DIRECTED HYPERGRAPHS:
THE GROUP OF THEIR COMPOSITION

By

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ABSTRACT

In this thesis we extend Sabidussi's theorems on the automorphism group of the wreath product of graphs to a special kind of relational systems. That is, we define directed hypergraphs and their wreath product and prove theorems giving necessary and sufficient conditions for the group of the product to be the wreath product of the groups of the components.

This also extends our own results on the groups of the wreath products of directed graphs and hypergraphs.

There is no excuse for intellectual laziness.

L. Berggren

This thesis would not have
been possible without those
who make mathematics fun:

Vašek Chvátal
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Robert Woodrow

They may not know it but it
is their attitude that creates.

There are those who
listen patiently even
to the most ridiculous
and reply with same.

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et surtout

à Dominique

dont les ongles repoussent

Světlo přichází za soumraku
A jako vichřice vášeň a pláč

Jak snadné pak předvídat minulost
Jak snadné pak spát

A na zítřek vzpomínat jako už kdysi
Tenkrát
Kdy v záblescích tmy se vše zdálo tak
Jasné

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CHAPTER I

INTRODUCTION

One of the sources of graph theory is organic chemistry. Enumerating all possible distinct isomers of the saturated hydrocarbons C_nH_{2n+2} with a given number n of carbon atoms led Cayley [4], [5] to research on trees. In such a project (enumeration) an important part is played by the automorphisms of objects under consideration. With Pólya's theorem [14] the role of the automorphism group became crucial (the theorem permits counting the number of equivalence classes induced by a permutation group). It comes as no surprise that the relationship between graphs and groups has often been considered as a problem of independent interest. Two obvious questions have been considered, reconsidered and ramified: given a graph, what is its automorphism group?; given a group, is there a graph which has it for its group of automorphisms? The second question was answered in the affirmative for finite groups (Frucht [7], [8]) and it is even known that requiring the graphs to satisfy some specified properties (connectivity, chromatic number, degree of regularity) does not decrease the number of them with automorphism groups isomorphic to a given one - it is infinite (Sabidussi [17]). Finding the group of a given graph can prove difficult. A useful technique is to reduce the graph in question to simpler ones about which more information is available. This thesis is a contribution to such an approach albeit in a more general setting and generalizing an approach taken for graphs by Sabidussi. Let us now establish a working language by beginning with a few definitions.

* * *

If V is a set we denote by $\text{card}\, V$ its cardinality, by $V^{(2)}$ the set of two-element subsets of V, by $P(V)$ its power set. A graph $G = (V,E)$ consists of a set V of points (vertices) and a set $E \subseteq V^{(2)}$ of edges. If, in place of a subset of $V^{(2)}$, we consider a subset A of $V^2 \setminus \{(x,x) | x \in V\}$ the result is a directed graph (digraph); we will denote it by $D = (V,A)$. A hypergraph $H = (V,F)$ is obtained similarly - the edges making up F are non-empty subsets of V. It should be noted that our graphs and digraphs are sometimes known as simple ([3]), they can be infinite (unlike those in [12]) and hypergraphs need not satisfy $\bigcup_{e \in F} e = V$ (the way those in [2] do).

Let us - in this paragraph - call a g-structure any of the three graphs just defined. Let $S_1 = (V_1,E_1)$

and $S_2 = (V_2,E_2)$ be two g-structures of the same kind (both graphs or di - or hyper - graphs). An <u>isomorphism</u> of S_1 with S_2 is a bijection $\alpha : V_1 \to V_2$ such that $\alpha(e) \in E_2$ if and only if $e \in E_1$; write $S_1 \simeq S_2$ and define $\alpha(e) = \{\alpha(x) | x \in e\}$ for graphs and hypergraphs and $\alpha(x,y) = (\alpha(x),\alpha(y))$ for digraphs. An <u>automorphism</u> of a g-structure S is an isomorphism of it with itself. It is an elementary exercise to show that the set of all automorphisms of S forms a group under composition. It is called the automorphism group of S and denoted by $\mathfrak{A}(S)$.

As is often the case when dealing with structures (especially finite ones) it is useful and interesting to explore operations which permit constructing new structures from old. Hanani's proof of the existence of Steiner triple systems is a good example of the usefulness of this approach. Thus, several operations were defined - the union, the join, the product, the composition of two, usually disjoint, graphs. And, naturally, the question of finding the (automorphism) group of the result in terms of the groups of the components was posed in each case. For the first three operations the answers were fairly easy (see, for example [12; 165-166]), the fourth problem was settled only at the second attempt ([13],[15]). Since this thesis is concerned with extensions of this result, let us consider the theorem obtained by Sabidussi in [15]. To allow us to do this a few more definitions are needed.

The neighbourhood $N(x)$ of a point $x \in V$ in a graph $G = (V,E)$ is the set $\{y | \{x,y\} \in E\}$. The closed neighbourhood of x is $\overline{N(x)} = \{x\} \cup N(x)$. A graph G is said to be connected if for every non-trivial partition of V into $X \cup Y$ there is an $\{x,y\} \in E$ with $x \in X$, $y \in Y$. The complement of G is the graph $\overline{G} = (V,\overline{E})$ where $\overline{E} = \{\{x,y\} \in V^{(2)} | \{x,y\} \notin E\}$. Given two graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ with $V_1 \cap V_2 = \emptyset$, define their composition (wreath, lexicographic product) $G_1[G_2] = (V,E)$ by $V = V_1 \times V_2$ and

$$E = \{e \in V^{(2)} | \pi_1(e) \in E_1, \text{ or } \operatorname{card} \pi_1(e) = 1 \text{ and } \pi_2(e) \in E_2\}.$$

As usual, $\pi_i(e)$ is the i^{th} projection of e (on V_i).

The following definition will be needed throughout the thesis.

Definition

Let $\mathfrak{A}$ and $\mathfrak{B}$ be permutation groups on disjoint sets X and Y, respectively. The composition of $\mathfrak{A}$ and $\mathfrak{B}$ ($\mathfrak{A}$ around $\mathfrak{B}$, the wreath product of $\mathfrak{A}$ by $\mathfrak{B}$), $\mathfrak{A}[\mathfrak{B}]$ acts on $X \times Y$ and consists of pairs $(\alpha, \{\beta_x\}_{x \in X})$, with $\alpha \in \mathfrak{A}$, and each $\beta_x \in \mathfrak{B}$. The action is defined by

$$(\alpha,\{\beta_x\}_{x\in X})(x,y) = (\alpha(x),\beta_x(y)).$$

Note that $(\alpha,\{\beta_x\}_{x\in X})^{-1} = (\alpha^{-1},\{\beta^{-1}_{\alpha^{-1}(x)}\}_{x\in X})$ and $(\alpha,\{\beta_x\}_{x\in X})(\alpha',\{\beta'_x\}_{x\in X}) = (\alpha\alpha',\{\beta_{\alpha'(x)}\beta'_x\}_{x\in X})$.

We will say of two permutation groups $\mathfrak{A}$ and $\mathfrak{B}$ on (not necessarily disjoint) sets X and Y respectively that they are identical, $\mathfrak{A} \equiv \mathfrak{B}$, if there is an isomorphism $h : \mathfrak{A} \to \mathfrak{B}$ and a bijection $g : X \to Y$ such that $g(\alpha(x)) = h(\alpha)(g(x))$. In this work we always have $X = Y$ and $\mathfrak{A} \subset \mathfrak{B}$ (see Lemma 1, p.23). This allows us to treat identity of the two groups as their equality, although we still write $\mathfrak{A} \equiv \mathfrak{B}$.

Theorem 1

If $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ are graphs on disjoint vertex sets and if V_2 is finite then $\mathfrak{A}(G_1[G_2]) \equiv \mathfrak{A}(G_1)[\mathfrak{A}(G_2)]$ if and only if

(1) if there are $u \neq v$ in G_1 with $N(u) = N(v)$ then G_2 is connected.

(2) If there are $u \neq v$ in G_1 with $\overline{N(u)} = \overline{N(v)}$ then $\overline{G_2}$ is connected.

It is not difficult to see the necessity of the conditions; the sufficiency requires some work.

Although in practice most graphs considered are finite it is interesting to see what happens when the vertex sets are infinite. The best result so far is also due to Sabidussi [16] ("SC" means "Sabidussi Condition", see also Chapter II, page 17).

