplus A_i$, where the A_i are unique up to isomorphism. □

The next Corollary provides a partial generalization of a characterization of right noetherian rings by Matlis [20] and Papp [27], who proved the implications between (1) and (2) below, respectively.

<u>Corollary 2.8</u>: *The following are equivalent:*

(1) R is right noetherian.

(2) Every injective right R-module is a direct sum of indecomposable submodules.

(3) Every continuous right R-module is a direct sum of indecomposable submodules.

<u>Proof</u>: (1) ⟹ (2): Let M be any continuous R-module. That R is right noetherian, implies that R has right Krull dimension. Thus Corollary 2.7 yields that $\sum_{i \in I} \oplus A_i \subset' M$, where the A_i are indecomposable continuous submodules of M. Consequently, $E(M) = E(\sum_{i \in I} \oplus A_i) = \sum_{i \in I} \oplus E(A_i)$ holds. However, since M is continuous (hence π-injective), Proposition 1.19 can be applied, and we obtain $M = \sum_{i \in I} \oplus (M \cap E(A_i))$. Here, each $E(A_i)$ is uniform since A_i is so, and therefore, $M \cap E(A_i)$ is uniform.

The converse follows trivially from the implication (2) to (1), and the fact that every injective module is continuous. □

Remark 2.9: In the above decompositions of a continuous module (even in cases where $M_2 = 0$), the essential inclusion of $\sum_{i \in I} \oplus A_i$ in M_1 cannot be further 'improved' or 'strengthened', without additional assumptions, in the following sense:

a) If R is a ring such that for each continuous R-module M, $M_2 = 0$ and we get equality instead of inclusion - that is, $M = M_1 = \sum_{i \in I} \oplus A_i$ holds, then we have seen by Corollary 2.8 that this subsequently forces R to be right noetherian as every injective R-module is continuous.

b) The continuous module M_1 is not necessarily "minimal" over the direct sum $\sum_{i \in I} \oplus A_i$, in the sense that there may exist continuous modules properly contained in M_1 and essential over the same direct sum $\sum_{i \in I} \oplus A_i$. We exhibit this by the next example.

First, we give the following result by Utumi [30].

Theorem 2.10: *Let R be a regular ring. Then the following are equivalent:*

(1) R is right continuous.

(2) There exists a self-injective overring T, with $J(T) = 0$, and such that every idempotent of T lies in R.

Example 2.11: Let $\{D_\alpha\}_{\alpha\in I}$ be an infinite family of division rings, and P_α be proper subdivision rings of D_α. Put $M = \prod_{\alpha\in I} D_\alpha$, and $R = \prod_{\alpha\in I}(D_\alpha, P_\alpha) = \{x = (x_\alpha) \in M \mid x_\alpha \in P_\alpha$ for all $\alpha \in I$ except for a finite number$\}$. Then $\bigoplus_{\alpha\in I} D_\alpha \subset' R \subsetneqq' M$ holds. Note that M is a ring. Therefore, M can be viewed as a module over itself as well as over the ring R; and M is injective in both instances. Also, $\bigoplus_{\alpha\in I} D_\alpha$ is the socle of M.

The ring R is a regular ring, the overring M is self-injective and $J(M) = 0$; furthermore, every idempotent of M is a tuple $x = (x_\alpha)$, where $x_\alpha = 1$ or 0, hence obviously x is an element of R. Consequently, Theorem 2.10 implies that R_R is a continuous module.

Next, we point out the following facts:

i) Since each D_α is an indecomposable summand of M_R, and since $M_R \supset' \bigoplus_{\alpha\in I} D_\alpha$ holds, $\bigoplus_{\alpha\in I} D_\alpha$ is one of the choices for the direct sum $\sum_{i\in I} \oplus A_i$ of indecomposable summands, in the decomposition of M as a continuous R-module.

ii) As shown above, R_R is a continuous module. Then again, since each D_α is an indecomposable summand of R_R and R_R is essential over $\bigoplus_{\alpha\in I} D_\alpha$; it follows that $\bigoplus_{\alpha\in I} D_\alpha$ can again be one of the choices for the sum $\sum_{i\in I} \oplus A_i$ of indecomposable summands in the decomposition of the continuous module R_R.

iii) We claim that $\bigoplus_{\alpha\in I} D_\alpha$ is the only choice of $\sum_{i\in I} \oplus A_i$ of indecomposable summands of M, over which M is essential; Indeed, if $\sum_{\ell\in L} \oplus A_\ell$ be another maximal direct sum of indecomposable summands of M, such that M is essential over it, then, since $\underset{\alpha \in I}{\oplus} D_\alpha$ is the socle of M, $\underset{\alpha \in I}{\oplus} D_\alpha \subset' \sum_{\ell \in L} \oplus A_\ell \subset' M$, holds.

Now, for each $\ell \in L$, $A_\ell \cap (\underset{\alpha \in I}{\oplus} D_\alpha) \neq 0$ holds, by the essentiality of M over $\underset{\alpha \in I}{\oplus} D_\alpha$. We pick an element x in this intersection, and an index $\beta \in I$, for which it has a nonzero component x_β. Then, multiplying by all $r \in R$, such that r_β is arbitrary, and $r_\alpha = 0$, for all $\alpha \neq \beta$, we see that $D_\beta \subset A_\ell$, and hence $\sum_{\ell \in L} \oplus A_\ell = \underset{\alpha \in I}{\oplus} D_\alpha$ holds true.

In view of our observation that R_R is a continuous module, properly contained in the continuous module M, essential over the same maximal direct sum $\underset{\alpha \in I}{\oplus} D_\alpha$ of indecomposable summands; and in view of (iii) above, which says that $\underset{\alpha \in I}{\oplus} D_\alpha$ cannot be replaced by another sum $\sum_{\ell \in L} \oplus A_\ell$ of indecomposable summands, we conclude that our assertion 2.9(b) holds true.

On the other hand, we remark that M, considered as an injective module, is a 'minimal' injective module over the direct sum $\underset{\alpha \in I}{\oplus} D_\alpha$, since it is its injective hull.

CHAPTER III

DIRECT SUMS OF CONTINUOUS MODULES

§1. Introduction

It is a known fact that a finite direct sum of injective modules is always injective. Furthermore, if R is right noetherian, then any arbitrary direct sum of injective right R-modules is also injective. However, these properties fail for quasi-injective modules. Mohamed and Bouhy [21] showed that they also fail to hold for continuous modules. In this chapter, we study direct sums of indecomposable continuous modules and give a necessary and sufficient condition for a finite direct sum to be continuous. We then proceed to look at infinite direct sums of continuous modules and give a result in this situation.

§2. Finite Direct Sums

The following example [21] shows that, even over noetherian rings, a finite direct sum of indecomposable continuous modules need not be continuous.

Example 3.1: Let $F = \mathbb{Z}/\langle 2\rangle$ (or any finite field) and let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where $\mathbb{Z}$ is the ring of integers. The ring R is artinian, and the modules $A = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$, and $B = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, are continuous (even quasi-injective), as R-modules. However, $R = A \oplus B$ is not continuous, since the module A_R is isomorphic to the submodule $C_R = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ of R, while C_R is not a summand of R_R.

A closer look at the above example reveals that, although the module B is A-injective (being injective itself), the converse is not true: Indeed, if f is the isomorphism from B to $E(A) = \begin{pmatrix} 0 & 0 \\ F & F \end{pmatrix}$, given by $f\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$, then $f(B) \nsubseteq A$, and this implies that A is not B-injective. This observation will be useful for the main result of this chapter. First we prove a lemma:

<u>Lemma 3.2</u>: *Let* $N = \sum_{i=1}^{t} \oplus A_i$, *where each* A_i *is uniform and* A_j-*injective for all* $j \neq i$. *Then, every nonessential submodule of* N *is contained in a proper summand of* N.

<u>Proof</u>: Let A be a nonessential submodule of N. Then $A \cap A_k = 0$, for some k. For, suppose for all i, there are nonzero elements $a_i \in A \cap A_i$. Then $\sum_{i=1}^{t} \oplus a_i R \subset' \sum_{i=1}^{t} \oplus A_i = N$, as the A_i are uniform. This contradicts the nonessentiality of A. Consider the homomorphism $\sigma = 1 - \pi_k : A \to \sum_{i \neq k} \oplus A_i$, where π_k is the projection to A_k. This is a monomorphism, since $\sigma(a) = a - a_k = 0$ implies $a = a_k \in A \cap A_k = 0$. Now as A_k is A_i-injective for all $i \neq k$, A_k is $\sum_{i \neq k} \oplus A_i$-injective (see Chapter I). Hence, there exists a homomorphism $\phi : \sum_{i \neq k} \oplus A_i \to A_k$ such that $\phi\sigma = \pi_k|_A$. Thus, for any $a = \sum_{i=1}^{t} a_i \in A \cap \sum_{i=1}^{t} \oplus A_i$, we get $a_k = \pi_k(a) = \phi\sigma(a) = \phi(a - a_k)$. We define a submodule C of N by $C = \{b + \phi(b) \mid b \in \sum_{i \neq k} \oplus A_i\}$, and we claim that $N = C \oplus A_k$. Now, since

any $x \in N = \sum_{i=1}^{t} \oplus A_i$ can be written as $x = b + x_k$, with $b \in \sum_{i \neq k} \oplus A_i$, and $x_k \in A_k$, we have $x = (b + \psi(b)) + (x_k - \psi(b)) \in C + A_k$. This implies $N = C + A_k$. Next, if $a_k \in C \cap A_k$, then $a_k = b + \phi(b)$, for some $b \in \sum_{i \neq k} \oplus A_i$. Thus, $b = a_k - \phi(b) \in A_k \cap \sum_{i \neq k} \oplus A_i = 0$, and, therefore, $N = C \oplus A_k$. We further show that $A \subset C$, completing the proof: Indeed, $a = \sum_{i=1}^{t} a_i \in A$ implies $\sigma(a) = a - a_k \in \sum_{i \neq k} \oplus A_i$, hence $a = (a - a_k) + a_k = \sigma(a) + \phi\sigma(a) \in C$, by definition of C. □

We now give the main theorem of the chapter, which provides a necessary and sufficient condition for a finite direct sum of indecomposable continuous modules to be continuous.

<u>Theorem 3.3</u>: *Let* $M = \sum_{i=1}^{n} \oplus A_i$, *where the* A_i *are indecomposable submodules. Then* M *is continuous iff each* A_i *is continuous and* A_j*-injective for all* $j \neq i$.

<u>Proof</u>: Let $M = \sum_{i=1}^{n} \oplus A_i$, A_i indecomposable, be continuous. Then each A_i is continuous, being a summand, and is also A_j-injective for all $j \neq i$ (see Chapter I).

Conversely, let all A_i be continuous and A_j-injective for all $j \neq i$. We note that as each A_i is continuous and indecomposable, it becomes uniform and has a local endomorphism ring. Next, among all the decompositions of M into indecomposable summands, we choose a decomposition $M = \sum_{i=1}^{n} \oplus A_i$ with minimal t such that $A \subset \sum_{i=1}^{t} \oplus A_i = N$, say. It should be pointed out here, that all decompositions of M into indecomposable summands are isomorphic, by the Krull-Schmidt-Azumaya theorem. We now claim that $A \subset' \sum_{i=1}^{t} \oplus A_i$: If not, then by the above lemma, A is contained in a proper summand S of N, that is $N = S \oplus S'$, and $A \subset S$, $S' \neq 0$. The direct decomposition length $\ell(S)$ of S is strictly less than the length t of N. But this contradicts the minimality of t, with $A \subset \sum_{i=1}^{t} \oplus A_i$. This proves the claim, and verifies the first condition of continuity, as N is a summand of M.

We proceed to prove the second condition of continuity now, namely, that for any summand B of M, any monomorphism $f: B \to M$ splits.

Claim: Let C be an indecomposable summand of M, and $f: C \to M$ be a monomorphism. Then f splits.

Proof of the claim: As C is an indecomposable summand, there is an isomorphism $\phi: A_k \cong C$, for some k. Consider,

$A_k \xrightarrow{\phi} C \xrightarrow{f} M = \sum_{i=1}^{n} \oplus A_i \xrightarrow{\pi_i} A_i$. As f is monomorphism, one obtains

$\bigcap_{i=1}^{n} \mathrm{Ker}\,\pi_i f = 0$: $x \in \bigcap_{i=1}^{n} \mathrm{Ker}\,\pi_i f$ implies $\pi_i f(x) = 0$, for all i,

hence $\sum_{i=1}^{n} \pi_i f(x) = f(x) = 0$, which yields $x = 0$.

Consequently, as C is uniform, being continuous and indecomposable, this yields $\mathrm{Ker}\,\pi_i f = 0$, for some i. For this i, $\pi_i f\phi: A_k \to A_i$ is a monomorphism. Now, in case $i = k$, $\pi_i f\phi: A_i \to A_i$ is a monomorphism from the indecomposable continuous module A_i into itself, and hence, it is an isomorphism. On the other hand, if $i \neq k$, then A_k is A_i-injective and therefore, $\pi_i f\phi$, being a monomorphism, splits, as the following diagram commutes:

$$\begin{array}{ccc} A_k & \xrightarrow{\ \pi_i f\phi\ } & A_i \\ \| & \overset{\psi}{\nearrow} & \\ A_k^* & & \end{array} \quad , \quad (\psi \text{ exists as } A_k \text{ is } A_i\text{-injective})$$

Consequently, in both cases, $(\pi_i f)$ and $(\pi_i f)^{-1}$ are isomorphisms. Define $g: M \to C$ by $g|_{A_i} = (\pi_i f)^{-1}$, and $g|_{\sum_{j \neq i} \oplus A_j} = 0$. One observes that $g \circ f = I_C$. Since, for any $c \in C$, one may write $f(c) = a + b$ where $a \in A_i$ and $b \in \sum_{j \neq i} \oplus A_j$; one obtains

$$\begin{aligned} g \circ f(c) &= g(f(c)) = g(a + b) = (\pi_i f)^{-1}(a) \\ &= (\pi_i f)^{-1}(\pi_i(a + b)) = (\pi_i f)^{-1}(\pi_i f(c)) \\ &= (\pi_i f)^{-1}(\pi_i f)(c) = c . \end{aligned}$$

Therefore, $f: C \rightarrowtail M$ splits as claimed.

Next, let B be an arbitrary summand of M, that is, $M = \sum_{i=1}^{n} \oplus A_i = B \oplus B'$, for some $B' \subset M$. Then $B = \sum_{i=1}^{t} \oplus A_i'$ for some $t \leq n$, where $A_i' \cong A_{P(i)}$, and P is a one to one map into the set $\{1, 2, 3, \ldots, n\}$. This follows from the fact that all A_i have local endomorphism rings, hence M has a semiperfect endomorphism ring and, therefore, any idempotent of the endomorphism ring of M can be written as a finite sum of indecomposable orthogonal idempotents.

Now, to establish that the monomorphism $f: B \rightarrowtail M$ splits, we proceed by induction, over the direct sum decomposition length t of the summand B. If $t = 1$, the summand B is indecomposable, and we have already established the claim. Next, assume that any monomorphism of a summand of M of direct decomposition length less than t, into M, splits. Let $B = B_1 \oplus B_2$, where B_1 is a summand of length $t - 1$, and B_2 is indecomposable. Let e_i denote the injections from B_i to B. The monomorphism $fe_1: B_1 \rightarrowtail M$ splits by the assumption of induction; hence $M = M_1 \oplus M_2$ where $M_1 = \operatorname{Im} fe_1$. Let π_i denote the projections of M onto M_i, and let $f_{ij} = \pi_i f e_j: B_j \to M_i$; then f can be expressed by the matrix

$$f = \begin{bmatrix} f_{11} & f_{12} \\ 0 & f_{22} \end{bmatrix} : B = B_1 \oplus B_2 \to M = M_1 \oplus M_2$$

Define $\theta: B \to B$ by the matrix $\begin{bmatrix} 1 & -f_{11}^{-1} f_{12} \\ 0 & 1 \end{bmatrix}$. Then

$$f\theta = \begin{bmatrix} f_{11} & f_{12} \\ 0 & f_{22} \end{bmatrix} \begin{bmatrix} 1 & -f_{11}^{-1} f_{12} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} f_{11} & 0 \\ 0 & f_{22} \end{bmatrix} ; B \to M$$ holds. Clearly,

$f_{22} = \pi_2 f e_2: B_2 \to M_2$ is a monomorphism; and as B_2 is indecomposable, it splits. Let $f'_{22}: M_2 \to B_2$ be the splitting homomorphism, and define $\phi: M \to B$ by the matrix $\begin{bmatrix} f_{11}^{-1} & 0 \\ 0 & f'_{22} \end{bmatrix}$. The matrix multiplication then yields that $\theta\circ\phi\circ f = I_B$; thus f splits as required. This shows that $M = \sum_{i=1}^{n} A_i$ is continuous. □

§3. Infinite Direct Sums

Next, we look at infinite direct sums of indecomposable continuous R-modules and obtain the following result:

Theorem 3.4: *Let R be right noetherian; and let $M = \sum_{i\in I} \oplus A_i$ be a direct sum of indecomposable continuous R-modules. Then M is continuous iff it is π-injective.*

Remark 3.5: Note that $M = \sum_{i\in I} \oplus A_i$ π-injective, implies that A_i is A_j-injective, for all $j \neq i$.

Proof: If M is continuous, it is obviously π-injective. To prove the reverse implication, we observe that the first condition of continuity follows from the π-injectivity of M: Let $N \subset M = \sum_{i\in I} \oplus A_i$ be a submodule. Then $E(M)$, the injective hull of M, has a summand E_1 which is essential over N. Thus, by π-injectivity, $E_1 \cap M$ is a summand of M, which is essential over M.

Next, for the second condition, consider a submodule N of M, such that N is isomorphic to a summand of M. We have to show that N is itself a summand of M.

Claim: $N = \sum_{j\in J} \oplus A_j^*$ for a subset J of I, where $A_j^* \cong A_j$.

Proof of claim: As N is a submodule of M, $E(N)$ is a summand of $E(M) = \sum_{i\in I} \oplus E(A_i)$, where each $E(A_i)$ is uniform since A_i is so. Again, as R is right-noetherian, $E(N) = \sum_{k\in K} \oplus B_k$ such that for each $k \in K$, there exists $i_k \in I$, with $B_k \cong E(A_{i_k})$ (by Azumaya's Theorem). However, since N is isomorphic to a summand S, say, of the π-injective module M, it follows that N is itself π-injective. This yields $N = N \cap E(N) = \sum_{k\in K} \oplus (N \cap B_k)$, where, for each k, $N \cap B_k$ is isomorphic to a submodule of $E(A_{i_k})$, and thus is uniform. Also S is a summand of M, $M = S \oplus S'$, for some $S' \subset M$. The fact that N is isomorphic to S, implies that M is isomorphic to the external direct sum $N \oplus S'$.

