

**HERMANN COHEN'S *DAS PRINZIP DER INFINITESIMAL-*
*METHODE UND SEINE GESCHICHTE***

**HERMANN COHEN'S *DAS PRINZIP DER INFINITESIMAL-
METHODE UND SEINE GESCHICHTE***

By

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Abstract

In his 1883 book *Das Prinzip der Infinitesimal-Methode und seine Geschichte*, Hermann Cohen gives a detailed history of calculus combined with an argument for its philosophical significance. Through his historical account Cohen seeks to provide the “critical grounding” of the concept of the infinitesimal and to show that the infinitesimal is a necessary presupposition for mathematical natural science.

From its earliest reception Cohen’s book has faced harsh criticism, and there is no doubt that this poor reception is due in part to Cohen’s difficult writing style. The difficulty of the book has also led to a general lack of detailed discussion of its contents, despite the fact that the work is almost universally regarded by commentators as marking an important transition in Cohen’s thought from his early interpretation of Kant towards his own system of philosophy.

The purpose of this thesis is to provide a detailed account of the main thrust of Cohen’s historical argument in which he identifies Leibniz and Newton as systematic forerunners of Kant. Understanding this argument requires that it be situated within the context of *Erkenntniskritik*, Cohen’s neo-Kantian system of philosophy. Thus, I begin by discussing *Erkenntniskritik* and aspects of Cohen’s interpretation of Kant that are particularly relevant to his critical grounding of the infinitesimal. This discussion is followed by a presentation of Cohen’s historical argument focusing on his treatment of Leibniz, Newton, Galileo, and Kant. Lastly, I consider Bertrand Russell’s and Gottlob Frege’s well-known criticisms of Cohen’s book.

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List of Abbreviations

CPR—Kant, Immanuel. Critique of Pure Reason. Trans. Norman Kemp Smith. New York: St. Martin's Press, 1929.

KTE—Cohen, Hermann. Kants Theorie der Erfahrung. Hildesheim: G. Olms, 1987.

PIM—Cohen, Hermann. Das Prinzip der Infinitesimal-Methode und seine Geschichte: ein Kapitel zur Grundlegung der Erkenntniskritik (1883). Frankfurt am Main: Suhrkamp Verlag, 1968.

TF—Cohen, Hermann. "Zur Kontroverse zwischen Trendelenburg und Kuno Fischer" (1871). Hermann Cohens Schriften zur Philosophie und Zeitgeschichte. Ed. Albert Görland and Ernst Cassirer. Berlin: Akademie Verlag, 1928. 229-275.

Introduction

In his 1883 book *Das Prinzip der Infinitesimal-Methode und seine Geschichte* (*The Principle of the Infinitesimal Method and its History*, hereafter cited as PIM), Hermann Cohen presents a history of calculus and an argument for its philosophical grounding. However, the PIM cannot be viewed solely as a treatise on the history and philosophy of mathematics. This fact is partly made evident by the fact that as a treatise on the foundations of calculus, Cohen's book is sharply out of step with its time. The second half of the nineteenth century saw dramatic developments in the foundations of mathematics, predominated by the pursuit of clarity and rigor and the move away from geometrical arguments in favor of purely arithmetical ones. Finding a secure ground for calculus was a primary concern of mathematicians of the time and the field was revolutionized by the work of Weierstrass, Dedekind, Cantor, and others. Within this context Cohen's proposed grounding appears odd and out of date. First of all, he makes no mention of these new developments in mathematics. Instead, he endeavors to establish the foundations of calculus through a critical assessment of its historical development. He operates under the working supposition that calculus's true foundations can only be discovered by revealing the connections and conceptual developments that lead to its discovery. Setting Cohen further at odds with the prevailing spirit of mathematics at the time is the fact that his discussion is couched in Kantian terminology and, most crucially, that he seeks to ground calculus through a justification of infinitesimals, or infinitely small quantities. The ontological status of infinitesimals had been *the* major source of the

controversy surrounding calculus since its discovery, and many theorists felt that the major achievement of the new mathematics of the time was to have finally done away with these ambiguous mathematical entities.

Given these discrepancies between the PIM and contemporary trends within mathematics at the time, if one were to read the book solely as a treatise on the philosophy of mathematics it would appear to be little more than an anomaly resulting from a naïve philosopher, without the requisite aptitude in mathematics, treading into foreign territory. However, as this thesis is partly intended to show, the PIM cannot be read in this way and the scope of Cohen's argument is much broader than the philosophy of mathematics. This fact begins to become clear when we call attention to the oft-overlooked subtitle of the book, which reads: "A Chapter in the Foundation of *Erkenntniskritik*." *Erkenntniskritik* refers to Cohen's neo-Kantian system of philosophy, and the PIM cannot be understood outside the context of this larger project and philosophical perspective. As Gregory Moynahan writes in his essay on the work's place within Cohen's thought, it was "intended to form something of a 'popularization' of Cohen's ideas...[and] formed the germ of Cohen's own philosophy as put forth in the second edition of his *Kants Theorie der Erfahrung* two years later" (35). What Moynahan observes here is crucial to the purpose of this thesis. The PIM is universally recognized by commentators as containing the germ of Cohen's systematic philosophy, and that of the Marburg School of neo-Kantianism that he founded. The philosophy of the Marburg School ultimately encompassed diverse subject matter, including philosophies of science, aesthetics, law, ethics, and religion. In spite of renewed interest in Cohen and the

Marburg school generally, and the PIM's wide recognition as containing the germ of Cohen's later thought, there is very little written directly about the work.¹ This surprising lack of discussion may be partly explained by the text's difficult style, which as Moynahan explains, has been regarded from its earliest reception by critics as being "extremely difficult and occasionally obtuse" (35).

In my view, the reasons for the difficulty of PIM are twofold. First, there is the inherent difficulty and complexity of the history of calculus. Cohen's historical account requires specialized knowledge of seventeenth-century science and mathematics, which he does not provide in the book. Second, Cohen's arguments are based on a systematic approach to philosophy. He makes reference throughout the book to conceptual aspects of *Erkenntniskritik* without taking the time to clearly and explicitly spell out exactly what this neo-Kantian framework entails. This crucial framework was only explicitly discussed by Cohen two years later in the second edition of *Kants Theorie der Erfahrung* (*Kant's Theory of Experience* hereafter cited as KTE).

This thesis is a presentation of Cohen's argument in the PIM that resolves the two interpretive difficulties I here described. Hence, in chapter 1, entitled "Questions of Objectivity," I provide context for the PIM by discussing Cohen's unique notion of objectivity. This notion dictates that the objects of our knowledge do not correspond to determinate "things-in-themselves," but are "constructed" by means of synthetic relations. To explain this notion, I begin with a brief sketch of Cohen's first published

¹ Aside from Moynahan's paper, the other notable exception to this rule within the English literature on Cohen is Lydia Patton's dissertation, "Hermann Cohen's History and Philosophy of Science." In chapter 3, entitled "The Infinitesimal Method," Patton discusses primary aspects of Cohen's argument in the PIM with a view towards his later philosophy of science.

paper on Kant interpretation: “Zur Kontroverse zwischen Trendelenburg und Kuno Fischer” (“On the Trendelenburg-Fischer Debate”). Published in 1871, this paper appeared shortly before the first edition of KTE. In it Cohen seeks to both clarify and resolve the dispute between Adolf Trendelenburg and Kuno Fischer over the proper interpretation of Kant’s transcendental aesthetic. As we will see, this debate sets the stage for Cohen’s interpretation of Kant and development of *Erkenntniskritik*. *Erkenntniskritik* is central to the PIM, but Cohen only vaguely explains this Kantian approach to philosophy. Thus, the tenets of this schema will be the second topic of discussion in this chapter. I conclude this chapter by returning to the Trendelenburg-Fischer article to highlight comments Cohen makes regarding the history of philosophy, its importance and how it should be practiced.

In chapter 2, entitled “Cohen’s Neo-Kantianism,” I further supplement the presentation of Cohen’s system of philosophy in the PIM by taking a detailed look at aspects of his interpretation of Kant. The bulk of this interpretation is contained in the two editions of KTE, and providing a close reading of these lengthy works is beyond the scope of this project. I therefore highlight the main aspects of this interpretation relevant to the PIM by drawing from and consolidating the partial accounts given in a number of secondary sources. The chapter’s primary focus is to present Cohen’s appropriation of Kant’s transcendental method of philosophy and his reorientation of this method towards a philosophy of science. The topics of discussion will be: Kant’s “new” concept of experience; Cohen’s redefinition of the Kantian terms “thought” and “intuition”; the importance of the category of reality and the “anticipations of perception” for Cohen’s

argument in the PIM; and the connection he draws between the principle of continuity and the “unity of consciousness.” Lastly, I consider in what sense Cohen’s interpretation can be considered a “return to Kant.”

Chapter 3, entitled “The History of the Infinitesimal Method,” is a presentation of what I consider to be the main thrust of Cohen’s historical argument in the PIM.

Throughout his historical discussion, Cohen extensively quotes Leibniz, Newton, and others to reveal the motivations behind their discovery of calculus. Throughout this chapter I therefore cite these passages and discuss Cohen’s interpretation of them. My discussion focuses on Cohen’s treatment of Leibniz, Galileo, Newton, and Kant, and is intended to illustrate Cohen’s critical grounding of the concept of the infinitesimal. I show how he tries to provide this justification by interpreting the historical evidence through the lens of *Erkenntniskritik*. In doing so, I highlight the ways in which Cohen’s approach is based on Kant’s “transcendental logic” and how his approach differs from the attempts to ground the infinitesimal through “general logic.”

The PIM has been subject to harsh criticism since its publication. In chapter 4, entitled “Cohen’s Critics,” I present two well-known criticisms of the book: Bertrand Russell’s critique in *The Principles of Mathematics* and Gottlob Frege’s 1885 review. These critiques of the PIM are thematically presented and are critically discussed to highlight both conceptual aspects of the Cohen’s book, and to highlight some methodological considerations. For example, Cohen’s notions of pure and applied mathematics are contrasted with Frege’s and Russell’s within the context of their respective philosophical viewpoints. Setting Cohen up in opposition to his critics is also a

useful means for understanding the polemical dimension to the PIM and the equally polemical dimensions to its reception. There is certainly much more than the ontological status of the infinitesimal at stake in this exchange. The chapter concludes with a direct assessment of the viability of Cohen's position in the PIM based on Russell's and Frege's criticisms.

Chapter 1 — Questions of Objectivity

The purpose of this chapter is to introduce Cohen's notion of objectivity. Overcoming subjective-objective dualism is a central concern in Cohen's thought. As Andrea Poma observes, it was the potential Cohen saw in Kant's transcendental philosophy to break down the supposed opposition between subjective impressions of objects and the objective world as it exists in itself that motivated Cohen to adopt what he considered to be the Kantian method of philosophizing (Poma, 4). The notion of objectivity plays a central role in the first paper Cohen published dealing with Kantian philosophy: his 1871 article "Zur Kontroverse zwischen Trendelenburg und Kuno Fischer" ("On the Trendelenburg-Fischer Debate").² The debate in question centered on how to properly interpret Kant's "transcendental aesthetic" and how space and time as a priori intuitions should be understood. Commentators disagree as to whether Cohen's interjection can be considered decisive in the debate; however, it is widely recognized that the Trendelenburg-Fischer paper is important when it comes to understanding Cohen's thought. In his book *The Rise of Neo-Kantianism*, Klaus Christian Köhnke situates the development and dissemination of the neo-Kantianism within the intellectual climate of nineteenth-century Germany. Köhnke argues that Cohen's paper acquires

a special significance for having first introduced a novel interpretation of Kantian philosophy: it contains a first sketch of the Marburg picture of Kant, and [...the] specific understanding of Kant, associated with Marburg to no small degree, answered questions which precisely

² This paper first appeared in *Zeitschrift für Völkerpsychologie und Sprachwissenschaft*, 7: 249-96. I cite from the version in *Hermann Cohens Schriften zur Philosophie und Zeitgeschichte*, Ed. Ernst Cassirer and Albert Görland (Berlin: Akademie-Verlag, 1928): 229-75.

[the Trendelenburg-Fischer] debate had thrown open. (176)

Köhnke is here referring to questions that pertain to the nature of objectivity, and more generally, to the relevance of Kant's thought to the natural sciences of the latter half of the nineteenth century. I will provide a brief overview of Cohen's argument from this paper and will then discuss his notion of objectivity, which became one of the central tenets of the Marburg School. Lastly, I will discuss Cohen's views on the importance of the history of philosophy. This topic is also broached by Cohen in the Trendelenburg-Fischer paper, and his unique understanding of objectivity plays a central role in what he prescribes.

1.1 *The Trendelenburg-Fischer Debate*

Cohen's paper "On the Trendelenburg-Fischer Debate" is a response to the famous dispute between the Aristotelian Adolf Trendelenburg and the neo-Hegelian philosopher Kuno Fischer. The dispute began in 1862 when, in the second edition of his *Logische Untersuchungen*, Trendelenburg critiqued a form of Hegelian logic presented in Fischer's *Logik und Metaphysik*. However, it was with Trendelenburg's 1867 treatise "Über eine Lücke in Kants Beweis von der ausschliessenden Subjektivität des Raumes und der Zeit", ("On the Gap in Kant's Proof of the Exclusively Subjective Nature of Space and Time") that the dispute became focused on the proper interpretation of Kant's transcendental aesthetic (Köhnke, 170). It was also at this time that Trendelenburg and Fischer became personally hostile towards each other. Thereafter the discussion descended further and further into confusion and partisanship, with both sides resorting to personal attacks,

tangential embellishments, and polemical rants, as indicated by the titles of some of the interjections, such as Trendelenburg's *Kuno Fischer and his Kant* (1869) and Fischer's *Anti-Trendelenburg* (1870) (Köhnke, 169). Köhnke notes that although there are approximately fifty brochures, treatises, and reviews centering on the dispute from various commentators, Cohen's contribution in 1871 stands out for focusing on the factual content at hand. The central issue, Köhnke claims, was a disagreement over "whether the transcendental aesthetic was to be understood as a realist or an idealist theory of experience" (168). Cohen's own interpretation of Kant was not fully presented until the first edition of KTE was released later in 1871. His critical responses to both Trendelenburg and Fischer are illustrative because these replies set the stage for Cohen's new interpretation, so much so that, as Köhnke points out, an anonymous critic commenting on KTE remarked that Cohen's argument that Kant's primary epistemological significance was having discovered a new theory of experience, "seemed to come right out of the Fischer-Trendelenburg contention" (176).

Cohen's paper centers on two main questions: 1) whether Trendelenburg had indeed established that Kant had left a gap in his proof of the exclusive subjectivity of space and time; and 2) whether Fischer had incorporated something un-Kantian into his presentation of Kantian doctrine. He would also ultimately answer what he called a "secondary question" of whether Fischer was successful in demonstrating that Trendelenburg's supposed gap does not exist (TF, 231/2).³ With regard to this last

³ My citations from Cohen's Trendelenburg-Fischer paper refer to two sources: the first page reference refers to the edition in *Schriften zur Philosophie und Zeitgeschichte*; and the second page reference refers to Patton's unpublished translation of the paper, which I have read in parallel with the German. Unless stated otherwise, the English translations of the article given in this chapter are from her draft.

question, Cohen rules that Fischer did not succeed in demonstrating a flaw in Trendelenburg's argument.⁴ However, Cohen was not in total agreement with Trendelenburg, who argued that when Kant claimed to have proved the "exclusive subjectivity" of space and time, he only considered two possibilities: that space and time are merely objective or that they are merely subjective. Trendelenburg claimed that a "gap" existed in Kant's proof because the latter had not considered a third possibility that space and time could be purely subjective and yet also be true of the objective world (TF, 232, 3). As a result of this oversight, Trendelenburg argued, Kant's proof is incomplete.

For the sake of clarity, I would like to quickly review Kant's position before addressing Cohen's assessment of Trendelenburg's argument about it. In the first *Critique*, Kant defines the "transcendental aesthetic" as "the science of all principles of *a priori* sensibility" (CPR, B35). The term "sensibility" is defined as "the capacity for receiving representations through the mode in which we are affected by objects" (B33). Kant argues that our representations are conditioned, *a priori*, by intuition. As the science of the *a priori* principles of intuition, the purpose of the transcendental aesthetic is to isolate sensibility (from the understanding) and identify the two pure forms of intuition: space and time (B36). Kant claims that space and time are subjective conditions of sensibility, without which our representations would not be possible. Space, for example, conditions our "outer sense," by means of which "we represent to ourselves objects as

⁴ It should be noted that Cohen had a personal stake in the debate when he argued in favor of Trendelenburg in this regard. Trendelenburg was influential in the Berlin intellectual scene in which Cohen was involved, the journal in which Cohen's paper appeared was closely connected with Trendelenburg, and Cohen himself had studied with Trendelenburg. In contrast, Cohen had no such personal connection with Fischer and expressed a very negative view of Fischer's lectures after having listened in on them. (Poma, 4–5).

outside us, and all without exception in space. In space their shape, magnitude, and relation to one another are determined and determinable” (B37). Time conditions our inner sense, “by means of which the mind intuits itself or its inner state” (B37).