Theorem 2

If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are disjoint graphs and if G_2 satisfies

SC_G : $\text{card}(N(x) \cap N(y)) < \text{card } V_2$ for all $x \neq y$ in V_2

then $\mathcal{O}(G_1[G_2]) \equiv \mathcal{O}(G_1)[\mathcal{O}(G_2)]$ if and only if (1) and (2) of Theorem 1 hold.

The importance of the $\overrightarrow{SC}$ lies in the fact that it allows the proof of the sufficiency.

One could, as Foldes did in 1975 ([6]), ask about analogous theorems for digraphs, hypergraphs and, more generally, relational systems. This requires three things (a sequence well-known in mathematics).

(1) Appropriate definitions that reduce to those already in existence for graphs,

(2) correct conditions which become those of Theorems 1 and 2,

(3) a proof of the necessity and the sufficiency of these conditions (should this be the case).

Let us consider our work of [9] and [10]. The terminology is that of [11] and this thesis and is slightly different from that of [9] and [10].

Digraphs

Let $D = (V,A)$ be a directed graph and write, for simplicity, xy instead of $(x,y) \in V^2$. Put

$$N^+(u) = \{v \in V \mid uv \in A\} \qquad \overline{N^+(u)} = \{u\} \cup N^+(u)$$

$$N^-(u) = \{v \in V \mid vu \in A\} \qquad \overline{N^-(u)} = \{u\} \cup N^-(u)$$

$$N(u) = (N^+(u), N^-(u)) \qquad \overline{N(u)} = (\overline{N^+(u)}, \overline{N^-(u)})$$

$$N(u)\backslash X = (N^+(u)\backslash X, N^-(u)\backslash X).$$

Say that u and v are <u>equivalent</u> if $N(u) = N(v)$ and that they are <u>strongly equivalent</u> if $\overline{N(u)} = \overline{N(v)}$. We call D <u>connected</u> if for each non-trivial partition of V into $X \cup Y$ there is an $xy \in A$ with $x \in X$, $y \in Y$ or $x \in Y$, $y \in X$. The <u>complement</u> $\bar{D}$ of D is the digraph $(V,\bar{A})$ with $\bar{A} = \{uv \in V^2\backslash\{(x,x) \mid x \in V\} \mid uv \notin A\}$.

A <u>Z-chain</u> C in D is a subgraph induced by a set of vertices indexed by the integers such that $C = (V',A')$ and $V' = \{v_i \in V \mid i \in Z\}$, $A' = \{v_iv_j \mid i < j\}$ and $N(v_i)\backslash V' = N(v_j)\backslash V'$ for $i,j \in Z$, ("induced" means $A' = A \cap V'^2$). Let now $D_1 = (V_1,A_1)$, $D_2 = (V_2,A_2)$ and $V_1 \cap V_2 = \emptyset$. The <u>dijoin</u> (directed join) $D_1 \vec{\vee} D_2$

of D_1 to D_2 is the digraph (V,A) obtained by putting $V = V_1 \cup V_2$ and $A = A_1 \cup A_2 \cup \{uv | u \in V_1, v \in V_2\}$. A digraph $D = (V,A)$ is a bijoin if there are non-trivial partitions $V = X \cup Y = X' \cup Y'$ such that $D = D\langle X\rangle \vec{v} D\langle Y\rangle = D\langle X'\rangle \vec{v} D\langle Y'\rangle$ and $D\langle X\rangle \simeq D\langle Y'\rangle$, $D\langle X'\rangle \simeq D\langle Y\rangle$. We define $D\langle U\rangle = (U, A \cap U^2 \setminus \{(x,x) | x \in U\})$ as the subgraph of D induced by $U \subseteq V$.

With D_1 and D_2 as above we define the wreath product $D = D_1[D_2] = (V,A)$ of D_1 around D_2 by $V = V_1 \times V_2$ and $(u,x)(u',x') \in A$ if and only if either $uu' \in A_1$, or $u = u'$ and $xx' \in A_2$. The result of [10] extending Theorem 1 is the following.

Theorem 3

If D_1 and D_2 are disjoint digraphs and if V_2 is finite then $\mathfrak{A}(D_1[D_2]) \equiv \mathfrak{A}(D_1)[\mathfrak{A}(D_2)]$ if and only if

(1) if there are $u \neq v$ in D_1 which are equivalent then D_2 is connected

(2) if there are $u \neq v$ in D_2 which are strongly equivalent then $\overline{D_2}$ is connected

(3) if there is a Z-chain in D_1 then D_2 is not a bijoin.

This theorem implies the result of [1]. We did not generalize Theorem 2 in [10]; it is, however, true that the following holds.

Theorem 4

If D_1 and D_2 are disjoint digraphs and if D_2 satisfies

SC_D : $\mathrm{card}(\{z \mid xz \in A_2 \text{ or } zx \in A_2\} \cap \{z \mid yz \in A_2 \text{ or } zy \in A_2\}) < \mathrm{card}\, V_2$

for any $x \neq y$ in V_2

then $\mathfrak{A}(D_1[D_2]) \equiv \mathfrak{A}(D_1)[\mathfrak{A}(D_2)]$ if and only if (1), (2) and (3) of Theorem 3 hold.

This will be a corollary of the results of Chapter III.

Hypergraphs

Let $H = (V,F)$ be a hypergraph. We say that H is connected if for every non-trivial partition of V into $X \cup Y$ there is an $e \in F$ with $e \cap X \neq \emptyset \neq e \cap Y$. One would now expect a definition of a complement of H and its connectedness; this is not what is needed. We say that H is anti-connected if for every non-trivial partition of V into $X \cup Y$ either there are $x \in X$, $y \in Y$ with $\{x,y\} \notin F$ or there is an $e \subset V$, $e \cap X \neq \emptyset \neq e \cap Y$, $\mathrm{card}\, e \geq 3$. If H is a graph then it is anticonnected exactly when its complement is connected. We will use the symbol $\overline{H}$ to denote

something else for hypergraphs, namely a (possibly new) hypergraph $(V,\overline{F})$ obtained by defining
$\overline{F} = F \cup \{\{u\} | u \in V\}$.

If $e \in F$, $u,v \in V$ then $e_{u,v}[v,u]$ is the set obtained by replacing v by u and u by v in e. We say that $u \neq v$ in V are equivalent in H if $e_{u,v}[v,u] \in F$ exactly when $e \in F$, $e \in P(V)$, and no edge contains $\{u,v\}$. They are strongly equivalent in H if $e_{u,v}[v,u] \in F$ if and only if $e \in F$, $e \in P(V)$, $\{u,v\} \in F$ and no edge contains $\{u,v\}$ properly. The points $u,v \in V$ are similar if there is an $h \in \mathfrak{A}(\overline{H})$ such that $h(u) = v$.

The wreath product of two disjoint hypergraphs $H_1 = (V_1,F_1)$ and $H_2 = (V_2,F_2)$ is the hypergraph $H = H_1[H_2] = (V,F)$ given by $V = V_1 \times V_2$ and, for $e \in P(V)$, $e \in F$ if and only if $\pi_1(e) \in F_1$, or card $\pi_1(e) = 1$ and $\pi_2(e) \in F_2$. The theorem given in [9] is this.

Theorem 5

If H_1 and H_2 are disjoint hypergraphs and if V_2 is finite then $\mathfrak{A}(H_1[H_2]) \equiv \mathfrak{A}(H_1)[\mathfrak{A}(H_2)]$ if and only if

A. (1) If there are $u \neq v$ in H_1 which are equivalent then H_2 is connected.

(2) If there are $u \neq v$ in H_1 which are strongly equivalent then H_2 is anti-connected.

(3) If there are $u \neq v$ equivalent in $\overline{H}_1$ but not in H_1 and if H_2 is not connected with a partition $X \cup Y = V_2$ then there are $x \in X$, $y \in Y$ such that $\{x\},\{y\} \notin F_2$.

(4) If there are $u \neq v$ strongly equivalent in $\overline{H}_1$ but not in H_1 and if H_2 is not anti-connected with a partition $V_2 = X \cup Y$ then there are $x \in X$, $y \in Y$ such that $\{x\},\{y\} \notin F_2$.

B. If there is a $u \in V_1$, with $\{u\} \in F_1$ then

(1) if there are x, y in H_2 which are similar then $\{x\} \in F_2$ if and only if $\{y\} \in F_2$.

(2) If there is a v in V_1, $\{v\} \notin F_1$ and if u and v are similar then there is an $x \in V_2$ with $\{x\} \notin F_2$.

As with digraphs this can be extended to generalize Theorem 2. And, as with Theorem 4, the result will be a corollary of those in Chapter III.