Again, by the same arguments as above, one obtains $S' = \sum_{\ell\in L} \oplus C_\ell$, for some uniform submodules C_ℓ. Consequently, M has two decompositions into uniform submodules $M = \sum_{i\in I} \oplus A_i \cong \sum_{k\in K} \oplus (N \cap B_k) \oplus \sum_{\ell\in L} \oplus C_\ell$. Hence, by Krull-Schmidt theorem, and by pairing off suitable summands, we get a bijection from J to K, where J is a subset of I, such that $\sum_{j\in J} \oplus A_j \cong \sum_{k\in K} \oplus (N \cap B_k) = N$, where each uniform summand on the right is isomorphic to one of the uniform summands on the left. Thus,

$N \cong \sum_{j\in J} \oplus A_j^*$, where $A_j^* \cong A_j$ for all $j \in J$. This proves the claim.

Now, there arise two cases:

Case I. J is finite: Since N is isomorphic to $\sum_{j\in J} \oplus A_j = S$, as was just shown, it suffices to prove that every monomorphism $f: S = \sum_{j\in J} \oplus A_j \to M$ splits. The proof for this, follows exactly on the same lines as in the previous theorem, where we first established the splitting when S is indecomposable, and then proved it for any finite index set J , by induction on the direct sum decomposition length of S . However, note that when S is indecomposable summand, and $\phi: A_k \to S$ is an isomorphism for some k ; then the proof of the claim that there exists some α , such that, $\pi_\alpha f: S \to A_\alpha$ is a monomorphism, has to be modified as follows: $\bigcap_{j\in J} \mathrm{Ker}\pi_j f = 0$ holds, since $\pi_j f(x) = 0$, for all $j \in J$ implies that $0 = \sum_{j\in J} \pi_j f(x) = f(x)$, and therefore, $x = 0$. Now, pick a nonzero finitely generated submodule X of S . Then $f(X)$ is contained in a finite direct sum $\sum_{i\in F} \oplus A_i$ (where F is a finite subset of I). Consider $X \xrightarrow{f|_X} \sum_{i\in F} \oplus A_i \xrightarrow{\pi_i} A_i$; then $\bigcap_{i\in F} \mathrm{Ker}(\pi_i f|_X) = \bigcap_{i\in F} (\mathrm{Ker}(\pi_i f) \cap X) = 0$. But since S is uniform (being indecomposable), X is uniform, and thus $\mathrm{Ker}\pi_\alpha f \cap X = 0$, for some $\alpha \in F$. Again, as $\mathrm{Ker}\pi_\alpha f$ is a submodule of the uniform module S , and X is nonzero, we obtain $\mathrm{Ker}\pi_\alpha f = 0$, and thus $\pi_\alpha f: S \to A_\alpha$ is a monomorphism.

<u>Case II</u>. J is infinite: Let $N \subset M$ be such that $N \cong \sum_{j\in J} \oplus A_j$, where J is an infinite subset of I. Then we know that $N = \sum_{j\in J} \oplus A_j^*$, $A_j^* \cong A_j$. We show that N is a summand of M. Consider the set of all finite subsets F of J. This is an updirected set with inclusion as the ordering. Put $N_F := \sum_{j\in F} \oplus A_j^* \cong \sum_{j\in F} \oplus A_j$. Then $N = \bigcup_{F\subset J} N_F$ and by case I, one obtains that N_F is a summand of M. Hence $M = N_F \oplus K_F$, for some $0 \neq K_F \subset M$. We point out, now, that since $M \supset N \supset A_j^*$ holds for all $j \in J$, we can choose an injective hull for each A_j^*, $j \in J$, inside $E(N)$. Hence $\sum_{j\in F} E(A_j^*)$ is contained in $E(N)$. We show by induction, that $\sum_{j\in F} E(A_j^*)$ is a direct sum: Assume that $\sum_{j\in F'} E(A_j^*)$, is a direct sum for every proper subset F' of F. Hence for each $k \in F$, the sum $\sum_{j\in F\setminus\{k\}} E(A_j^*)$ is direct. We show that $E(A_k^*) \cap (\sum_{j\in F\setminus\{k\}} \oplus E(A_j^*)) = 0$. Suppose not, then let $a_k = \sum_{j\in F\setminus\{k\}} a_j$ be a nonzero element where $a_j \in E(A_j^*)$ for all $j \in F$. Now, as $0 \neq a_k \in E(A_k^*)$ is a nonzero element, there exists a nonzero element $0 \neq r \in R$, such that $0 \neq a_k r \in A_k^*$ holds. Again, as $\sum_{j\in F\setminus\{k\}} \oplus E(A_j^*)$ is essential over $\sum_{j\in F\setminus\{k\}} \oplus A_j^*$, there exists a $0 \neq s \in R$, such that $0 \neq (\sum_{j\in F\setminus\{k\}} a_j r)s \in \sum_{j\in F\setminus\{k\}} \oplus A_j^*$. But then this yields that $0 \neq a_k rs = (\sum_{j\in F\setminus\{k\}} a_j r)s$ is a nonzero element in $A_k^* \cap \sum_{j\in F\setminus\{k\}} A_j^*$, which is a contradiction. This shows that $\sum_{j\in F} \oplus E(A_j^*)$ is a direct sum, and $\sum_{j\in F} \oplus E(A_j^*) \subset \sum_{j\in P} \oplus E(A_j^*)$ for any finite subset P

of J containing F. Set $E(N_F) := \sum_{j\in F} \oplus E(A_j^*)$, for any finite subset F of J. Then $E(M) = E(N_F) \oplus E(K_F)$, since $M = N_F \oplus K_F$. But M is π-injective, therefore, $M = [M \cap E(N_F)] \oplus [M \cap E(K_F)]$. Comparing with $M = N_F \oplus K_F$, one observes that, $N_F = M \cap E(N_F)$: $x \in M \cap E(N_F)$ implies $x = n + k$, where $n \in N_F$, $k \in K_F$; hence, $x - n = k \in [M \cap E(N_F)] \cap K_F = 0$, thus $x = n \in N_F$, and consequently, $M \cap E(N_F) \subset N_F$. The reverse inclusion is obvious.

We may, therefore, write

$$M \cap [\bigcup_{F\subset S} E(N_F)] = \bigcup_{F\subset J} [M \cap E(N_F)] = \bigcup_{F\subset J} N_F = N ,$$

where the union is taken over all finite subsets F of J. Next, one obtains

$$\bigcup_{F\subset J} E(N_F) = \bigcup_{F\subset J} [E(\sum_{j\in F} \oplus A_j^*)] = \bigcup_{F\subset J} [\sum_{j\in F} \oplus E(A_j^*)] = \sum_{j\in J} \oplus E(A_j^*)$$

and $E(N) = E(\sum_{j\in J} \oplus A_j^*) = \sum_{j\in J} \oplus E(A_j^*)$; as R is right noetherian.

Consequently, $\bigcup_{F\subset J} E(N_F) = E(N)$ is injective. Thus, $E(M) = (\bigcup_{F\subset J} E(N_F)) \oplus B$, for some $B \subset E(M)$. Again, since M is π-injective, we get

$$M = [M \cap (\bigcup_{F\subset J} E(N_F))] \oplus [M \cap B]$$

$$= [\bigcup_{F\subset J} (M \cap E(N_F))] \oplus [M \cap B] = N \oplus [M \cap B]$$

This shows N is a summand of M, and establishes the continuity of M. □

CHAPTER IV

CONTINUOUS HULLS

§1. Introduction

Eckmann and Schopf [9] established the existence, and uniqueness up to isomorphism, of the injective hull of a module. This result proved to be a major step in the study of injective modules, and provided impetus to further work in this field.

Quasi-injective hulls were studied by Johnson and Wong [17]. They showed the existence of the quasi-injective hull and described it explicitly in terms of the endomorphism ring of the injective hull.

Goel and Jain [11], and L. Jeremy [16], independently studied the equivalent concepts of π-injectivity and quasi-continuity, respectively, and proved that π-injective (quasi-continuous) hulls exist and are unique up to isomorphism. An explicit description of the π-injective hull in terms of the idempotents of the endomorphism ring of the injective hull was also given.

In this chapter, we propose to study different possible definitions of continuous hulls. Though the existence of continuous hulls for arbitrary modules has not yet been established, we do give, in detail, an explicit description of continuous hulls for certain classes of cyclic modules, and a necessary and sufficient condition for the existence of a continuous hull for arbitrary cyclic modules, over a commutative ring.

Recall that a module is injective if and only if it has no proper essential extension, and that every module can be embedded in an injective module. A module E is said to be an *injective hull* of a module M if E is a minimal injective module containing M.

Theorem 4.1: (Eckmann and Schopf [9]). *Let M be a module; then:*

1) *Any injective module containing M contains an injective hull of M.*
2) *A module $E \supseteq M$ is an injective hull of M if and only if E is a maximal essential extension of M.*
3) *If $\phi: M \to M$ is an R-isomorphism, and E and E′ are two injective hulls of M, then ϕ can be extended to an isomorphism $\hat{\phi}: E \to E'$.*

We remark that in view of (3), any two injective hulls of a module M are isomorphic under an isomorphism which maps M identically.

We use the notation $E = E(M)$ for the injective hull of M, as mentioned in Chapter I.

Definition 4.2: The *quasi-injective hull* is defined to be a minimal quasi-injective extension of a module.

The following theorem gives their explicit description.

Theorem 4.3: (Johnson and Wong [17]). *Every right R-module M has a unique minimal quasi-injective essential extension given by $K \cdot M$, where $K = Hom_R(E(M), E(M))$.*

Goel and Jain [11], proved the following two results.

Theorem 4.4: *Let M be any R-module, $K = Hom_R(E(M),E(M))$ and let V be the subring of K generated by all the idempotents in K. Then M is π-injective if and only if M is a (V,R)-submodule of M.*

Definition 4.5: The minimal π-injective essential extension of M in $E(M)$ is called *π-injective hull* of M, and is denoted by M^{π}.

Proposition 4.6: *Each R-module M has a unique π-injective hull in $E(M)$, which is given by $M^{\pi} = V \cdot M$, where V is the subring generated by all idempotents in $K = Hom_R(E(M),E(M))$.*

§2. Continuous Hulls: Possible Definitions

With these preliminaries in mind, we now attempt to define and describe continuous hulls. The definitions of injective, quasi-injective and π-injective hulls, and their properties listed above, serve as a guide. At least the following three types may be proposed:

Definition 4.7: Let M be an R-module. A module C containing M is a *type I-continuous hull* if

i) C is continuous; and

ii) there does not exist any other continuous module C' such that $M \subset C' \subsetneq C$.

(That is, C is a minimal continuous module C' containing M.)

More stringent definitions would be the next two:

<u>Definition 4.8</u>: A module C containing M is a *type II-continuous hull* of M if

i) C is continuous; and

ii) for any other continuous module C' containing M, there exists a monomorphism $\alpha: C \rightarrowtail C'$, which is the identity on M.

<u>Definition 4.9</u>: A module C containing M is a *type III-continuous hull* of M (with respect to a given injective hull E of M), if

i) C is continuous; and

ii) whenever there is a continuous module C', such that $M \subset C' \subset E$, then $C \subset C'$.

We remark here that this definition is independent of the choice of E, in the following sense: if E' is another injective hull of M, and if $\alpha: E \to E'$ is an isomorphism over M, then $\alpha(c)$ is a type III-continuous hull of M in E', and it is independent of the choice of α.

Next, we shall study the relationship between these three types of continuous hulls, and derive some of their properties.

<u>Proposition 4.10</u>: *The above defined continuous hulls are in the following relationship:*

$$\textit{Type III} \Rightarrow \textit{Type II} \Rightarrow \textit{Type I}$$

<u>Proof</u>:

i) Type III $\Rightarrow$ Type II; Let C be a type III-continuous hull of M. Thus there is an injective hull E of M, such that

$M \subset C \subset E$. Let C' be another continuous module containing M. By continuity of C', we get a summand C_1' of C', such that $M \subset' C_1'$. Now, consider $I_M: M \to M$, the identity map on M. Then I_M can be extended to a homomorphism $\phi: C_1' \to E$, by injectivity of E. As $M \subset' C_1'$, ϕ is a monomorphism, since $\operatorname{Ker}\phi \cap M = 0$ implies $\operatorname{Ker}\phi = 0$. Hence $C_1' \cong \phi(C_1')$ holds. Thus $\phi(C_1')$ is a continuous module and $M = \phi(M) \subset \phi(C_1') \subset E$. This yields $C \subset \phi(C_1') \twoheadrightarrow C_1' \subset C'$, and therefore there exists the monomorphism $\phi^{-1}|_C: C \rightarrowtail C'$, showing that C is of type II.

ii) Type II $\Rightarrow$ Type I: Let C be a type II-continuous hull of M. We wish to show that C is of type I. Let C' be a continuous module such that $M \subset C' \subset C$. As C' is continuous, there exists a summand C_1' of C' such that C_1' is essential over M, and $C' = C_1' \oplus C_2'$. Now, C being of type II, there exists a monomorphism $\alpha: C \rightarrowtail C_1'$, which is identity on M. Thus $C \cong \alpha(C)$, and $M = \alpha(M) \subset' \alpha(C) \subset' C_1' \subset C$ holds. By continuity of C, there exists a submodule X of C such that $C = \alpha(C) \oplus X$. Applying modular law, one gets $C_1' = \alpha(C) \oplus (X \cap C_1')$, since $\alpha(C) \subset C_1' \subset C$.

Now, as C_1' is essential over M and $X \cap M = 0$, one obtains $X \cap C_1' = 0$, and hence $C_1' = \alpha(C) \cong C$ holds. But $C \cong C_1'$ yields that C is essential over M, as C_1' is so. This implies $X = 0$ since $X \cap M = 0$. Hence $C = \alpha(C)$ and therefore, one obtains $C = C'$. □

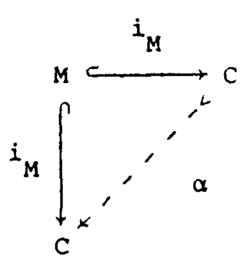
<u>Corollary 4.11</u>: *Any continuous hull of M is essential over M.*

Proof: Let C be a type I-continuous hull of M. By continuity there exists a summand C_1 of C, such that $M \subset' C_1 \subset C$, but this contadicts the fact that C is of type I, unless $C = C_1$. Hence C is essential over M. □

We remark here that we do not have any proofs or counter examples to decide whether the above implications do, or do not hold in the reverse directions. The existing examples*, that we know, are all of type III.

Lemma 4.12: *A type II-continuous hull of a module M is unique upto isomorphism over M.*

Proof: Let C be a continuous hull of M of type II, and let C' be another such a continuous hull of M. Then there exists a monomorphism $\alpha: C \rightarrowtail C'$, such that the diagram



commutes, where i_M is the inclusion map of M into C.

Therefore, $C \cong \alpha(C) \subset C'$ holds. However, as the type II-continuous

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* Dr. B.J. Müller has recently constructed an example of a cyclic, uniform, nonsingular module which does not have a type III or type II continuous hull but does possess a type I-continuous hull.

hull C' is of type I, as a consequence of Proposition 4.10 (ii), we conclude $\alpha(C) = C'$. Thus $C \cong C'$ holds. □

<u>Lemma 4.13</u>: *Whenever a module M has a type II (or type III) - continuous hull C_2, as well as a type I-continuous hull C_1, then C_1 is isomorphic to C_2 over M, and hence becomes a type II (respectively, type III) -continuous hull.*

<u>Proof</u>: There exists a monomorphism $\alpha: C_2 \rightarrowtail C_1$ over M, since in both cases, C_2 is of type II. This yields $M \subset \alpha(C_2) \subset C_1$. But then $\alpha(C_2) = C_1$, as C_1 is of type I; and hence α is an isomorphism $\alpha: C_2 \to C_1$ over M. Consequently, C_1 is of the same type as C_2. □

Looking at the above possible definitions of continuous hulls, a natural question arises, namely, which one of these definitions will be most suitable, and serve our purpose best as the continuous hull of a module?

We do not see how one could hope to prove the important property of uniqueness up to isomorphism for the type I-continuous hulls. Therefore, we consider these hulls unsuitable. Type II-continuous hulls have the property of uniqueness up to isomorphism. However, in all our examples and theorems, whenever a type II-continuous hull exists, we find that it is automatically of type III. Therefore, we find it convenient to select the stronger definition of type III-continuous hull to serve our purpose.

From now on, the term 'continuous hull' will mean a type III-continuous hull, unless specified otherwise.

§3. Continuous Hulls For Uniform Cyclic Modules

Definition 4.14: An element $a \in R$ is said to *act regularly* on an R-module M, if $ma = 0$ implies $m = 0$, for all $m \in M$. The set of all such elements of R, for a given module M, will be denoted by $\mathfrak{C}(M)$.

Theorem 4.15: *Let R be commutative, and let $M = cR$ be a uniform cyclic R-module. Then the quotient module*

$$Q(M) = \{\tfrac{m}{b} \mid m \in M,\ b \in \mathfrak{C}(M)\}$$

is the continuous hull of M.

Proof: As $M = cR$ holds, we obtain $Q(M) = \{\frac{cr}{b} \mid r \in R,\ b \in \mathfrak{C}(M)\}$. We claim that $Q(M)$ is continuous:

First, one may easily observe that $Q(M)$ is essential over M: For, let $0 \neq \frac{cr}{b} \in Q(M)$ be any element; then $0 \neq b \cdot \frac{cr}{b} = cr \in M$. Hence $M \subset Q(M) \subset E(M)$ holds; and the first condition of continuity follows trivially.

Next, let $V_R \subset Q(M)_R$ be any submodule, and let $\phi\colon Q(M)_R \to V_R$ be an isomorphism. We wish to show that $V = Q(M)$. Let $\phi(c) = v = \frac{cr}{b} \in V$. Our first claim is that $\phi(Q(M)) = vQ(R)$, where $Q(R) = \{\frac{r}{s} \mid r,s \in R,\ s \in \mathfrak{C}(M)\}$ is the corresponding quotient ring of R.