After identifying space and time as the pure forms of our intuition, Kant asks the question:

What, then, are space and time? Are they real existences? Are they only determinations or relations of things, yet such as would belong to things even if they were not intuited? Or are space and time such that they belong only to the form of intuition, and therefore to the subjective constitution of our mind, apart from which they could not be ascribed to anything whatsoever? (B37)

To answer this question, Kant undertakes a transcendental analysis of both space and time. In doing so, he claims to show that they both are purely subjective conditions. His argument for the pure subjectivity of space, for example, proceeds by showing that space is an a priori necessary requirement for representation and that spatial relations are not empirical concepts derived from our experience because these spatial relations must necessarily be presupposed a priori to our representations bearing any order. (B38) He concludes that it is “solely from the human standpoint that we can speak of space, of extended things, etc. If we depart from the subjective condition under which alone we can have outer intuition, namely liability to be affected by objects, the representation of space stands for nothing whatsoever” (B42).

In his assessment of Trendelenburg’s gap, Cohen observes that Trendelenburg uses the terms “pure subjectivity” and “a priori” in relation to the intuitions of time and space in the same way—“both mean that they presuppose no empirical perceptions, no experience” (TF, 233/4). In contrast, for Trendelenburg, for space and time to be “purely

objective” means they are grounded only in things and known only empirically (TF, 233/4). Cohen further observes that Trendelenburg stands in agreement with Kant insofar as he shows space and time to be purely subjective, and that the intuitions of space and time cannot be acquired purely from experience and that they cannot be grounded in things (TF, 234/4). Cohen claims that Trendelenburg disagrees with Kant insofar as Kant takes the proof that space and time are purely subjective as proof that they are merely, or exclusively, subjective. Trendelenburg’s problem is, as Cohen puts it, that “*pure* subjectivity *cannot* mean that space and time are real *only and exclusively in us*, and that they cannot be valid for anything outside us” (TF, 235/5). For Trendelenburg, if space and time cannot be true of anything outside our subjective representations, then it would be possible for something real to exist that is beyond the realm of our subjectivity. How then could we know anything of this objective reality? If space and time are exclusively subjective, then what assurance is there that our subjective representation of an object is true of the object in itself? That is why, Cohen argues, Trendelenburg seeks to advocate a third option: that space and time are simultaneously “necessary to the representation, real in the things” (TF, 237/6). Trendelenburg sought to fulfill this third option, as Patton explains in her dissertation, by deriving an account of time and space based on the physiological sensory apparatus and psychological processes of the individual subject (79). From this standpoint of parallelism, space and time could be seen as purely subjective, because they are a condition for the possibility of experience insofar as they result from the subject’s biological structures and cognitive faculties, and yet can also be

representative of the objective world, which has determinate characteristics that are represented in the cognitive processes of the subject.

Cohen sides with Trendelenburg insofar as Cohen is concerned to stave off interpretations of Kant like that of Fischer, who argues that pure intuitions are, as Köhnke writes, “perceptions without a given object” (174). Such interpretations negate the role of pure intuitions as the conditions for the possibility of experience. And as Patton points out, these kinds of interpretations were thought by Cohen to lead towards the view that the “objects of cognition are mere phenomena.”⁵ Cohen is certainly correct in ruling that Fischer did put something “un-Kantian” into his reading of Kant, given that Kant clearly states in the opening lines of the transcendental aesthetic that intuition is that mode of knowledge through which it immediately relates to objects (CPR, B33).

Cohen disagrees with Trendelenburg’s notion of objectivity. Recall that Trendelenburg had defined knowledge as being purely objective when it is derived purely from things, or empirically. Cohen argues that Trendelenburg has a different notion of objectivity and of experience than Kant does; however, Cohen does not develop this in a positive sense until later that year in the first edition of KTE, where he argues that Kant’s primary epistemological significance is the discovery of a “new” concept of experience. This new concept of experience and other aspects of Cohen’s interpretation of Kant are the subject matter for Chapter 2; however, before moving on I would like to address other aspects of Cohen’s methodology that come to the fore in the Trendelenburg-Fischer paper.

⁵ Patton “The Critical Philosophy Renewed,” 112.

1.2 *Erkenntniskritik*

Cohen differs from Trendelenburg not only in his interpretation of Kant, but also in the methodology to which Trendelenburg subscribes. Trendelenburg is representative of the first wave of *Erkenntnistheorie*, a movement which, Patton explains, was characterized by the desire to “give an explanation of space, time and the categories of thought as the results of a natural, causal process,” and also “to banish dialectic from logic, by restricting logic to reflections on the data of empirical science.”⁶ In order to understand the difference between this approach to philosophy and Cohen’s it is useful to consider the statements of Paul Natorp on the methodology of the Marburg School which he and Cohen co-founded. In a paper on the Marburg school’s interpretation of Kant, Natorp claims that one of the primary presuppositions behind the approach of Trendelenburg and other exponents of *Erkenntnistheorie* was that one could “begin from fundamental and irreducible concepts and indemonstrable propositions and aim at judgments of identity (‘analytic’ in Kant’s sense).”⁷ This approach, Natorp asserts elsewhere, presupposes that the object as determinately given “as a product of finite and hence exhaustible elements.”⁸ The task of acquiring knowledge then consists in drawing out these elements, which are already present in the object, through a sufficiently penetrating analysis. Kant defines analytic judgments as ones for which “the predicate B belongs to subject A, as something which is (covertly) contained in its concept” (CPR, B10). Given that

⁶ “Hermann Cohen’s History and Philosophy of Science,” 29.

⁷ Natorp “Kant und die Marburg Schule,” 196, as cited by Patton “Hermann,” 79.

⁸ Natorp, *Plato’s Theory of Ideas*, 343, as cited by Veal, 13.

Trendelenburg's goal is to arrive at knowledge through a series of analytic judgments, it becomes understandable that he would want to secure the mutual subjectivity and objectivity of space and time. In order for such judgments of identity to form a secure basis for knowledge, there must be some assurance that the object given in subjective representation corresponds to a real object. Since objects in the subject's representation are given in space and time, space and time must be able to lay some claim to objectivity. Otherwise, Cohen asks in paraphrasing Trendelenburg's position, "how does the object wander over into your representation?" (TF, 235/ 5) As I have discussed, Trendelenburg desired to establish a foundation of knowledge based on a causal sense-physiological account. Through such an account he hoped to secure the mutual subjectivity and objectivity of space and time. Space and time are subjective insofar as they result from the sensory apparatus of the subject. They are considered objective, because a causal account could be given explaining the real physical process that underlies the subjective representation of real objects.

The Marburg School's critical approach to knowledge presupposes an entirely different relation between subject and object than that advocated by Trendelenburg. In contrast to such positions, Natorp explains that the critical view of the Marburg School "takes its stand in the path (of knowledge) and not in the goal (the object), asserting that being—namely the being of knowledge—is to be understood by appeal to knowledge and denying that knowledge is to be understood by appeal to being—as if we were already in the possession of being."⁹ This means that the critical approach inaugurated by Cohen

⁹ Natorp, *Plato's* 343, as cited by Veal, 14.

does not require that one posit an object in itself to which knowledge must correspond; rather, knowledge itself becomes the starting point of critical inquiry. For Cohen, the facts of natural science expressed in systems such as Newtonian physics are of particular interest, and he argues that studying the principles on which their validity rests calls for a distinct area of research.

This distinct area of research is what Cohen calls *Erkenntniskritik*.¹⁰ In the PIM, he states that *Erkenntniskritik* takes scientific facts as its starting point, and seeks to establish the a priori principles that allow for the conditions for the possibility of natural science and on whose foundation the validity of science rests (§9, 49). Patton explains in her dissertation that one of the key tenets of *Erkenntniskritik* is that it “treats the axioms of a theory as synthetic rather than analytic propositions: they are principles that govern the construction of a theory, rather than first principles of a deduction” (81). Thus, the task of philosophy should be to undertake a transcendental analysis that deals with the internal validity of principles and laws governing the construction of a scientific theory and its objects.

The notion that scientific systems are grounded in synthetic judgments and that the objects of scientific systems are constructed go hand in hand. Kant defined synthetic judgments as those in which the predicate “B lies outside the concept A” (CPR, B10). Such judgments, for Kant, are not judgments of identity, but are “ampliative” because they add something to the concept of the subject in question that is not to be found by a

¹⁰ Various translations exist in the English literature for the term *Erkenntniskritik*. Moynahan translates the term as the “critique of knowledge” (43). In her dissertation, Patton translates it as “knowledge-critique,” (“Hermann” 16) and subsequently in her later work, for example her translation of Cassirer’s essay, “Hermann Cohen and the Renewal of Kantian Philosophy,” it appears as “cognition-critique” (97).

mere process of analytic deduction (CPR, B11). In addition, Kant argues that while all judgments a posteriori are synthetic (for example that all bodies are heavy), there are also a priori synthetic judgments, to which all mathematical propositions belong and which are contained as basic principles in physics (CPR, B14-18). For Cohen, the objects of a scientific theory result from constructing a series of synthetic judgments. Patton cites the example of planetary orbits to illustrate what Cohen has in mind. She argues that Kepler's achievement in describing planetary orbits as elliptical is an example of such synthetic reasoning:

But what *is* an ellipse? It is an ideal geometrical form—ultimately, an idea. Kepler's achievement was to construct a set of relations between that idea and the mechanical estimates of planetary motions so he could prove that the idea of an ellipse gives a law-like estimate of the motion of the planets around the sun. After all, what else would an "elliptical orbit" be? We surely don't want to insist that the concept of an ellipse is substantial and made manifest many times in the orbits of planets. Rather, we can say that the constructible geometrical shape "ellipse" serves as a law or rule according to which we may interpret or measure our celestial observations. As a result, the ellipse gives us a principle of order *a priori* according to which we may measure or estimate the mechanical forces of planetary orbits. ("Hermann" 76)

The relationship Cohen sees between geometry and nature comes into focus through this example. The planets are observable phenomena in the sky, but as Patton argues, without the "elliptical orbit" hypothesis to order our observations, "which posits the right kind of law-like relation between the theory and the phenomena, we would never be able to identify elliptical orbits as scientific *facts*" ("Hermann" 75). The planetary orbits and the planets themselves insofar as astronomy is concerned, become objects of science through a synthetic process of reasoning. As Patton notes, elliptical orbits do not exist manifestly in the heavens ready to be identified; rather, the idea of the ellipse provides the basis for

the construction of a systematic approach through which planetary observations are brought into a unified picture.

One also sees the centrality of geometrical forms to Cohen's philosophy of science in the example of elliptical orbits. In the PIM, he comments on the connection between objectivity and geometry:

The objectivity of things rests in the first instance on their geometrical ideality and idealizability [*Idealisierbarkeit*]. If there were no conic section, then Kepler's planetary orbits could not be established and the planets themselves could not be objectified out of and within those orbits. In this way, pure mathematical intuition forms the basis of all cognition of nature [*Naturerkennens*].¹¹ (§ 28, 68)

Here one sees that, for Cohen, the objectivity of things lies in geometrical form, and that pure mathematical intuition, on which geometry is based, forms the basis of the synthetic reasoning through which science of nature is constructed. In the first *Critique*, Kant claims that for synthetic judgments to be possible, "I must have besides the concept of the subject something else (X), upon which the understanding may rely, if it is to know that a predicate, not contained in this concept, nevertheless belongs to it" (CPR, A8). Cohen identifies the something else (X) that facilitates synthetic judgments as intuition. Alberto Coffa notes that Cohen recognizes two meanings of synthetic judgment in the *Critique*: 1) having a predicate that is not thought in the subject, and 2) having intuition as the ground of the synthesis. He argues that 1) is the nominal definition and 2) is the real one. Coffa explains the terms nominal and real in this context as being analogous to Wolff's explanation that in defining a clock, the nominal definition would be "a machine that

¹¹ This is my modified version of this passage in Patton's partial draft translation. Her draft covers sections 1-30, 32, 35, 37, 39, 40-41, of the PIM. Translations from the PIM are my own unless stated otherwise. My thanks to Dana Hollander for her advice on this and other translations.

shows hours,” while the real definition would “point out its structure.” Hence definition 2) of synthetic judgments is the real definition, because it “gives an account of the causes or sources of the features ascribed in the nominal” (Coffa, 58).

A picture of Cohen’s notion of objectivity is now starting to come into focus. Natural phenomena are objective only insofar as they become objects of mathematical natural science through a process of synthesis by which scientific systems are constructed. Unlike Trendelenburg, Cohen does not look for the assurance that an analytic judgment (or judgment of identity) concerning an object of representation corresponds to a *real* object. Rather, his critical approach focuses on scientific theories and the validity and coherence of the principles through which scientific theories are constructed.

Werner Flach explains in his introductory essay to the PIM, that Cohen’s *Erkenntniskritik* is a Kantian schema under which the categories each correspond to a fundamental principle (*erkenntniskritischer Grundsatz*). The PIM is dedicated to the transcendental investigation of the category of reality and its corresponding fundamental principle: the anticipations of perception (or principle of intensive magnitude). This is done, Flach asserts, in order to legitimate their corresponding basic concept (*erkenntniskritischer Grundbegriff*), which is the concept of the infinitesimal (Flach, 14). These issues along with other aspects of Cohen’s interpretation of Kant will be discussed in greater detail in Chapter 2. For now it is sufficient to point out that Cohen sees his *Erkenntniskritik* as a continuation of the Kantian method of philosophizing and as the approach required for seeking out the fundamental laws of thought that are necessary

presuppositions for the possibility of natural science. He cites the concept of the infinitesimal as one of these necessary presuppositions and argues that the battles over its logical justification show the necessity “to complement logic through a related but distinct area of research” (PIM, § 1, 43). This distinct area of research is, of course, *Erkenntniskritik*, which offers an alternative path not only to *Erkenntnistheorie*, but also to subjective idealism. For Cohen, subjective idealism is problematic because it views things as ideas that have no claim on being outside of the human mind (PIM, § 88, 188). In contrast, although he argues that all things are founded in ideas, Cohen’s *Erkenntniskritik* is grounded in the “facts” of the natural sciences, “in which,” he says, “things alone are given and susceptible to philosophical questioning” (PIM, § 88, 189).

A final aspect of *Erkenntniskritik* I would like to mention is its methodological opposition to accounts of knowledge based on individual psychology. As Cohen writes in the PIM, *Erkenntniskritik* “is not simply directed towards the knowing mind (*erkennenden Geist*), rather it is directed towards the content of knowledge” (§ 13, 52). Cohen argues that the basic principles of knowledge are not predetermined by some element of the psyche. Psychology, he claims, is based on a faulty presupposition because it seeks to break down and describe consciousness from its elements (§ 7, 47). For such an account to be possible, one must assume that consciousness is determinately given in itself and subject to a science that could break it down into constituent structures and processes. However, Cohen’s view is that such elements would only ever remain hypothetical, since consciousness is such that nothing operating within it could ever uncover “that with which it truly begins and wherein it originates” (PIM, § 7, 47). Cohen

rejects any such hypothetical account of consciousness, or any other account grounded in individual psychology. For how could such an approach ever secure the grounding of objective knowledge? How can a science be developed to scientifically determine whether objective knowledge is possible, when that science must already presuppose objectivity? Would this not be analogous to someone, for example, writing a legal document certifying her own authority to write legal documents, in order to assure herself that her legal documents can make claim to legality? Both cases are circular in the same way, because in both cases one presupposes that one has authority to act, even though the action is intended to establish this authority in the first place. *Erkenntniskritik*, by contrast, avoids this circularity through its critical approach. By taking mathematical natural science as its starting point, it begins with a matter of fact “that is objectively given and grounded in principles” (PIM, § 7, 48). Hence, it does not need to posit hypothetical elements. This objective content can be located in systems of knowledge, such as Newtonian physics, and *Erkenntniskritik* only presumes to investigate the principles and laws from which their validity hangs.