Theorem 6

If H_1 and H_2 are disjoint hypergraphs and if H_2 satisfies

$$SC_H : \text{card}(\{z \mid \{x,z\} \subseteq e \in F_2\} \cap \{z \mid \{y,z\} \subseteq e \in F_2\}) < \text{card } V_2$$

for all $x \neq y$ in V_2

then $\mathfrak{A}(H_1[H_2]) \equiv \mathfrak{A}(H_1)[\mathfrak{A}(H_2)]$ if and only if A and B of Theorem 5 hold.

For relational systems the problem is more complicated. To begin with, it is not clear what appropriate (that is, reducible to the previous ones) definitions should be in general. In the cases considered - graphs, digraphs, hypergraphs - we had at most one relation of each size (cardinality) in each of the g-structures S_1, S_2 and, consequently, could define a unique relation in $S_1[S_2]$ for each size occurring in either S_1 or S_2. Should - for example - the edges of graphs G_1, G_2 be coloured with at least two colours, how would we colour the edges of $G_1[G_2]$? We could use a new colour c_i for each colour c used in G_i $(i = 1,2)$ or use the colours of the appropriate projections. But unless the second scheme is used and some colour appears in both G_1 and G_2 we will have $\mathfrak{A}(G_1[G_2]) \equiv \mathfrak{A}(G_1)[\mathfrak{A}(G_2)]$ since we want automorphisms to preserve colours. In any event the

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results thus obtained do not seem to be leading in the right direction and will not be considered here. That is, of course, not to say that the extension to relational systems cannot be provided; we are merely reporting our (present) inability to do so.

It is with this in mind that we define directed hypergraphs - they are relational systems with the property that for each ordinal μ there is at most one relation in V^{μ} (where V is the underlying set). But - surprise (?) - even this restricted definition does not allow for a clear-cut extension of our (and Sabidussi's) previous theorems. The problem comes from constants (i.e. $\{x\}^{\mu} \in R$ for some $R \subseteq V^{\mu}$) and although we restrict somewhat the class from which the "outside" (i.e. H_1) hypergraph can be taken by introducing a Technical Condition the best we can do is produce a set of necessary and a set of sufficient conditions. These sets, in general, are not the same. Nonetheless, they are sufficiently similar to coincide in the cases of graphs, digraphs and hypergraphs as well as in some other special cases. As the difficulties will become apparent in the proofs there is no need to expand on the matter now. Rather, we proceed with the results.

CHAPTER II

DEFINITIONS

It may seem pretentious to devote a whole chapter to definitions. The hope is that it will make the reading easier by providing a reference section.

Let us first settle on a hierarchy of set operations: $\times$ over $\cup$, $\cap$, $\setminus$; $\setminus$ over $\cup$, $\cap$. That is, $A \times B \cup C = (A \times B) \cup C$; $A \times B \setminus C = (A \times B)\setminus C$, etc. Let us also agree to omit parentheses whose absence will not create confusion: $f((a,b)) = f(a,b)$.

Let A and B be disjoint sets and let I be a set of ordinals not containing zero. Put

$$F(A;I) = \bigcup_{\mu \in I} A^{\mu}$$

and

$$C(A;I) = \bigcup_{\mu \in I} \bigcup_{x \in A} \{x\}^{\mu}.$$

The size of $f \in F(A;I)$, denoted by $|f|$, is the element of I for which $f \in A^{|f|}$. The underlying set of f is the set $[f] = \{x \in A | f(i) = x$ for some $i < |f|\}$. If $f \in F(A \times B;I)$ denote by $\pi_A(f)$ the projection of f on A, similarly for π_B.

Let $e \in F(A;I)$, $u \neq v \in A$. A function $f \in A^{|e|}$

could have, among others, the following properties.

(1) $f(i) = e(i)$ if $e(i) \notin \{u,v\}$.

(2) $f(i) \in \{u,v\}$ if $e(i) \in \{u,v\}$.

(3) If there are $i < j < |e|$ with $f(i), f(j) \in \{u,v\}$ then $\{u,v\} \subseteq [f]$.

We define

$$e(u,v) = \{f \in A^{|e|} \mid f \text{ satisfies (1), (2) and (3)}\}$$

$$e^*(u,v) = \{f \in A^{|e|} \mid f \text{ satisfies (1) and (2)}\}.$$

A <u>directed hypergraph</u> (diaper) $H = (V,E;I)$ consists of a set V of vertices (points) and a set $E \subseteq F(V;I)$ of edges with I a set of ordinals not containing zero. Let $A \subseteq I$. We say that H is <u>A-split</u> if there is a partition of V into $X \cup Y$ such that $e \in E$ if and only if $|e| \in A$ for any $e \in F(V;I)$ with $[e] \cap X \neq \emptyset \neq [e] \cap Y$. For example, a digraph $D = (V,E;\{2\})$ is connected if and only if it is not $\emptyset$-split.

Still with $A \subseteq I$ we define $\overline{H} = \overline{H}(I)$, $H(A) = (V,E \cup F(V;A);I)$ and $\overline{H}(A) = (V,E \cup C(V;A);I)$. For convenience we will denote $E \cup F(V;A)$ by $E(A)$ and $E \cup C(V;A)$ by $\overline{E}(A)$. If $X \subseteq V$ define $E\langle X\rangle = E \cap F(X;I)$ and, with this, the <u>subdiaper induced by X</u>, $H\langle X\rangle = (X,E\langle X\rangle;I)$.

For simplicity we will often abuse notation and write $\{x\}^\mu$ for the (unique) function in $\{x\}^\mu$ and, in particular, $\{x\}$ for $\{x\}^1$. Also, if $f \in V^2$ with $f(0) = x \neq y = f(1)$ we will write simply xy for f.

Let $x \in V$. The neighbourhood of x in H is the set $N(x) = \bigcup_{\substack{e \in E \\ x \in [e]}} [e] \setminus \{x\}$. The set of constants of x is $J(x) = \{\mu \in I \mid \{x\}^\mu \subseteq E\}$. A diaper H satisfies the Technical Condition, TC, if $J(x) \in \{A,B\}$ for all $x \in V$ and some $A,B \subset I$ with $A \neq B$ unless $A = B = \emptyset$.

As for g-structures we define an isomorphism of $H_1 = (V_1,E_1;I_1)$ and $H_2 = (V_2,E_2;I_2)$ as a bijection $\alpha : V_1 \to V_2$ such that $e \in E_1$ if and only if $\alpha(e) \in E_2$, with $\alpha(e)(i) = \alpha(e(i))$, $i < |e|$. For the purposes of this thesis we need not distinguish between an isomorphism - in case I_1 can differ from I_2 - and a strong isomorphism requiring $I_1 = I_2$. We write $H_1 \simeq H_2$, as usual. An automorphism of H is, naturally, an isomorphism of it with itself. The group formed by the automorphisms of H will, again, be denoted by $\mathfrak{A}(H)$.

Let $u \neq v \in V$. Put
$B(u,v) = \{\mu \in I \mid [e] = \{u,v\}$ for some $e \in V^\mu \cap E\}$. We will call u and v similar (via α), $u \sim v$, in H if $\alpha(u) = v$ for some $\alpha \in \mathfrak{A}(\bar{H})$. They will be called equivalent, $u \equiv v$, in H if

(1) $e(u,v) \subset E$ for each $e \in E$

(2) $e^*(u,v) \subset E$ for each $e \in E$, $[e] \neq \{u,v\}$.

A diaper H satisfies the Sabidussi Condition, SC, if for any pair of distinct points u and v in V we have $\mathrm{card}(N(u) \cap N(v)) < \mathrm{card}\, V$. If, for $u \neq v$, this does not hold we say that u and v have a large neighbourhood intersection.

Let $A \cup \{2\} \subseteq I$, $A \cap \{1,2\} = \emptyset$, $B \subseteq A \cup \{1\}$. Following convention denote by Z the set of integers. An (A,B)-Z-chain in H is the subdiaper $C = (V',E';A \cup \{2\})$ induced by V' such that

$$V' = \{v_i \mid i \in Z\}$$

$$E' = \{v_i v_j \mid i < j \in Z\} \cup C(V',B) \cup F(V',A) \setminus C(V',A)$$

and such that $v_i \equiv v_j$ in ${}_C H^{ij}$ whenever $i \neq j$. We define ${}_C H^{ij} = (V, {}_C E^{ij}; I)$ by

$${}_C E^{ij} = E \cup \{v_m v_n \mid \min(i,j) \leq n < m \leq \max(i,j)\}.$$

If $B = \emptyset$ we talk about an A-Z-chain, if $A = B = \emptyset$ about a Z-chain.