It is a known fact that for any two quotient modules $Q(A)$ and $Q(B)$, every R-homomorphism $\phi: Q(A) \to Q(B)$ is a $Q(R)$-homomorphism; hence, in our case, $\phi: Q(M) \to V \subset Q(V)$, is a $Q(R)$-homomorphism. Thus, for any $q = \frac{cx}{t} \in Q(M)$, where $x \in R$, and $t \in \mathfrak{C}(M)$

$$\phi(q) = \phi(\frac{cx}{t}) = \phi(c)\frac{x}{t} \in v\,Q(R)\ ,$$

hence $V = \phi(Q(M)) \subset v\,Q(R)$ holds. Conversely, let $v\frac{r}{s} \in vQ(R)$, where $r \in R$ and $s \in \mathfrak{C}(M)$; then

$$v\frac{r}{s} = \phi(c)\frac{r}{s} = \phi(\frac{cr}{s}) \in \phi(Q(M))$$

consequently, one obtains $V = \phi(Q(M))$ as claimed.

Next, we claim $v\,Q(R) = Q(M)$: Indeed, let $v\frac{s}{t} \in vQ(R)$, be an arbitrary element, where $t \in \mathfrak{C}(M)$, $s \in R$. We have $v\frac{s}{t} = \frac{cr}{b} \cdot \frac{s}{t} = \frac{crs}{bt} \in Q(M)$, since $bt \in \mathfrak{C}(M)$. Therefore, $v\,Q(R) \subset Q(M)$. Conversely, let $q = \frac{cx}{p} \in Q(M)$, $x \in R$, $p \in \mathfrak{C}(M)$. Then $q = \frac{cx}{p} = \frac{cr}{b} \cdot \frac{bx}{pr} = v\frac{bx}{pr}$ holds. Now since $\phi: c \longmapsto v = \frac{cr}{b}$ is a monomorphism, we obtain $r \in \mathfrak{C}(M)$. For, let $mr = 0$, where $m = cs \in cR = M$; then we get $0 = (cs)r = (cr)s$, as R is commutative.

$$0 = \frac{crs}{b} \Rightarrow vs = 0 \Rightarrow cs = 0 \quad (\text{since } \phi \text{ is mono})$$
$$\Rightarrow m = cs = 0 \Rightarrow r \in \mathfrak{C}(M)\ .$$

Thus we conclude that $pr \in \mathfrak{C}(M)$, hence $q = v\frac{bx}{pr} \in v\,Q(R)$. This shows that $Q(M) = v\,Q(R)$, and establishes the claim. Therefore, $V = \phi(Q(M)) = v\,Q(R) = Q(M)$ demonstrates that $Q(M)$ has the second condition of continuity.

We now show that $Q(M)$ is the continuous hull of M in the given injective hull $E(M)$. One can easily deduce from the previous argument, that $Q(M) = c \cdot Q(R)$ holds, by taking $V = M$ and $\phi = 1_M$, the identity on M. Hence, $M \subset c \cdot Q(R) \subset E(M)$ holds. Let C be another continuous overmodule in $E(M)$. It follows that C is a $Q(R)$-module: For, let $\frac{a}{b} \in Q(R)$, where $b \in \mathfrak{C}(M) \subset \mathfrak{C}(E(M))$. Then the continuity of C yields $C = Cb$, as Cb is isomorphic to C. This gives $C \cdot \frac{a}{b} = Cb \cdot \frac{a}{b} = Ca \subset C$, proving our claim. We, therefore, obtain

$$Q(M) = c \cdot Q(R) \subset C \cdot Q(R) \subset C .$$

Thus $Q(M)$ is contained in any other continuous overmodule of M in $E(M)$. □

The following is an example of a continuous hull of a uniform cyclic module, which is different from its quasi-injective and injective hulls.

<u>Example 4.16</u>: Let $\mathbb{Z}$ and $\mathbb{Q}$ stand for the ring of integers, and the field of rationals, respectively. Consider the ring

$$R = \{\text{Finite sums} \sum_{i\in[0,\infty)} \alpha_i x^i \,|\, \alpha_i \in \mathbb{Z}\}/\langle x\rangle .$$

This is uniform as an R-module: Indeed, let $a = \sum_{\ell=1}^{n} \alpha_\ell x^{i_\ell}$ and $b = \sum_{k=1}^{m} \beta_k x^{j_k}$, $\alpha_\ell, \beta_k \in \mathbb{Z}$, $0 \le i_\ell$, $j_k < 1$, with $\alpha_1 \ne 0$, $\beta_1 \ne 0$, be two nonzero elements of R. We can represent these as

$$a = x^{i_1} \sum_{\ell=1}^{n} \alpha_\ell x^{i_\ell - i_1}, \quad b = x^{j_1} \sum_{k=1}^{m} \beta_k x^{j_k - j_1}, \text{ where } 0 \leq i_\ell - i_1 < 1,$$

and $0 \leq j_k - j_1 < 1$. Let $i_1 \leq j_1$; then $x^{j_1} = x^{i_1} x^{j_1 - i_1}$. Now, consider the product $x^{j_1}\left(\sum_{\ell=1}^{n} \alpha_\ell x^{i_\ell - i_1}\right)\left(\sum_{k=1}^{m} \beta_k x^{j_k - j_1}\right) = bs = ar$, where

$s = \left(\sum_{\ell=1}^{n} \alpha_\ell x^{i_\ell - i_1}\right) \in R$, and $r = x^{j_1 - i_1}\left(\sum_{k=1}^{m} \beta_k x^{j_k - j_1}\right) \in R$. The product $bs = ar$ is non-zero, since $\alpha_1, \beta_1 \neq 0$, and $0 \neq x^{j_1}\alpha_1\beta_1$ is the lowest term of the product.

Next, consider the ring

$$Q = \{\text{Finite sums } \sum_{i \in [0,\infty)} \alpha_i x^i \mid \alpha_i \in \mathbb{Q}\} <x>$$

Obviously, $R \subsetneq Q$. We show that Q is the total quotient ring of R:

First, we show that all regular (non-zero divisor) elements of Q are invertible in Q, hence Q is its own total quotient ring.

Notice that $N(Q)$, the set of all nilpotent elements of Q, is the set $N = \{\text{Finite sums } \sum_{\ell=1}^{n} \alpha_\ell x^{i_\ell} \mid i_\ell > 0\}$. Indeed, if $i_1 > 0$, then $i_1 m \geq 1$, for some positive integer m. Now, any element $a = \sum_{\ell=1}^{n} \alpha_\ell x^{i_\ell} \in Q$, can be represented by $a = x^{i_1} b$, where $i_1 > 0$ is the smallest power of x occurring, and $b \in Q$. Hence,

$a^m = (x^{i_1}b)^m = x^{i_1 m}b^m = 0$; and so $N \subset N(Q)$ holds. Furthermore, if $i_1 = 0$, then $\sum_{\ell=1}^{n} \alpha_\ell x^{i_\ell} = \alpha_1 + \sum_{\ell=2} \alpha_\ell x^{i_\ell}$, a sum of an invertible element α_1, and a nilpotent element, b say. This is invertible and its inverse is given by

$$(\alpha_1 + b)^{-1} = (\alpha_1(1 + \alpha_1^{-1}b))^{-1} = \alpha_1^{-1}(1 - \alpha_1^{-1}b - (\alpha_1^{-1}b)^2 - \dots - (\alpha_1^{-1}b)^{m-1}),$$

where m is the index of nilpotency of b.

Hence, the elements of the local ring Q, are either nilpotent or invertible. This shows that the Jacobson radical of Q is the nil-radical. Further, as regular elements of Q cannot be nilpotent, they are invertible in Q. Thus Q is its own total quotient ring.

By the above discussion, it also follows that the total quotient ring of R is contained in Q. We further show that it is actually equal to Q: Let $t \in Q$, then one can write $t = \frac{r}{d}$, where $r \in R$, and $d \in \mathbb{Z}$, since, $t = \sum_{\ell=1}^{n} \alpha_\ell x^{i_\ell}$, $\alpha_\ell = \frac{\alpha'_\ell}{\beta_\ell} \in Q$ and $\alpha'_\ell, \beta_\ell \in \mathbb{Z}$, implies that $t = \frac{1}{d}\sum \alpha'_\ell x^{i_\ell}$ where $0 \neq d$ is the greatest common divisor of β'_ℓs. Now $0 \neq d \in \mathbb{Z}$ is regular in $\mathbb{Z} \subset \mathbb{Q}$, thus $d^{-1} \in Q$. Hence, Q is contained in the total quotient ring of R. This establishes the claim. Our Theorem 4.15 applies now, and we conclude that Q is the continuous hull of R_R.

Next, we claim that Q is not injective. Consider the ideals

$$x^{\frac{1}{n}}R = \{\text{Finite sums} \sum_{k,\text{finite}} \alpha_k x^{i_k} \mid i_k \geq \tfrac{1}{n}\}$$

of R, and define the map

$$\phi = \bigcup_{n=1}^{\infty}\phi_n : \bigcup_{n=1}^{\infty} x^{\frac{1}{n}}R \to Q\,, \quad \text{where } \phi|x^{\frac{1}{n}}R = \phi_n \text{ is}$$

given by the multiplication by $r_n = 1 + x^{\frac{1}{2}} + \ldots + x^{\frac{n-2}{n-1}} + x^{\frac{n-1}{n}}$.

Note that ϕ is well defined as $\phi_{n+1}\Big|x^{\frac{1}{n}}R = \phi_n$. However, there is no element $q \in Q$, for which $\phi(x) = x \cdot q$, for all $x \in \bigcup_{n=1}^{\infty} x^{\frac{1}{n}}R$. Since, in that case, q would have to be an infinite sum, and such a q does not lie in Q. Consequently, ϕ cannot be extended; showing that Q is not injective.

§4. Continuous Hulls For Non-Singular Cyclic Modules

It is a well-known fact that if R is commutative and non-singular, then the injective hull of R is its maximal quotient ring, and is a commutative, regular ring (Lambek [19]).

Lemma 4.17: *Let R be a nonsingular ring. Let S be an intermediate ring between R, and $E(R_R) = Q$, the maximal right quotient ring of R. If S_R is π-injective, then any R-submodule of S which is R-isomorphic to an R-summand of S, becomes a cyclic S-submodule, and the R-isomorphism becomes an S-isomorphism.*

Proof: Let $A_R \subset S_R$ be any submodule, and let $\phi: X_R \to A_R$, be an R-isomorphism for a summand X_R of S_R, such that $X_R \oplus X'_R = S_R$. Now

$Q = E(R_R) = E(S_R) = E(X) \oplus E(X') = eQ \oplus (1 - e)Q$ where $e^2 = e \in Q$. By π-injectivity of S, we obtain $S = (S \cap eQ) \oplus (S \cap (1 - e)Q)$. We now show that $S \cap eQ = eS$: As S is π-injective, $eS \subset S$ for all $e^2 = e \in Q$; hence, $eS \subset S \cap eQ$ holds. Conversely, $x = e\hat{r} \in S \cap eQ$ implies that $ex = e\hat{r} = x \in eS$ and therefore, $S \cap eQ = eS$. This shows that $S = eS \oplus (1 - e)Q$. Comparing this with the representation $S = X_R \oplus X'_R$, one can easily obtain that $X = eS$.

Now since $\phi: eS \to A_R$ is an isomorphism, we get $\phi(eS) = A$. Next, we claim that ϕ is a S-homomorphism.

<u>Proof of claim</u>: Let $s \in S \subset Q$, then there exists an essential right ideal D of R, such that $sD \subset R$. Thus for all $d \in D$,

$$[\phi(xs) - \phi(x)s]d = \phi(xs)d - \phi(x)sd$$
$$= \phi(xsd) - \phi(xsd) = 0 .$$

Hence, $[\phi(xs) - \phi(x)s]D = 0$, but since R is nonsingular, this yields $\phi(xs) = \phi(x)s$, for all $s \in S$.

Therefore, $A = \phi(eS) = \phi(e)S$ holds, implying that A is a cyclic S-submodule of S. This completes the proof. □

We may now proceed to prove the existence of continuous hulls for another class of cyclic modules, namely the nonsingular cyclic modules, over a commutative ring R. First, a definition and a useful lemma.

Definition 4.18: A regular ring is said to be *strongly regular*, if all idempotents lie in the centre.

Note that whenever $uvu = u$ holds, uv and vu become idempotents, hence in a strongly regular ring, lie in the centre. This implies that $u = uvu = u^2v = vu^2$ holds for such rings.

The following lemma is well-known [19].

Lemma 4.19: *Let R be a strongly regular ring. Then for every element $a \in R$, there exists a unique $x \in R$, such that $axa = a$ and $xax = x$.*

Proof: Let $a \in R$, be any element. As R is strongly regular, there exists $y \in R$ such that $aya = a = a^2y$. Set $s = yay$. Then

$$axa = a(yay)a = (aya)ya = aya = a,$$

and

$$xax = (yay)a(yay) = y(aya)(yay) = y(aya)y$$
$$= yay = x.$$

This x is unique: for, let x' be another element with $x'ax' = x' = ax'^2 = x'^2a$, and $ax'a = a = a^2x' = x'a^2$ then,

$$x = ax^2 = (x'a^2)x^2 = x'(a^2x)x = x'ax = (x'^2a)ax$$
$$= x'^2(a^2x) = x'^2a = x'.$$

This completes the proof. □

Proposition 4.20: *Let R be commutative. Then for every nonsingular cyclic R-module, and for every subring T of the endomorphism ring of* *its injective hull* E*, there exists a module* $X \subset E$*, containing the given cyclic module, and satisfying the following:*

i) X *is continuous;*

ii) $TX \subset X$*;*

iii) X *is absolutely minimal satisfying i) and ii); i.e., if* C *is any other module with the above properties of* X*, then* $X \subset C$*.*

Proof: Let $\bar{R} = R/I$ be the nonsingular cyclic R-module. We know that $E(\bar{R}_{\bar{R}})$ is a regular ring, since $\bar{R}$ is a nonsingular ring. Also $E(\bar{R}_{\bar{R}})$ as an R-module, is the quasi-injective hull of $\bar{R}_R$, and thus is equal to $\text{ann}_{E(\bar{R}_R)} I$, that is

$$\bar{R} \subset E(\bar{R}_{\bar{R}}) = \text{ann}_{E(\bar{R}_R)} I \subseteq E(\bar{R}_R) = E .$$

Again, since $\bar{R}$ is nonsingular, $E(\bar{R}_R)$ is nonsingular as an R-module.

We claim that $E(\bar{R}_{\bar{R}}) = E(\bar{R}_R)$: Let $e \in E(\bar{R}_R)$. This implies $eL \subset \bar{R}$, where L is an essential ideal of R.

$$eL \cdot I = 0 \Rightarrow (eI)L = 0 \quad \text{(as } R \text{ is commutative)}$$

$$\Rightarrow eI \subset Z(E_R) = 0 \quad \text{(since } E \text{ is nonsingular)}$$

$$\Rightarrow e \in \text{ann}_{E(\bar{R}_R)} I = E(\bar{R}_{\bar{R}}) .$$

Thus, E is itself a commutative regular ring.

Now, let T be any subring of $\text{Endo}(E_R) \cong E$. We show the existence of a module X, which has the required properties of the hypothesis.

Let Π be the subring of $E(\bar{R})$, generated by all the idempotents of $E(\bar{R})$; then $\Pi\bar{R} = (\bar{R})^{\pi}$, is the π-injective hull of $\bar{R}$. Now, let R' be any regular subring of E, containing $T\Pi\bar{R}$, that is $\bar{R} \subset T\Pi\bar{R} \subset R' \subset E = E(\bar{R})$. Then since R' contains Π, R'_R is π-injective; because for any idempotent $e^2 = e \in E$, $eR' \subset R'$.

We can further show that R' is continuous as an R-module: One can easily observe that R'_R satisfies the first condition of continuity as it is π-injective. Indeed, let $A_R \subset R'_R \subset E(R'_R)$. Now, as $E(R'_R)$ is continuous, there exists a summand E_1 of $E(R'_R)$ which is essential over A. This implies that $R' \cap E_1$ is a summand of R', which is essential over A, showing the first condition of continuity. To show the second condition, let $A_R \subset R'_R$ and let A_R be isomorphic to a summand of R'. Applying Lemma 4.17, we see that A_R is a cyclic R'-submodule of R', that is, a principal ideal of R'. R' being regular, it follows that A_R is a summand of R'_R. Thus R'_R is continuous.

Next, consider the family

$$\mathcal{F} = \{R'_i \mid T\Pi\bar{R} \subset R'_i \subset E(\bar{R}), \text{ and } R'_i \text{ is a regular subring of } E\}$$

then $\mathcal{F} \neq \emptyset$ since $E = E(\bar{R}) \in \mathcal{F}$. Set $\bigcap_{R'_i \in \mathcal{F}} R'_i = X$, then X is a regular ring:

Let $a \in X$, thus $a \in R'_i$, for all R'_i. Since R'_i are regular subrings of commutative (hence strongly) regular ring E, by Lemma 4.19 there exists a unique x with $axa = a$ and $xax = x$. As x is uniquely determined inside E, we get $x \in R_i$ for all $R_i \in \mathcal{F}$, and therefore, $x \in X$. Thus X is the smallest regular ring containing $T\Pi\bar{R}$. We see that

i) X is continuous as an R-module, by the above discussion; and

ii) $T \subset X$ implies, $TX \subset X \cdot X \subset X$, since X is a ring.

iii) To show the third property, let C be a continuous R-module such that $\bar{R} \subset C \subset E = E(\bar{R}_{\bar{R}}) = E(\bar{R}_R)$ with $TC \subset C$.

We note, since $C \subset E(\bar{R}_{\bar{R}}) = \text{ann}_E(I)$ hence $CI = 0$, thus C is an $\bar{R}$-module. Therefore, it does not make any difference whether we consider C as an $\bar{R}$-module or as an R-module.

Our aim, now, is to show that $X \subset C$:

Define $S =: \{a \in E \mid aC \subset C\}$. It can be easily seen that S is a ring. Also $S \supset \bar{R}$, since C is an $\bar{R}$-module; $S \supset T$, since $TC \subset C$; and $S \supset \Pi$, since C is continuous, hence invariant under every projection of E. Thus $T\Pi\bar{R} \subset S \subset E$.

Claim: S is a regular ring.