1.3 *Ideas and History*

Towards the end of “On the Trendenburg-Fischer Debate,” Cohen begins to comment on the importance of the history of philosophy and how it should be practiced. The remarks are appropriate to the subject of the paper, since he was ruling over the controversy about what is the correct interpretation of Kant’s transcendental aesthetic. I would like to

address some of Cohen's general comments expressed here concerning methodology because historical interpretation and system building are central to Cohen's philosophy of science. Recognizing the role that the history of philosophy plays in Cohen's thought is crucial to understanding the PIM given that the presentation of the history of the infinitesimal method and the reflections on the significance of the infinitesimal for *Erkenntniskritik* are inexorably intertwined.

Patton provides a clue as to what influenced Cohen's hybrid systematic-historical approach to knowledge when she points out that his article on Trendelenburg and Fischer appeared in a journal dedicated to publishing articles on *Völkerpsychologie*, a movement founded by Moritz Lazarus and Heymann Steinthal, the latter of whom she singles out as one of Cohen's professors who had the most significant influence on him ("Hermann" 9). Furthermore, she points out that Steinthal and Lazarus operated under the assumption that "ideas, taken independently of history, and history reciprocally determine each other," and that "one should study what is given in history (*real* facts and events) to have access to real content, but that content is partly a result of reasoning" ("Hermann" 67).¹² Cohen's historical method for analyzing scientific theories bears traces of this notion. In the PIM, for example, there is constant interplay between systematic reflection and consideration of primary sources in an attempt to trace conceptual connections between the work of different thinkers (such as Galileo and Leibniz for example).

For Cohen, a clear historical reconstruction is always constructed with a view towards the "source" [*Springpunkt*] or basic idea of the philosophical system at hand.

¹² For a more detailed account of Steinthal's and Lazarus' ideas and influence on Cohen, see chapter 2 of Patton's dissertation, entitled "Genetic and Critical Methods in History," 59-82.

This basic idea defines a thinker's contribution to philosophy at a given time, and that idea links certain thinkers together insofar as their philosophical systems or solution to a given problem can be seen to have been born out of the same basic insight. But Cohen asks: "How does one grasp the basic idea?" (TF, 271/23) It may be true that if one were to look at a historical account of philosophy, one might be able to discern what its author took to be the basic question; but in the original act of writing history, how does the historian locate the basic idea from which a system of philosophy emanates? And furthermore, how does the reader judge between two conflicting historical accounts? The recognition of this tension is clearly evident in Cohen's writing on methodology in the history of philosophy:

The path that the historian of philosophy has to take is determined by our relative conception of objectivity. The more he takes a systematic view of the texts he represents, the clearer his work will turn out in terms of documentary faithfulness, not only in systematic clarity. (TF, 272/24)

The relative conception of objectivity used by Cohen stems from the fact that a given text, and even the history of philosophy itself, is presented as a mosaic of facts and ideas, and the assemblage and presentation of this raw material must be "freely constructed" by the interpreter.

Here again we see the interplay of objectivity and construction in Cohen's thought. He says that there simply is no single basic idea in a text, and that it is seductive and illusory to think that there is one objective historical description. In this case, deriving an historical interpretation of a given philosophical system would be simply a matter of analysis and drawing out the "correct" answer that is already at hand in the text. Cohen holds that this is not the case; rather, the interpreter performs an act of synthesis in

drawing connections, highlighting particular passages, and putting forward conclusions. He argues that no philosophical problems, and particularly contemporary ones, are represented free of the influence of the author's systematic aims and perspective (TF, 272/23). Hence, there is always systematic partisanship at play behind any act of historical interpretation and in the approach to any philosophical problem. It follows from this that when presented with conflicting approaches to a given problem, or conflicting historical interpretations of a great thinker, one must judge between them based on their systematic clarity, coherence, and effective power to resolve the problem at hand. It is apparent for Cohen that partisanship is to be expected, and that part of being an philosophical historian is to put oneself "audaciously in the middle of conflicting parties" (TF, 272/23).

Given this inherently partisan conception of the history of philosophy, and indeed philosophy generally, it can be concluded that Cohen expects that there is no one resolution to a given philosophical problem. One can read the history of philosophy as a series of systematic approaches to some basic question, each one growing from another, such that the "faults in one system would hold the kernel of the new one!" (TF, 271/23) This promise of ever-improving solutions to a problem is the ultimate benefit Cohen attributes to studying the history of philosophy. Although there is no final resolution to a given philosophical problem, we can always strive towards greater clarity, coherence, and explanatory power:

The more we go into the systematic difficulties, *the more independently we correct our work on the great thinkers*, the more clearly will the elements of analysis come together, the more determinately will historical development branch off, the less doubtful will dialogue become.

(TF, 273/24)

Historical analysis and reflection is a key aspect of Cohen's critical approach to philosophy, because the fundamental philosophical questions are still open and must be traced back through the great thinkers in light of contemporary concerns in order that the systems and the sought after solutions progressively improve. One such fundamental question for Cohen is how to establish the foundations of natural science. The PIM, in which he attempts to show Leibniz and Newton as the systematic forerunners of Kant is written toward this end.

Chapter 2: Cohen's Neo-Kantianism

The purpose of this chapter is to give a detailed account of Cohen's neo-Kantianism. In particular, it will focus on how Cohen retools Kantian terminology and orients it towards a philosophy of science, which is based on his notion that objects of science are constructed. My exposition begins with Cohen's argument that Kant's transcendental aesthetic presents a "new" theory of experience. Next, I discuss how Cohen defines the Kantian terms "intuition" (*Anschauung*) and "thought" as "elements of a scientific method." As I remarked in chapter 1 (p. 13), the PIM focuses on the "anticipations of perception" and the category of reality. These topics will be discussed in greater detail in this chapter, along with Cohen's notion of continuity as the "unity of consciousness."

Cohen puts forward his interpretation of Kant in the PIM and in the two editions of KTE. KTE was Cohen's first book-length publication; it was published in its first edition in 1871, shortly after the paper on the Trendelenburg-Fischer debate, while the second revised edition appeared in 1885.¹³ It was through this work that Cohen presented his positive interpretation of Kant that he had withheld from the Trendelenburg-Fischer paper. Although Cohen would not fully develop his own independent system of critical idealism until later in his career, it is generally recognized that these works, especially the

¹³ In the words of Andrea Poma, the second edition of KTE is "far more than a mere revision and correction of the 1871 version." I wish to avoid a lengthy discussion of the differences between the two. For our purposes it suffices to say that the second edition was the result of further historical research Cohen did during the intervening years, and that it included a deeper analysis of the relationship between thought and intuition as well as of things-in-themselves (something which Cohen did not consider at all in the first edition). For a more detailed discussion of the differences between the first and second editions see Poma, chapters 1 and 3.

second edition of KTE and the PIM, contain the beginnings of his independent and original philosophical perspective that would not be fully developed in his 1902 work, *Logik der reinen Erkenntnis*.¹⁴

2.1 *The New Concept of Experience*

Kant begins the *Critique of Pure Reason* by observing that “there can be no doubt that all our knowledge begins with experience [...]but though our knowledge begins with experience, it does not follow that it all arises out of experience” (B1). The knowledge that does not arise out of experience is a priori knowledge, and the relationship between a priori knowledge and experience is a central question in the *Critique*. As Alan Richardson explains, the relationship between a priori knowledge and experience is problematic in Kant’s view. Richardson argues this is the case because “all knowledge arises from experience but experience itself is explainable only through a transcendental account of the a priori forms of the intuition and understanding” (55). In the *Critique*, Kant defines experience not as lived experience or empirical perception, but as “knowledge of objects” (CPR, B1). Cohen seizes this definition, identifying the best knowledge of objects we have as the knowledge of the mathematically based physical sciences. In Cohen’s time, this type of knowledge was epitomized by Newtonian mechanics, for no other system in human history had enjoyed such long success and achieved such tangible and seemingly universal results. Thus if, as Cohen believed, the task of transcendental philosophy was to begin with experience, then a suitably critical

¹⁴ For such an account, see Flach, 28-32.

transcendental system of philosophy would have to begin with what Cohen called the “fact of science.” Cohen asserts this methodological principle in the opening passages of the PIM:

The critique of reason is the critique of knowledge [*Erkenntnis*] or of science. The critique discovers the pure in reason, insofar as it discovers the conditions of certainty on which knowledge as science rests. (§ 8, 48-49).

Recall that in our discussion of the Trendelenburg-Fischer debate, we established that Cohen sought to refute the causal and psychological justifications of science that dominated early trends in *Erkenntnistheorie*. Furthermore, Cohen saw Kant as presenting a *method* of philosophizing. This is reflected in Cohen’s critique of science, in that this method begins with the undeniable “fact” that there is scientific knowledge and proceeds from there to establish the a priori conditions for the possibility of this knowledge.

In his 1912 paper on the contributions made by Cohen to the “renewal” of Kantian philosophy, Ernst Cassirer argues that by focusing on the “facts of science” Cohen’s philosophy takes a “strictly objective turn” (97). However, one must carefully delineate exactly what kind of objectivity Cassirer ascribes to Cohen. In his refutation of Trendelenburg’s position regarding the purely subjective status of space, Cohen had argued that Trendelenburg misunderstood what it meant for space to be a priori. Trendelenburg had wanted to secure the objectivity of the sciences by showing, contrary to the view he ascribed to Kant, that our representations of objects are not *merely* subjective, but are the result of the subject’s access to the real objective characteristics of the world external to us. Such a viewpoint presupposes a conceptual framework in which there is the *objective* world fully determined in itself, and our *subjective* impressions of

this objective reality. Such a conceptual framework was present in the popular trends in *Erkenntnistheorie* that Cohen sought to reject. As was discussed in the previous chapter, his thought is marked by his desire to abandon the subjective-objective paradigm. This move, Poma argues, was motivated by Cohen's desire to find an alternate path that would lead beyond speculative metaphysics, naïve materialism, and psychological subjectivism (Poma, 4).

Cohen found his alternate path in Kant's transcendental philosophy and Cohen's idea that the real object of science is constructed through a process of synthetic reasoning. With this idea, the a priori becomes the formal conditions that govern the construction of experience:

Experience itself becomes a concept we have to construct [*construiren*] in pure intuition and pure thought: the formal conditions of its possibility, space, time, and synthetic unity, count henceforth as a priori, because with them we *construct experience*, because they are formal constituents of experience.¹⁵

To better understand what Cohen is getting at here, let us take Newtonian mechanics as an example of a system of scientific knowledge. The Newtonian system represents a unified set of synthetic relations insofar as it is constructed in an internally coherent fashion according to a set of fundamental principles and axioms. For Cohen synthetic unity—an axiomatic system derived without recourse to contingent matters of fact—is an a priori condition for the possibility of experience because, in his view, any system of mathematical science must be constructed in this way (Merrick, 55). The concept of space used in Newtonian mechanics is Euclidean space. Euclidean space is

¹⁵ KTE, 104 as cited in Richardson, 60.

itself an a priori form of space, because “everything that can be known about this spatial structure is obtained from a set of fundamental principles laid down by the practitioners of geometry, without having recourse to any empirically verified, contingent matter of fact”(Merrick, 55).¹⁶ Furthermore, because Newtonian mechanics is based on Euclidean space, Euclidean space can be seen as a formal condition for the possibility of experience, insofar as experience is constructed within a Newtonian framework. The same can be said about Newton’s concept of absolute time.

2.2 — *Thought and Intuition Rethought*

In the PIM Cohen advocates caution in dealing with the term “intuition” (*Anschauung*). He argues that one cannot blame Kant for the lack of clarity associated with the term and its frequent misuse because its meaning is naturally interwoven with other Kantian concepts, and because no one who forges a new concept is capable of demonstrating its meaning fully and completely (§ 23, 61). In building a system of philosophy, he claims, fundamental concepts must be placed, like stones, carefully beside one another and “polished.” Caution must be exercised when using a technical term that it is not used in a similar manner to common parlance [*Umgangssprache*]; it should not be used as though its meaning were self-evident. Such caution, in Cohen’s view, had not been exercised with the term “intuition,” and one of his objectives in redefining the term is to distance it from loose usage that is “erratic from author to author, and often so within the same

¹⁶ I am deeply indebted to Merrick’s presentation of these issues in her paper, “What Frege Meant When He Said: Kant is Right about Geometry,” for my understanding of this aspect of Cohen’s thought. The paper contains an interesting discussion in which Merrick argues that Frege shared the understanding of Kantian intuition developed by Cohen and the Marburg School.

author's [usage]" (PIM, § 22, 61). Consequently, Cohen seeks to return the term to its proper systematic meaning, by distinguishing it from the physiology of perception (Patton, "Critical" 113), or from some vague mental faculty or mental state by which one represents an object in one's mind's eye or "pictures something."

Cohen's reading of Kant is focused on the significance of his critical philosophy for scientific knowledge and scientific methodology. To Cohen, intuition is not a mental representation of an object; rather, intuition should be thought of as a "means of knowledge" (*Erkenntnismittel*), or as an "element of a method" (PIM, §24, 62). Thought too must be understood in a similar manner as oriented towards scientific knowledge and method. Cohen identifies both thought and intuition as "critical abstractions" (*erkenntniskritische Abstraktionen*), and he argues that it is a mistake to hold that just because all knowledge must be presentable to intuition, all other "means, conditions, and foundations of knowledge" must be placed under it (PIM, §21, 59). For him, both thought and intuition are "sources of knowledge" (*Quellen der Erkenntnis*), and each individual instance of objective knowledge, or each case where something has been objectified in mathematical physical science, is the result of a combination of both of these sources.

Now let us consider the example of Newtonian physics in order to demonstrate intuition as an element of a scientific method. If we identify pure intuition, following Merrick's reading of Cohen, as "a method, *i.e.*, the method of generating and cognizing the spatial structure(s) studied within geometry and, more importantly, the spatial structure presupposed by the natural sciences," then intuition would give "the basic concepts, axioms, and procedure of traditional Euclidean geometry" (Merrick, 59-60)

because the spatial structure employed in Newtonian physics is Euclidean space, and the points, lines, curves, straight lines, etc. used in Newtonian physics all derive their meanings from the Euclidean framework. Thought, on the other hand, would encapsulate the procedures for performing calculations, drawing connections between concepts, and arriving at conclusions employed by the physicist.¹⁷

An important consequence of Cohen's definition of intuition as a method of generating spatial structures is that Cohen could potentially be able to incorporate non-Euclidean geometries into his account of science. Although Cohen was aware of the non-Euclidean models of geometry that existed during his time, he saw Euclidean geometry as being of prime importance because it provided the spatial structure necessary for the Newtonian system. However, Cohen died before 1919, when Einstein unveiled his general theory of relativity, which was a new scientific system based on a non-Euclidean model of space, or space-time to put it more accurately. Einstein's achievement signaled a paradigm shift that ended the long run of Newtonian physics as the premier scientific theory. Kant's system has often been seen as antiquated by the development of non-Euclidean systems because of his adherence to space as the a priori, Euclidean, form of sensible intuition. However, through his retooling of the a priori as the formal condition for the construction of experience, Cohen shows that the Kantian *method* of philosophy and Kantian terminology need not be limited to Euclidean space. There is nothing in

¹⁷ It may strike one here that the distinction presented here between thought and intuition as independent sources of knowledge is blurry at best. For how can one separate the procedure for calculation in a physical theory from the spatial structures from which they derive their physical meaning? Analytic geometry certainly presupposes that the same relations can be expressed both algebraically and geometrically. Hence, what need is there to distinguish them as separate elements of a method when the unity between them is so fundamental? Given this difficulty, it is not surprising that, as many commentators have noted, Cohen abandoned the distinction between thought and intuition altogether in his later work.

Cohen's definition of intuition as a formal condition for constructing experience that would prohibit him from adopting relativity theory. In this case, pure intuition would become the method of cognizing and generating spatial structures except, instead of those structures inherent in Euclidean geometry, the structure would be a Riemannian manifold, necessary for modeling the curved structure of space-time.

2.3 *The Category of Reality and the Anticipations of Perception*

Cohen's redefinition of intuition as an element of a method for cognizing and generating spatial structures means that science can potentially be understood as a free enterprise. Because the objects of science are constructed, the process of refining our scientific methods and understanding can be thought of as an open-ended task of deriving increasingly better scientific systems. For Cohen, the physicist's task is not to describe the world conceived of as a thing-in-itself, something that, as Merrick eloquently puts it, "is out there and is what it is entirely independently of our tools of cognizing it" (47). Intuition is not descriptive in the sense that it is a faithful copy of things as they are independent of science; rather, intuition is prescriptive in the sense that a scientist prescribes certain spatial relationships that have objective meaning within the context of scientific practices. Thus, the scientist's primary concern and the task of the critical philosopher concerned with the problem of science should be the evaluation of the logical structures of principles and ideas for their internal coherence and validity. Neither the scientist nor the philosopher should be presupposing and speculating about the material determinateness of things.