Given two diapers $H_1 = (V_1,E_1;I_1)$, $H_2 = (V_2,E_2;I_2)$, $V_1 \cap V_2 = \emptyset$, and a set A of ordinals not containing 1 and 2 we define the A-dijoin of H_1 to H_2, written $H_1 \vec{v}_A H_2 = (V,E;I)$, by

$$V = V_1 \cup V_2$$

$$I = I_1 \cup I_2 \cup A \cup \{2\}.$$

$$E = E_1 \cup E_2 \cup \{xy \mid x \in V_1, y \in V_2\} \cup F(V;A) \setminus (F(V_1;A) \cup F(V_2;A)).$$

A diaper $H = (V,E;I)$ is called an A-bijoin if there are non-trivial partitions $V = X \cup Y = X' \cup Y'$ with

$$H = H\langle X\rangle \vec{v}_A H\langle Y\rangle = H\langle X'\rangle \vec{v}_A H\langle Y'\rangle$$

and

$$H\langle X\rangle \simeq H\langle Y'\rangle, \quad H\langle X'\rangle \simeq H\langle Y\rangle.$$

If $A = \emptyset$, we omit the subscript, write $H_1 \vec{v} H_2$ and say "dijoin"; similarly for "bijoin".

If H_1 and H_2 are as above define their wreath (lexicographic) product or composition $H = H_1[H_2] = (V,E;I)$ as follows.

$$V = V_1 \times V_2$$

$$I = I_1 \cup I_2$$

$$E = \{e \in F(V;I) \mid \pi_1(e) \in E_1, \text{ or } \pi_1(e) \in C(V_1;I) \text{ and } \pi_2(e) \in E_2\}$$

with $\pi_i(e) = \pi_{V_i}(e)$ for simplicity. In the sequel H_1, H_2, H will be those just defined unless indicated otherwise. Further, we will denote by $\mathfrak{A}_i$ and $\mathfrak{A}$ the groups $\mathfrak{A}(H_i)$ and $\mathfrak{A}(H)$, respectively.

Let $\alpha \in \mathfrak{A}$, $u \in V_1$. Denote by $I_\alpha(u)$ the image of u under α, that is, the set

$\{v \in V_1 | \alpha(\{u\} \times V_2) \cap \{v\} \times V_2 \neq \emptyset\}$. This set can be partitioned into $\mathcal{O}_\alpha(u) = \{v \in \mathcal{I}_\alpha(u) | \alpha^{-1}(\{v\} \times V_2) \subseteq \{u\} \times V_2\}$ and $\mathcal{N}_\alpha(u) = \mathcal{I}_\alpha(u) \setminus \mathcal{O}_\alpha(u)$; these are called "onto" and "not-onto", respectively. We say that <u>α preserves copies</u> if card $\mathcal{I}_\alpha(u) =$ card $\mathcal{I}_{\alpha^{-1}}(u) = 1$ for all $u \in V_1$ (i.e. $\mathcal{I}_\alpha(u) = \mathcal{O}_\alpha(u)$ and card $\mathcal{I}_\alpha(u) = 1$). If every α preserves copies then so does $\mathfrak{A}$.

We are now ready for the important chapter.

CHAPTER III

MAIN THEOREMS

As we already know, this chapter aims to provide some necessary and some sufficient conditions for the group of the composition of two diapers to be the composition of the groups of the components. . We also know we should not expect the necessary conditions to be sufficient and vice versa. With this in mind we state the conditions.

A. (1) If there are $u \neq v$ in V_1 with $u \equiv v$ in H_1 then $H_2(J(u))$ is not $B(u,v)$-split.

(2) If there are $u \neq v$ in V_1 with $u \equiv v$ in $\overline{H}_1(J(u)\Delta J(v))$ but $J(u) \neq J(v)$ and if $H_2(J(u))$ is $B(u,v)$-split by $V_2 = X \cup Y$ then there are $\mu, \nu \in J(u)\Delta J(v)$ and $e \in X^{\mu}$, $f \in Y^{\nu}$ with $e,f \notin E_2$.

(3) If there are $u \neq v$ in H_1 with $u \equiv v$ in $\overline{H}_1(J(u)\Delta J(v))$ but $J(u) \neq J(v)$ and if $H_2(J(u))$ is $B(u,v)$-split by $V_2 = X \cup Y$ then for each $\mu \in J(u)\Delta J(v)$ there are $e \in X^{\mu}$, $f \in Y^{\mu}$ with $e,f \notin E_2$.

B. If there is a $u \in V_1$ with $J(u) \neq \emptyset$ then

(1) $\mathfrak{A}(H_2(J(u))) \subseteq \mathfrak{A}_2$.

(2) If there is a $\beta \in \mathfrak{A}(\overline{H}_1(I_1))\setminus\mathfrak{A}_1$ then there are $u,v \in V_1$, similar via β, and $\mu \in J(u)\Delta J(v)$, $e \in V_2^{\mu}$ such that $e \notin E_2$.

(3) If there is a $\beta \in \mathfrak{A}(\overline{H}_1(I_1))\setminus\mathfrak{A}_1$ then there are $u,v \in V_1$, similar via β, and, for each $\mu \in J(u)\Delta J(v)$ there is an $e \in V_2^{\mu}$ such that $e \notin E_2$.

C. If there are $u \in V_1$ and $A \subseteq I$ such that $H_2(J(u))$ is an A-bijoin then u lies in no (A,B)-Σ-chain in H_1.

We propose the following theorems (and a host of corollaries afterward).

Theorem 7

If $\mathfrak{A} \equiv \mathfrak{A}_1[\mathfrak{A}_2]$ then A(1), A(2), B(1), B(2) and C hold.

Theorem 8

If H_1 satisfies the TC and H_2 the SC then $\mathfrak{A} \equiv \mathfrak{A}_1[\mathfrak{A}_2]$ whenever A(1), A(3), B(1), B(3) and C hold.

We will prove Theorem 7 directly and Theorem 8 with the help of a series of lemmas.

Proof of Theorem 7

By contradiction; assuming each of the conditions in turn to be false we construct mappings $\alpha : V \to V$ such that $\alpha \in \mathfrak{A}\backslash\mathfrak{A}_1[\mathfrak{A}_2]$.

A. (1) If there are u, v as described and $H_2(J(u))$ is $B(u,v)$-split with a partition $V_2 = X \cup Y$ we define α by

$$\left.\begin{array}{l}\alpha(u,x) = (v,x)\\ \alpha(v,x) = (u,x)\end{array}\right\} \text{ if } x \in X$$

$$\alpha(w,z) = (w,z) \quad \text{otherwise.}$$

(2) Let u, v be as described. If, for each $\mu \in J(u)\Delta J(v)$ and all - without loss of generality - $e \in X^{\mu}$, we have $e \in E_2$, define α as in (1).

B. (1) If there is a $g \in \mathfrak{A}(H_2(J(u))\backslash\mathfrak{A}_2$ put $\alpha(u,x) = (u,g(x))$ and, for $w \neq u$, $\alpha(w,x) = (w,x)$.

(2). If there is a $\beta = \mathfrak{A}(\overline{H}_1(I_1))\backslash\mathfrak{A}_1$ and if for all u, v similar via β, all $\mu \in J(u)\Delta J(v)$ and all $e \in V_2^{\mu}$ we have $e \in E_2$, define α by $\alpha(w,x) = (\beta(w),x)$.

C. If there are u and A as described and u lies in an (A,B)-Z-chain on $\{v_i | i \in Z\}$ in H_1, define α as follows. Let $V_2 = X \cup Y = X' \cup Y'$ be the partitions with which $H_2(J(u))$ is an A-bijoin and let $g : X \to Y'$, $h : Y \to X'$ be the isomorphisms required by the definition of an A-bijoin. Put

$$\begin{aligned} \alpha(v_i,x) &= (v_i,g(x)) && \text{if } x \in X \\ \alpha(v_i,y) &= (v_{i+1},h(y)) && \text{if } y \in Y \\ \alpha(w,z) &= (w,z) && \text{otherwise.} \end{aligned}$$

To verify that $\alpha \in \mathfrak{A} \setminus \mathfrak{A}_1[\mathfrak{A}_2]$ in each case is routine.

□

We now begin the sequence of lemmas needed to prove Theorem 8.

Lemma 1

$\mathfrak{A}_1[\mathfrak{A}_2] \subseteq \mathfrak{A}$.