Proof of the claim: Let $a \in S$, so we obtain $b \in E$ such that $aba = a^2b = a$, and $bab = b^2a = b$. Thus $ba = f = f^2$ and $fC = baC \subset C$, where fC is a summand of C. Also, $aC \subset C$, since $a \in S$. Now, consider the map $aC \to baC = fC$, defined by $ac \mapsto bac$. This is a well defined R-homomorphism. It is mono: For, $bac = 0$ implies that $ac = (aba)c = 0$. Consequently, we have $aC \cong baC = fC$, and continuity of C implies that aC is a summand of C. Let $aC = \varepsilon C$, where $\varepsilon = \varepsilon^2 \in E(\bar{R})$. Then $C = \varepsilon C \oplus (1 - \varepsilon)C = aC \oplus (1 - \varepsilon)C$. It follows that

$$\begin{aligned} bC &= (ba)C \oplus b(1 - \varepsilon)C \\ &= fC \oplus b^2a(1 - \varepsilon)C \\ &\subset C + b^2(1 - \varepsilon)aC \\ &= C + b^2(1 - \varepsilon) \cdot \varepsilon C = C . \end{aligned}$$

Thus $b \in S$, hence S is a regular ring. Therefore, $X \subset S \subset SC \subset C$, shows that X is the smallest submodule of E, satisfying i) and ii). □

Remark 4.21: In the above proof we note that X is not only an R-submodule of E, but has a ring structure; in fact, it is a regular ring.

As a corollary to this proposition, we derive the following result about continuous hulls of nonsingular cyclic modules.

Corollary 4.22: *Let R be commutative. Then every nonsingular cyclic R-module has a continuous hull, which is a regular ring.*

Proof: Let $\bar{R} = R/I$ be the nonsingular cyclic R-module. In the above proposition, take the subring T of the $\mathrm{end}_R E = E(\bar{R})$, as equal to $\bar{R}$ itself. Then X, the smallest regular subring of E, containing $\pi\bar{R} = (\bar{R})^{\pi}$, the π-injective hull of $\bar{R}$, becomes the continuous hull of $\bar{R}$. □

Below, we construct an example which shows that there exist continuous hulls for nonsingular commutative rings which are different from both the injective hulls and the π-injective hulls.

<u>Example 4.23</u>: Let $\{F_\alpha, \alpha \in I\}$ be a family of fields. We have $1 = (1,1,\dots,1) \in \prod_{\alpha\in I} F_\alpha$. Consider the ring R generated by $\bigoplus_{\alpha\in I} F_\alpha$ and 1. It can be easily seen that πF_α is the injective hull of $\bigoplus_{\alpha\in I} F_\alpha$ as an R-module. We also obtain that $\bigoplus_{\alpha\in I} F_\alpha$ is the smallest essential ideal of R; since any essential ideal of R contains the socle $\bigoplus_{\alpha\in I} F_\alpha$ of R. R is nonsingular, since $r = (r_\alpha) \in Z(R)$, the singular ideal of R, implies that $r \cdot (\bigoplus_{\alpha\in I} F_\alpha) = 0$. Therefore, $r_\alpha \cdot e_\alpha = 0$, for all $\alpha \in I$, where e_α has 1 as the α^{th} component and zero elsewhere. Thus $r_\alpha = 0$, for all $\alpha \in I$.

For subrings S_α of F_α, we introduce the notation:

$$\prod_{\alpha\in I}^{f}(F_\alpha, S_\alpha) = \{(x_\alpha) \in \prod_{\alpha\in I} F_\alpha \,|\, \{\alpha \in I: x_\alpha \notin S_\alpha\} \text{ and } \{x_\alpha: \alpha \in I\} \text{ are finite sets}\}.$$

Now, to construct R^π, the π-injective hull of R, we look at all nonzero idempotents of $\prod F_\alpha$, which are tuples having 1 at some places and zero elsewhere. Thus the π-injective hull of R is $R^\pi = \prod_{\alpha\in I}^{f}(F_\alpha, \rho_\alpha)$ in our notation, where ρ_α is the prime subring of F_α, that is the subring generated by $1 \in F_\alpha$.

Next, by our Corollary 4.22, the continuous hull of R will be the smallest regular ring containing R^π. A regularity inverse for an element $(x_\alpha) \in \prod_{\alpha\in I}^{f}(F_\alpha, \rho_\alpha)$ is obtained by putting 0, whenever $x_\alpha = 0$; and x_α^{-1}, whenever $x_\alpha \neq 0$. Adjoining all these regularity inverses to R^{π}, we get the regular ring $R^* = \prod\limits_{\alpha\in I}^{f}(F_\alpha, \Phi_\alpha)$, where Φ_α is the prime subfield of F_α. R^* is the continuous hull of R and is a proper subring of $\prod\limits_{\alpha\in I} F_\alpha$ if $\Phi_\beta \subsetneqq F_\beta$ for some $\beta \in I$. One obtains

$$\bigoplus_{\alpha\in I} F_\alpha \subset R = \langle \bigoplus_{\alpha\in I} F_\alpha, 1\rangle \subsetneqq \prod_{\alpha\in I}^{f}(F_\alpha, \rho_\alpha) \subset \prod_{\alpha\in I}^{f}(F_\alpha, \Phi_\alpha) \subset \prod_{\alpha\in I} F_\alpha .$$

Now, if the characteristic of F_α is prime for all α, then the prime subfield Φ_α is the same as the prime subring ρ_α, and therefore, one has $R^{\pi} = R^*$. However, if there exists $\beta \in I$, such that $\text{char}.F_\beta = 0$, then prime subfield Φ_α of F_α is $\mathbb{Q}$, the field of rationals and $R^{\pi} \subsetneqq \Pi^{f}(F_\alpha, \Phi_\alpha)$.

We now look at noncommutative, nonsingular rings, where we obtain a restricted kind of continuous hull for R. Namely, we can show that there exists a continuous R-module, which is an R-bimodule and is smallest among all continuous R-bimodules containing R, inside an injective hull of R. The following result, due to Utumi [31], will be used for this purpose.

Theorem 4.24: *Every regular right self-injective ring S is decomposed into the direct sum of two ideals A and B, such that A is strongly regular, and B is generated by idempotents.*

Our result for non-commutative, non-singular rings can be formally stated as follows:

Theorem 4.25: *Let R be an arbitrary (not necessarily commutative) non-singular ring. Then there exists a right continuous R-bimodule*

${}_R R^*_R \subset E(R_R)$ *, containing* R*, which has the following property: If* ${}_R M_R$ *is any right continuous R-bimodule such that* $R \subset {}_R M_R \subset E(R_R)$ *, then* $R^* \subset M$ *.*

Remark 4.26: Note that the R-bimodule R^* above, has similar properties as the module x in the Proposition 4.20: R^* is right continuous, invariant under the subring R of the endomorphism ring of $E(R)$ acting on the left, and absolutely minimal with respect to these two properties.

Proof of Theorem: As remarked earlier, the injective hull $E(R)$ of R is a regular ring since R is nonsingular. By Theorem 4.24, we can write $E = E(R) = S_1 \oplus S_2$, where $S_1 = eE$ is strongly regular, and $S_2 = (1 - e)E$ is generated by idempotents of E , $e^2 = e \in E$. Let R^π , be the π-injective hull of R , then we can get $R^\pi = (R^\pi \cap eE) \oplus (R^\pi \cap (1 - e)E)$. Since $(1 - e)E$ is generated by idempotents and R^π contains all idempotents of E , one obtains $(1 - e)E \subset R^\pi$. Hence $R^\pi \cap (1 - e)E = (1 - e)E$; and $R^\pi = eR^\pi \oplus (1 - e)E$.

Next, any regular ring R' , such that $R \subset R^\pi \subset R' \subset E$ holds, becomes continuous as in Proposition 4.20. Hence, R' can be decomposed as $R' = eR' \oplus (1 - e)E$. The existence of the smallest regular ring containing R^π is obtained again by applying the previous argument. This becomes the required right continuous R-bimodule ${}_R R^*_R$. Indeed, if ${}_R M_R$ is any right continuous R-bimodule over R , inside E ; then by above $M = eM \oplus (1 - e)E$. The arguments of Proposition 4.20 apply again to show that the ring

$$S = \{a \in eE \mid a(eM) \subset eM\}$$

is a regular ring, contained in eM. However, we need eM to be an R-bimodule to show that S contains the ring R, and hence contains R^{π}. This, then, shows that $R^* \subset M$ holds true, completing the proof. □

§5. Continuous Hulls for Arbitrary Cyclic Modules

Recall the following:

Definition 4.27: The *closure of a submodule* N in M is $cl_M(N) = \{x \in M \mid (N:x)$ is essential in $R\}$, where $(N:x) = \{r \in R \mid xr \in N\}$. Then $cl_M(0)$ is the *singular submodule* of M, denoted by $Z(M)$, and $cl_M(cl_M(0))$ is the *torsion submodule* of M, denoted by $Z^*(M)$. A module is called *closed* if it is equal to its own closure. Goldie [12], proved that for any module N, $clcl(N)$ is closed, that is, $cl(clcl(N)) = clcl(N)$.

The following theorem was shown by Harada [14], and Cusick [8], and is easy to prove.

Theorem 4.28: *Let E be an injective R-module, then $E = E_1 \oplus E_2$, where $E_1 = Z^*(E)$ is singular, and E_2 is a nonsingular submodule of E.*

Proof: We observe that $Z^*(E)$ becomes injective: Let F be the injective hull of $Z^*(E)$. Then every nonzero element of F can be brought down into $Z^*(E)$ by multiplication with an essential right ideal of R. Hence it belongs to $Z^*(E)$, implying that $Z^*(E) = F$. □

Now, we prove the following lemma, which will be of great importance for our main result.

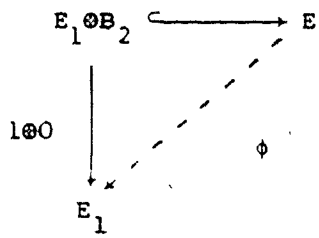
<u>Lemma 4.29</u>: *Let X be π-injective and $E(X) = E_1 \oplus E_2$, where $E_1 = Z^*(E)$ is singular and E_2 is nonsingular. Define $X_i = E_i \cap X$, $i = 1, 2$. Then X is continuous iff X_i are continuous, $i = 1, 2$.*

<u>Proof</u>: We first note that $X_1 = Z^*(X)$ is a singular and X_2 is a nonsingular module, because of the above decomposition of $E = E(X)$. Now, the first condition of continuity follows trivially as before, from the π-injectivity of X. To check the second condition, consider a summand B of X, and a monomorphism $B \rightarrowtail X$; we want to show that f splits.

As B is a summand of X, it is itself π-injective. We obtain $B = B_1 \oplus B_2$, as above with $B_1 = Z^*(B)$ singular, and B_2 nonsingular. The B_i are summands of B, hence of X. Notice that as B_1 is the singular submodule of B and X_1 is the singular submodule of X, we have $B_1 \subset X_1$, and hence B_1 is a summand of X_1. As B_1 is singular, $f(B_1)$ $(\cong B_1)$ is also singular. This implies $f(B_1) \subset X_1 = Z^*(X)$; thus $f|_{B_1} : B_1 \rightarrowtail X_1$ is a monomorphism. But X_1 is continuous, hence $f|_{B_1}$ splits.

Therefore, one gets $X_1 = f(B_1) \oplus C_1$, for some submodule C_1 of X_1. Consider B_2, a summand of X, and let $\pi_i : X \to X_i$, $i = 1, 2$ be the projections. Then $\pi_2 f|_{B_2} : B_2 \to X_2$ is a monomorphism since $f(B_2) \cong B_2$ is nonsingular, and $\operatorname{Ker}\pi_2 = X_1$ is singular, thus $f(B_2) \cap \operatorname{Ker}\pi_2 = 0$.

Note similarly that $B_2 \cap E_1 = 0$, since B_2 is nonsingular and E_1 is singular. Consider the following commutative diagram



(Here ϕ exists, since E_1 is injective.)

Now, let $K = \operatorname{Ker}\phi$, then $E = E_1 \oplus K$: One observes that $E_1 \cap K = 0$, follows trivially, since $\phi|_{E_1} = I_{E_1}$. Further, since $1 \oplus 0$ is epi, $E = E_1 + B_2 + \operatorname{Ker}\phi$; and as $\phi(B_2) = 0$ one gets $E = E_1 + K$. Thus $E = E_1 \oplus K$. Consequently, we obtain $X = X_1' \oplus X_2'$, where $X_1' = X \cap E_1 = X_1$, and $X_2' = X \cap K$, by π-injectivity of X. Therefore, B_2 is a summand of X_2', and $X_2' \cong X_2$; as $X = X_1 \oplus X_2 = X_1' \oplus X_2'$, and $X_1 = X_1'$. We claim that $X_2 = \pi_2 f(B_2) \oplus C_2$, for some submodule C_2 of X_2. This follows since $\pi_2 f|_{B_2} : B_2 \rightarrowtail X_2 \cong X_2'$ is a monomorphism, and X_2, (and hence) X_2' are continuous, thus $\pi_2 f|_{B_2}$ splits, which yields $X_2 = \pi_2 f(B_2) \oplus C_2$.

Finally, we show that $X = X_1 \oplus f(B_2) \oplus C_2$. Let $x = x_1 + x_2 \in X_1 \oplus X_2 = X_1 \oplus \pi_2 f(B_2) \oplus C_2$, where $x_i \in X_i$. Then

$$\begin{aligned} x = x_1 + x_2 &= x_1 + \pi_2 f(b_2) + c_2, \quad \text{for some } b_2 \in B_2,\ c_2 \in C_2 \\ &= x_1 - \pi_1 f(b_2) + f(b_2) + c_2, \quad \text{since } f(b_2) = \pi_1 f(b_2) + \pi_2 f(b_2) \\ &\in X_1 \oplus C_2 + f(B_2), \quad \text{as } x_1 - \pi_1 f(b_2) \in X_1 \end{aligned}$$

Now let $y \in (X_1 \oplus C_2) \cap f(B_2)$; then $y = f(b_2) = x_1 + c_2$, where $b_2 \in B_2$, $c_2 \in C_2$, $x_1 \in X_1$. This implies

$f(b_2) = \pi_1 f(b_1) + \pi_2 f(b_2) = x_1 + c_2 \in X_1 \oplus X_2$, hence $\pi_1 f(b_1) = x_1$, and $\pi_2 f(b_2) = c_2$, by the uniqueness of representation in a direct sum. Therefore, we get $c_2 = \pi_2 f(b_2) \in C_2 \cap \pi_2 f(B_2) = 0$, which gives $c_2 = 0 = \pi_2 f(b_2)$. But $\pi_2 f|_{B_2}$ is mono, and therefore, one gets $b_2 = 0$, implying $f(b_2) = 0$. Thus, $(X_1 \oplus C_2) \cap f(B_2) = 0$. Consequently,

$$\begin{aligned} X &= X_1 \oplus f(B_2) \oplus C_2 = f(B_1) \oplus C_1 \oplus f(B_2) \oplus C_2 \\ &= f(B_1) \oplus f(B_2) \oplus C_1 \oplus C_2 = f(B_1 \oplus B_2) \oplus C_1 \oplus C_2 \\ &= f(B) \oplus C_1 \oplus C_2 \ . \end{aligned}$$

Hence f splits; this demonstrates the second condition of continuity for X. □

To find the continuous hull for an arbitrary cyclic R-module $M = mR$, for a commutative ring R, we consider the following situation again, $M = mR \subset E(M) = E = E_1 \oplus E_2$, where $E_1 = Z^*(E)$ is singular and E_2 is nonsingular. Let $m = e_1 + e_2$ where $e_i \in E_i$, $i = 1, 2$, then, there exists an embedding: $M = mR \rightarrowtail e_1R \oplus e_2R \subset E_1 \oplus E_2 = E$, defined by $mr \longmapsto (e_1 + e_2)r = e_1r + e_2r$.

<u>Theorem 4.30</u>: *Let R be commutative, and let $M = mR$ be any cyclic R-module. Let $E = E(M) = E_1 \oplus E_2$, where $E_1 = Z^*(E)$ is singular, and E_2 is nonsingular. Further, let $m = e_1 + e_2$, where $e_i \in E_i$, $i = 1, 2$. Then M has a continuous hull in E if and only if $e_1R + \mathrm{ann}_{E_1}(e_2^{\circ})$ has a continuous hull in E_1.*

<u>Proof</u>: Given that $e_1R + \text{ann}_{E_1}(e_2^\circ)$ has a continuous hull X_1 in E_1, we show that $mR \cong \bar{R}$ has a continuous hull in E.

Let $\pi_i: E \to E_i$, $i = 1, 2$; be the respective projections and let T be the subring of the $\text{Endo}_R E_2$, which is generated by all $\pi_2\pi|_{E_2}$, where $\pi^2 = \pi \in \text{Endo}_R E$. Since e_2R is nonsingular as an R-module, being a submodule of one such, one can apply Proposition 4.20. Thus, there exists an R-module X_2, such that $e_2R \subset X_2 \subset E_2$, and it has the following properties

i) X_2 is continuous;

ii) $TX_2 \subset X_2$; and

iii) X_2 is absolutely minimal such. That is, whenever there exists a module M such that $e_2R \subset M \subset E_2$ holds, satisfying i) and ii) above, then $X_2 \subset M$ holds. One also obtains $X_2 \subset \text{ann}_{E_2} e_2^\circ$, since $\text{ann}_{E_2} e_2^\circ$ is quasi-injective as an R-module, and is therefore one of these competing M's.

Now, form the sum $X = X_1 \oplus X_2$. We claim that X is π-injective: Consider any $\pi = \pi^2 \in \text{Endo}_R E$, where $E = E(X)$. As we have $M = e_1R \oplus e_2R \subset' X = X_1 \oplus X_2 \subset' E = E_1 \oplus E_2$, hence $E(X_1) = E_1$ and $E(X_2) = E_2$, and also, since there is no map from the singular submodule E_1 to the nonsingular submodule E_2, we obtain $\pi|_{E_1} \in \text{Endo}_R E_1$ and hence $\pi(X_1) \subset X_1$, because X_1 is continuous. Consider $\pi_1\pi|_{E_2}: E_2 \to E_1$, then $\pi_1\pi(X_2) \cdot e_2^\circ = 0$ since, $X_2 \subset \text{ann}_{E_2}(e_2^\circ)$, hence $\pi_1\pi(X_2) \subset \text{ann}_{E_1}(e_2^\circ) \subset X_1$, by assumption on X_1.