Cohen's system raises an important question: if the objects of science are not grounded in relation to the determinate characteristics of things as they exist in themselves, then how are we to be sure that our scientific systems bear any reference to reality? Holzhey notes that one of the key ways in which Cohen's thought diverges from Kant's is in how Cohen answers this question. For Kant, the sensible-material component of experience is essential if knowledge was to have reference to reality (Holzhey, 15). For Cohen, by contrast, the sensible-material component of experience is subsumed under mathematical and logical thought:

Sensation itself is not an object, but only a type of relationship of consciousness with its content, whose aim is the determination of this content as an object.¹⁸

The object with which Cohen is concerned is not an object whose passively received impression is conveyed to the subject through sensation. Rather, sensation does play a role in the determination of an object, but the object is *real* only insofar as it becomes an object of the mathematical sciences. Recall the example of Kepler's planetary orbits discussed in chapter 1 (p. 11). In that case, the celestial observations of the planets presented the problem to be resolved: deriving a law-like explanation that renders these observations into a unified picture. For Cohen, the planets only become real objects in the first instance of their geometric representation. To put it another way, the planets only become real objects when a law-like explanation for their motions is determined. Hence, reality is derived through geometrical form and the construction of law-like scientific explanations that incorporate the sensible-material observations available. This is the

¹⁸ KTE, 553 as cited by Poma, 43.

meaning behind the oft-cited passage from the PIM: “Stars are not *given* in the sky, but in the reason of astronomy.”¹⁹ The objects of science, into which the sensible content of experience has already been incorporated, are what Cohen considers philosophically significant.

Cohen claims in KTE that “All knowledge is related to *objects*. The knowledge is called *intuition*, which is related to objects *immediately*.”²⁰ The intuition to which Cohen refers here is not sensible intuition, but pure mathematical intuition. Holzhey argues that for Cohen it is the category of reality “which ensures that consciousness has any reference to an x as a given, i.e. to an intuition” (15). As I mentioned in chapter 1 (p. 14), there is a corresponding basic principle for each category in Cohen’s system of philosophy, and corresponding to the category of reality is Kant’s “anticipations of perception.” In the *Critique*, Kant defines this principle: “in all appearances, the real that is an object of sensation has intensive magnitude, that is, a degree” (CPR, B207). In his appropriation of this principle, Cohen renames it the “principle of intensive magnitude” and, according to Holzhey, defines it in the following way: “an intensive magnitude must be attributed to all real objects of perception” (14). Key to Cohen’s reinterpretation of the principle of the anticipations of perception is his removal of the emphasis on the “object

¹⁹ PIM, § 88, 189 as cited by Holzhey, 19. Veal observes that Cohen makes an implicit reference to Plato’s *Republic*, 529a-e, here (Veal, 12). In this passage Socrates explains to Glaucon why astronomy should be part of a philosopher’s education. Socrates recommends the study of astronomy insofar as it leads reason beyond mere appearances to contemplating the true realities of number and form that guide celestial movement. This passage in particular stands out: “The stars that decorate the sky, though we rightly regard them as the finest and most perfect of visible things, are far inferior, just because they are visible, to the true realities; that is, to the relative velocities, in pure numbers and perfect figures, of the orbits and what they carry in them, which are perceptible to reason and thought but not visible to the eye” (Plato, 529c-d). It is beyond the scope of my project to discuss in detail Plato’s role in Cohen’s thought, although it is widely recognized that Plato is an extremely important figure for Cohen—perhaps as much as Kant. For detailed accounts see Poma, Chapter 2 and Albertini, 21-25.

²⁰ KTE, 15 as cited in Patton, “Critical” 114.

of sensation” in favor of the “real object of perception.” With this move, the “real” object loses its basis in reference to sensation. Aside from this conceptual shift of emphasis, the “principle of intensive magnitude” maintains more or less the same structure as the “anticipations of perception.”

In the *Critique*, Kant says that “the real which corresponds to sensations in general, as opposed to negation = 0, represents only that something the very concept of which includes being, and signifies nothing but the synthesis of an empirical consciousness in general” (CPR, B217). Here Kant defines the “real” (which corresponds to sensation) by setting up in contraposition with “negation = 0” because the total absence of sensation in an instant would amount to a representation of the instant as “empty” (B209). Between the reality of sensation and negation, Kant claims, there is a continuity “of many possible intermediate sensations [or degrees] the difference between any two of which is always smaller than the difference between the given sensation and zero, or complete negation” (B210). Kant argues that this means the “real” always has magnitude, but this magnitude does not proceed from the synthesis of parts into a whole because sensation takes place in an instant and not through a succession of instants—hence, the magnitude of sensation is not an extensive magnitude. Rather, the “real” itself, Kant says, can only be thought of as an intensive magnitude, “the apprehension of which is not successive, but instantaneous” (B210) and which can be “apprehended only as unity” (B210). For Kant, all that can be known a priori about what is real in appearances is grounded in intensive magnitude. “Everything else,” he claims, “must be left to experience” (B218). Kant cites changes of states as an illustration of this principle. For

example, one cannot know a priori what the temperature of a given kettle will be—that must be left to empirical observation. What can be known a priori, according to Kant, is that any change in the temperature will be a continuous change, in which between any two temperatures there will be an infinite gradation of intermediate degrees of temperature.

Just as Kant grounds the real objects of sensation in intensive magnitude, Cohen seeks to ground the real objects of mathematical science in intensive magnitude. For Kant, the objects of empirical intuition are extensive magnitudes, which are grounded in the intensive reality of sensation. For Cohen, the objects of pure mathematical intuition are extensive magnitudes, which are grounded in intensive magnitude. The category of reality has its corresponding basic principle (*erkenntniskritischer Grundsatz*): the “principle of intensive magnitude.” In turn, there is a basic concept (*erkenntniskritischer Grundbegriff*) that is schematized through each basic principle. The basic concept that corresponds with the principle of intensive magnitude is the concept of the infinitesimal. It is through the concept of the infinitesimal that intensive magnitude is conceived of as the origin (*Ursprung*) of extensive magnitudes, and the infinitesimal as intensive magnitude becomes, in a manner of speaking, the representative (*Vertreter*) of reality:

The infinitely small means, as intensive magnitude, reality in the determinate and pregnant sense that it supplies to natural science the real that is presupposed and sought in all natural science, [i.e.,] that it makes up and constitutes the real.²¹

²¹ PIM, § 92, 196; Hollander’s translation.

Like Kant, Cohen maintains the idea that reality is grounded in intensive magnitude.

However, for Cohen it is intensive magnitude, thought of purely in mathematical terms that engenders finite mathematical quantities and geometrical forms that are the basis of reality. Furthermore, continuity becomes, for Cohen, an overarching feature of consciousness.

2.4 — Continuity as the Unity of Consciousness

The infinitesimal, thought of as intensive magnitude, is a necessary presupposition for natural science because it serves as the origin for the objects of science. The objects of science are constructed as extensive magnitudes; however, the concept of extensive magnitude itself, in Cohen's view, requires further grounding. This is because, as Poma notes in identifying Cohen's argument, the objects of science are given as finite unities, which are "compared with other unities by means of a number schema" (42). Cohen argues that number is the first instance of legitimation and constitution of synthetic knowledge, or scientific knowledge, because objects become measurable and comparable through numbers (PIM, § 78, 166). But then the problem arises: what is the origin of these discrete finite unities? For example, I can understand the number 10 as a unity made up of two groups of 5, or as 10 individuals. Hence, I could say that the origin of 10 lies in collections of smaller unities. But if I reduce 10 to ten individual ones, then what is the origin of those ones? Is it two halves, four quarters? This problem of grounding is why Cohen calls finite magnitudes merely arbitrary comparative unities, because there is no fixed ground for the comparison. He argues that the status of finite numbers is further

called into question by the irrational numbers, because here the determinateness of the finite discretion is broken (PIM, § 43, 85). This is something the Greeks were aware of, and the secret that $\sqrt{2}$ was an inexhaustible, “imperfect” number cost one of Pythagoras’s followers his life.²² Cohen argues that in irrational numbers, one sees the “impulsive” and “legitimizing” idea of the principle of continuity, because “unity, insofar as it contains the infinite within it, is no longer discrete; rather, it is continuous determinateness” (PIM, § 44, 86). He adds that to go from the finite to the infinite by deduction results in “wonder,” which is clarified as soon as one goes in reverse, deductively, from the infinite to the finite (PIM, § 44, 86). For Cohen, the infinite must be thought of purely as an absolute continuous unity that can be determined to whatever degree is required. For example, π can be determined to any required number of digits, but not exhaustively. This inexhaustibility demonstrates the need for discrete finite unities to be grounded in an absolute continuous unity (such as the real number continuum).

Cohen identified continuity as the basic form (*Grundgestalt*) of the “unity of consciousness,” which, Cassirer noted, “is [for Cohen] only another expression for the unity of synthetic principles on whose validity rests on the possibility of experience, and thus, the possibility of objectivity in general” (98). Furthermore, Cohen says in the PIM that our faculty “to form generic terms (*Gattungsbegriffe*), which appear to be negative, yet which are just as fruitful and necessary for scientific knowledge as positive ones” (§ 41, 81), is due to the continuity of consciousness, because it is continuity that leads to the “limiting judgment.” The limiting judgment, he notes, appears merely negative and

²² An account of this story is given in Rudy Rucker’s *Infinity and the Mind*, 61-62. Rucker’s book is an excellent introduction for non-mathematicians to issues concerning the infinite.

unhelpful in certain contexts, such as when one says “The understanding is not a table,” or when one employs the judgment “not human” (*Nicht-Menschen*) in order to convey the concepts “three-sided, melancholy, and sulfuric acid” (§ 41, 81). By contrast, there are cases, Cohen argues, in which the limiting judgment is indispensable. The example Cohen gives is the question: “What is *Geist*?” In answering this question, we find that it is impossible to do so without negating the meaning of the concept with recourse only to positive judgments. Rather, it is far more helpful to delineate the concept of *Geist* by differentiating it from with the concepts that are its opposite. Instead of saying “*Geist* is x,” I say “*Geist* is non-y.” Cohen answers the question as follows: “*Geist* is immaterial; *Geist* belongs to that class of thought determinations [*Gedankenbestimmungen*] that are characterized by the exclusion of material things” (§41, 82). Kant calls this mode of judgment the “infinite judgment” because it delineates a potentially infinite class of concepts through a negative predicate.²³

Cohen identifies a further example of the usefulness of the limiting judgment that is of particular relevance to the topic of natural science: imponderables. Take the relation between rest and motion for example. The two appear to be opposite and incommensurable, because it is inconceivable how motion could ever come from rest, or the total absence of motion.²⁴ However, when rest is defined in terms of motion, as an

²³ For a detailed discussion of the infinite judgment and its relation to Cohen’s mature philosophy of science, see Amos Funkenstein, “The Persecution of Absolutes: On the Kantian and Neo-Kantian Theories of Science.”

²⁴ There exist a number of treatments of the paradoxes involved in conceptualizing motion. One such treatment is called the “Paradox of Half-distances.” The problem is presented as such: consider an object moving from a given point A to another point B. Now consider that there exists a third point, C, between them. In order for the object to move from A to B, it must first move from A to C. However, before it can cover the distance between A to C, it must cover the distance between A and D, a point lying half way

infinitely small amount of motion, a positive reciprocity between rest and motion is established. The total absence of motion becomes a negative limiting concept that can never be reached, and motion is established as consisting of a continuum of degrees starting from rest (an infinitely small degree of motion). One can think of the point of rest as the intensive moment from which motion is engendered. Thus, while motion is measured as an extensive magnitude, the theoretical precondition for this state of affairs lies in seeing its “origin” in intensive magnitude (“rest”) and this is made possible by setting motion up in contraposition with its opposite, the complete negation of motion.

The limiting judgment plays a central role in the PIM because Cohen holds that “the infinitesimal is the most instructive example of the fruitfulness, the plan, and the merit of the limiting judgment” (§ 42, 83). This is because the infinitesimal is “initially only characterized by its difference from the finite” (§ 42, 83). Cohen argues that much of the confusion surrounding the concept of the infinitesimal lies in not understanding the proper balance between thought and intuition. All finite magnitudes are given in intuition; however, the infinitesimal is not given in intuition, although Cohen argues that it is a type of magnitude. The infinitesimal is a basic concept of *Erkenntniskritik* that corresponds to the category of reality and the basic principle of intensive magnitude (or Kant’s anticipations of perception). Intensive magnitude is defined as an absolute qualitative unity, which can really only be conceived in a limitative way in opposition to discrete finite quantity. This is analogous to defining the infinite as “unbounded” in opposition to bounded finite magnitudes.

between those two. This reasoning can be carried on ad infinitum, such that the object can never be thought to cover any distance, because it must always first traverse an even smaller one.

Cohen argues that thought and intuition combine in every instance of knowledge. Intensive magnitude, or reality, is a condition for possibility of knowledge, but one must recognize it as coming purely from the side of thought (§ 22, 60). One sees the necessity of this balance, he claims, in the problem of the tangent, insofar as “the point on the tangent, which unites the different motions of a point into *one* direction, produces a curve in that direction.”²⁵ That is to say that intensive magnitude intervenes in the moment represented as the point on the tangent to a given curve is where the direction of the curve is “generated.” But in order to recognize this, one must be able to consider the point on the curve and the point on the tangent to be one single point, whereas in intuition the two can only be represented as approaching each other to within a given margin of error. Hence, although the infinitesimal is not able to be represented in intuition—i.e. it is not a finite part of the function—it “engenders” the function. Thus, one must be careful to differentiate between the content of knowledge and the formal conditions of that knowledge. That is to say the infinitesimal is a necessary presupposition for the possibility of the objectification of motion, but it is not itself an object of science, or a constituent part of a given function.

2.5 — *A Return to Kant?*

Before outlining Cohen’s history of the infinitesimal method, which will be the subject of the next chapter, I would like to address the question of whether Cohen’s interpretation of Kant is a faithful historical reconstruction of the latter’s work. In chapter 1, I discussed

²⁵ PIM, § 39, 80 as cited by Holzhey 15.

the inherent tension within Cohen's approach to the history of philosophy resulting from the desire to understand previous thinkers, such as Kant, "in their own words" while simultaneously asserting that an historical account always involves an appropriation of the work in question in service of the particular goals and intentions of the interpreter within. Now that the key aspects of Cohen's neo-Kantian system have been elucidated, it must be determined whether Cohen overcame the contradictory nature of his historical project with regard to Kant. Here I would like to present a particularly convincing argument put forward by Köhnke, that due to his own systematic partisanship, Cohen does not provide a faithful reading of the historical Kant. Köhnke argues that Cohen

1) claims to have deduced from the *Critique of Pure Reason* its opposite, a "critique of experience" on whose account he 2) pursues, and was obliged to pursue, a course of argumentation wholly independent of the *Critique*. In doing this he 3) selects his quotations simply according to their usefulness for his own theory without paying the regard to the relative value of the role they play in the *Critique*. Ignoring their original argumentative function he 4) employs these quotations, and even mere parts of sentences, as dogmas and as wholly objective autonomous statements claiming a terminological and semantic identity in his own work and Kant's. (178)

Köhnke asserts that Cohen identifies the *Critique* as a "critique of experience" because in reading it this way he offers his own negative critique of empiricism, positivism, and materialism. Kant's critique, however, deals primarily with the question of 'pure reason', i.e. it is a critique of ideas that transcend all experience. Hence it seems that Cohen, preoccupied with furthering his own methodological goals, appears to have misunderstood (or perhaps consciously misrepresented) the "meaning, objective, tendency, and intention" of Kant's work (180).

In line with his primary concern in the Trendelenburg-Fischer debate, Cohen's interpretation of the *Critique* as a critique of experience centers on the Kantian notion of aprioricity. Köhnke identifies Cohen's starting point in this regard as Kant's argument that "space is not an empirical concept abstracted from external experience" (180). By this, Kant meant that we do not derive our notion of spatial relations and representations purely from the external world; rather, our "outer experience itself is possible at all only through that representation" (CPR, B38). Köhnke explains that Kant's formulation implies that "as an act of both the understanding and of the senses, experience is already founded on the concepts and forms of intuition" (Köhnke, 181). Cohen, Köhnke asserts, takes considerable liberty when he completes Kant's statement that "space is not an empirical concept abstracted from external experience," by saying that "it is, rather, space which constructs the external objects from which experiential impressions proceed" (Köhnke, 180). Here begins Cohen's notion that the object of experience is actively produced and that all of experience is constructed. For Kant, experience is the result of the interplay between the a priori forms of sensibility and the senses themselves—it is a blend of the a priori and a posteriori. For Cohen, as Köhnke makes clear, the a priori element becomes the supreme contributor to experience and "the realm of the a posteriori, the givenness of the object is completely lost" (Köhnke, 181). As we have already seen, for Cohen, the object of experience, or of natural science, has its origin in the mathematical category of reality, grounding in the principle of intensive magnitude as a fundamental law of thought. The being of real objects is entirely dependent on them

being actively produced as mathematical constructions, and they exist only to the extent that we conceive them (Köhnke, 182).