Proof

Let $\alpha \in \mathfrak{A}_1$, $\beta_x \in \mathfrak{A}_2$ for $x \in V_1$. Clearly $(\alpha, \{\beta_x\}_{x \in V_1}) \in \mathfrak{A}$.

□

Lemma 2

If $\mathfrak{A}$ preserves copies and B(1), B(3) hold then $\mathfrak{A} \subseteq \mathfrak{A}_1[\mathfrak{A}_2]$.

Proof

Let $\alpha \in \mathfrak{A}$. Since it preserves copies we can define $\alpha_1 : V_1 \to V_1$ and, for each $u \in V_1$, $\alpha_u : V_2 \to V_2$ by

$\alpha_1(u) = v$ if and only if $I_\alpha(u) = \{v\}$

$\alpha_u(x) = y$ if and only if $\alpha(u,x) = (v,y)$.

We claim of course, that $\alpha_1 \in \mathfrak{A}_1$ and $\alpha_u \in \mathfrak{A}_2$ for each u.

(i) Let $e \in E_1$ and consider $\alpha(e)$. If $[e] \neq \{u\}$ for any $u \in V_1$ then $\alpha_1(e) \in E_1$ (since edges "between copies" in H can only come from edges in H_1). If $[e] = \{u\}$ for some $u \in V_1$ then $u \sim \alpha_1(u)$ in $\overline{H}_1(I_1)$. If $\alpha_1 \notin \mathfrak{A}_1$ and if $J(u) \neq J(\alpha_1(u))$ then B(3) guarantees that for each $\mu \in J(u)\Delta J(\alpha_1(u))$ there is an $f \in V_2^\mu$, $f \notin E_2$. Now for each $\mu \in J(u)\Delta J(\alpha_1(u))$ we have $(\alpha_1(u) \times V_2)^\mu \subseteq E$ since α is an automorphism. Therefore $J(u) = J(v)$. Thus $\alpha_1(e) \in E_1$.

(ii) Consider any α_u. If $J(u) = \emptyset$ there is nothing to prove. If $J(u) \neq \emptyset$ then $\alpha_u \in \mathcal{O}\!\mathcal{l}(H_2(J(u))$ and, by B(1), $\alpha_u \in \mathcal{O}\!\mathcal{l}_2$.

□

With the above lemmas in mind we can devote the rest of this section to proving that the SC on H_2, the TC on H_1 and A(1), A(3), B(3) and C imply that $\mathcal{O}\!\mathcal{l}$ preserves copies. To simplify the proofs we make two easy but important remarks.

(1) If $e \in E$, $\text{card}[\pi_1(e)] > 1$ then $\pi_1(e) \in E_1$.

(2) If $e \in F(V;I)$, $\text{card}[\pi_1(e)] > 1$ then $e \in E$ if and only if $f \in E$ whenever $\pi_1(f) = \pi_1(e)$, $f \in F(V;I)$.

With these we can formulate an argument which will appear, in different forms, many times. Let $\alpha \in \mathcal{O}\!\mathcal{l}$, $u \in V_1$ be such that $\text{card}\, I_\alpha(u) > 1$. Let $(w,z) \in V\backslash\alpha(\{u\} \times V_2)$ and, for $v \in I_\alpha(u)$, let $(v,x_v) \in \alpha(\{u\} \times V_2)$. If $e \in E$ is such that $[e] \cap \alpha(\{u\} \times V_2) \neq \emptyset$ and $(w,z) \in [e]$ then $f \in E$ for any $f \in V^{|e|}$ such that $f(i) \in \{(v,x_v) | v \in I_\alpha(u)\}$ if $(w,z) \neq e(i) \in I_\alpha(u) \times V_2$ and $f(i) = e(i)$ otherwise. To see this, let f be any such function. Consider first e^* obtained by putting $e^* = e$ if $\text{card}[\pi_1(e)] = 1$ and, if $\text{card}[\pi_1(e)] > 1$, by putting

$e^*(i) = (v, x_v)$ whenever $(w,z) \neq e(i) \in \{v\} \times V_2$ and $v \in I_\alpha(u)$, $e^*(i) = e(i)$ otherwise. Since $\alpha^{-1}(e^*) \in E$ and $\text{card}[\pi_1(\alpha^{-1}(e^*))] > 1$ we have $\pi_1(\alpha^{-1}(e^*)) \in E_1$. But clearly $\pi_1(\alpha^{-1}(e^*)) = \pi_1(\alpha^{-1}(f))$ and, hence, $f \in E$.

This kind of argument can vary: we can begin with an edge $\bar{e} \in E$ before picking a "representative" $e \in E_1$ (i.e. $e \in V^{|\bar{e}|}$ with $\pi_1(e) = \bar{e}$); the "representative" can be chosen with care to do a particular job; the argument can be used many times over; it can be augmented by references to remarks (1) and (2) to conclude, for example, that a particular $e \in F(V;I)$ is an edge, etc. In each case we will simply refer to a ping-pong argument. We note that in some cases the axiom of choice may be needed; we will mention this again in the end of the thesis.

Lemma 3

Let $\alpha \in \mathcal{O}\!\mathcal{L}$, $u \in V_1$, $\text{card}\, I_\alpha(u) > 1$. If H_2 satisfies the SC then either $O_\alpha(u) = \emptyset$ or $E_1\langle I_\alpha(u)\rangle \backslash F(I_\alpha(u);\{1\}) = F(I_\alpha(u);J(u)\backslash\{1\})$.

Proof

The lemma says that if - for α and u given - $O_\alpha(u) \neq \emptyset$ then all edges in $H_1\langle I_\alpha(u)\rangle$ of size at least two have sizes from $J(u)$ and, by the remarks, all such edges are present. We will, therefore, show that if $\mathcal{O}_\alpha(u) \neq \emptyset$ and $e \in E_1\langle \mathcal{I}_\alpha(u)\rangle$, $|e| \geq 2$ then $|e| \in J(u)$. The rest is clear.

It is evident that if V_2 is finite then $\mathcal{O}_\alpha(u) = \emptyset$. Suppose, then, that V_2 is infinite, $\mathcal{O}_\alpha(u) \neq \emptyset$ and let $e \in E_1\langle \mathcal{I}_\alpha(u)\rangle$, $|e| \geq 2$ and $|e| \notin J(u)$. We proceed in three steps.

(i) If there is a $v \in [e] \cap \mathcal{N}_\alpha(u)$ then we can find an $x_v \in V_2$ such that $(v,x_v) \notin \alpha(\{u\} \times V_2)$ and an $f \in V^{|e|}$ such that $f(i) = (v,x_v)$ for exactly one $0 \leq i < |e|$ and $f(i) \in \alpha(\{u\} \times V_2)$ otherwise (this may require the axiom of choice). Let $X = \{x \in V_2 \,|\, \alpha(u,x) \in \{v\} \times V_2\}$, $Y = V_2 \backslash X$. By ping-pong arguments it follows that for any $x \in X$, $y \in Y$ there is an $e_{xy} \in E_2$ such that

$$e_{xy}(i) = \begin{cases} x & \text{if } e(i) = v \\ y & \text{otherwise.} \end{cases}$$

To see this, consider $\alpha^{-1}(f)$. Since $\alpha^{-1}(f(i)) \in \{u\} \times V_2$ unless $f(i) = (v,x_v)$ there is, for each $y \in Y$, an $f_y \in V^{|e|}$

such that $f_y(i) = \alpha^{-1}(f(i))$ if $f(i) = (v,x_v)$ and $f_y(i) = (u,y)$ otherwise. Now $\pi_1(f_y) = \pi_1(\alpha^{-1}(f))$ and so both f_y and $\alpha(f_y)$ are in E. Define $\hat{e}_{xy}$ by

$$\alpha(\hat{e}_{xy}(i)) = \begin{cases} \alpha(u,x) & \text{if } e(i) = v \\ \alpha(u,y) & \text{otherwise.} \end{cases}$$

We have $\hat{e}_{xy} \in E$ since $\pi_1(\hat{e}_{xy}) = \pi_1(\alpha(f_y))$. Putting $e_{xy} = \pi_2(\hat{e}_{xy})$ and remembering that $|e| \not\subset J(u)$ we obtain the desired edge. Hence $\text{card}(N(x) \cap N(x')) \geq \text{card}\, Y$ and $\text{card}(N(y) \cap N(y')) \geq \text{card}\, X$ for any $x,x' \in X$, $y,y' \in Y$. Since either X or Y has the cardinality of V_2, either X or Y consists of exactly one point (lest SC be violated).