Further, $\pi_2\pi(X_2) \subset TX_2 \subset X_2$, by ii) above, therefore, for any $x = x_1 + x_2 \in X$, where $x_i \in X_i$, $i = 1, 2$;

$$\begin{aligned}\pi(x) = \pi(x_1 + x_2) &= \pi(x_1) + \pi(x_2) \\ &= \pi(x_1) + \pi_1\pi(x_2) + \pi_2\pi(x_2) \\ &= [\pi(x_1) + \pi_1\pi(x_2)] + \pi_2\pi(x_2) \in X_1 + X_2 = X\end{aligned}$$

which yields $\pi(X) \subset X$. This shows that X is π-injective as it is invariant under every idempotent $\pi^2 = \pi \in \mathrm{Endo}_R E$. One can apply now Lemma 4.29 to obtain X is continuous.

We now show that X is contained in any other continuous R-module Y, whenever $M = mR \subset Y \subset E = E_1 \oplus E_2$ holds. Let Y be such a continuous R-module. Then $Y = Y_1 \oplus Y_2$, where $Y_i = Y \cap E_i$, $i = 1, 2$; is continuous, being a summand.

Note that $e_i = \pi_i(m) \in \pi_i(Y) \subset Y \cap E_i = Y_i$ implies that $e_iR \subset Y$. Now, $\pi_2\pi(Y_2) \subset \pi_2\pi(Y) \subset \pi_2(Y) \subset Y$, by continuity of Y and $\pi_2\pi(Y_2) \subset E_2$ implies that $TY_2 \subset Y \cap E_2 = Y_2$. This yields, $X_2 \subset Y_2$ since X_2 was smallest such module by its construction.

Let $a \in \mathrm{ann}_{E_1}(e_2^\circ)$; then the map $e_2R \to aR \subset E_1$, given by $e_2r \longmapsto ar$, is well defined since $e_2r = 0$ implies $r \in e_2^\circ$ and hence $ar = 0$. We can extend this map to $\phi: E_2 \to E_1$ by injectivity of E_1, and since Y_1 is Y_2-injective, $\phi(Y_2) \subset Y_1$. Hence $a = \phi(e_2) \in Y_1$, which yields $e_1R + \mathrm{ann}_{E_1}(e_2^\circ) \subset Y_1$. But X_1 is continuous hull of $e_1R + \mathrm{ann}_{E_1}(e_2^\circ)$ by hypothesis, thus $X_1 \subset Y_1$. Therefore, $X = X_1 \oplus X_2 \subset Y_1 \oplus Y_2 = Y$; proving that X is the continuous hull of $M = mR$.

Conversely, assume that mR has a continuous hull Y in E, then we show that a continuous hull of $e_1R + \mathrm{ann}_{E_1}(e_2^\circ)$ exists. As above, we obtain $Y = Y_1 \oplus Y_2$, $e_iR \subset Y_i$, $TY_2 \subset Y_2$, $e_1R + \mathrm{ann}_{E_1}(e_2^\circ) \subset Y_1$, and that each Y_i is continuous, and $X_2 \subset Y_2$ by construction of X_2. Let C be any continuous R-module such that $e_1R + \mathrm{ann}_{E_1}(e_2^\circ) \subset C \subset E_1$. Consider the direct sum $C \oplus X_2$; this is π-injective: As $C \subset E_1$, C is singular. By its continuity, $\pi(C) \subset C$, for all $\pi^2 = \pi \in \mathrm{Endo}_R E$, because $\pi|_{E_1} \in \mathrm{Endo}_R E_1$. Again, $\pi_1\pi(X_2) \subset \mathrm{ann}_{E_1}(e_2^\circ) \subset C$ and $\pi_2\pi(X_2) \subset X_2$ yields $\pi(C \oplus X_2) \subset C \oplus X_2$.

Hence $C \oplus X_2$ is π-injective and therefore continuous, again by Lemma 4.29; with $mR = e_1R \oplus e_2R \subset C \oplus X_2 \subset E$. However, since $Y = Y_1 \oplus Y_2$ is the continuous hull of mR, one obtains $Y = Y_1 \oplus Y_2 \subset C \oplus X_2$, and therefore, $Y_2 = X_2$, and $Y_1 \subset C$. This shows that Y_1 is the continuous hull of $e_1R + \mathrm{ann}_{E_1}(e_2^\circ)$ in E_1, and completes the proof of the theorem. □

CHAPTER V

RINGS FOR WHICH EVERY CONTINUOUS MODULE IS QUASI-INJECTIVE

§1. Introduction

The fact that every quasi-injective module is continuous leads to the natural question: For what classes of rings does every continuous module become quasi-injective? In Chapters I and IV, we have given explicit examples where the concept of continuity differs from that of quasi-injectivity; this establishes that continuity is a non-trivial concept. The need for the study of rings for which every continuous module is quasi-injective, arises also because of the fact that, for such rings, the existence and uniqueness of the continuous hull becomes obvious, namely, the quasi-injective hull.

In this chapter, we verify that, for large classes of rings - for example, commutative noetherian ones, and commutative semiprimary ones with $J^2 = 0$, this property holds true. We also give some characterizations for commutative chain rings, commutative perfect rings, and finitary rings with the above property. A necessary and sufficient condition for uniform continuous modules to be quasi-injective is also given.

§2. Commutative Noetherian Rings

It is a known fact (due to Matlis [20]) that, if R is a commutative noetherian ring, then every indecomposable injective module

E corresponds to a prime ideal P, in a one to one correspondence such that $E \cong E(R/P)$.

<u>Remark 5.1</u>: In the above situation $E \cong E(R/P)$ is an $R_{\bar{P}}$ as well as an $\hat{R}_P$-module, where $\hat{R}_P$ is the completion of R_P. Furthermore, E is artinian as an R_P-module.

<u>Lemma 5.2</u>: *Let* $E = E(R/P)$ *be an indecomposable injective module. Then* $X \subset E$ *is quasi-injective if and only if* X *is an* R_P-$(\hat{R}_{\bar{P}})$ *module.*

<u>Proof</u>: Matlis ([20], Theorem 3.7) shows that the endomorphism ring of E is $\hat{R}_P$. Therefore, and since X is quasi-injective if and only if it is closed under all endomorphisms of its injective hull, it follows that X is quasi-injective if and only if it is an $\hat{R}_P$-submodule of E. As every element of E is annihilated by some power of the prime ideal P ([20], Theorem 3.4), the $\hat{R}_P$-submodules of E are the same as the R_P-submodules. □

Next, we prove the following result about uniform continuous modules over a commutative, noetherian ring.

<u>Proposition 5.3</u>: *Let* R *be a commutative, noetherian ring. Let* $E = E(R/P)$ *be an indecomposable injective* R*-module. Then any continuous module* $X \subset E$ *is quasi-injective.*

Proof: Let $X \subset E$ be continuous. It would be sufficient, in view of the above lemma, to show that X is an R_P-module. Let $s = \frac{a}{b} \in R_P$ be any nonzero element, where $0 \neq a \in R$ and $b \in R \backslash P$. Then $b \notin P$ implies that it is invertible in R_P. This implies that b acts monomorphically on X, and $X \cong Xb \subset X$ holds. But X is continuous, hence $X = Xb$. Consequently, $Xs = X\frac{a}{b} = (Xb)\frac{a}{b} = Xa \subset X$ holds. Therefore, X is an R_P-module, as claimed, and hence is quasi-injective. □

Corollary 5.4: *Let R be a commutative noetherian ring. Let X and Y be continuous submodules of $E(R/P)$. Then there exists a monomorphism $f: X \rightarrowtail Y$ if and only if $X \subset Y$ holds.*

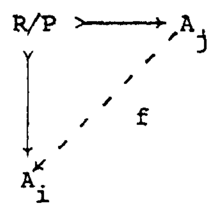
Proof: Let $f: X \rightarrowtail Y$ be a given monomorphism. By the above proposition, X is quasi-injective, being continuous. Therefore, X is invariant under the extension of f to $E(R/P)$. This implies that $f(X) \subset X$ holds. However, as X is indecomposable and continuous, $f(X) = X$. On the otherhand, $f(X) \subset Y$ holds by hypothesis. Therefore, $X \subset Y$ holds true.

The converse is obvious. □

Lemma 5.5: *Let R be a commutative, noetherian ring. Let A_i, A_j be submodules of $E(R/P)$, such that $A_i \oplus A_j$ is continuous; then $A_i = A_j$.*

Proof: We consider the following two possibilities:

Case I. P is a maximal ideal: Let P be maximal, then $E(R/P)$ is artinian. Hence, $A_i, A_j \subset E(R/P)$ implies that $Soc(A_i) \cong R/P \cong Soc(A_j)$. The following diagram commutes



(f exists, since A_i is A_j-injective, as $A_i \oplus A_j$ is continuous.)

The above $f: A_j \to A_i$, is a monomorphism, since $f|_{R/P}: R/P \rightarrowtail A_j$ is an essential monomorphism. Thus, by corollary, $A_j \subset A_i$ holds. Similarly, $A_i \subset A_j$ holds and this implies $A_i = A_j$ as required.

Case II. P is not maximal: Let P be a prime ideal, not necessarily maximal. Then we localize R at P, so that PR_p is the maximal ideal of R_p. We also know that $E(R/P)$ is isomorphic to $E(R_p/PR_p)$. Further, as A_i, A_j are continuous submodules of $E(R/P)$, they are quasi-injective (by Proposition 5.3), and hence R_p-submodules of the R_p-module $E(R_p/PR_p)$. Thus this case reduces to Case I and hence $A_i = A_j$. □

Finally, we come to the main theorem of this section.

Theorem 5.6: *Let R be a commutative, noetherian ring. Then every continuous R-module is quasi-injective.*

Proof: Let M be any continuous R-module. Then by Corollary 2.8 (Chapter II), one obtains $M = \sum_{i\in I} \oplus A_i$, where the A_i are continuous, indecomposable, and A_j-injective for all $i \neq j$. Again, since each $A_i \subset E(R/P)$, for some prime ideal P, we obtain by Proposition 5.3, that A_i is quasi-injective for all $i \in I$.

$E(M) = E(\sum_{i\in I} \oplus A_i) = \sum_{i\in I} \oplus E(A_i)$ holds, as R is noetherian.

Next, let $\phi \in Hom_R(E(M),E(M))$; we show that $\phi(M) \subset M$ holds: As $\phi \in Hom(E(M),E(M)) = Hom(\sum_{i\in I} \oplus E(A_i), \sum_{j\in J} \oplus E(A_j))$, $\phi = (\phi_{ij})$, where $\phi_{ij}: E(A_j) \to E(A_i)$. Thus, $\phi|_{A_j}$ maps A_j into A_i, because A_i is A_j-injective for all $i,j \in I$. That is, $\phi_{ij}(A_j) \subset A_i$ holds. Now, let $m \in M = \sum_{i\in I} \oplus A_i$ be any element. Then $m = \sum_{i=1}^{n} a_i$, for some n, (say), where $a_i \in A_{k_i}$.

$$\phi(m) = \phi(\sum_{i=1}^{n} a_i) = (\phi_{ij})(\sum_{i=1}^{n} a_i) = ((\sum_j \phi_{ij} a_j)_i)$$

But by above $\sum \phi_{ij} a_j \in A_{k_i}$, for all i, and hence $\phi(m) \in \sum_{i=1}^{n} \oplus A_{k_i} \subset M$. This shows that M is invariant under any endomorphism of its injective hull; and therefore, is quasi-injective. □

§3. Rings With Condition(*).

Definition 5.7: We say that a ring R satisfies *condition(*)* if $Q(\bar{R})$ is self-injective for all uniform factorrings $\bar{R}$ of R, where $Q(\bar{R})$ is the total quotient ring of $\bar{R}$.

We will show, in this section, that this condition is necessary and sufficient for every uniform continuous R-module to be quasi-injective. We also show that for some classes of rings with condition(*) every continuous module is quasi-injective. First, we prove the following related proposition in this direction.

Proposition 5.8: *Let E be a uniform, injective module over a commutative ring R. Then the following are equivalent:*

1) Every continuous submodule of E is quasi-injective.

2) $(\mathrm{Endo}_R E)e = eQ$, for all $e \in E$, where

$$Q = \{\tfrac{a}{b} \mid a,b \in R,\ b \text{ acts regularly on } E,\ \text{i.e. } b \in \mathfrak{C}(E)\}.$$

Proof: (1) $\Rightarrow$ (2): Assume (1) holds, and let $e \in E$. Then $eQ = Q(eR) = \{\frac{er}{b} \mid er \in eR,\ b \in \mathfrak{C}(eR) = \mathfrak{C}(E)\}$: Indeed, $eq \in eQ$, implies $eq = e\frac{a}{b} = \frac{ea}{b} \in Q(eR)$, and conversely $x \in Q(eR)$ implies $x = \frac{er}{b}$, $b \in \mathfrak{C}(E)$, hence $x = e\frac{r}{b} \in eQ(R)$. But as eR is uniform, cyclic, by Theorem 4.20 (Chapter IV) the continuous hull of eR exists and is equal to $Q(eR)$. Thus eQ is continuous and hence quasi-injective. This implies that $\phi(eQ) \subset eQ$, for all $\phi \in \mathrm{Endo}_R E$. Consequently, $\phi(e) = eq$, for some $q \in Q$, and so (2) holds true.

(2) $\Rightarrow$ (1): Let A be any arbitrary continuous submodule of E. Therefore, A is continuous and uniform. This implies $A\frac{1}{b} \subset A$, for all $b \in \mathfrak{C}(A) = \mathfrak{C}(E)$. Indeed, Ab is isomorphic to A, and $Ab \subset A$, thus one obtains $Ab = A$. Hence, for each element $a \in A$, there exists an element $a' \in A$, with $a'b = a$, thus $a' = \frac{a}{b} \in A$. This implies that $A\frac{1}{b} \subset A$ holds, and therefore, A is a Q-module.

Next, let $\phi \in \mathrm{Endo}_R E$ be any element and $a \in A \subset E$. Then by (2) $\phi(a) = eq$, for some $q \in Q$. But since A is a Q-module, this yields that $\phi(A) \subset A$, hence A is quasi-injective. □

Remark 5.9: The ring $Q = \{\frac{a}{b} | a,b \in R,\ b \in \mathfrak{C}(E)\}$ in the above proposition is a local ring with the unique maximal ideal $\mathfrak{M}(Q) = \{\frac{a}{b} \in Q | a \notin \mathfrak{C}(E)\}$.

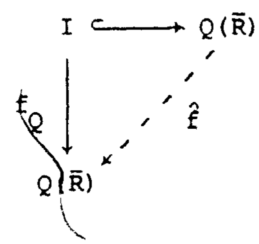
The existence of continuous hulls for uniform cyclic modules over commutative rings, established in Chapter IV (Theorem 4.20), is again used in the main theorem of this section, which provides a useful criterion for any uniform continuous R-module to be quasi-injective.

Theorem 5.10: *Let R be a commutative ring. Then the following are equivalent:*

(1) Every uniform continuous R-module is quasi-injective.

(2) R satisfies the condition(*).

Proof: (1) ⇒ (2): $Q = Q(\bar{R})$, the total quotient ring of the uniform factorring $\bar{R}$ of R, is the continuous hull of $\bar{R}_R$, as $\bar{R}_R$ is a uniform cyclic R-module. Then (1) implies that $Q(\bar{R})$ is quasi-injective as an R-module. Let I be an ideal of Q. Then every R-homomorphism, and hence every Q-homomorphism $f: I \to Q(\bar{R})$, can be extended to an R-homomorphism $\hat{f}: Q(\bar{R}) \to Q(\bar{R})$ such that the diagram

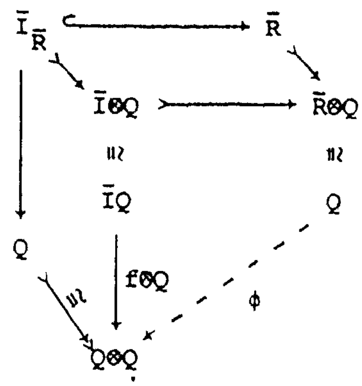


commutes. But the R-homomorphism $\hat{f}: Q(\bar{R}) \to Q(\bar{R})$ is trivially a Q-homomorphism. Hence $Q(\bar{R})$ is self-injective, for all uniform factorrings $\bar{R}$ of R. Thus R satisfies condition(*).

(2) ➡ (1): Let R satisfy condition(*). Let $xR = R/x^{\circ} = \bar{R}$ be a uniform cyclic R-module. Then $Q(\bar{R})$, the continuous hull of $\bar{R}_R$ is self-injective.

Claim: Q is quasi-injective as an R-module:

Proof of Claim: For this, it will be enough to show that Q is injective as an $\bar{R}$-module. Let $f: \bar{I}_{\bar{R}} \to Q_{\bar{R}}$ be an $\bar{R}$-homomorphism from an ideal $\bar{I}$ of $\bar{R}$ into Q. Since $Q(\bar{R})$ is the localization of $\bar{R}$ at the multiplicative set $\mathfrak{C}(\bar{0})$, of regular elements of $\bar{R}$ (which, by commutativity, is trivially a reversible, Ore set), it is well-known that Q becomes $\bar{R}$-flat and $Q \otimes_{\bar{R}} Q \cong Q$ holds. Then, as Q is $\bar{R}$-flat, we obtain that the map $\bar{I}Q \cong \bar{I} \otimes Q \rightarrowtail \bar{R} \otimes Q \cong Q$ is a monomorphism. The maps $\bar{I} \rightarrowtail \bar{I} \otimes Q$ and $\bar{R} \rightarrowtail \bar{R} \otimes Q$ are defined naturally. Define $f \otimes Q: \bar{I} \otimes Q \to Q \otimes Q$ in the usual way. Then $f \otimes Q$ is a Q-map and hence there exists a homomorphism $\phi: Q \to Q \otimes Q$, such that the following diagram commutes.



The composition of the monomorphism $\bar{R} \rightarrowtail \bar{R} \otimes Q$, with ϕ and the inverse isomorphism from $Q \otimes Q$ to Q, gives us the required extension of f. Consequently, $Q(\bar{R})$ is $\bar{R}$-injective, and hence quasi-injective as an R-module. This proves the claim.

Next, take an arbitrary uniform continuous R-module A. We show that A is quasi-injective. Let $x \in A \subset E(A)$; and $\phi \in \mathrm{Hom}_R(E(A), E(A))$ be arbitrary. It would be sufficient for us to show that $Q(xR) \subset A$ holds for all $x \in A$. Since then, $\phi(Q(xR)) \subset Q(xR) \subset A$ holds, as $Q(xR)$ is quasi-injective as R-module by the above argument.