Unlike Cohen, Kant does not hold that all experience is constructed or even that science is constructed from synthetic a priori principles. Köhnke notes that for Kant, pure mathematics is the only field in which a priori knowledge is attained *purely* through construction. Furthermore, the purpose of transcendental philosophy is to inquire into what makes possible the apriority of synthetic a priori judgments by determining the “preconditions, possibilities and limits of ‘pure reason’” (Köhnke, 184). By contrast, Cohen takes the a priori itself to have an “autonomous power” or active agency whereby we generate the content of experience. Köhnke observes that Cohen quotes the following statement from Kant’s preface to the second edition of the *Critique* no fewer than twelve times: “that we know of things only the a priori we ourselves introduce into them” (Köhnke, 184). Here is Kant’s statement in its original context:

As regards objects which are thought solely through reason, and indeed as necessary, but which can never—at least not in the manner in which reason thinks them—be given in experience, the attempts at thinking them (for they must admit of being thought) will furnish an excellent touchstone of what we are adopting as our new method of thought, namely, that we can know *a priori* of things only what we ourselves put into them. (CPR, Bxviii)

Köhnke argues that when the entire sentence is cited, we see that Kant is referring to objects that “can in no way be given in experience,” i.e. the objects of metaphysics, such as God. Hence, it is clear that Cohen has taken this passage out of context and applied it to experience, or science, when it was originally intended as a remark on the limits of experience and what can be known.

Although Köhnke does not mention it directly, I find Kant's footnote to the above passage particularly illustrative, in which he claims that:

This [new method of thought], modeled on that of the student of nature, consists in looking for the elements of pure reason in *what admits of confirmation or refutation by experiment*. Now the propositions of pure reason, especially if they venture beyond all limits of possible experience, cannot be brought to the test through any experiment with their *objects*, as in natural science. In dealing with those *concepts* and *principles* which we adopt *a priori*, all that we can do is to contrive that they be used for viewing objects from two different points of view—on one hand, in connection with experience, as objects of the sense and of the understanding, and on the other hand, for the isolated reason that strives to transcend all limits of experience, as objects which are thought merely. (CPR, Bxix)

I cite this passage at length because it illustrates clearly that the *new* concept of experience which Cohen ascribes to Kant, is indeed a *new* concept—but it is Cohen's, and not original to Kant. Kant claims above that a priori concepts and principles must be viewed from two perspectives: in connection with experience, and as objects that transcend the limits of experience. The first case refers to objects of sense and understanding, while the second would refer to objects of pure thought, such as geometrical forms. In the case of science, Kant states, it is ultimately the experiment which resolves any conflict that could arise between these perspectives. Consider again the example of Kepler's planetary orbits. We established earlier that there are two components at play: the planets as they are observed as sensible-material phenomena, and the mathematical form of the ellipse, that brings the phenomena into a unified picture and allows the movement of the planets to be understood in terms of law-like forces. The difference seen here between Kant's and Cohen's accounts of science is that ultimately for Kant it is the sensible-material object against which the scientific explanation must be

verified. For example, Kepler's elliptical orbits have no reference to a real object in experience, unless they prove to conform to the planets given as objects of sensation. He holds that this reference to an accessible object is what separates a priori concepts and principles in physics from those in speculative metaphysics, whose objects lie beyond experience. Obviously for Cohen, a scientific hypothesis must explain empirical observations, but he does not acknowledge that there is a sensible-material object; rather, objectivity only enters from the side of pure mathematical synthetic reasoning. The planets are not objects for him until the first instance of their geometrical idealizability. Unlike Kant, for Cohen, there *are no* objects beyond experience (or at least no philosophically significant ones), but his concept of experience—a unified set of a priori synthetic relations that make up mathematical science—is very different from Kant's.

I have presented here Köhnke's arguments for suggesting that Cohen's reinterpretation of Kant goes much further than to clarify or present the historical Kant, and I think it has been clearly demonstrated that Kant's new theory of experience is really Cohen's. I do not, however, bring these arguments forward in order to cast Cohen in a negative light. Recalling Cohen's views on how the history of philosophy should be practiced, it follows naturally that his interpretation of Kant goes beyond a mere textual recreation. Köhnke is correct in suggesting that Cohen's partisan aims dominated his reading of Kant. Köhnke successfully demonstrates that Cohen's reading of the *Critique* does not necessarily live up to his requirement of documentary faithfulness expressed in the Trendelenburg-Fischer article. I think this requirement is extremely important if one is to accept Cohen's notion of "relative objectivity" in the history of philosophy.

Otherwise, as is what happened in the Trendelenburg-Fischer controversy, a philosopher such as Kant can be evoked in name only while conceptual clarity and historical rigor are sacrificed to superficial rhetoric and confused partisanship.

That said, I think Köhnke overlooks that Cohen views the tension between historical reconstruction and furthering resolutions to contemporary philosophical problems as a constructive tension. For Cohen, what makes the history of philosophy important is the ability of the interpreter to take up what she sees as the “basic idea” contained in a previous thinker’s work and build on it in the hopes of developing a more complete solution. Thus it is fitting that Cohen takes the transcendental method of philosophy and Kantian terminology and retools them based on a new understanding of objectivity, which he sees as necessary to procure a proper understanding of the sciences. Cohen does so because he takes up the Kantian system within the context of the late nineteenth century, when the goals of the highly successful empirical sciences and that of philosophy were becoming increasingly intertwined. As a result, the “basic idea” Cohen finds in Kant’s first *Critique* is the germ of his new critical approach, on which the influential Marburg School would ultimately be founded. Had Cohen not been bold in his appropriation and retooling of the Kantian transcendental method, this would not have been possible. Hence, I argue that the good in his interpretation of Kant outweighs the bad.

Chapter 3: The History of the Infinitesimal Method

In this chapter I lay out the main features of Cohen’s history of the infinitesimal method, focusing primarily on his arguments concerning the contributions of Leibniz, Galileo, Newton, and Kant.²⁶ The argument Cohen advances throughout his historical presentation is particularly difficult to follow for three reasons: it narrates the historical development of the calculus, which in itself involves difficult conceptual content; this already difficult content is used in order to advance a broader systematic argument, which involves the intricate schema of *Erkenntniskritik*; and the first two difficulties are further compounded by Cohen’s chaotic and at times polemical writing style.

Cohen’s systematic goal must be kept in the forefront while reading his historical presentation. This goal is to establish the “critical grounding” (*erkenntniskritische Begründung*) of the concept of the infinitesimal. Cohen argues that the justification of the concept of the infinitesimal is one that lies beyond the boundaries of traditional logic. Rather, he claims, the terminological precision that comes with transcendental analysis is required in order to establish the infinitesimal as a necessary presupposition of all natural science. Recall that in Cohen’s view, Kant provided the proper understanding for science through his new concept of experience. According to this new understanding, not only are “experience” and “mathematical natural science” synonymous terms, but all objects

²⁶ Cohen’s presentation is not limited to these four figures. Along with an assessment of their roles in the development of the concept of the infinitesimal, Cohen also discusses the ideas of Euler, Lagrange, Carnot, Wolff, Baumgarten, Lambert, Bendavid, E. G. Fischer, Fries, Apelt, Herbart, Hegel, and Cournot. It is beyond the scope of my argument to address Cohen’s views on each of these thinkers’ roles in the history of the infinitesimal. In some cases he brings them up to illustrate and refute a deficient understanding of the infinitesimal or of intensive magnitudes. For example Euler’s notion of the infinitesimal as 0 (null) is rejected (§ 69).

of science, and hence experience, are constructed through a process of synthetic reasoning. The critical meaning of the infinitesimal, Cohen argues, shines through in the basic problem of mechanics, because it is within the field of mechanics that the “things of nature” become determined as “real objects.” He claims that this determination and objectification of physical bodies requires the broadening of the concept of magnitude to include infinitesimal, intensive magnitude. Thus, the supposition of infinitesimal magnitudes is a necessary step in the “making real” (*Realisierung*) of physical bodies. It is through the methods of calculus that the things of nature attain objective being, and Cohen argues that this motivation can be clearly made valid by looking to the discoverers of the calculus:

For no other purpose did Leibniz introduce his [differential] and Newton his [fluxion] than for the expansion and fixation of that experience which as natural science, Kant makes into the problem of the critique of reason [*Vernunftkritik*]. (§ 15, 55)

By investigating the “scientific relationships” that led to the discovery of the differential and the fluxion, Cohen endeavors to secure the critical grounding of the infinitesimal—a grounding through which the infinitesimal is schematized as a basic concept corresponding to the category of reality and is made valid through the principle of intensive magnitude. To do so, Cohen traces a complex path through the ideas of Leibniz and Newton towards those of Kant; however, the stakes for Cohen go beyond historical curiosity. For him, the critical grounding and justification of the infinitesimal is, as the subtitle of the PIM, “A Chapter in the Foundation of *Erkenntniskritik*,” indicates, a part of the larger furthering and justification of *Erkenntniskritik*.

3.1—Leibniz's Grounding of the Calculus

As one of the founders of the calculus, Leibniz assumes central importance in Cohen's historical presentation. Cohen argues that it is a mistake to assert that Leibniz's most worthwhile contribution to the invention of the calculus consisted primarily in the introduction of convenient notation, while Newton's methods rested on a more certain logical foundation (§ 16, 56).²⁷ Contrary to this view, Cohen seeks to demonstrate in his presentation of Leibniz's discovery of the differential (dx) that the systematic justification and positive meaning of the infinitesimal is to be found more clearly in Leibniz's thought than in Newton's. Cohen argues that although Leibniz did not develop the terminological precision that marks the critical turn in philosophy, the basic idea behind the critical grounding of the infinitesimal—bridging the gap between the things of nature and geometry—can be found by looking to the general context of Leibniz's thought, and at the “direction” of the problem of analysis, which Leibniz's differential calculus is intended to resolve.

For the sake of clarity, I would like to digress briefly from Cohen's presentation to discuss the problem of analysis and explain what the differential is within this context. The geometrical problem of analysis involves determining the rate of change, or the difference in y over the difference in x (symbolized as dy/dx), of a given curve at a given point, (x, y) . The rate of change corresponds to the slope of the line which is tangent to the curve at the given point. The problem is: what kind of quantities are dy and dx , if they

²⁷ Here Cohen is responding to K. I. Gerhardt the editor of Leibniz's, *Mathematische Schriften*. Cohen cites material from this collection frequently throughout his presentation of Leibniz's thought.

represent the change in y and the change in x at a single, instantaneous, point on the curve? Remember that dy and dx are defined in terms of differences in the values of x and y . To determine the average rate of change of a function between two distinct points, (x_1, y_1) and (x_2, y_2) is a straightforward matter: I take the difference in the y co-ordinates and divide it by the difference in the x co-ordinates, and the result is a ratio of finite numbers: $dy/dx = [(y_2 - y_1)/(x_2 - x_1)]$. However, in the case of any function that is not linear, such as $y = x^2$, average rates of change do not prove terribly useful, because the rate of change itself is variable. More useful in these cases is the calculation of the rate of change at a single point. The problem is that if a single point were plugged into the equation above, the result would be undefined, $0/0$. Leibniz's solution to the problem is to "approach" a single point such that dx , for example, is thought of as being arbitrarily close to 0, such that the error between dx and 0 is not nothing, but can be shown to be smaller than any quantity one may wish to assign. This is in concurrence with Leibniz's adherence to the syncategorematic understanding of the infinite. This understanding is in accordance with Archimedean Axiom, which holds that, given two magnitudes having a ratio, one can find a multiple of either which will exceed the other. This rules the possibility that there exist infinitesimals, or magnitudes that are smaller than any other finite magnitude, because taken in reverse, it means that one can always find a multiple of either that will be smaller than the other. This conception of infinitesimals allows Leibniz to treat them as though they are infinitely small, albeit fictions, and to work with them as arbitrarily small finite magnitudes.

The basis for Leibniz's differential is the law of continuity, which, as Cohen observes, Leibniz considers to be an ideal metaphysical law and necessary idea, which nevertheless applies intact to nature (§ 52, 104). Another name Leibniz uses for this law is the "principle of general order." Cohen draws attention to Leibniz's 1687 letter to Bayle, in which Leibniz says that this principle "has its origin in the infinite and is absolutely necessary in geometry, but it is effective in physics as well."²⁸ The principle of general order, as stated by Leibniz as part of a reply to Father Malebranche over a controversy concerning Descartes' physics, is as follows:

When the difference between two instances in a given series or that which is presupposed can be diminished until it becomes smaller than any given quantity whatever, the corresponding difference in what is sought or in their results must of necessity also be diminished or become less than any given quantity whatever. Or to put it more commonly, when two instances or data approach each other continuously, so that one at last passes over into the other, it is necessary for their consequences or results (or the unknown) to do so also. This depends on a more general principle: that, *as the data are ordered, so the unknowns are ordered also.* (351)

One can see here how this principle applies to the differential, because it ensures that as the x and y values approach the point of the tangent the values of dx and dy will proceed towards zero. The principle is necessary because as the tangent point is approached there is an inexhaustible continuum of degrees of difference to be "traversed." Since all the degrees in this progression cannot be determined explicitly, a general principle is required to ensure that the relation of data to unknowns will hold.

²⁸ Letter of Mr. Leibniz on a General Principle Useful in Explaining the Laws of Nature though a Consideration of the Divine Wisdom; to Serve as a Reply to the Response of the Rev. Father Malebranche, in *Gottfried Wilhelm Leibniz: Philosophical Papers and Letters*, 351, as cited in PIM, § 53, 106.

Given the importance of the principle of general order to the grounding of the differential, I would like to look further into Leibniz's explication of it in his letter to Father Malebranche, where he illustrates the principle through two examples. The first is a geometric example of an ellipse converging on a parabola:

We know that a given ellipse approaches a parabola as much as is wished, so that the difference between ellipse and parabola becomes less than any given difference, when the focus of the ellipse is withdrawn far enough from the first focus, for then the radii from that distant focus differ from parallel lines by an amount as small as can be desired. (352)

Leibniz then observes that a parabola can then be considered as an ellipse whose foci are an infinite distance from each other. In his editorial note on the paper Loemker observes that to stave off criticism concerning the use of the term "infinite," Leibniz argues that a parabola can be viewed as a "figure which differs from some ellipse by less than any given difference" (352). Furthermore, Leibniz observes that as the second focus is drawn farther away and the ellipse converges on the parabola, it does so through a continuous progression of innumerable intermediary figures, which means that there is no sudden jump or gap that must be traversed between successive figures; rather, there is an ideal continuum of figures between the ellipse and parabola for which the parabola is one extreme that can be approached to within whatever degree of error is desired.

The second example Leibniz gives for the principle of general order is a case where rest can be considered as "an infinitely small velocity or infinite slowness" (352).

Leibniz writes:

Therefore whatever is true of velocity or slowness in general should be verifiable also of rest taken in this sense, so that the rule for resting bodies must be considered as a special case of the rule for motion [...]

Likewise equality can be considered an infinitely small inequality, and inequality can be made to approach equality as closely as we wish.
(353)

One sees here the analogy between the two examples. In the geometric case, the parabola was considered as a special limiting case for the continuum of figures as the foci of a given ellipse are drawn further apart.²⁹ In the example of motion, rest is the analogous special case. In both cases the special case is an ideal that can be approached to within any amount of error desired. That means that one does not have to posit the *actual existence* of an infinitely small velocity, only that there is a continuum of innumerable finite velocities that can be determined to approach rest to any error desired.

The principle of general order provides the basis for this solution, and Cohen notes that this understanding suffices for the purposes of mathematical rigor. He quotes Leibniz saying that when dealing with an infinitely small quantity, all one must show is “that the ‘error’ is less than the one an adversary wanted to assign, and that consequently we could not assign any.”³⁰ Interestingly, Cohen follows his citation with the following comment:

With this forbearance on the mathematical proof we step onto the slippery path of the *so-called* logical grounding [*Begründung*] of the differential, which principally consists of the cancellation of the mistake by yet other mistakes, whereas a few lines before the critical [*erkenntniskritische*], realising [realisierende] grounding had been found. (§ 54, 110)

This passage brings out an important feature of Cohen’s investigation of the infinitesimal method, because it highlights the difference he sees between the logical grounding of the

²⁹ It is noteworthy that the other extreme of such a continuum would be a circle: the case where the two foci perfectly coincide.