(ii) Thus, if both $O_\alpha(u)$ and $N_\alpha(u)$ are non-empty, v as in (i) and $w \in O_\alpha(u)$ then for some $x \in V_2$ we have $X = \{\alpha^{-1}(v,x)\}$. Consider now

$H_u = (\{u\} \times V_2, \{f \in F(\{u\} \times V_2; I) \mid \pi_2(f) \in E_2\}; I)$.

This diaper is isomorphic to H_2 as is H_w defined similarly. Let $\beta : \{u\} \times V_2 \to \{w\} \times V_2$ be an isomorphism of H_u and H_w. The points $\alpha^{-1}(v,x)$

and $\alpha^{-1}\beta\alpha(v,x)$ have a large neighbourhood intersection (and are distinct), contradicting the SC.

(iii) If $N_\alpha(u) = \emptyset$ then $[e] \subseteq O_\alpha(u)$ and picking any $w \in O_\alpha(u)$ we can induce a partition $X \cup Y$ of V_2 as in (i). As there we deduce that one of X, Y has the cardinality of V_2 and, that, therefore, the SC cannot hold.

□

Lemma 4

If H_2 satisfies the SC, $\alpha \in \mathfrak{A}$, $u \in V_1$, card $I_\alpha(u) > 1$ then $v \equiv v'$ in $\overline{H}_1(J(v)\Delta J(v'))(\{2\})$ for any $v,v' \in I_\alpha(u)$.

Proof

$(\overline{H}_1(J(v)\Delta J(v'))(\{2\}) = H'(\{2\})$ with $H' = \overline{H}_1(J(v)\Delta J(v')))$. Let $v \neq v' \in I_\alpha(u)$, $e \in \overline{E}_1(J(v)\Delta J(v'))(\{2\}) = E_1^*$. If either $[e] \cap \{v,v'\} = \emptyset$ or $[e] \subset I_\alpha(u)$, $|e| \in J(u) \cup \{2\}$ there is nothing to prove as clearly $e(v,v') \subseteq E_1^*$ (or $e^*(v,v') \subseteq E_1^*$, as the case may be). If $[e] \nsubseteq I_\alpha(u)$ then $e^*(v,v') \subseteq E_1^*$ since $\alpha \in \mathfrak{A}$. For the rest - $|e| \geq 3$, $[e] \subseteq I_\alpha(u)$, $[e] \cap \{u,v\} \neq \emptyset$ - we consider two cases, according to Lemma 3.

(i) $\mathcal{O}_\alpha(u) \neq \emptyset$. Then $|e| \in J(u)$, a situation already dealt with.

(ii) $\mathcal{O}_\alpha(u) = \emptyset$. Then each $\{w\} \times V_2$ contains a point $(w,x_w) \notin \alpha(\{u\} \times V_2)$. For $w \in \mathcal{I}_\alpha(u)$ let $(w,x^w) \in \alpha(\{u\} \times V_2)$. Suppose first that $[e] \nsubseteq \{v,v'\}$. Let $\hat{e} \in E$ be given by

$$\hat{e}(i) = \begin{cases} (e(i),x^{e(i)}) & \text{if } e(i) \in \{v,v'\} \\ (e(i),x_{e(i)}) & \text{otherwise} \end{cases}$$

(clearly $\pi_1(\hat{e}) = e$). It is then easy to see that for any $\bar{e} \in e^*(v,v')$ there is an $\bar{\bar{e}} \in E$ with $\pi_1(\alpha(\bar{\bar{e}})) = \bar{e}$: just put $\bar{\bar{e}}(i) = \alpha^{-1}((e(i),x_{e(i)}))$ if $e(i) \notin \{v,v'\}$ and $\bar{\bar{e}}(i) \in \alpha^{-1}(\bar{e}(i),x^{\bar{e}(i)})$ otherwise and observe that $\pi_1(\alpha^{-1}(\hat{e})) = \pi_1(\bar{\bar{e}})$. Hence $e^*(v,v') \subseteq E_1^*$. If $[e] \subseteq \{v,v'\}$, consider $\hat{e}_0 \in E$ defined by

$$\hat{e}_0(i) = \begin{cases} (e(0),x_{e(0)}) & \text{if } i = 0 \\ (e(i),x^{e(i)}) & \text{if } i > 0. \end{cases}$$

Looking at $\alpha^{-1}(\hat{e}_0)$ we note that for any $\bar{e}_0 \in e^*(v,v')$ with $\bar{e}_0(0) = e(0)$ we have an $\bar{\bar{e}}_0 \in E$ such that

$\pi_1(\alpha^{-1}(\hat{e}_0)) = \pi_1(\bar{\bar{e}}_0) = \pi_1(\alpha^{-1}(\bar{e}_0))$ and that, therefore, $\bar{e}_0 \in E_1^*$. But of course there is nothing special about $i = 0$ in the definition of $\hat{e}_0$: for each $\mu < |e|$ we can define an $\hat{e}_\mu$ by

$$\hat{e}_\mu(i) = \begin{cases} (e(\mu), x_{e(\mu)}) & \text{if } i = \mu \\ (e(i), x^{e(i)}) & \text{otherwise.} \end{cases}$$

If we now write $e_\mu^*(v,v')$ for the set of $\bar{e}_\mu \in e^*(v,v')$ with $\bar{e}_\mu(\mu) = e(\mu)$ we can deduce that $e^*(v,v') \subseteq E_1^*$ quite simply. Let $\bar{e} \in e^*(v,v')$ and pick $\mu < \nu < |e|$. Let $\bar{e}_\mu \in e_\mu^*(v,v')$ be such that $\bar{e}_\mu(i) = \bar{e}(i)$ except possibly at $i = \mu$. If $\bar{e}_\mu(\mu) = \bar{e}(\mu)$ there is nothing more to do. Otherwise consider $\bar{e}_\mu$ in place of e - this will yield $(\bar{e}_\mu)_\nu^*(v,v') \subseteq E_1^*$. Clearly $\bar{e} \in (\bar{e}_\mu)_\nu^*(v,v')$ which completes the proof.

□

Lemma 5

Let α, u, $I_\alpha(u)$ and H_2 be as in Lemma 4 and suppose that H_1 satisfies the TC. If A(1) and A(3) hold then $I_\alpha(u) = \{v,v'\}$ for some $v \neq v' \in V_1$ and exactly one of vv' and $v'v$

is in E_1.

Proof

Let $v \neq v' \in I_\alpha(u)$. Consider the following partitions of V_2 into $X_v \cup Y_v$ and $S_v \cup T_v$.
$X_v = \{x \in V_2 | \alpha(u,x) \in \{v\} \times V_2\}$, $Y = V_2 \backslash X_v$,
$S_v = \{x \in V_2 | \alpha^{-1}(v,x) \in \{u\} \times V_2\}$, $T_v = V_2 \backslash S_v$.
It is routine to verify that $H_2(J(u))$ is $B(v,v')$-split by $X_v \cup Y_v$. We claim that $H_2(J(u))$ is also $B(v,v')$-split. This is almost trivial in case $O_\alpha(u) \neq \emptyset$ since then $B(v,v') = J(u)\backslash\{1\} = J(v)\backslash\{1\}$ and so $\overline{H}_2(\{1\})(J(u)) = \overline{H}_2(\{1\})(J(v))$ and the partition $X_v \cup Y_v$ will work. If $O_\alpha(u) = \emptyset$ then use $S_v \cup T_v$: let $f \in F(V_2, I_2 \cup J(v))$ be such that $[f] \cap S_v \neq \emptyset \neq [f] \cap T_v$ and define $\hat{f}$ by $\hat{f}(i) = (v,f(i))$. Consider any f' satisfying

$$\alpha^{-1}(f'(i)) = \alpha^{-1}(f(i)) \qquad \text{if } f(i) \in T_v$$
$$\alpha^{-1}(f'(i)) \in \alpha^{-1}(\{v'\} \times V_2) \times \{u\} \times V_2 \qquad \text{if } f(i) \in S_v.$$

Clearly $\pi_1(\alpha^{-1}(\hat{f})) = \pi_1(\alpha^{-1}(f'))$ and $[\pi_1(f')] = \{v,v'\}$. Now, since card $\pi_1(\alpha^{-1}(\hat{f})) > 1$, we have $f \in E_2(J(v))$ if and only if $\hat{f} \in E$ if and only if $\alpha^{-1}(f') \in E$ if and only if $f' \in E$ if and only if $|f| = |f'| \in B(v,v')$.