Now we have $Q(xR) = \{\frac{xr}{s} | r,s \in R,\ s \in \mathfrak{C}(xR)\}$. But since A is uniform, $xR \subset' A$, and hence $\mathfrak{C}(xR) = \mathfrak{C}(A)$ holds. The continuous R-module A becomes a $Q(R)$-module, by the arguments used in Proposition 5.8. Then $Q(xR) = xQ(R) \subset AQ(R) \subset A$. Consequently, $Q(xR) \subset A$ holds for all $x \in A$. One, therefore, concludes that A is quasi-injective. □

<u>Definition 5.11</u>: A ring R is called *finitary* if every cyclic R-module is of finite uniform dimension.

Theorem 5.12: *Let R be a commutative finitary ring. Then every continuous R-module is quasi-injective if and only if R satisfies the condition(*).*

Proof: Let R satisfy the condition(*). Let M be a continuous R-module and $x \in M$ be arbitrary. Then xR is of finite uniform dimension; and $E(xR) \subset E(M)$ implies that $E(M) = E_0 \oplus E(xR)$, for some $E_0 \subset E$. But as xR has finite uniform dimension, one obtains $E(xR) = \sum_{i=1}^{n} \oplus E_i$, for some n, where E_i is uniform for each i. Therefore, $E(M) = E_0 \oplus \sum_{i=1}^{n} \oplus E_i = \sum_{i=0}^{n} \oplus E_i$, holds. By continuity of M, one obtains $M = \sum_{i=0}^{n} \oplus (M \cap E_i) = \sum_{i=0}^{n} \oplus M_i$, where $M_i = M \cap E_i$, for each i. Note that the above implies that M_i is M_j-injective for all $j \neq i$, and since each M_i, $i \neq 0$, is uniform and continuous, and therefore, is quasi-injective by condition(*). Next, let $\phi: E(M) \to E(M)$ be any homomorphism, then $\phi = (\phi_{ij})$, where $\phi_{ij}: E_j \to E_i$, for all $i,j = 0,\ldots,n$. Now, $x = \sum_{k=0}^{n} x_k$, where $x_0 = 0$, $x_i \in M_i$, $i = 1,\ldots,n$. Since, $x \in xR \subset \sum_{i=1}^{n} \oplus M_i$. Then

$$\phi(x) = (\phi_{ij})_{i,j=0}^{n} \left(\sum_{k=0}^{n} x_k\right) = \left(\sum_{k=0}^{n} \phi_{ik}(x_k)\right)_{i=0}^{n}$$

holds. However, we have $\phi_{ik}(x_k) \in M_i$, since M_i is M_k-injective for all i, k, except if $i = k = 0$; and $\phi_{00}(x_0) = \phi_{00}(0) = 0 \in M_0$. This implies that $\phi(x) \in M$. Consequently, M is quasi-injective. □

Remark 5.13: Note that in view of Proposition 5.3, a commutative noetherian ring automatically satisfies condition(*). Moreover, every noetherian ring (even every ring with Krull dimension) is finitary. The above theorem, therefore, gives a generalization of Theorem 5.6.

Next, we attempt to look at continuous modules over arbitrary commutative rings satisfying the condition(*). We know, by the decomposition theorem (2.5), that every continuous module M can be decomposed as $M = M_1 \oplus M_2$, with $M_1 \supset' \sum_{i \in I} \oplus A_i$, where the A_i are indecomposable summands, such that every finite direct sum of them is still a summand; and M_2 has no non-zero uniform submodule.

If the ring R satisfies the condition(*), each uniform continuous module, and hence, each A_i is quasi-injective. We show by an example at the end of the chapter that condition(*) is not sufficient to ensure that M_1 is quasi-injective. However, if we put a restriction on the 'types' of the A_i, then we do obtain a result in this direction.

Definition 5.14: We regard the modules X and Y to be of the *same type* (denoted $X \sim Y$) if $E(X) \cong E(Y)$.

The following well-known result ([2], Exercise 17, P.191) is stated without proof, for future reference.

Lemma 5.15: *Let* $M = \sum_{i=1}^{n} \oplus M_i$. *Then* M *is quasi-injective if and only if* M_i *is* M_j*-injective for all* i, j.

Theorem 5.16: *Let R be a commutative ring with condition(*). Let M be a continuous R-module with $M \supset' \sum_{i \in I} \oplus A_i$, where each A_i is continuous, indecomposable, and A_j-injective for all $j \neq i$; further let there be only a finite number of types of the A_i. Then M is quasi-injective.*

Remark 5.17: This theorem applies in particular to M_1, in the above mentioned decomposition theorem, provided the number of types of the A_i is finite. This is true since each $A_i \oplus A_j$ is a summand of M_1, and therefore, A_i is A_j-injective for all $j \neq i$.

Proof of the theorem: Since there are only a finite number of types of the A_i, we can partition the index set I as $I = I_1 \dot\cup I_2 \dot\cup I_3 \dot\cup \ldots \dot\cup I_n$, such that $E(A_i) \cong E(A_j)$ for all $i,j \in I_k$. Then $E(M) = E(\sum_{i \in I} \oplus A_i) = E(\sum_{i \in I_1} \oplus A_i) \oplus \ldots \oplus E(\sum_{i \in I_n} \oplus A_i)$. Continuity of M yields $M = [M \cap E(\sum_{i \in I_1} \oplus A_i)] \oplus \ldots \oplus [M \cap E(\sum_{i \in I_n} \oplus A_i)]$.

One can easily check that for any two modules X_i and X_j whenever X_i is X_j-injective for $i \neq j$, then $E(X_i) \cong E(X_j)$ implies that $X_i \cong X_j$. (In fact, the isomorphism of the injective hulls induces the required isomorphism.) Therefore, since for all $i,j \in I_k$, $E(A_i) \cong E(A_j)$ holds, and since A_i is A_j-injective for all $i \neq j$, it follows that $A_i \cong A_j$. Hence, all A_i belonging to the same type are isomorphic. Also, note that each A_i, being continuous and uniform, is quasi-injective, by the assumption condition(*) of the theorem.

Next, consider the summand $[M \cap E(\sum_{i\in I_k} \oplus A_i)]$ of the module M. There are two possibilities:

<u>Case I</u>. I_k is infinite: Then we can partition I_k as $I_k = I_k' \dot{\cup} I_k''$, such that cardinality (I_k') = cardinality (I_k''). This implies that $\sum_{i\in I_k'} \oplus A_i \cong \sum_{i\in I_k''} \oplus A_i$, and hence $E(\sum_{i\in I_k'} \oplus A_i) \cong E(\sum_{i\in I_k''} \oplus A_i)$.

We can write

$[M \cap E(\sum_{i\in I_k} \oplus A_i)] = [M \cap E(\sum_{i\in I_k'} \oplus A_i)] \oplus [M \cap E(\sum_{i\in I_k''} \oplus A_i)]$. This is true since $M \cap E(\sum_{i\in I_k} \oplus A_i)$ is continuous, being a summand of M, and its injective hull is $E(\sum_{i\in I_k} \oplus A_i) = E(\sum_{i\in I_k'} \oplus A_i) \oplus E(\sum_{i\in I_k''} \oplus A_i)$, because $\sum_{i\in I_k} \oplus A_i \subset' M \cap E(\sum_{i\in I_k} \oplus A_i)$.

Now, since $E(\sum_{i\in I_k'} \oplus A_i) \cong E(\sum_{i\in I_k''} \oplus A_i)$ holds, and since $M \cap (\sum_{i\in I_k'} \oplus A_i)$ and $M \cap E(\sum_{i\in I_k''} \oplus A_i)$ are injective relative to each other, one obtains that $M \cap E(\sum_{i\in I_k'} \oplus A_i)$ is isomorphic to $M \cap E(\sum_{i\in I_k''} \oplus A_i)$, in view of the above observation. This implies that each of these summands is quasi-injective and hence $[M \cap E(\sum_{i\in I_k} \oplus A_i)]$ is quasi-injective by Lemma 5.15.

Case II. I_k is finite: Then $[M \cap E(\sum_{i \in I_k} \oplus A_i)]$ can be written as

$M \cap E(\sum_{i \in I_k} \oplus A_i) = M \cap [\sum_{i \in I_k} \oplus E(A_i)] = \sum_{i \in I_k} \oplus [M \cap E(A_i)]$. The last equality follows again by continuity of $M \cap E(\sum_{i \in I_k} \oplus A_i)$, and by the fact that its injective hull is equal to $E(\sum_{i \in I_k} \oplus A_i) = \sum_{i \in I_k} \oplus E(A_i)$, since $M \cap E(\sum_{i \in I_k} \oplus A_i)$ is essential over $\sum_{i \in I_k} \oplus A_i$.

Now, each $M \cap E(A_i)$ is uniform and continuous, hence quasi-injective by condition(*). Consequently, $M \cap E(\sum_{i \in I_k} \oplus A_i)$ is quasi-injective, being a finite direct sum of such modules, again by Lemma 5.15. Another application of Lemma 5.15 yields that the module M is quasi-injective. □

§4. Some Special Classes Of Rings.

In this section we characterize some special classes of rings for which every continuous module is quasi-injective. First we prove the following result for future reference.

Lemma 5.18: *Let R be a commutative local ring with a nonzero socle S. Then R coincides with $Q(R)$, the total quotient ring of R.*

Proof: Let J be the Jacobson radical, the unique maximal ideal of R. We know that J is given by the intersection of annihilators of simple modules. Now, let $0 \neq a \in R$, then: Either $a \in J$, in which case $aS = 0$, and hence a is not regular; or $a \notin J$, and then a is invertible as R is a local ring and all non-invertible elements lie in J. Thus, every regular element of R is invertible, which implies that $R = Q(R)$. □

Definition 5.19: An ideal I of R is said to be *completely meet irreducible* if $I = \cap C_i$, C_i ideals, implies $I = C_i$, for some i. In this case, $\bar{R} = R/I$ is called a *completely meet irreducible ring.*

Theorem 5.20: *Let R be a commutative perfect ring. Then the following are equivalent:*

1) *Every continuous R-module is quasi-injective.*
2) *Every uniform continuous R-module is quasi-injective.*
3) *All completely meet irreducible factorrings $\bar{R}$ of R are self-injective.*
4) *All completely meet irreducible factorrings $\bar{R}$ of R are artinian.*

Proof: A commutative perfect ring R is a finite direct sum of local rings; therefore, the category of all R-modules is the product of the categories of modules over these local rings. Consequently, we may, without loss of generality, assume that R itself is actually local. (1) $\Longleftrightarrow$ (2): One direction is obvious. For the converse, let every uniform continuous R-module be quasi-injective, and let M be any arbitrary continuous R-module. As we take R to be local, there exists only one isomorphism class of simple R-modules. Perfectness of

R yields that $E(M)$ is essential over its socle S, a direct sum of simple modules S_i, $i \in I$; we, therefore, obtain $E(M) = E(S) = E(\sum_{i \in I} \oplus S_i)$, where $S_i \cong S_j (\cong R/J)$ for all $i, j \in I$. This, then, is a special case of Theorem 5.16. Indeed, S_i is continuous and S_j-injective for all i, j, since S_i is simple and hence quasi-injective. Note that there is only one type of the S_i, as each $S_i \cong S_j$. Theorem 5.16, therefore, implies that M is quasi-injective.

(2) $\Longleftrightarrow$ (3). Let every uniform continuous R-module be quasi-injective. Let $\bar{R}$ be any completely meet irreducible factorring of R. $\bar{R}$ is uniform and has a simple socle. By the previous Lemma (5.18), it follows that $\bar{R} = Q(\bar{R})$; and hence $\bar{R}$ is continuous, since $Q(\bar{R})$ is the continuous hull of uniform cyclic R-module $\bar{R}$, by Theorem 4.20. The assumption implies that $\bar{R}$ is quasi-injective as an R-module, and hence injective as an $\bar{R}$-module. Therefore, $\bar{R}$ is self-injective.

Conversely, due to Theorem 5.10, to show that every uniform continuous R-module is quasi-injective, it is enough to show that condition(*) holds, that is, that $Q(\bar{R})$ is self-injective for all uniform factorrings $\bar{R}$.

Let $\bar{R}$ be any uniform factorring of R. Then, since R is perfect, $\bar{R}$ has a nonzero socle and therefore, $\bar{R}$ is completely meet irreducible. By the assumption (3) and the Lemma 5.18, $Q(\bar{R}) = \bar{R}$ is self-injective for all completely meet irreducible factorrings $\bar{R}$, and therefore, for all uniform factorrings $\bar{R}$. This shows that (2) holds.

(3) $\Longleftrightarrow$ (4): Let $\bar{R}$ be self-injective, for every completely meet irreducible factorring $\bar{R}$ of R. Since $\bar{R}$ is self-injective and perfect, it is quasi-Frobenius. Therefore, $\bar{R}$ is artinian. Conversely,

if $\bar{R}$ is artinian and completely meet irreducible, then, by Vamos ([33], Exercise 6.1, P.184), it follows that $\bar{R}$ is self-injective. □

Corollary 5.21: *Let R be a commutative perfect ring with $J^2 = 0$, where J is the Jacobson radical of R. Then every continuous R-module is quasi-injective.*

Proof: Let $\bar{R} = R/I$ be any completely meet irreducible factorring of R. Perfectness of $\bar{R}$ implies $\bar{R} = \sum_{i=1}^{n} \oplus \bar{R}_i$, where each $\bar{R}_i$ is a local ring. But since $\bar{R}$ is completely meet irreducible, it has a simple socle and therefore, $\bar{R} = \bar{R}_j$ for some j. Thus $\bar{R} = \frac{R}{I}$ becomes local; we show that it is artinian.

Let M be the maximal ideal of R containing the ideal I. We claim that $\bar{M} = \frac{M}{I}$ is semisimple: We observe that $R/J+I$ is simple, because it is local and semisimple, being isomorphic to $\frac{R/I}{J+I/I} = \frac{\bar{R}}{\bar{J}}$ and to $\frac{R/J}{J+I/J}$, respectively. This yields that $M = J + I$. Consequently, $\bar{M} = \frac{M}{I} = \frac{J+I}{I} = \bar{J}$ holds; hence $\bar{M}\bar{J} = \bar{J}^2 = \overline{J^2} = \bar{0}$ implies that $\bar{M}$ is semisimple. Since $\bar{R}$ is completely meet irreducible, $\bar{M}$ is zero or simple. Therefore, the composition length of $\bar{R}$ is less than or equal to 2, and $\bar{R}$ is artinian. The above theorem yields that every continuous R-module is quasi-injective. □

Example 5.22: Consider the ring $R = \frac{F[x_n]}{\langle x_n x_m x_k \rangle}$, $n,m,k \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural members. Let I be the ideal $I=\langle x_n x_{n+1} - x_1 x_2, x_k x_\ell \rangle$, where $n,k,\ell \in \mathbb{N}$ and $|k - \ell| \neq 1$. Consider $\bar{R} = R/I$, then $\bar{R}$ is completely meet irreducible and has a simple socle. But $\bar{J}$ is infinitely generated by the $\bar{x}_n$, $n \in \mathbb{N}$, and thus $\bar{R}$ is not artinian. Thus by Theorem 5.18, there exists a continuous R-module which is not quasi-injective. Actually, since $\bar{R}$ as an R-module is local, uniform and completely meet irreducible by Lemma 5.18, $\bar{R} = Q(\bar{R})$. This implies that $\bar{R}$ is continuous as an R-module. However, $\bar{R}_R$ is not quasi-injective. Since, if so, then $\bar{R}$ is self-injective (as an $\bar{R}$-module), and hence quasi-Frobenius. But this implies that $\bar{R}$ is artinian, a contradiction.

Definition 5.23: A module M is called *linearly compact* if any family of cosets having the finite intersection property has nonempty intersection. Equivalently, M is linearly compact if every system of finitely solvable congruences $X \equiv X_i (\mathrm{mod} N_i)$, $i \in I$, N_i submodules of M, is solvable.

Definition 5.24: A commutative ring is said to be a *chain* ring (also called *valuation* ring, but not necessarily a domain), if the lattice of its ideals is linearly ordered.

Remark 5.25: A linearly compact chain ring is also known as a *maximal valuation* ring.

Definition 5.26: A chain ring R is *restricted linearly compact* (also known as an *almost maximal valuation ring*), if every proper epimorphic image $\bar{R}$ of R is linearly compact.

H. Boerner [6], proved some results, for rings, not necessarily commutative, whose commutative specializations are as follows:

Theorem 5.27 [6]: *Let* R *be a chain ring and* $Q = Q(R)$ *be the total quotient ring of* R. *Then the following are equivalent:*

i) Q *is self-injective.*

ii) R *is linearly compact with respect to annihilators.*

iii) Q *is linearly compact and* $Qa = Qa^{\circ\circ}$, *for all* $a \in Q$ *(where* $X^{\circ} = \{r \in R | Xr = 0\}$, *is the annihilator of* X*).*

Theorem 5.28 [6]: *For a commutative ring* R, *the following are equivalent:*

i) Uniform Dimension $(R) < \infty$, *and every indecomposable finitely generated R-module is uniserial.*

ii) Every finitely generated R-module is a direct sum of uniserial R-modules.

iii) R *is a finite direct sum of restricted linearly compact chain rings.*

Theorem 5.29 [6]: *A commutative ring* R *is restricted linearly compact if and only if every epimorphic image of* R *is linearly compact with respect to annihilators.*

Finally, we give a characterization for chain rings having the property that every continuous module is quasi-injective.

Theorem 5.30: *For a commutative chain ring R, the following are equivalent:*

1) R is a restricted linearly compact ring.

2) Every uniform continuous R-module is quasi-injective.

3) Every continuous R-module is quasi-injective.

Proof: (1) ⇒ (2): Let R be restricted linearly compact. Then every epimorphic image $\bar{R}$ of R is linearly compact with respect to annihilators, by Theorem 5.29. Consequently, Theorem 5.27 implies that $Q(\bar{R})$ is self-injective. Thus Q is injective as a Q-module, where $Q = Q(\bar{R})$. It follows by the claim in Theorem 5.10 that Q is quasi-injective as an R-module. Thus Theorem 5.10 also implies that every uniform continuous R-module is quasi-injective (since R satisfies condition(*)).

(1) ⇒ (3): Let R be a restricted linearly compact chain ring. Hence, by the above argument, every uniform continuous module is quasi-injective. Now, let M be any continuous module. By Theorem 5.28, one obtains that every finitely generated (and hence in particular, every cyclic) R-module is a finite direct sum of uniform modules. This, obviously, implies that R is a finitary ring. Consequently, application of Theorem 5.12 yields that M is quasi-injective.

(3) ⇒ (2) is obvious.

(2) ⇒ (1): Let us assume that every uniform continuous R-module is quasi-injective. Thus R satisfies condition(*). We wish to show that R is restricted linearly compact.