³⁰ Leibniz, “Replique” 190, as cited in PIM, § 54, 110. This and the subsequent passages Cohen quotes in French and Latin from Leibniz and Newton have been translated for me by Richard Arthur unless explicitly stated otherwise.

differential and the critical grounding (*erkenntniskritische Begründung*). Cohen chastises Leibniz for trying to justify the differential merely on the grounds that it inserts no undue error into calculations. Cohen charges that in doing so Leibniz is entering into a fruitless line of discourse. This charge should strike one as odd, given that the calculus should be founded in a mathematically rigorous way. Cohen's argument is that Leibniz protected the differential from objections by claiming that it was only infinitely small in the syncategorematic sense: finite but inexhaustibly, or incomparably small. Cohen also cites other examples of Leibniz referring to infinitesimals as incomparably small with respect to other finite quantities like “the distance to the fixed stars in relation to the diameter of a musket ball [*Kugel*]” (PIM, § 57, 118). In making these comparisons, Cohen claims that Leibniz abandoned the positive and original character of the concept of the differential. Here the differential is still thought of as infinitely small in a sense, but is only a useful fiction, or represents a convenient way of speaking about a method that relies entirely on finite quantities. From Cohen's perspective this covers over the true significance of the differential, as a new type of quantity.

What Cohen sees lacking in the “logical grounding” of the differential is the systematic justification of the concept. In the above-cited passage where Cohen says that “a few lines earlier the critical, realising grounding [of the differential] was found” by Leibniz, Cohen is referring to passages from the same response to Bayle he cites in the PIM just prior:

The “actual” phenomena of nature are and must be adjusted in such a way that nothing will be encountered “where the law of continuity” (which I have introduced) and “all the other” most exact “rules” of mathematics are violated. And even more than this, “things can only

be rendered intelligible by rules... Thus, although mathematical meditations are “ideal,” this does not at all diminish their usefulness, because “actual things” do not deviate from their rules; and one can say, in fact, that it is in this that the reality of phenomena insists, which distinguishes them from “dreams.”³¹

Here Cohen sees the critical potential in Leibniz’s thought that he is contemplating the relation between mathematical ideality and the phenomenal world. Under Cohen’s notion of objectivity, the phenomena of nature become real objects only in the first instance in which they are made intelligible through a mathematically based scientific theory. In recognizing a connection between the reality of phenomena and the fact that actual things in nature adhere to ideal rules, Cohen argues that Leibniz makes an advance over Descartes—who as Cohen notes also labored to distinguish consciousness of reality from dreams—towards a firm critical foundation for philosophical speculation (§ 54, 109).

What is missing in Leibniz’s approach, Cohen argues, is the “methodological determination of basic concepts” (§ 54, 109). This methodological determination and definition of these basic concepts, Cohen says, is the essential beginning of *Erkenntniskritik*. Leibniz, in Cohen’s view, did not develop in his systematic approach to philosophy the critical precision needed to provide a proper grounding for the concept of the infinitesimal. Instead, Cohen cites Leibniz as saying that mathematicians should not be allowed to enter into metaphysical discussions, so as “not to become entangled with the ‘real existence’ (!) of ‘points,’ of ‘indivisibles,’ of the ‘infinitely small,’ and of

³¹ “*Replique*” 189-90, as cited in PIM, § 54, 108-9. The quotes within the quotes denote emphasis added by Cohen.

infinities ‘in all rigor.’”³² Hence, on Cohen’s account, Leibniz instead treads the “slippery path” of seeking the logical grounding of the concept of differential.

Cohen writes in the opening passages of the PIM that “so long as the grounding of the concept of the infinitesimal is attempted merely in logic, the deficiency of such a grounding must remain—despite the innumerable attempts at its logical justification that have been undertaken, always anew, since the invention of the calculus ” (§ 1, 43). He sees the way out of this continual cycle of failed attempts as leading directly through *Erkenntniskritik*. The proper grounding of the concept of the infinitesimal, he argues, lies in its correspondence with the principle of intensive magnitude and the category of reality (§ 18, 57). To understand this distinction Cohen makes between logic and *Erkenntniskritik*, it is helpful to recall the distinction Kant makes in the *Critique* between “transcendental” and “general” logic. General logic, Kant writes, “has nothing to do with the origin of knowledge, but only considers representations, be they originally *a priori* in ourselves or only empirically given, according to the laws which the understanding employs when, in thinking, it relates them to one another” (B80). By contrast, Kant says that transcendental logic “concerns itself with the laws of understanding and of reason solely in so far as they relate *a priori* to objects” and “should determine the origin, the scope, and objective validity of such knowledge” (B81). Cohen draws a similar contrast between logic and *Erkenntniskritik*. He argues that to seek out a logical grounding for the differential (or the concept of the infinitesimal generally) is to confuse the content of knowledge with its formal conditions. The infinitesimal is not itself an object of

³² “*Replique*” 190, as cited in PIM, § 54, 109. Cohen’s emphasis added.

knowledge, but rather a necessary presupposition for mathematical natural science. To see this, one must approach its grounding from a critical perspective, because its justification lies in its systematic correspondence with the category of reality. This systematic justification concerns itself with the conditions for the possibility of mathematical science, and hence lies beyond the scope of logic.

3.2—*The Pure Form of Movement*

Cohen seeks to demonstrate the correspondence between the concept of the infinitesimal, the category of reality, and the principle of intensive magnitude. However, in order to see this critical meaning of the differential, the investigation must begin with the application of the differential to mechanics. Cohen cites Leibniz' treatise, "On the Use of Geometry in its Present State and on Further Developments," where Leibniz writes:

Physics then, insofar as the mind can carry it out, ends in geometry and we cannot thoroughly understand any phenomenon in bodies before we have derived the first ideas of figure and motion.³³

Cohen cites a number of examples of Leibniz asserting this relationship between physics and geometry in order to show that the seeds of *Erkenntniskritik* were latent in Leibniz's thought. Just as with Cohen's "historical" presentation of Kant's thought, one must be careful in discerning how much license Cohen is taking in interpreting the significance of Leibniz's statements. There is indeed clear documentary evidence that Leibniz considers the differential to be one of the "first ideas" necessary for physics that he mentions in the

³³ "*Disseratio*," 325, as cited in PIM, § 56, 115. The full Latin title is: "Disseratio Exoterica de Statu praesenti et Incrementis Novissimis deque usu Geometriae," which Leibniz later revised to: "De usu Geometriae Statu praesenti ac novissimis ejus incrementis."

above passage. For example, Cohen cites a letter from Leibniz to Wallis where the former writes:

Now it is no wonder that certain problems can be considered solved after the reception of my calculus that could not have been hoped for before, those especially which deal with the transition from geometry to nature.³⁴

What should not be accepted immediately, however, is the notion that Leibniz saw the same order of priority between mathematics and nature that Cohen does concerning the status of real objects. True, Leibniz does observe that it is through geometry that things become intelligible, and indeed, that it is in their adherence to eternal laws that actual things are discernible from dreams. Leibniz does not, however, argue that it is through our created mathematical entities that things in nature *become* real objects. In Cohen's defense, he does not argue that Leibniz held this view; rather, Cohen observes that Leibniz tried to show that monads were "the anchorage of reality" (§ 51, 102) and that his failure to separate the concept of "substance" from that of "reality" as independent categories of understanding—the way Kant would later define them—prevented him from articulating the critical understanding of natural science. That said, Cohen does read the kernel of the critical position into Leibniz's thought, because, Cohen writes, "this equivalence between the differential and the intensive [*des Intensiven*] is expressed by Leibniz, is well known to his correspondents, and..." (§ 19, 58).

But even if, as Cohen claims, the equivalence between the differential and the intensive is present in Leibniz's work, then it is still questionable whether this equivalence can be shown to have anything to do with Cohen's notion of intensive

³⁴ "Leibniz an Wallis" 24, as cited in PIM, § 57, 117.

magnitude as a basic principle of *Erkenntniskritik*. In fact, there is no indication of a necessary connection in Leibniz's thought between intensive magnitude and reality. Rather, it seems that Leibniz sees in the fact that his ideal rules are confirmed in nature in the completely reverse order from what Cohen has in mind—that ideal rules can make some claim on reality because they themselves are confirmed *by* nature.

Regardless of whether the connection between his own and Leibniz's ideas is truly present, Cohen's motivations for trying to advance such a connection are clear. There is an inherently partisan aspect to Cohen's views on the history of philosophy, and he is trying to advance a methodological argument against those who would justify the differential on logical grounds. What could be better in this case than to show that the seeds of one's position are to be found in the original, impulsive idea behind the differential itself? This basic idea, which Cohen seeks to locate in Leibniz's thought, is that reality itself is grounded in mathematical entities, and the ultimate justification of the differential is intimately tied with the project of *Erkenntniskritik*.

Two meanings of the differential are identified in Cohen's presentation of Leibniz: the geometrical and the mechanical. Both meanings are conveyed by Leibniz in a response to Nieuwentijt, who disputed the existence of the so called "higher-order" differentials. For Leibniz, higher-order differentials are the differentials of differentials. For example, if dx is the first-order differential, the second-order differential is ddx . If the first-order differential is thought of as an infinitely small difference of finite magnitudes, then the second-order differential is the difference of the difference. In his response to Nieuwentijt, Leibniz explains the geometrical meaning, that "differentials of the first

degree refer to tangents or directions of a line, but higher-order differentials refer to oscular or curved lines.”³⁵ Just as a tangent line to a curve touches the curve at a single point, so an osculation can be thought of as an infinitely small curved line “kissing” the curve at a single point of contact. Leibniz goes on to explain that since a curve changes the inclination of its direction at every single point, the angle of the tangent changes “insensibly” or by “incomparably small differences” (PIM, § 57, 116).³⁶ For Cohen, the mechanical interpretation of the differential bears the more important meaning of the concept than the geometrical, because it is this meaning that relates directly to physics. Again he cites Leibniz’s response to Nieuwentijt, where Leibniz gives further justification of the existence of higher-order differentials:

But the truth and use of these successive differentiations is confirmed in things themselves. For as [...] I noted elsewhere, ordinary quantity, first differential, and second differential are as motion (x) and speed (dx) and sollicitation (ddx), which is the element of speed. By motion a line is described, by velocity the element of a line, and by sollicitation (the beginning of descent due to gravity or the motion by a centrifugal endeavor) the element of an element.³⁷

Under the mechanical interpretation the first-order differential corresponds to speed, or an infinitesimal element of motion, and the second-order differential corresponds to sollicitation. Sollicitation is an infinitesimal element of speed. Just as speed is thought of as the element of change in position, sollicitation is thought of as the element of change in this change. The mechanical meaning of the differential is paramount for Cohen. It is

³⁵ “Responsio,” 320, as cited in PIM, § 57, 116.

³⁶ A consequence of Leibniz’s law of continuity is that between any two points, there are a potentially infinite number of intermediate points. Hence, in traversing a curve between any two distinct points, A and B, the slope of the tangent can be thought to traverse a potentially infinite number of intermediate degrees of inclination between the angle of inclination at A and the angle of inclination at B.

³⁷ “Responsio,” 325, as cited in PIM, § 57a, 119-120.

through this interpretation that the infinitesimal becomes a necessary presupposition for natural science. In his essay on Cohen's contribution to the renewal of Kantian philosophy, Cassirer eloquently expresses the importance of the infinitesimal in Cohen's thought when he writes that "without [the infinitesimal], it would be impossible to describe rigorously the concept of motion as natural science presupposes it, let alone to have a conceptual command of the lawfulness of motions" (99).

Interpreting the first-order differential as the speed of a moving body and the second-order differential as the solicitation, or change in speed, allows for the motion of a body being acted on by a constant force, for example gravity, to be represented by a curve. The calculus allows for the analysis of this motion. Through the calculus the instantaneous velocity of the body at any given time, or any given point on the curve, can be determined. Cohen quotes Leibniz in this regard, from his 1698 letter to Schulenburg, where Leibniz writes: "it is established that there is no better way 'to open the door from geometry to nature' than by infinitely many intermediate 'degrees' in every change."³⁸ The mechanical meaning of the differential and the geometrical meaning are both intimately related to the establishment of a physics of motion. The geometrical curve is the mathematical representation of an infinite number of intermediate degrees of change, while the mechanical interpretation of the differential is the recognition of the geometrical form's ability to render the phenomenon of change in nature intelligible. This is the basis for the conceptual command of motion. Leibniz's calculus provides the method for making the indeterminate forces acting on an object in motion determinable.

³⁸"Zwei," 240, as cited in PIM, § 57, 121.

But it is not the only “first idea of figure and motion” that forms the basis for Leibniz’s views on physical principles. Hence, we must take a closer look, as Cohen does, at Leibniz’s theories of dynamics. As Leroy Loemker explains, Leibniz’s mature theory of dynamics is presented in his 1695 treatise, “Specimen dynamicum” (435).

Cohen cites Leibniz from this work:

We have suggested elsewhere that there is something besides extension in corporeal things, namely, a force of nature implanted everywhere in it by its Author, a force which [...] is moreover equipped with an endeavor or striving [...] This striving frequently presents itself to the senses, and in my opinion should be understood by reason as present everywhere in matter, even where it does not appear to sense [...] Certainly this force [...] must constitute the innermost nature of bodies [...] And so there is nothing real in motion except that momentaneous [state] which must consist in a force striving towards change. Therefore, whatever there is in corporeal nature apart from the object of geometry, i.e. extension, reduces to this force.³⁹ (§ 58, 123)

In this passage Leibniz identifies the need for the recognition of something aside from extension in order to conceptualize movement, and that it is in this other element that bodies “strive” or “endeavor” towards change. The significance Cohen assigns to this passage is that there is an element to motion that is not represented purely through extension, and hence, in Kantian terminology, that the conceptualization of motion requires the supposition of an element that is not given in intuition (*Anschauung*). For Cohen, all extensive magnitudes are quantities given in intuition. However, the objectification of the endeavor that impels a moving body towards motion requires the supposition of a different kind of magnitude. Hence, Cohen argues, the presupposition of intensive, qualitative magnitudes becomes necessary for the science of motion.

³⁹ Here I give a translation provided to me by Richard Arthur. A full translation of the treatise can be found in Loemker’s volume, *Philosophical Papers and Letters*, 435-452.

Cohen presents the above-cited passage from “Specimen dynamicum” along with short excerpts from Leibniz’s 1671 treatise “The Theory of Abstract Motion, or, The Universal Reasons for Motions, Independent of Sense and Phenomena.”⁴⁰ In this treatise, Leibniz lays out a set of “predemonstrable foundations,” which as the longer title indicates are intended as a priori axioms for the understanding of motion. Cohen draws attention to numbers 4), 5), and 18) in particular:

4) *There are indivisibles or unextended things*, otherwise neither the beginning nor the end of a motion or body is intelligible.⁴¹

5) *A point is not that which has no part*, nor that whose part is not considered; but that which *has no extension* [...]

18) *One point is greater than another, one endeavor is greater than another, but one instant is equal to another*, whence time is expounded by a uniform motion in the same line, although its parts do not cease in an instant, but are indistant.⁴²

By Cohen’s account, the significance of Leibniz’s “Theory of Abstract Motion,” is that it provides an ideal model of motion in which the path of any motions must be conceived as beginning from an inextensive point. Arthur explains in a note to the text that Foundation 18, “has its origin in Galileo’s analysis of fallings bodies” (Arthur, “Theory” 429

⁴⁰ This treatise appeared along with the “Theory of Concrete Motions, or, A Hypothesis About the Reasons for the Phenomena of Our World,” as part of a larger work entitled “A New Physical Hypothesis.” Partial translations can be found in Arthur’s collection, *The Labyrinth of the Continuum*, 338-343, and Loemker’s above-mentioned collection as well, 139-145. My citations are from Arthur’s translation.

⁴¹ Leibniz gives the following demonstration for this foundation: “any space, body, motion, and time has a beginning and an end. Let that whose beginning is sought be represented by the line *ab*, whose midpoint is *c*, and let the midpoint of *ac* be *d*, that of *ad* be *e*, and so on. Let the beginning be sought to the left, on *a*’s side. I say that *ac* is not the beginning, since *dc* can be taken away from it without destroying the beginning; nor is *ad*, since *ed* can be taken away, and so on. Therefore nothing is a beginning from which something on the right can be taken away. But that from which nothing having extension can be taken away is unextended. Therefore the beginning of a body, space, motion, or time (namely, a point, an endeavor, or an instant) is either nothing, which is absurd, or is unextended, which was to be demonstrated” (“Theory” 339).