So, by A(1), $v \not\equiv v'$ in H_1 and, since $v \equiv v'$ in $\overline{H}_1(J(v)\Delta J(v'))(\{2\})$, either $J(v) \neq J(v')$ or $vv' \in E_1$ exactly when $v'v \notin E_1$. If card $I_\alpha(u) \geq 3$ then $wz \in E_1$ for all $w \neq z \in I_\alpha(u)$ whenever $wz \in E_1$ for some $w \neq z \in I_\alpha(u)$, by ping-pong arguments. Hence either card $I_\alpha(u) = 2$ or $wz \in E_1$ if and only if $zw \in E_1$ for all $w \neq z \in I_\alpha(u)$. But the latter implies that card $I_\alpha(u) = 2$ since the TC holds for H_1 and, consequently there are at most two points with distinct sets of constants. So we have that $I_\alpha(u) = \{v,v'\}$ in any case and $J(v) \neq J(v')$. By the TC again, $J(u) = J(v)$ (without loss of generality). Hence $H_v(J(v))$ is $B(v,v')$-split by $X_v \cup Y_v$. Also, $H_{v'}(J(v'))$ is $B(v,v')$-split by $S_{v'} \cup T_{v'}$ (analogous to $S_v \cup T_v$). By A(3) there are $e \in Y_v^\mu$, $f \in S_{v'}^{\mu'}$ such that $e,f \notin E_2$ for all $\mu \in J(v)\Delta J(v')$. Let $e_u \in (\{u\} \times V_2)^\mu$ and $f_{v'} \in (\{v'\} \times V_2)^\mu$ be such that $\pi_2(e_u) = e$ and $\pi_2(f_{v'}) = f$. Clearly $e_u \notin E$, and $f_{v'} \in E$ if $\mu \in J(v')\backslash J(v)$ while $e_u \in E$ and $f_{v'} \notin E$ if $\mu \in J(v)\backslash J(v')$. But in the former case $\alpha(e_u) \in E$ and in the latter $\alpha^{-1}(f_{v'}) \in E$, neither of which is possible.

We conclude that, without loss of generality, $vv' \in E_1$, $v'v \notin E_1$ and $v \equiv v'$ in $H_1^{01} = (V_1, E_1 \cup \{v_1v_0\}; I_1)$, with $v_0 = v$, $v_1 = v'$.

□

Lemma 6

Let $G = (U,F;J)$ be a diaper satisfying the SC and let $U = X \cup Y = X' \cup Y'$ be non-trivial partitions. Let also $G^0 = G^0\langle X\rangle \vec{v}\, G^0\langle Y\rangle = G^0\langle X'\rangle \vec{v}\, G^0\langle Y'\rangle$, where G^0 is obtained from G by omitting all edges of size three or more and all constant edges. Then

(1) If $X \neq X'$ then U is finite.

(2). If card X = card X' then $X = X'$.

(3) If G^0 is in fact a bijoin with these partitions then it is isomorphic to $Z_n[K]$ for some digraph K and some (positive) integer n. Define

$$Z_n = (\{v_i \mid 0 \le i < n\}, \{v_iv_j \mid 0 \le i < j < n\}; \{2\}).$$

Proof

(1) If U is infinite and $X \neq X'$ then, since the SC holds, X (without loss of

generality) contains exactly one point. Also, one of X', Y' is a one-element set, say $\{y\}$. Clearly $x \neq y$ (by assumption if card $X' = 1$ and from the fact that G^0 is a dijoin with each of these partitions otherwise). But it is also clear that card$(N(x) \cap N(y))$ = card U, contradicting the SC.

(2) If card X = card X' and $X \neq X'$ then there are $x \in X \setminus X'$ and $y \in X' \setminus X$ such that $xy, yx \in F$, contradicting the definition of a dijoin.

(3) Suppose G^0 is a bijoin with the given partitions. Then U is finite (by (1) if $X \neq X'$ and by the SC if $X = X'$ since that means card X = card $Y = 1$) and $G^0\langle X\rangle \simeq G^0\langle Y'\rangle$ and $G^0\langle Y\rangle \simeq G^0\langle X'\rangle$. Without loss of generality assume card $X \geq$ card Y. It is easy to see (argument of (2)) that $Y \subseteq Y'$ and $X' \subseteq X$. We will now proceed by induction on card U. A bijoin must have at least two points and the case card $U = 2$ is trivial. Suppose the claim is true for all bijoins on less than card U vertices and consider two cases.

(i) $X \cap Y' = \emptyset$. Then
card X = card Y = card X' = card Y' = = 1/2 card U and, by (2), $X = X'$, $Y = Y'$. We can take $n = 2$ and $K \simeq G^0\langle X\rangle$.

(ii) $X \cap Y' \neq \emptyset$. Then $X \cap Y' = X \backslash X' = Y' \backslash Y$ and we have $X = X' \cup (X \cap Y')$, $Y' = Y \cup (X \cap Y')$. In fact, we have more: $G^0\langle X\rangle$ is a bijoin since $G^0\langle X'\rangle \vec{v}\ G^0\langle X \cap Y^0\rangle = G^0\langle X\rangle \simeq G^0\langle Y'\rangle = = G^0\langle X \cap Y'\rangle \vec{v}\ G^0\langle Y\rangle$. By induction hypothesis $G^0\langle X\rangle \simeq Z_p[K]$ for some p and K. To complete the proof we only need to show that $G^0\langle X \cap Y'\rangle \simeq Z_m[K]$ for some m and the same K. But that is clear from the recursive construction implicit in the argument.

□

Lemma 7

If H_2 satisfies the SC and H_1 the TC and if A(1), A(3) and C hold then $\mathfrak{A}$ preserves copies.

Proof

If not then there are α_1, u, $I_\alpha(u) \not\subseteq \{v,v'\}$ as in Lemmas 3-5. We have, without loss of generality,

$vv' \in E$ and $v'v \notin E$. Let us denote by H'_w the subdiaper of H induced by $\{w\} \times V_2 = V_w$, that is, isomorphic to $H_2(J(w))$. Let us also put $p_0 = u$, $r_0 = v$, $r_1 = v'$ and $P_0 = \{p_0\}$, $R_0 = \{r_0, r_1\}$. We will

(i) for $n < \omega$ construct sets P_n and R_n of points in V_1 so that $P_n = \{p_i \mid -n \leq i \leq n\}$, $R_n = \{r_i \mid -n \leq i \leq n+1\}$, $p_i p_j \in E$ and $r_i r_j \in E$ if and only if $i < j$.

(ii) Put $P = \bigcup_{n<\omega} P_n$, $R = \bigcup_{n<\omega} R_n$ and show that these are underlying sets for $(B(v,v'),J(u))$-Z-chains in H_1, provided that V_1 is infinite.

(iii) Show that H'_u is a $B(v,v')$-bijoin.

(iv) Conclude that α preserves copies.

This is how.

(i) Suppose we have P_n and R_n such that

(a) $p_i p_j \in E_1$ if and only if $-n \leq i < j \leq n$

(b) $r_i r_j \in E_1$ if and only if $-n \leq i < j \leq n+1$

(c) each V_{p_i} and each V_{r_i} is partitioned (non-trivially): $V_{r_i} = X_i \cup Y_i$, $V_{r_i} = X'_i \cup Y'_i$ so that $\alpha(X_i) = Y'_i$, $\alpha(Y_i) = X'_{i+1}$, $-n \leq i \leq n$.