Claim: If every completely meet irreducible factorring $\bar{R}$ of R is linearly compact, then R is restricted linearly compact.

<u>Proof of claim</u>: Let $\{C_\alpha\}$, $\alpha \in I$, be all completely meet irreducible ideals of R. Then $0 = \bigcap_{\alpha \in I} C_\alpha$ holds. Now, if I is a nonzero ideal contained in all C_α then $I \subset \bigcap_{\alpha \in I} C_\alpha = 0$, a contradiction. Hence, there exists a completely meet irreducible ideal C_0, which is contained in I, since R is a chain ring. Therefore, R/C_0 is completely meet irreducible factor and $R/C_0 \twoheadrightarrow R/I$ is a well defined epimorphism. This proves that any proper $\bar{R}$ is linearly compact and hence R is restricted linearly compact.

Next, let $\bar{R}$ be any completely meet irreducible factorring of R. Then $\bar{R} = Q(\bar{R})$ by Lemma 5.18. However, since R satisfies condition(*), this yields that $(\bar{R} =)\ Q(\bar{R})$ is self-injective. Consequently, Theorem 5.27 (iii) implies that $\bar{R}$ is linearly compact. This proves that R is restricted linearly compact. □

§5. <u>An Example</u>

Recall Theorem 2.10, due to Utumi, which states that a regular ring R is right continuous if and only if there exists an overring T, which is self-injective, with $J(T) = 0$, and such that every idempotent of T lies in R. This will be a useful result in determining the continuous modules in the next example.

We now give a slight variation of Example 2.11, for the commutative case, which exhibits a ring R such that:

i) If M is a continuous R-module, with $M' \supset \sum_{i \in I} \oplus A_i$, where each A_i is an indecomposable summand of M and A_j-injective for $j \neq i$, then M need not be quasi-injective even if R satisfies the condition(*).

ii) R has lots of continuous modules but very few quasi-injectives. So this example provides a ring which, although it satisfies condition(*) - and therefore, has every uniform continuous module quasi-injective - is far from having every continuous module quasi-injective.

Example 5.30: Let $\{F_\alpha\}_{\alpha\in I}$ be a family of fields and let $P_\alpha \neq F_\alpha$ be proper subfield for each α. Put $S = \prod_{\alpha\in I} F_\alpha$ and let $R = \prod_{\alpha\in I}(F_\alpha, P_\alpha) = \{(x_\alpha) \in S \mid x_\alpha \in P_\alpha$ for all $\alpha \in I$, except for a finite number of $\alpha\}$. Note that S is the injective hull of R, as an R-module.

i) R, a regular subring of S, satisfies the condition(*): Since, if $\bar{R}$ is a uniform factor, then regularity of $\bar{R}$ implies that $\bar{R}$ is a field: Indeed, let $0 \neq \bar{a} \in \bar{R}$ be any nonzero element. Then there exists an element $\bar{b} \in \bar{R}$, such that $\bar{a}\bar{b}\bar{a} = \bar{a}$, thus $(\bar{a}\bar{b}-\bar{1})\bar{a} = 0$ holds. This implies that $\bar{a}\bar{R} \cap (\bar{a}\bar{b}-\bar{1})\bar{R} = \bar{0}$, hence uniformity of $\bar{R}$ yields that $(\bar{a}\bar{b}-\bar{1})\bar{R} = 0$, since otherwise $\bar{a} = 0$, a contradiction. Therefore, $\bar{a}\bar{b} = \bar{1}$, and hence $\bar{R}$ is a field. Consequently, $\bar{R} = Q(\bar{R})$ is self-injective for every uniform factorring $\bar{R}$.

R_R is a (right) continuous ring in S, by the previous mentioned theorem, as it contains all idempotents of S, and $J(S) = 0$. However, R_R is not quasi-injective, since R is not self-injective. The module R_R is essential over its socle $\bigoplus_{\alpha\in I} F_\alpha$, and each F_α is a summand of R. Further, each F_α is F_β-injective being injective ($\bigoplus_{\alpha\in I} F_\alpha$ is even quasi-injective), and $R \supset \bigoplus_{\alpha\in I} F_\alpha$ provides the desired example.

ii) We will show that every ideal M of R is continuous, and that all M such that $\underset{\alpha\in I}{\oplus} F_\alpha \subsetneqq M \subset R = \underset{\alpha\in I}{\Pi}(F_\alpha, P_\alpha)$ are continuous but not quasi-injective. (In fact, an ideal M is quasi-injective iff $M \subseteq \underset{\alpha\in I}{\oplus} F_\alpha$).

It is easy to observe that every ideal M of R is π-injective As all idempotents of the ring S lie in R, R contains all idempotents of every overring S' contained in S. That M is an R-ideal, implies, then, that it is invariant under all idempotents of the endomorphism ring of its injective hull, and hence, that it is π-injective.

Claim: There is a one to one, inclusion preserving correspondence between proper ideals M of R, and filters on the index set I.

Proof of Claim: For any proper ideal M of R, define a filter $\mathcal{F}_M$ on the index set I, by $\mathcal{F}_M = \{I_m : m \in M\}$, where $I_m = \{\alpha \in I \mid m_\alpha = 0\}$ is the zero set of m. That this forms a filter, is easy to check:

1) $I_m \neq \phi$ holds, since every element $m = (m_\alpha) \in M$ has at least one component equal to zero. Indeed, otherwise we obtain $(1,1,\ldots,1) = (m_\alpha)(m_\alpha^{-1}) \in mR \subset M$; and this contradicts that M is a proper ideal.

2) Let $I_m \in \mathcal{F}_M$, and let $I_m \subset J$. Consider $r = (r_\alpha) \in R$ such that $r_\alpha = 0$ if $\alpha \in J$, and $r_\alpha = 1$ if $\alpha \notin J$. Then $m' = mr$ is an element of M such that $m'_\alpha = 0$ precisely if $\alpha \in J$. This shows $J = I_{m'} \in \mathcal{F}_M$.

3) Let I_m and $I_{m'}$ be in $\mathcal{F}_M$. Each nonzero component m_α or m'_α can be replaced by $1 \in F_\alpha$, without loss of generality. Then,

trivially, we obtain $I_m \cap I_{m'} = I_{m+m'-mm'}$, where $m + m' - mm' \in M$.

Conversely, given a filter $\mathfrak{F}$ on the index set I, we can construct an ideal $M_{\mathfrak{F}}$ of R corresponding to the filter. Let $M_{\mathfrak{F}}$ be the collection of all elements of R, for which elements of $\mathfrak{F}$ are zero sets. We show that $M_{\mathfrak{F}}$ is an ideal: Let $x = (x_\alpha)$, $y = (y_\alpha) \in M_{\mathfrak{F}}$; then the zero set I_{x+y} of $x + y$ contains the intersection $I_x \cap I_y \in \mathfrak{F}$. Consequently, $I_{x+y} \in \mathfrak{F}$, and therefore, $x + y \in M_{\mathfrak{F}}$. Next, let $r = (r_\alpha) \in R$, then I_{xr} contains $I_x \in \mathfrak{F}$. Hence $I_{xr} \in \mathfrak{F}$, and so $xr \in M_{\mathfrak{F}}$. This shows that $M_{\mathfrak{F}}$ is an ideal of R.

It is easy to verify that this is a one to one correspondence between proper ideals of R and filters on the index set I. It can also be easily seen that $A \subset B$ iff $\mathfrak{F}_A \subset \mathfrak{F}_B$ for any two ideals A and B of R.

Now, to show the second condition of continuity for M, we establish an even stronger fact: If A and B are subideals of M, and if $A \cong B$, then $A = B$. Let ϕ be the isomorphism from A to B. Then ϕ can be extended to an endomorphism of the injective module $S = \prod_{\alpha \in I} F_\alpha$, and therefore, can be given by multiplication by an element $x = (x_\alpha) \in S$. For $a \in A$, we have $\phi(a) = ax = b$, for some $b \in B$. This implies that $I_a \subset I_{\phi(a)} = I_b$ holds. A similar application of ϕ^{-1} to the element b implies $I_b \subset I_a$. Hence, $I_a = I_b$ and therefore, the filters $\mathfrak{F}_A$ and $\mathfrak{F}_B$ are identical; consequently, $A = B$ holds.

In view of the above, it is evident that if A is isomorphic to a summand B of M, then $A = B$, and consequently, A is itself a summand of M. Thus, the second condition of continuity holds true.

Next, if M is such that $\bigoplus_{\alpha \in I} F_\alpha \subsetneqq M \subset R$ then M is not quasi-injective: Indeed, then there exists an $m = (m_\alpha) \in M$ such that $m \notin \bigoplus_{\alpha \in I} F_\alpha$, and hence the set $I \setminus I_m$ is infinite. We construct $x = (x_\alpha) \in S$, such that

$x_\alpha = 0$ if $m_\alpha = 0$; and $x_\alpha = \dfrac{a_\alpha}{m_\alpha}$ if $m_\alpha \neq 0$, where $a_\alpha \in F_\alpha \setminus P_\alpha$.

Then $x(m) = mx = (m_\alpha x_\alpha) \notin R = \prod_{\alpha \in I}(F_\alpha, P_\alpha)$, and hence $mx \notin M$. This shows that M is not invariant under the endomorphisms of its injective hull S. Hence M is not quasi-injective.

Note, on the other hand, that $\bigoplus_{\alpha \in I} F_\alpha$, and its subideals being semisimple, are quasi-injective.

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On the Existence of Continuous Hulls,

by

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ABSTRACT.

We discuss possible definitions for a continuous hull of a module, we prove existence for cyclic modules with uniform singular submodule over commutative rings, and we give an example of a module over a non-commutative ring which fails to have a continuous hull.

ON THE EXISTENCE OF CONTINUOUS HULLS

by

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1. Introduction. The injective hull $E(M)$ of a module M is, by definition, a minimal injective extension of M. Here, "minimal" can be taken to mean either that no intermediate module $M \subset X \subsetneq E(M)$ is injective, or that $E(M)$ is contained in any other injective extension of M up to an isomorphism over M. The injective hull is also characterized as a maximal essential extension; and it is unique up to isomorphism over M.

Analogous hulls exist for several generalizations of the concept of injectivity, namely for N-injectivity with respect to a module N, in particular quasi- (or self-) injectivity [8], for π-injectivity (or quasi-continuity) ([3], [6], [7]), and for $\underline{F}$-injectivity with respect to a torsion theory $\underline{F}$ [12]. These hulls are defined by the corresponding minimality properties. They are all submodules of the injective hull, and therefore essential over M. Moreover, they have a striking uniqueness property: as submodules of a fixed injective hull, they are absolutely unique, not only unique up to isomorphism.

Perhaps the most interesting generalization of the concept of injectivity, however, is that of continuity ([14], [7], [10], [11]): A module is called continuous if (i) every submodule is essential in a summand, and (ii) every submodule isomorphic to a summand is itself a summand.

Nothing seems to be known so far about the existence of continuous hulls, and not even a definition appears to have been proposed. We discuss three such definitions, and the obvious implications between them. We settle, for this paper, on the strongest one, and we prove existence for various cyclic modules over commutative rings. We also provide an example where no continuous hull exists.

All our rings have identity elements, and all modules are unitary right-modules. Homomorphism and endomorphism rings act on the left.

We recall, for the reader's convenience, that a module M is called π-injective [3] or quasi-continuous ([6], [7]), if for every direct sum of submodules $A \oplus B \subset M$, the projection $A \oplus B \to A$ extends to an endomorphism of M. Every continuous module is π-injective. Theorem 1.1 in [3] states that a module M is π-injective, if and only if M is closed under all idempotents of endo E(M), if and only if every decomposition $E(M) = \Sigma \oplus E_i$ induces a decomposition $M = \Sigma \oplus (M \cap E_i)$. As an immediate consequence, every π-injective module satisfies the condition (i) of the definition of a continuous module.

2. The Concept of a Continuous Hull.

DEFINITIONS. Let M be a module, with an injective hull E, and let H be a continuous overmodule of M.

(I) H is called a <u>type I continuous hull</u> of M, if $M \subset X \subset H$ for a continuous module X implies $X = H$.

(II) H is called a <u>type II continuous hull</u> of M, if for every continuous overmodule X of M, there exists a monomorphism $\mu : H \to X$ over M.

(III) H is called a <u>type</u> III <u>continuous</u> <u>hull</u> of M (in E), if $M \subset H \subset E$, and if $H \subset X$ holds for every continuous module $M \subset X \subset E$.

REMARK. The definition (III) is independent of the choice of E, in the following sense: if E' is another injective hull of M, and if $\varphi : E \to E'$ is an isomorphism over M, then $\varphi(H)$ is a type III continuous hull of M in E', and it is independent of the choice of φ.

We note that a type III continuous hull is, obviously, uniquely determined as a submodule of a fixed injective hull.

PROPOSITION 1. <u>Every type III continuous hull is type II, and every type II continuous hull is type I. All continuous hulls are essential over M. If a type II continuous hull exists, then it is isomorphic over M to every type I continuous hull.</u>

PROOF. Type III implies type II: Let $M \subset H \subset E$ be a type III continuous hull, and let $M \subset X$ be a continuous overmodule. As $E(X) = E(M) \oplus E'$ holds, with $E(M) \cong E$, and as X is π-injective, we have a decomposition $X = X_1 \oplus X_2$ with $X_1 = X \cap E(M)$ and $X_2 = X \cap E'$. Moreover, $E(X_1) = E(M)$ and $E(X_2) = E'$. If $\varphi: E(M) \to E$ is an isomorphism over M, then $\varphi(X_1) \cong X_1$ is continuous, hence contains H by the type III condition. Then, $\varphi^{-1}|H:H \to X_1 \subset X$ is the required monomorphism.

Type II implies type I: Let H be a type II continuous hull of M, and let $M \subset X \subset H$ be a continuous module. There is a decomposition $X = X_1 \oplus X_2$ such that M is essential in X_1. By the type II condition, there is a monomorphism $\mu: H \to X_1$ over M. The submodule $\mu(H) \cong H$ of H is a summand of H, hence of X_1. By essentiality, we conclude $\mu(H) = X_1$. Consequently, H is also essential over M, and $M \subset \mu(H) \subset X \subset H$ implies $\mu(H) = H$ hence $X = H$.

Essentiality of the continuous hulls: If $M \subset H$ is a type I continuous hull, then $E(H) = E(M) \oplus E'$ holds, and therefore $H = H_1 \oplus H_2$ with $H_1 = H \cap E(M)$ and $H_2 = H \cap E'$. Thus, we have $M \subset H_1 \subset H$, which implies that $H_1 = H$ and $E' = 0$.

Uniqueness up to isomorphism: If $M \subset H_i$ are two type i continuous hulls (i = I, II), then there is a monomorphism μ: $H_{II} \to H_I$ over M. Since $\mu(H_{II}) \cong H_{II}$ is continuous, and since $M \subset \mu(H_{II}) \subset H_I$, we conclude $\mu(H_{II}) = H_I$. This completes the proof of the proposition.

We have, at present, no example of a module over a commutative ring, which fails to have a continuous hull. We mention that we have tried to determine those commutative rings for which every continuous module is quasi-injective, and that we have verified this property for a large number of rings, including all noetherian ones, and all semiprimary ones of radical square zero [11]. Thus, continuous modules and hulls become nontrivial only for fairly big commutative rings.

In Section 4, we shall provide an example of a module, over a non-commutative ring, which has no continuous hull of type II of III. This module possesses a continuous hull of type I, but it exhibits rather pathological behavior. We doubt, therefore, that the type I continuous hull will be a useful concept.

Because of this impression, and since we can prove the type III property whenever we can prove the existence of a continuous hull of any type, in a general situation, we find it convenient to refer here to type III continuous hulls simply as continuous hulls.

3. Existence Theorems. Our first goal is the existence of continuous hulls for non-singular cyclic modules. For a later application, we state the next lemma in a slightly more general form.

LEMMA 2. Let M be a non-singular cyclic module, over a commutative ring R. Let E be an injective hull of M, and let T be a subring of the endomorphism ring of E. Then, $E \cdot \mathrm{ann}_R(M) = 0$, and there exists a continuous module $M \subset X \subset E$ with $TX \subset X$, which is contained in any other continuous module Y with $M \subset Y \subset E$ and $TY \subset Y$.

COROLLARY 3. Every non-singular cyclic module over a commutative ring has a continuous hull.

PROOF OF THE COROLLARY. This is just the special case $T = R/\mathrm{ann}_R(E)$ of the lemma.

PROOF OF THE LEMMA. With $I = \mathrm{ann}_R(M)$, we have $M \cong \overline{R} = R/I$, as R-modules. It is well known, and easily checked, that $\overline{R}$ is also non-singular as $\overline{R}$-module. Moreover, the injective hulls of $\overline{R}$ as R- and as $\overline{R}$-module coincide: indeed, if we denote the former by E, then the latter is $\mathrm{ann}_E(I)$. And $e \in E$ implies $eL \subset \overline{R} \subset \mathrm{ann}_E(I)$, for some large ideal L of R. Hence, $0 = eLI = eIL$ holds, and therefore $eI \subset Z(E_R) = 0$, demonstrating $EI = 0$.

Since $\overline{R}$ is non-singular as $\overline{R}$-module, its injective hull E coincides with the maximal quotient ring, which is (von Neumann) regular and commutative. Moreover, this ring E can be identified with endo(E) via left-multiplication, and then it contains T as a subring. A second subring of E is $\overline{R}$, and a third one is the subring P generated by all the idempotents of E.

We claim that any regular ring $\overline{R}P \subset A \subset E$ is continuous as R- (or equivalently as $\overline{R}$-) module. The condition (i) follows immediately from $\overline{R}P \subset A$. For the condition (ii), we have to consider an R-submodule N of A, an R-decomposition $A = A_1 \oplus A_2$, and an R-isomorphism $\varphi: A_1 \to N$. Since A sits inside the maximal quotient ring E, any R-homomorphism between A-submodules of A is automatically an A-homomorphism. This observation, applied to the projections $A \to A_i$ viewed as maps $A \to A_iA$, yields $A_iA = A$. Then, its application to $\varphi: A_1 \to NA$ shows $NA = N$. We conclude that N is a principal ideal of the regular ring A, and therefore a summand. This proves our claim.

Next, we recall that in any commutative regular ring, for each element a there exists a unique element b with $aba = a$ and $bab = b$ (cf. [9], Exercise 3 on p. 36). This fact implies immediately that the intersection of any family of regular subrings of a commutative regular ring is again regular. In particular, there exists a unique smallest regular ring X with $\overline{R}PT \subset X \subset E$.