⁴² “Theory,” 339-42, as cited in PIM, § 58, 122.

endnote). Galileo presumes that the time of a body's fall is divided into equal parts and that this equality is preserved even for infinitely small moments. Hence, the "uniform motion" referred to in 18) is "that in which equal infinitesimals of space are accrued in equal moments," and "uniform acceleration" is "that in which equal infinitesimals of velocity are accrued in equal moments" ("Theory" 429n). As I will discuss in the next section of this chapter, the connection between Leibniz, Newton, and Galileo proves to be very significant in Cohen's account.

It is through the combination of ideas contained in "The Theory of Abstract Motion" and "Specimen dynamicum" that Cohen identifies the conceptual foundation of motion to which the critical justification of the differential is intimately linked. Cohen describes the positive meaning of the differential as "that new type of magnitude" which "is based in the intensive and mediated through the inextensive" (§ 58, 126). Under this conception, every motion is conceived of as beginning from an inextensive point, or instantaneous moment in which the intensive interjects through the body's "endeavor" or "striving" towards motion. Hence, the extensive magnitude traced by the path of motion, is engendered by these intensive endeavors at every "moment," or "point." For Cohen, the positive meaning of the differential is realized, when it is taken to be a real, variable magnitude that allows for the determination of the intensive changes, occurring within incomparably small, but equal, moments of time. Furthermore, Leibniz's calculus provides a method of calculation using differentials that can be applied generally to a limitless set of cases. However, Leibniz was not able to situate his notion of the

differential within a universal science of motion. This was Newton's achievement which will be discussed after a brief detour through Galileo.

3.3—*The Prototype of Natural Forces*

Cohen locates the beginnings of the universal science of motion in Galileo's "law of falling bodies," which Cohen calls the "prototype of forces of nature" (§ 48, 95). The law states that the distance covered by a falling body is proportional to the length of time of the fall squared. The reason Cohen identifies Galileo's work as the "prototype" of natural forces, is because of a particular conceptual advance found in Galileo's method: the use of ideal "limiting" cases. An example of such a case is "inertial motion," in which a body is conceived of moving in a perfectly uniform manner, with no change speed or direction. Such motion is not a fact of nature, because every body has some force acting upon it, such as gravitational force and atmospheric resistance.⁴³ Inertial motion is not something that "happens" in nature. Nevertheless it can be used as a limiting case through which the accelerated motion of bodies can be understood, insofar as a case of accelerated motion can be considered to deviate from the inertial one. Hence in any given moment forces act on a body such that it does not persist in the same direction or with the same speed it had in the previous moment. Cohen observes that this move to explain nature through ideal laws not found in sensual nature is the beginning of the foundation of natural science, which would be fulfilled through Newton and systematized by Kant (§ 48, 96).

⁴³ Arguably this is true even in deep space, because there is no limit to the effects of gravitational forces.

It should be noted that Galileo does not conceive of infinitesimals as Leibniz or Newton does. Galileo maintained a notion of infinitely small “indivisibles.” However, Cohen argues that Galileo does make the distinction between intensive and extensive magnitudes, because velocity for Galileo is an intensive magnitude. Furthermore, Galileo’s conception of acceleration requires the presupposition of a continuum of equal, but indivisible and incomparably small, moments of time. The intensity of velocity is set in proportion to these moments in time and determines the extensive span of space covered during this moment. Hence, Cohen argues, in Galileo’s thought one sees intensive magnitudes engendering extensive ones through the presupposition of a time continuum.⁴⁴ Acceleration then measures the increase in velocity throughout the continuous progression of moments. Cohen argues that the critical justification of the infinitesimal requires that one not derive continuity from the divisibility of space, but rather, that one see space as engendered by intensive forces acting continuously in time. In this way, he says, reality is brought to nature (PIM, § 48, 98).

3.4—*Newton’s Fluxions*

While Cohen recognizes Galileo as providing the prototype for natural forces, it is Newton, he argues, who fulfills the task of providing a system for conceptualizing motion that proceeds from basic laws and axioms. Cohen commends Newton’s research method, saying that it is “one so sound and correct—we could better say it is the exemplar—that

⁴⁴ I should note that Galileo is not the only thinker to connect intension with continuous time. In § 46 of the PIM, he identifies Newton’s teacher Barrow as having made significant advances in this direction and as having influenced both Newton and Leibniz.

we must recognize Kant also as a Newtonian in this primary respect” (§ 64, 135). Here Cohen is referring to the fact that Newton not only presupposed space in such a way that differentiated it from empirical space, but also time. Newton does so by defining absolute space and time as follows in the *Principia*:

1. Absolute, true, and mathematical time, in and of itself and of its own nature, without reference to anything external, flows uniformly and by another name is called duration. Relative, apparent, and common time is any sensible and external measure precise or imprecise of duration by means of motion; such a measure—for example, an hour, a day, a month, a year—is commonly used instead of true time.

2. Absolute space, of its own nature without reference to anything external, always remains homogeneous and immovable. Relative space is any movable measure or dimension of this absolute space; such a measure or dimension is determined by our senses from the situation of the space with respect to bodies and popularly used for immovable space[...].
(408-09)

Although Cohen does not do so, I have chosen to cite these passages at length because of the importance of the concept of absolute space and time to Newtonian mechanics. We can see that in both cases Newton distinguishes absolute time and space from the relative versions that are sensually given. Absolute space is the immovable frame of reference to which all other spaces are relative. Because our current subject matter deals so intimately with the reality of mathematical entities, I feel it is important to highlight the mathematical nature of absolute space for Newton. Arthur explains that “Newton conceives of space as filled with ubiquitous figures—spheres, cubes, triangles, straight lines, and so forth [...] material bodies are simply ‘sensible’ or ‘corporeal

representations' of these underlying forms [...]."⁴⁵ Newton's notion that absolute, or real, space is filled with existent geometrical forms that underlie phenomena bears some resemblance to Cohen's notion that it is through mathematical synthesis that phenomena become real objects, and indeed, as we shall see, while Cohen does not explicitly refer to this passage, the root of Cohen's interest in Newton is partially unearthed here.⁴⁶ A full unearthing requires that we look more closely into absolute time, and with it, the heart of Newton's method of fluxions.

Absolute time, Newton says, is to be differentiated from relative time derived from motion, i.e. derived from clocks, or the motion of the sun, etc. Furthermore, absolute time itself flows without reference to anything external. Arthur takes up various objections to this notion in his discussion of Newton's absolute time, and indeed it is odd to think of the absolute flow of time without reference to anything. For example, when a river flows, we identify it as flowing relative to its banks. Arthur notes that in each case where we measure relative time against an equably moving body, such as the sun, we measure the equability of its motion against another equably moving body ("Newton's" 346). The fact that Newton's absolute time is not thought to equably flow relative to something else is not a contradiction; it is precisely what differentiates absolute time from empirical time. Arthur observes that this notion corresponds to the "neo-platonic [distinction] between pre-existent mathematical *form* and empirical *measure*"

⁴⁵ "Newton's," 342. I have found the detailed discussion of the relation between absolute time and Newton's fluxions in this paper to be an excellent supplement to Cohen's at times terse discussion of Newton. Furthermore, the historical accounts in each are similar in many respects. For example, both highlight Barrow's influence on Newton's calculus.

⁴⁶ Actually, there is no way Cohen could have cited the passages in which Newton fully articulated this view, because Arthur explained to me, the paper "De gravitatione etaequipondio fluidorum" ("On Gravity and Equilibrium of Fluids") was only unearthed in the 1960s.

(“Newton’s” 345). To relate this observation to Cohen’s position, we can say that the equable flow of absolute time is a formal condition, a priori, for the construction of Newtonian mechanics.

Cohen argues this very case, that Newton’s method of fluxions, and hence the method of calculation on which Newtonian mechanics is founded, presupposes and requires the notion of equably flowing time. Cohen quotes Newton from his 1671 treatise “Method of Fluxions,” saying: “I consider quantities as things generated by a continual increase such as the space which a moving body or any other moving thing describes.”⁴⁷ The germ of Newton’s critical significance in Cohen’s view, is just this notion—that the method of fluxions is one by which space, i.e. extensive magnitudes, are derived from the continuous flow of time. In order to see how this is the case, we must inquire in more detail into the method of fluxions, and for this I turn to Arthur’s account, which is more explicit than Cohen’s. Arthur explains Newton’s “flowing quantities” as follows:

The existence of *fluents* (i.e. flowing quantities) presupposes a temporal flux: a quantity cannot continually increase or diminish *in time* unless time itself undergoes a continual accretion. As time is generated in a constant flow, so are all other quantities, and the rates at which they change with respect to this constantly flowing time—their ‘velocities’ or *fluxions*—can be compared against each other. (“Newton’s” 334)

Here we learn that flowing quantities, or fluents, have a finite rate of change, or fluxion, at each instant. This is analogous to the notion that a moving body has an instantaneous velocity at a given moment in time. This is no surprise, since as Arthur shows, Newton was heavily influenced by his teacher Barrow, who applied Galilean ideas of motion to geometry. The advantage over Galileo’s conception of motion is that one does not have

⁴⁷ “Method” 34, as cited in PIM, § 64, 135.

to deal with the inherent contradictions that arise out of a continuum constructed from indivisibles. As Arthur argues, if a line is conceived of as a “trace of a continuously sliding instant,” then “the continuity assumption is built in: at every instant of time, each point or line has a well defined, finite instantaneous velocity, and the same for all other quantities generated in time” (“Newton’s” 341).

In Newton, Cohen sees the culmination of science that set the stage for Kant’s new concept of experience. In order to appropriate Newton’s calculus into his Kantian schema, Cohen reads intensive magnitudes into Newton’s instantaneous moment, arguing that Newton hides this meaning through his use of the method of “first and ultimate ratios” in the *Principia*.⁴⁸ As I. Bernard Cohen explains in his comprehensive introduction to the *Principia*, Newton’s method of first and ultimate ratios is intended as a rigorous theory of limits to avoid the use of actually infinitesimal moments of time (129). Newton sets out this method in a set of eleven Lemmas in Section 1 of the *Principia*. The first Lemma reads:

Quantities, and also ratio of quantities, which in any finite time constantly tend to equality, and which before the end of that time approach one so close to another that their difference is less than any given quantity, become ultimately equal. (433)

With this method, Newton begins with a ratio of finite quantities and does not have to posit the existence of infinitesimals. Under the concept of the limit, all that is asserted about the difference between the quantities is that for every difference there is one smaller as the ultimate limit is approached. We should be reminded here of our

⁴⁸ Cohen calls this method the “Methode der ersten und letzten Verhaeltnisse,” which translates more literally as the “method of first and last ratios.” Here I follow I. Bernard Cohen’s translation of the term in the *Principia* as the “method of first and ultimate ratios.”

discussion of Leibniz's syncategorematic definition of the differential: that the error can be shown to be smaller than any quantity an adversary may want to assign, and that hence there is none. Arthur notes, however, that Newton felt his method superior to Leibniz's algebraic method of differentials because Leibniz's method "depends on there being infinitesimals first (even if Leibniz regards them as fictions)," whereas Newton's method "makes no commitment to actual infinitesimals but only to limiting ratios, that is, limits of a sequence of ratios between two finite quantities" ("Newton's" 343). For this reason, many commentators have held that Newton's calculus rests on a more secure logical foundation than does Leibniz's.⁴⁹

Nevertheless, just as Cohen rejects Leibniz's logical grounding for the calculus, he also claims that the method of first and ultimate ratios contains his notion that intensive magnitudes engender extensive ones. Cohen argues that the key is in how one interprets the "first" and "ultimate" moments on which the method is based. It is illustrative first to see what Newton had to say in this regard, and then look at the analogy he draws between the first and ultimate ratios and the velocity of a moving body, while defending his method against possible objections:

It may be objected that there is no such thing as an ultimate proportion of vanishing quantities, inasmuch as before vanishing the proportion is not ultimate, and after vanishing it does not exist at all. But by the same argument it could equally be contended that there is no ultimate velocity of a body reaching a certain place at which the motion ceases; for before the body arrives at this place, the velocity is not the ultimate velocity, and when it arrives there, there is no velocity at all. But the answer is easy: to understand the ultimate velocity as that with which a body is moving, neither before it arrives at its ultimate place and the motion ceases, not

⁴⁹ Whether this is true is a matter of debate. For example, Arthur contests this conclusion in his article, "Leibniz's Syncategorematic Infinitesimals, Smooth Infinitesimal Analysis, and Newton's Proposition 6."

after it has arrived there, but at the very instant when it arrives, that is, the very velocity with which the body arrives at its ultimate place and with which the motion ceases. And similarly the ultimate ratio of vanishing quantities is to be understood not as the ratio of quantities before they vanish or after they have vanished, but the ratio with which they vanish. Likewise, also the first ratio of nascent quantities is the ratio with which they begin to exist [or come into being]. (*Principia* 442)

Cohen refers directly to the analogy drawn here by Newton between the first and ultimate ratios and the beginning and end velocities of a moving body (§ 65, 139). Cohen then comments that the ultimate ratio must be understood in intimate connection with, and as the same type of ratio as the first. He says:

We can regard the ultimate ratio accordingly as the end [*Endigung*] of the continuation of the first ratio. For without the first ratio one could not begin to mathematically describe and determine the generation [*Erzeugung*] of magnitudes. (§ 65, 139)

Cohen's argument is that the first, or nascent, ratio where quantities come into being is the same type of ratio as the ultimate, or evanescent, ratio. This is clearly demonstrated, he says, by the mechanical analogy made by Newton. If the ultimate ratio is thought of as analogous to the final velocity at the exact moment a body comes to rest, then the first ratio is analogous to the exact moment the motion begins. Upon recalling from our earlier discussions of Leibniz and Galileo that the law of continuity implies that rest can be thought of as infinitesimally small or intensive motion, it becomes clear what Cohen seeks to draw out of Newton's analogy. The path of motion both begins and ends with rest, or with an intensive magnitude mediated through an unextended point. Thus, if the first and ultimate ratios are truly analogous to the beginning and end of motion in the way Newton describes, then these limits of ratios become the bounds of quantities, like unextended points, where intensive magnitudes intervene.

It is apparent that the conclusions about the relation between extensive and intensive magnitudes Cohen draws from Newton's analogy are not also drawn by Newton. Newton says only that:

Those ultimate ratios with which quantities vanish are not actually ratios of ultimate quantities, but limits which the ratios of quantities decreasing without limit are continually approaching, and which they can approach so closely that their difference is less than any given quantity, but which they can never exceed and can never reach before the quantities are decreased indefinitely. (*Principia* 442)

Here Newton reasserts that the ultimate ratio is not an actual ratio, and by analogy, that the end velocity is not an actual ratio. Rather, the ultimate ratio is to be thought of as a limit that the quantities cannot reach until they are decreased indefinitely. This is a difficult notion because Newton's phrasing implies that there is some point at which the quantities can be considered to be "indefinitely decreased." However, that is not possible since the word "indefinite" implies that there is no point at which the quantities can be considered to be "decreased." Thus, the first and ultimate ratios are never reached, and by analogy, the beginning and end of motion never happen.

Although Newton argues that he is dealing with quantities that are not to be understood as "determinate in magnitude," but as "always decreased without limit" (*Principia* 443), I. Bernard Cohen observes that throughout the *Principia* Newton frequently assumes that tangents, arcs, chords, etc. "ultimately become equal" (129). Hence, many of Newton's demonstrations rely on the notion that the limit is approached closely enough such that there is a relationship of equality between limit and approaching quantities. Newton makes this connection himself, saying:

This matter [of quantities increasing indefinitely] will be understood more clearly in the case of quantities that are indefinitely great. If two quantities whose difference is given are increased indefinitely, their ultimate ratio will be given, namely the ratio of equality, and yet the ultimate or maximal quantities of which this is the ratio will not on this account be given. (*Principia* 443)

In this scenario, we are presented with two indefinitely increasing quantities, such that when they are “indefinitely increased” their ultimate ratio may be considered to be equality—i.e. ∞/∞ . This is because whatever difference there is between the quantities initially, it will become miniscule relative to their magnitudes as they are indefinitely increased. However, the question remains: how significant is it that the ratio between their differences will only ever approximate equality, albeit by a margin of error that can be made as small as is required?

In the PIM, Cohen presents an argument against the concept of the limit interpreted this way; however, I wish to postpone my assessment of it until chapter 4, because it relates directly to the criticisms of the PIM made by Bertrand Russell that I will be addressing there. My immediate concern is to consolidate my presentation of Cohen’s history and conclude with a discussion of Kant’s role in it.