To construct P_{n+1} we must add p_{n+1}, p_{-n-1} to P_n preserving the above properties. This is done as follows. Consider $\alpha^{-1}(Y'_{n+1})$. It must be disjoint from each V_{p_i} so far obtained. It follows from Lemmas 4 and 5 that there is a $p_{n+1} \in V_1 \setminus P_n$ such that $\alpha^{-1}(Y'_{n+1}) \subsetneq V_{p_{n+1}}$ and $p_{n+1} \equiv p_n$ in $H_1(\{2\})$. We put $X_{n+1} = \alpha^{-1}(Y'_{n+1})$, $Y_{n+1} = V_{p_{n+1}} \setminus X_{n+1}$ and note that $p_i p_{n+1} \in E_1$ for all $-n \le i \le n$, by ping-pong arguments, and - similarly - $p_{n+1} p_i \notin E_1$. This is easy to see: $r_i r_{n+1} \in E_1$ for $-n \le i \le n$ by assumption and so $(r_i, x_i)(r_{n+1}, y_{n+1}) \in E$ as well as $(r_i, y_i)(r_{n+1}, y_{n+1}) \in E$ for $x_i \in X'_i$, $y_i \in Y'_i$, $-n_i \le i \le n+1$. Any of these is carried to an edge by α^{-1} which in turn implies the claim. Consider now $\alpha^{-1}(X'_{-n})$. This is disjoint from V_{p_i} for $-n \le i \le n$ and from X_{n+1}. Further, $\alpha^{-1}(X'_{-n}) \cap Y_{n+1} = \emptyset$ since otherwise $p_{n+1} p_n \in E_1$. Hence a p_{-n-1} can be found in $V_1 \setminus (P_n \cup \{p_{n+1}\})$ such that $\alpha^{-1}(X'_{-n}) \subsetneq V_{p_{-n-1}}$. As before we conclude that $p_{-n-1} \equiv p_{-n}$ in $H_1(\{2\})$ and that

$p_{-n-1}p_i \in E_1$ for $-n \le i \le n+1$ while $p_ip_{-n-1} \notin E_1$. This gives P_{n+1}.
R_{n+1} is constructed - mutatis mutandis - analogously, giving r_{-n-1}, r_{n+2}, X'_{-n-1}, Y'_{-n-1}, X'_{n+2}, Y'_{n+2} with $r_{-n-1}r_i \in E_1$, $r_ir_{-n-1} \notin E_1$, $r_jr_{n+2} \in E_1$, $r_{n+2}r_j \in E_1$ for $-n \le i \le n+2$, $-n-1 \le j \le n+1$. Also $r_{-n-1} \equiv r_{-n}$ and $r_{n+1} \equiv r_{n+2}$ in $H_1(\{2\})$.

(ii) If V_1 is infinite and no contradiction has, therefore, appeared preventing the construction, let $P = \bigcup_{n<\omega} P_n$, $R = \bigcup_{n<\omega} R_n$. We have, by construction, $p_i \equiv p_j$ in $H_1\langle P\rangle H_1^{ij}$ and $r_i \equiv r_j$ in $H_1\langle R\rangle H_1^{ij}$ for all $i \neq j$. So both $H_1\langle P\rangle$ and $H_1\langle R\rangle$ are $(B(v,v'),J(u))$-Z-chains in H_1.

(iii) Consider $H^0_{p_0} = D$ (i.e. $H^0_{p_0}$ less edges of size three or more, as in Lemma 6). We have, by ping-pong arguments (since $r_0r_1 \in E_1$, $r_1r_0 \notin E_1$), $D = D\langle X_0\rangle \vec{v}\, D\langle Y_0\rangle$. Put $H_{r_0} = D'$; then $D' = D\langle X'_0\rangle \vec{v}\, D'\langle Y'_0\rangle$. Clearly $D \simeq D'$, thus
$H^0_2 = H^0_2\langle X\rangle \vec{v}\, H^0_2\langle Y\rangle = H^0_2\langle X'\rangle \vec{v}\, H^0_2\langle Y'\rangle$
with $X = \pi_2(X_0)$, $Y = \pi_2(Y_0)$, $X' = \pi_2(X'_0)$,

$Y' = \pi_2(Y_0')$. Now $X \neq X'$ if V_2 is infinite. Hence, by Lemma 6(1), V_2 is finite in any case. This means that card X_i = card X_j for all i and j and similarly for Y_i and Y_j, X_i' and X_j', Y_i' and Y_j'. Thus, by construction, $H_2^0\langle X\rangle \simeq X_2^0\langle Y'\rangle$ and $H_2^0\langle Y\rangle \simeq H_2^0\ X'$ and H_2^0 is a bijoin. But clearly, H_u is a $B(v,v')$-bijoin, from the preceding work (or directly by ping-pong arguments).

(iv) This contradicts C. So α must preserve copies.

□

CHAPTER IV

COROLLARIES AND COMMENTS

It is easy to see that the conditions A(2) and A(3), B(2) and B(3) become the same if, for some $\mu \in I_1$, $J(H_1) = \bigcup_{x \in V_1} J(x) \subseteq \{\mu\}$. This leads to the following

Corollary 1

If H_2 satisfies the SC and if card $J(H_1) \le 1$ then $\mathfrak{A} \cong \mathfrak{A}_1[\mathfrak{A}_2]$ if and only if A(1), A(2), B(1), B(2) and C hold.

Corollary 2

If H_2 satisfies the SC and $J(H_1) = \emptyset$ then $\mathfrak{A} \cong \mathfrak{A}_1[\mathfrak{A}_2]$ if and only if A(1) and C hold.

There is a host of corollaries to be obtained with the help of the following lemma.

Lemma 8

A diaper (V,E;I) satisfies the SC if one of the following holds

(1) V is finite,

(2) the set $E(x) = \{e \in \overline{E}\backslash C(V;I) | x \in [e]\}$ is finite,

(3) the number of vertices for which $E(x)$ is infinite is finite.

We will not list any of the corollaries available from Lemma 8 but restrict ourselves to saying that the results of [9], [10] and [15] (Theorems 1, 3 and 5) are among them. It is clear that Theorem 4 is a consequence of Corollary 2 and, from this, that Theorem 2 follows from the results of Chapter III. We will now point out how Theorems 5 and 6 and, hence, Theorems 1 and 2 in another way, can be obtained from the present work.

Let $H = (V,F)$ be an (ordinary) hypergraph. For each $e \in F$ (that is $\emptyset \neq e \subseteq V$) let $|e|$ be the least ordinal that well-orders e and let $o(e) = \{f \in e^{|e|} | f$ is a bijection$\}$. Put $E = \bigcup_{e \in F} o(e)$ and define $\hat{H} = (V,E;I)$ with $I = \{|e| | e \in F\}$. Corollary 1 now applies to the composition $\hat{H}_1[\hat{H}_2]$ of diapers obtained from given disjoint hypergraphs H_1 and H_2; all we need is a translation of the conditions. This is routine.

As we mentioned in Chapter III, the Axiom of Choice appears - possibly - many times. Though it is not clear that a proof of Theorem 8 cannot be found that does not make use of this axiom, we suspect that this may well be the case. At least it is far from obvious that the translation from hypergraphs to diapers (using the Well Ordering Axiom) can be achieved without it. It is hoped that further research will settle this question.

To conclude this chapter we provide an example showing that there are diapers H_1, H_2 such that H_1 satisfies the TC, H_2 does not satisfy the SC, the conditions of Theorem 8 hold and $\mathfrak{A}(H_1[H_2]) \not\cong \mathfrak{A}(H_1)[\mathfrak{A}(H_2)]$. The example is a simple case of the construction in [16] of lexicographically idempotent graphs. The fact that $\mathfrak{A}(H_1[H_2]) \not\cong \mathfrak{A}(H_1)[\mathfrak{A}(H_2)]$ was pointed out by Sabidussi in a private conversation.

Let X be a set of cardinality at least three and let $x_0 \in X$. Let Q denote the non-negative rationals. Define a graph $G = (V,E)$ by letting V be the subset of X^Q such that $f(a) = x_0$ for all but finitely many $a \in Q$ and by putting $\{g,f\} \in E$ if and only if exactly one of $g(a)$, $f(a)$ is x_0 with a being the least (in the natural order of Q) such that $f(a) \neq g(a)$. Now $G[G] \simeq G$: let $\alpha : Q \to [0,1) \cap Q$ and $\beta : Q \to [1,\infty) \cap Q$ be order-isomorphisms and map $(f,g) \to h$ by

$$h(a) = \begin{cases} f(\alpha^{-1}(a)) & \text{if} \quad a \in [0,1) \\ g(\beta^{-1}(a)) & \text{if} \quad a \in [1,\infty). \end{cases}$$

Call this mapping γ. Not only is this the required isomorphism, it also allows for many automorphisms of $G[G]$ which are not in $\mathfrak{A}(G)[\mathfrak{A}(G)]$. In fact, for any $0 \neq b \in Q$ the mapping $\alpha_b : G \to G$ is in $\mathfrak{A}(G)$ if $\alpha_b(f(a)) = f(ab)$. Combining α_b with γ we get $\beta_b = \gamma^{-1}\alpha_b\gamma$, an automorphism of $G[G]$. If $b \in (0,1)$, $\beta_b \notin \mathfrak{A}(G)[\mathfrak{A}(G)]$.

We end this thesis by mentioning that Sabidussi conjectures the following: *if* G *is a lexicographically idempotent graph (i.e.* $G[G] \simeq G$*) then* $\mathfrak{A}(G[G]) \neq \mathfrak{A}(G)[\mathfrak{A}(G)]$.

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