By the preceding consideration, the ring X is a continuous R-module. It satisfies $\overline{R} \subset X \subset E$ and $TX \subset X$; the latter since $T \subset \overline{R}PT \subset X$ holds. We have to compare X with any other continuous module Y satisfying $\bar{R} \subset Y \subset E$ and $TY \subset Y$. We define the subring $B = \{b \in E : bY \subset Y\}$ of E. Clearly, $\bar{R} \subset B$ and $T \subset B$ hold. The continuity, hence π-injectivity of Y, together with $E(Y) = E$, implies $P \subset B$; and we conclude $\bar{R}PT \subset B \subset E$. If we show that B is regular, we can deduce $X \subset B$, and consequently $X = X\bar{R} \subset BY \subset Y$, as stated in the lemma.

To this purpose, we consider any $b \in B$. There exists $e \in E$ with $beb = b$ and $ebe = e$. The element eb is idempotent hence contained in P, and ebY is a summand of Y. Multiplication by e defines an isomorphism from the submodule bY to the summand ebY of Y, since $eby = 0$ implies $0 = beby = by$. Therefore, there exists another idempotent $f \in P$ with $bY = fY$, and $Y = fY \oplus (1-f)Y$.
We obtain

$$eY = efY + e(1-f)Y = efY + ebe(1-f)Y = ebY + e^2(1-f)fy \subset Y.$$

This inclusion demonstrates $e \in B$, and shows that B is regular, as required.

REMARKS. We note that, according to our proof, the continuous hull of a non-singular cyclic module over a commutative ring is a (regular commutative) ring.

We also mention that our reasoning admits the following, somewhat disappointing, generalization to non-commutative right-non-singular rings R: there exists a regular overring which is continuous as an R-right-module, and which is contained in any right-continuous R-bimodule between R and the right-injective hull (which coincides with the maximal right-quotient ring). To show this fact, one uses that every right-selfinjective regular ring is the product of a strongly regular ring, and a regular ring which is generated by its idempotents ([13]), Theorem 3.2); for details see [11].

Our next aim is to reduce the existence problem for the continuous hull of an arbitrary cyclic module, to that for the continuous hull of a certain (not necessarily cyclic) singular module. Beforehand, we have to introduce some notation, and to prove an auxiliary lemma.

$Z^*(M)$ will denote the radical of the Goldie Torsion Theory, that is the second iterated singular submodule of M, which is explicitly defined by $Z^*(M)/Z(M) = Z(M/Z(M))$, where $Z(M)$ is the usual singular submodule. We call M <u>singular</u> if $Z^*(M) = M$ [4].

We recall that any injective module E has a decomposition $E = E_1 \oplus E_2$, where $E_1 = Z^*(E)$ and E_2 is non-singular ([1], [5]). It follows at once that any π-injective module X has a similar decomposition $X = X_1 \oplus X_2$ into a singular and a non-singular part; indeed if $E(X) = E_1 \oplus E_2$ as above, then $X_i = X \cap E_i$.

LEMMA 4. *If X is π-injective, and if these summands X_1 and X_2 are both continuous, then X is continuous.*

PROOF. Condition (i) follows immediately from π-injectivity. To verify condition (ii), we have to show that every monomorphism f from a summand B of X into X splits.

Such a summand B is also π-injective, hence has also a decomposition $B = B_1 \oplus B_2$ into a singular and a non-singular part. Both B_1 and B_2 are obviously summands of X. From $B_1 = Z^*(B_1) \subset Z^*(X) = X_1$, we obtain that B_1 is a summand of X_1, and that $f(B_1)$ is contained in X_1. Since $f|B_1$ is a monomorphism from B_1 into X_1, and since X_1 is continuous, we conclude that $X_1 = f(B_1) \oplus U_1$.

Let π_i be the projection $X \to X_i$. Then, $\pi_2 f|B_2 : B_2 \to X_2$ is a monomorphism, since $f(B_2) \cap \ker \pi_2 = f(B_2) \cap X_1 = 0$, because $f(B_2) \cong B_2$ is non-singular and X_1 is singular. Similarly, for the decomposition $E(X) = E_1 \oplus E_2$, we obtain $E_1 \cap B_2 = 0$. Hence, we have the homomorphism $1 \oplus 0 : E_1 \oplus B_2 \to E_1$, which extends by injectivity to a map $\varphi: E(X) \to E_1$. We verify immediately that $E(X) = E_1 \oplus \ker\varphi$ and $\ker \varphi \supset B_2$.

This induces a second decomposition $X = X_1' \oplus X_2'$, where $X_1' = X \cap E_1 = X_1$ and $X_2' = X \cap \ker \varphi \supset B_2$. Thus, B_2 is a summand of X_2'. With the natural isomorphism $X_2 \cong X/X_1 \cong X_2'$, we obtain a monomorphism $B_2 \to X_2 \cong X_2'$, which splits since X_2' is continuous. This yields $X_2 = \pi_2 f(B_2) \oplus U_2$, and we obtain $X = f(B_1) \oplus U_1 \oplus f(B_2) \oplus U_2 = f(B) \oplus U_1 \oplus U_2$. This completes the proof.

We consider now an arbitrary cyclic module $M \cong \overline{R} = R/\mathrm{ann}_R(M)$ over a commutative ring R, and we fix the following notation: The injective hull of $\overline{R}$ as R-module is $E = E_1 \oplus E_2$, decomposed into singular and non-singular part. $\overline{1} = e_1 + e_2$ is the corresponding decomposition of the generator $\overline{1}$ of $\overline{R}$.

We observe that E_i is the injective hull of e_iR $(i=1,2)$. A simple computation shows that $I = \mathrm{ann}_R(e_2)$ is given by $I/\mathrm{ann}_R(M) = Z^*(\overline{R})$. The submodule $e_1R + \mathrm{ann}_{E_1}(I)$ of E_1 appears to depend on the choice of the decomposition $E = E_1 \oplus E_2$, but actually it does not: if $E = E_1 \oplus E_2'$ is a second decomposition, with $\overline{1} = e_1' + e_2'$, then $\mathrm{ann}_R(e_2') = I$ and $e_1 - e_1' = e_2' - e_2 \in E_1 \cap \mathrm{ann}_E(I) = \mathrm{ann}_{E_1}(I)$, hence $e_1R + \mathrm{ann}_{E_1}(I) = e_1'R + \mathrm{ann}_{E_1}(I)$.

PROPOSITION 5. <u>A cyclic module</u> M <u>over a commutative ring</u> R <u>has a continuous hull if and only if the module</u> $e_1R + \mathrm{ann}_{E_1}(I)$ <u>has a continuous hull</u>.

PROOF. We apply Lemma 2 to the non-singular cyclic module e_2R, with injective hull E_2, and to the subring T of $\mathrm{endo}(E_2)$ generated by all $\pi_2\pi|E_2$, where π_i denotes the projection from E to E_i, and where π runs over all idempotents of $\mathrm{endo}(E)$. We obtain $E_2I = 0$, and the existence of a continuous module $e_2R \subset X_2 \subset E_2$ with $TX_2 \subset X_2$, which is contained in any other continuous module with the same properties.

We now assume that there exists a continuous hull $e_1R + \mathrm{ann}_{E_1}(I) \subset X_1 \subset E_1$. Lemma 4 shows that $X = X_1 \oplus X_2$ is continuous, provided it is π-injective. Therefore, we consider any $\pi = \pi^2 \in \mathrm{endo}(E)$, observing that $E = E(X)$. From $E_1 = Z^*(E)$ we obtain $\pi|E_1 \in \mathrm{endo}(E_1)$, and the continuity of X_1 implies $\pi(X_1) \subset X_1$. Any $\varphi \in \mathrm{hom}_R(E_2,E_1)$, and in particular $\varphi = \pi_1\pi|E_2$, maps X_2 into X_1, since $\varphi(X_2)I = \varphi(X_2I) \subset \varphi(E_2I) = \varphi(0) = 0$, hence $\varphi(X_2) \subset \mathrm{ann}_{E_1}(I) \subset X_1$. As $\pi_2\pi|E_2$ belongs to T, we have $\pi_2\pi(X_2) \subset TX_2 \subset X_2$. Together, we compute $\pi(X) = \pi(X_1) + \pi_1\pi(X_2) + \pi_2\pi(X_2) \subset X_1 + X_2 = X$. This demonstrates the desired π-injectivity of X.

We claim that X is a continuous hull of $\overline{R}$. For any continuous module $\overline{R} \subset Y \subset E = E_1 \oplus E_2$, we have the decomposition $Y = Y_1 \oplus Y_2$ with $Y_i = Y \cap E_i$. Moreover, we get $e_i = \pi_i(\overline{1}) \in \pi_i(Y) = Y_i$ hence $e_i R \subset Y_i$. As Y is π-injective, hence closed under projections of $E = E(Y)$, we deduce $\pi_2\pi(Y_2) \subset \pi_2\pi(Y) \subset \pi_2(Y) \subset Y \cap E_2 = Y_2$, hence $TY_2 \subset Y_2$. Thus, Y_2 is in competition with X_2, and we conclude $X_2 \subset Y_2$.

Next, we consider an arbitrary element $a \in \mathrm{ann}_{E_1}(I)$. We have $aI = 0$, hence $\mathrm{ann}_R(a) \supset I = \mathrm{ann}_R(e_2)$, and we obtain a natural map $e_2 R \to aR$. It extends by injectivity to some $\varphi \in \hom_R(E_2, E_1)$, with $a = \varphi(e_2) \in \varphi(Y_2)$. Since Y_1 is Y_2-injective, we get $\varphi(Y_2) \subset Y_1$, and we conclude that $\mathrm{ann}_{E_1}(I) \subset Y_1$. This shows $e_1 R + \mathrm{ann}_{E_1}(I) \subset Y_1$, and therefore $X_1 \subset Y_1$. Together, we have obtained $X = X_1 \oplus X_2 \subset Y_1 \oplus Y_2 = Y$. This shows our claim, and completes the proof in one direction.

For the converse, we are given the existence of a continuous hull Y for $\overline{R}$. Exactly as in the two preceding paragraphs, we obtain $Y = Y_1 \oplus Y_2$, $e_i R \subset Y_i$, $TY_2 \subset Y_2$, $X_2 \subset Y_2$, and $e_1 R + \mathrm{ann}_{E_1}(I) \subset Y_1$. We claim that Y_1 is a continuous hull for $e_1 R + \mathrm{ann}_{E_1}(I)$.

Let $e_1 R + \mathrm{ann}_{E_1}(I) \subset C \subset E_1$ be an arbitrary continuous module. Exactly as before for $X = X_1 \oplus X_2$, we obtain that $C \oplus X_2$ is π-injective hence continuous. We conclude that Y is contained in $C \oplus X_2$, and consequently $Y_1 = \pi_1(Y) \subset \pi_1(C \oplus X_2) = C$. This demonstrates the claim, and finishes the proof of the proposition.

Motivated by the condition in the proposition, we now show the following:

LEMMA 6. Let E be an indecomposable injective module, over a commutative ring R; let $e \in E$, and let I be an ideal of R. Then, $eR + \mathrm{ann}_E(I)$ has a continuous hull.

PROOF. Let C be the multiplicative set of those elements of R which operate regularly on E, and let R_C be the corresponding quotient ring. Then, E is an R_C-module.

Any continuous R-submodule X of E is also on R_C-submodule: indeed, each $c \in C$ defines an R-monomorphism $X \to X$ via multiplication, which splits by continuity, hence is onto since X is uniform. Therefore, X is C-torsionfree divisible hence an R_C-module.

In particular, $A = \mathrm{ann}_E(I)$ is an R_C-module, as it is quasi-injective and therefore continuous. We claim that $eR_C + A$ is also continuous. As this is trivially true if $e \in A$, we assume $e \notin A$, and we consider an arbitrary monomorphism φ from $eR_C + A$ to itself. It extends to an automorphism φ of E. We write $\varphi(e) = et + a$, with $t \in R_C$ and $a \in A$, and we let $\mu \in \mathrm{endo}(E)$ denote the multiplication by t.

If μ is a monomorphism, then clearly $t \in C$. If μ is not a monomorphism, then it belongs to the Jacobson radical of the local ring $\mathrm{endo}(E)$, and consequently $\varphi - \mu$ is an isomorphism. Now, for each endomorphism ψ of E we have $\psi(A) \subset A$, and if ψ is an isomorphism we get $\psi(A) = A$. Thus, for

$\psi = \varphi - \mu$, we obtain an element $b \in A$ such that $\varphi(b) - bt =$ $(\varphi - \mu)(b) = a$.

We put $e' = e - b$. Then, $eR + A = e'R + A$ holds, and therefore $e \notin A$ implies $e' \neq 0$.
Moreover, we have $\varphi(e') = \varphi(e) - \varphi(b) = et + a - \varphi(b) = et - bt = e't$. We deduce $t \in C$, as $0 \neq x \in E$ and $xt = 0$ yield $0 \neq xr = e'r'$ by essentiality of E over $e'R$, hence $0 = xtr = e'tr' = \varphi(e'r')$, hence the contradiction $e'r' = 0$, since φ is a monomorphism.

Thus, we have shown $t \in C$, whether μ is a monomorphism or not. We deduce $e = (\varphi(e) - a)t^{-1} \in \varphi(e)R_C + A$, and $\varphi(eR_C + A) = \varphi(e)R_C + A =$ $eR_C + A$, demonstrating that φ is onto. As $eR_C + A$ is uniform, this suffices to prove that it is continuous.

Now, if $eR + A \subset X \subset E$ is any continuous module, then $eR_C + A \subset XR_C = X$. This shows that $eR_C + A$ is a continuous hull for $eR + A$.

Specializing to $I = R$, we obtain $ann_E(I) = 0$ hence:

COROLLARY 7. Every uniform cyclic module over a commutative ring has a continuous hull.

We note that in this situation, the continuous hull is again a ring, namely the total quotient ring of $\overline{R} = R/ann_R(e)$.

Finally, we combine Proposition 5 with Lemma 6, the latter applied for E_1 instead of E, e_1 instead of e, and for the ideal I occuring in Proposition 5. We observe ~~the~~ $E_1 = Z^*(E) = E(Z^*(M)) = E(Z(M))$, since $Z(M)$ is essential in $Z^*(M)$ (cf. [1], [4]). We obtain the most general existence result of this paper:

THEOREM 8. _Every cyclic module over a commutative ring whose singular submodule is uniform, has a continuous hull._

4. Examples. We discuss two commutative rings which, if considered as modules over themselves, have continuous hulls that lie properly between the π-injective hulls and the (quasi-)injective hulls. The first one is uniform and singular, and illustrates Corollary 7. The second one is non-singular, and exemplifies Corollary 3.

We also exhibit a uniform non-singular cyclic left-module, over a non-commutative ring, which has no continuous hull of type II or III, but which possesses a (rather pathological) continuous hull of type I.

EXAMPLE 1. Let

$$R = \{\text{finite sums } \Sigma a_r x^r : a_r \in \mathbb{Z}, 0 \leq r \in \mathbb{R}\} / \langle x \rangle$$

be the factor ring, of the ring of all "polynomials" with integer coefficients in non-negative real powers x^r of an indeterminate x, modulo the ideal generated by $x = x^1$.

As module over itself, R is uniform and non-singular. Its π-injective hull is R itself. The continuous hull is the total quotient ring T of R

(cf. the note after Corollary 7), which is constructed like R except that rational coefficients are used instead of integer ones. The (quasi-) injective hull is still strictly larger, since, for instance, the homomorphism $\bigcup_{n=1}^{\infty} x^{1/n}R \to T$ which is given on $x^{1/n}R$ by multiplication with $1 + x^{1/2} + x^{2/3} + \ldots + x^{n-1/n}$, cannot be extended to R.

EXAMPLE 2. Let F_k $(k \in K)$ be an infinite family of fields of characteristic zero, and let R be the subring of the product ΠF_k generated by the sum $\oplus F_k$ and 1.

R is non-singular. For a common subring S of all the fields F_k, we introduce the notation

$\Pi_S F_k = \{(x_k) \in \Pi F_k : \{x_k : k \in K\}$ and $\{k : x_k \notin S\}$ are finite $\}$.

Then, the π-injective hull of R is $\Pi_{\mathbb{Z}} F_k$, the continuous hull is $\Pi_{\mathbb{Q}} F_k$ (cf. the construction in the proof of Lemma 2 and the subsequent Remark), and the (quasi-)injective hull is ΠF_k.

EXAMPLE 3. We consider a vectorspace V, over a field F, with basis elements v_m, w_k $(m, k \in \mathbb{N})$. We denote by V_n and W_n, the subspaces spanned by the v_m $(m \geqq n)$ and all the w_k, and by the w_k $(k \geqq n)$, respectively. We write Δ for the shifting operator $\Delta w_k = w_{k+1}$.

With this notation, our ring is

$R = \{\rho \in \mathrm{endo}(V) : \rho(v_m) \in V_m , \rho(w_k) = \Delta^k \rho(w_o) , (m,k \in \mathbb{N})\}$,

and our module is the R-left-module $M=W_o$.

Then, all the V_n and W_n are cyclic R-left-modules, namely $V_n = Rv_n$ and $W_n = Rw_n$. The V_n are the only R-submodules of V which are not contained in M. A simple calculation shows that $(\rho\sigma - \sigma\rho)(M) = 0$ holds for all $\rho, \sigma \in R$. As easy consequences, M and V are uniform and non-singular R-modules, M is isomorphic to $\bar{R} = R/I$ with $I = ann_R(M) = ann_R(w_o)$, and the factorring $\bar{R}$ is isomorphic to the polynomial ring $F[x]$.

Recalling that a uniform module is continuous if and only if every injective endomorphism is surjective, we check without difficulty that all V_n are continuous, but that M is not (since Δ provides an injective R-endomorphism which is not surjective). Consequently, M does not have a type III continuous hull (within an injective hull $E = E(M) = E(V)$, since such a hull would have to be contained in each V_n hence in $\cap V_n = M$). Neither can M have a type II continuous hull (since such a hull would have to be contained in V up to isomorphism over M, and hence would have to be isomorphic to some V_n, in contradiction to $V_n \supsetneqq V_{n+1} \supset M$).

There exists, however, a type I continuous hull of M, namely the quasi-injective hull $ann_E(I)$. (For a proof, view M as $\bar{R}$-module, and apply Corollary 7 and the subsequent observation). But, none of the continuous overmodules V_n contains a type I continuous hull of M (since the V_m $(m \geqq n)$ are the only intermediate modules).

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