3.5—*Kant and the Apriorization of Perception*

Cohen’s historical presentation of both Leibniz’s and Newton’s discoveries of the calculus is intended to advance the argument that:

With Leibniz as with Newton the concept of the infinitesimal is an expression of their systematic basic idea [*systematischer Grundgedanke*], their idealist understanding of reality. (§ 67, 142-43)

What Cohen refers to here is the impetus he recognizes in both thinkers' work to derive a mathematical science of nature. He presents a picture of Leibniz's differential calculus and Newton's method of fluxions whereby finite extensive quantities are created out of intensive magnitudes—an interpretation that is reinforced in both cases by the appeal to the mechanical interpretation of the methods. In both cases, the path of motion described by a moving body is determined to result from an indefinite gradation of imperceptible changes that occur everywhere along the curve. Leibniz's differential and Newton's fluxions are to be understood as means of calculation that allow for these degrees of change to be and made determinate at any particular instant chosen on the curve. Furthermore, Newtonian mechanics provides the means through a set of synthetic relations, or laws of motion, by which these degrees of change can be understood as resulting from the sum total of forces acting upon a moving body at any given instant in time—an instant that the calculus makes explicit. Thus, the method of calculation becomes the heart of a universal science of nature. Change in nature becomes comprehensible through mathematical forms, and moving bodies that were mere phenomenal things, become real objects.

Cohen views this development of mathematical science as a necessary precondition for the critical turn in philosophy, and it is Kant who systematizes Newton's and Leibniz's great achievements into a coherent philosophical perspective. The significance of the concept of the infinitesimal becomes clear, Cohen argues, when viewed in association with Kant's transcendental formulation, by which intensive magnitudes constitute the real in perception. Cohen writes:

If there were no reality grounded in the intensive infinitesimal, then sensation would be and would remain to be merely subjective, and there would be no means to objectify in it an a priori, factual, real element. (§ 60, 131)

One must be careful to situate this statement, and Kant's overall significance in Cohen's historical account, within the context of Cohen's reading of Kant as having discovered a new concept of experience. Under this new concept, Cohen argues, experience is itself constructed and is synonymous with mathematical natural science. Perception is philosophically significant within this schema inasmuch as there is a sense-material component to science. However, science is not based on subjective representations of empirical space. This is why Cohen asserts the importance of the terminological distinction between perception and intuition made by Kant (§ 78, 165-69).

Newton makes an important step in this direction by distinguishing absolute space and time from empirical space and time. However, Cohen says, it is Kant who defines space and time as pure forms of intuition and establishes space as the basis for geometrical knowledge and time as the basis for number and dynamics (§ 78, 165). The forms of intuition combine with thought in the first instance of legitimation and constitution of synthetic knowledge is constructed—the knowledge of objects (§ 78, 166). Kant says in the first *Critique*, that every instance of knowledge involves a combination of concept and intuition: “thoughts without intuitions are empty, intuitions without concepts are blind” (B75). For Cohen, to speak of knowledge is to speak of scientific knowledge, and the infinitesimal represents the combination of thought and intuition in each instance of scientific knowledge. Cohen views the significance of the calculus primarily through the lens of a specific problem: how do the things in nature,

become objects of science? In his introductory essay to the PIM, Werner Flach explains that the specific problem of physics for Cohen is one of realization of the “thing to be determined” (*das Zu-Bestimmende*) which appears in sensual perception, such that it becomes a “determined object” (*das Bestimmte*) (27). Cohen’s history of the calculus demonstrates that the goal of making motion determinate was the driving force behind its discovery. So on one level, the differential can be seen as a tool for mathematically measuring the instantaneous changes that occur in any case of accelerated motion. Like Kant, Cohen holds the view that all objects of intuition are extensive magnitudes, or as Kant puts it, magnitudes that “representation of parts makes possible, and therefore necessarily precedes, the representation of the whole” (CPR, B203). However, as the discussion of Leibniz’s dynamics is intended to show, conceiving of motion requires the supposition of intensive, qualitative magnitudes in order to capture the “impetus” that drives moving bodies. This requires the supposition of intensive quantities intervening through an extensive point, and Galileo provides the prototype for systematizing this by conceiving of space as being engendered by intensive magnitudes through uniform moments of time. But intensive magnitudes are not given in intuition and thus are a determination of thought.

The recognition of the infinitesimal as entering the equation of knowledge from the side of thought leads to the deeper meaning Cohen ascribes to it. The infinitesimal is, in his view, not just a tool of mathematical measurement, but corresponds to reality itself. This is the significance of the principle of intensive magnitude, or of Kant’s anticipations of perception, in Cohen’s schema. Recall our discussion, in chapter 2, Cohen’s notion of

the “unity of consciousness.” This unity is a kind of qualitative unity that is a necessary precondition for all understanding.⁵⁰ The anticipations of perception locate the reality of perception in its degree, or in qualitative unity. Cohen sees this as the precondition for any apriorization of nature. This particular aspect of his argument is undoubtedly the most obscure and difficult part of the PIM.

What does it mean to say that reality is intensive magnitude? This is an issue I will take up again at the end of chapter 4; however, to interpret what Cohen may have in mind, I wish to use the previously mentioned idea of Newton’s that absolute space is filled with “ubiquitous geometrical figures.” Arthur states:

Newton conceives space as filled with ubiquitous figures—spheres, cubes, triangles, straight lines, and so forth. [...] Material bodies are simply “sensible” or “corporeal representations” of these underlying forms, regions which God had endowed with the properties of impenetrability, movability, and sensibility. And according to Newton’s fluxional geometry [...] these moving figures trace out figures of a higher dimension by their continuous motion. If all mathematical quantities are so generated, every fluent will have a velocity or fluxion at every instant, and each may be regarded as “aggregated” out of its fluxions over time. (“Newton’s 342)

The picture of Newton’s ontology described by Arthur is analogous in certain respects to Cohen’s ontology in the PIM. In the above passage, the phenomenal world is broken down into three constituent aspects. First, material bodies are simply “corporeal representations” of the real underlying geometrical figures that fill absolute space. Second, there is a continuous motion or flow—the flow of absolute time—at every instant such that all bodies, both phenomenal and geometrical, are continuously generated. This generation occurs at the intersection of time and space, or the absolute

⁵⁰ For Kant, this is “transcendental apperception.”

“moment.” Lastly, there is God. For Newton, God is the creative determining force that imbues phenomenal bodies with their corporeality and is responsible for the absolute structures that underlie the phenomenal world. As I have said before, Cohen did not have access to Newton’s “De gravitatione.” However, I feel it would not be much of a stretch to think that Cohen picked up something of this neo-Platonic strain in Newton, especially given Cohen’s own well-documented Platonic leanings. Furthermore, the notion of space being generated out of time is highlighted in Cohen’s presentation of the history of calculus. It is important to note though, that I am intending to draw an analogy here between the constitutive elements of the phenomenal world in Newton’s and Cohen’s accounts. It would be false to assert that Cohen holds the same metaphysical notions Newton does. For example, in line with Kant, Cohen would not say anything about a creator God, (at least not within the context of the PIM). Nevertheless, there is an analogy that can be drawn between the two accounts. Cohen’s account is framed in Kantian concepts. Instead of absolute space, absolute time, and God underlying the phenomenal world, the reality of Cohen’s phenomenal world is grounded in the pure intuitions of space and time, and the thought determinations of a unified consciousness.⁵¹

In the PIM, Cohen clearly draws a connection between absolute space and time on the one hand, and the pure intuitions of space and time on the other, insofar as all are abstracted from empirical space and time (§ 64, 135). Newton’s absolute space and time allow for an a priori science of nature. Cohen’s view is that Kant systematizes this science of nature into a coherent philosophical perspective through his new concept of

⁵¹ In chapter 4, I will show that these “thought determinations” are limiting judgments.

experience. Absolute space and time are thus forerunners of the pure intuitions of space and time as formal conditions for the construction of experience. For Newton, God acts as the ultimate source of both the phenomenal and ideal realms. In “*De gravitatione*” he says:

Thus I have deduced a description of this corporeal nature from our faculty of moving our bodies, so that all the difficulties of the conception may at length be reduced to that; and further, so that God may appear (to our innermost consciousness) to have created the world solely by the act of will, just as we move our bodies by an act of will alone; and, besides, *so that I might show that the analogy between the Divine faculties and our own is greater than has formerly been perceived by Philosophers.* (141; my emphasis added)

Here Newton ties his view that quantities are generated by a continuous motion with the notion of divine will. The idea is that the source or originary impetus behind being must be understood by analogy with our own faculties. Under Cohen’s Kantian schema this analogy seems to be fulfilled. Instead of an absolute God as the source of phenomena and the ideal realm of figures, it is a unified consciousness that operates in this role. This idea of unified consciousness is one that is inherently tied with Cohen’s notion of objectivity and the new concept of experience. The separation between subject and determinate world in Newton’s account requires an external absolute source of being. Kant’s critical philosophy provides the means for Cohen to break down this separation and make consciousness the ultimate source.

What can this analogy now reveal about the nature of Cohen’s intensive reality and the apriorization of perception? Reality, for Newton, resided in ideal forms generated out of the flow of absolute time and the creative will of God. For Cohen, the objects of spatial intuition are also generated. This generation occurs out of the thought

determinations of consciousness acting within time (or the sense of inner intuition). In Newton's absolute "moment" there is the intersection of absolute space, absolute time, and God's active will. For Cohen, there is a similar absolute moment, in which thought and intuition combine to determine an object of knowledge. For Cohen, this is the moment of "realization" and also the critical meaning of the infinitesimal. Intensive reality does not refer to a unified plenum or to things-in-themselves. Rather, I think Cohen intends it as the potential determinateness or potential objectivity of things. This potential extends to perception insofar as the things of nature can be rendered determinate through the combination of pure thought and intuition. This is the apriorization of perception, and perception is only considered in the knowledge equation inasmuch as there is a sense-material component to science—the unified product of consciousness. For Newton, the phenomenal world is ideal form imbued with corporeal characteristics. For Cohen, the phenomenal world is ideal form with a sense-material component to it. As such, for Cohen, there is no reality outside of objectivity, experience outside of science, or being outside of consciousness. That is what Cohen's history and critical justification of the infinitesimal is intended to show.

Chapter 4: Cohen's Critics

Now that Cohen's argument for the critical grounding of the infinitesimal has been presented, it must be made subject to critique. The issues surrounding the foundations of the calculus have always stimulated controversy, and Cohen's treatment of the problem is by no means exempt. To frame and guide the discussion, I will be considering two well-known criticisms of the PIM: Bertrand Russell's refutation of the book, which appeared in his 1903 work, *The Principles of Mathematics* and Gottlob Frege's 1885 review.⁵²

In *The Principles of Mathematics*, Russell endeavors to show that mathematics is purely an extension of formal logic. His critique of the PIM is contained within a chapter entitled "Philosophical Arguments Concerning the Infinitesimal," in which Russell seeks to differentiate and defend his philosophy of mathematics from those of other "standard" philosophical writers. Among the most contentious issues Russell treats is the proper understanding of the infinite and infinitesimals. He singles the PIM out as a particularly instructive example of a flawed interpretation "not only because it deals explicitly with our present theme [the status of infinitesimals], but also because, largely owing to its historical excellence, certain very important mathematical errors, which it appears to me to contain, have led astray other philosophers who have not an acquaintance with modern

⁵² The full title reads, "Rezension von: H. Cohen, Das Prinzip der Infinitesimal-Methode und seine Geschichte," and it appeared in *Zeitschrift für Philosophie und philosophische Kritik*, 87: 324-329. It has been subsequently reprinted in the collection *Kleine Schriften*. My citations are from Hans Kaal's English translation found in *Collected Papers on Mathematics, Logic, and Philosophy*. I give the original *Zeitschrift* pagination, which Kaal provides in the margins of his translation, and which is also provided in the *Kleine Schriften*.

mathematics at first hand” (338). In his “*Rezension*,” Frege also argues that Cohen demonstrates a lack of adequate mathematical knowledge and mathematical facility in the PIM (324). Unlike Russell, Frege does not present his critique of the PIM within the context of a larger program for the philosophy of mathematics. Frege does, however, share with Russell the view that arithmetic and formal logic are fundamentally identical. Given that Russell and Frege both critique the PIM from roughly the same systematic viewpoint it is not surprising that their criticisms overlap in a number of instances. Due to this high degree of congruence between the two critiques, treating them each separately would be redundant. Thus, in what follows I will be highlighting aspects of both arguments against the PIM according to the following themes: 1) concerns over the style, methodology, and execution of the PIM; 2) how mathematical methods and entities should be grounded; 3) whether Cohen misunderstands the concept of the limit; and 4) whether Cohen succeeds in demonstrating the correspondence between infinitesimals and the category of reality.

4.1 *Conflicting Styles*

First I would like to address the overarching theme of Cohen’s writing style and method of argument. There is no doubt that it colors, at least partially anyway, Frege’s reception of the PIM in particular. As he opens his review, Frege apologizes for what may be his “defective understanding” of Cohen’s argument. If this is the case, Frege claims, he will “be excused because of Cohen’s style of writing, which is by no means distinguished for its clarity and which is sometimes even illogical” (325). This is a statement that anyone

who has toiled through the PIM will read sympathetically; however, I think there is a deeper methodological criticism of Cohen to be found here as well, which reveals just how drastically at odds Frege and Russell are with Cohen regarding what constitutes a proper philosophical argument. As Frege says:

What I miss here is that striving for precision of expression and logical irreproachability which alone could guarantee clarity of thought in investigations of this kind. And such a guarantee would deserve much more credit than the guarantee Cohen derives from agreement with the historical development of the problem. His opinion that historical insight alone can first disclose what has a claim to being a logical presupposition of [natural] science is an erroneous one. On the contrary, those logical foundations are perhaps always discovered only later on, after a considerable amount of knowledge has been accumulated. From the logical point of view, the historical starting point appears as something accidental. (325)

Frege's methodological disagreement with Cohen concerning the role that the history of philosophy should play in contemporary discussion is apparent in this statement. In chapter 1, I discussed Cohen's view that the history of philosophy is necessarily a partisan subject, and that one should not hesitate to evoke great figures, such as Kant, in advancing one's own point of view. Furthermore, Cohen sees philosophical aims progressing through continual refinement as a "basic idea" is taken up in successive systematic approaches that build upon their predecessor's failings. Thus, it is natural for Cohen that one's approach to a philosophical problem should begin with historical considerations because the very nature of a given problem is inseparable from its history. Frege's view of the history of philosophy is implicit in his rejection of its relevance to contemporary problems: namely that the history of a problem is a contingent matter of fact that has nothing to do with the logical structure of a problem and its appropriate

solution. For Frege, what secures one's position and makes for a good solution to a philosophical problem is not an appeal to Kant or Plato, but logical clarity and argumentative precision. Russell maintains similar argumentative standards to Frege. Russell claims that, for Cohen, the need for infinitesimals in calculus is not even open to question, and that "no arguments whatever are brought up to support it" (339). This is certainly an extremely strong claim by Russell and might lead us to question whether he read the PIM seriously.⁵³ However, I feel that making such an assumption would be misguided since it would overlook the strong argument Russell is making concerning methodology—that Cohen's combined systematic historical argument for the justification of the infinitesimal is not a *valid* form of argument.

I do not wish to render decisive judgment on either side at this time, apart from saying that Russell and Frege are justified in their attack on Cohen's execution of his argument in the PIM. This poor execution is not only due to Cohen's chaotic, and at times polemical, style. The complexity of the combined systematic-historical approach requires a great deal of conceptual and terminological clarity that is not present in the PIM. For example, the notion of *Erkenntniskritik* itself and what it entails is only vaguely defined. Execution aside, the fundamental difference of methodological presuppositions will continue to inform our discussion of the other themes.

⁵³ I have been fortunate to have access to the Bertrand Russell Archive at McMaster University where I was able to view Russell's own copy of the PIM, which he dated April 1898. Russell has underlined and highlighted passages throughout the book. There is also the occasional note written in the margins, most of which are short; a couple illegible. Throughout this chapter, I will insert some of these comments when they correspond with the topic under discussion. For now, I will point out that in § 1 of the PIM, where Cohen claims that the concept of the infinitesimal is a "basic concept of mathematical natural science" (43), Russell simply underlined the sentence and wrote, "No."

4.2 *An Appropriate Grounding for the Calculus*

A central aspect of Cohen’s critical grounding of the calculus that Frege and Russell criticize is the way in which Cohen not only appeals to the history of the calculus to substantiate his position, but also the fact that Cohen ultimately seeks to validate the infinitesimal through its application to physics. *Erkenntniskritik* is itself entirely geared towards uncovering the critical foundations of experience, which is considered to be synonymous with natural science. The critical grounding of the infinitesimal entails showing it to be a necessary presupposition of natural science. Thus, for Cohen, the infinitesimal, and with it the calculus, have no meaning purely in a mathematical sense without reference to their application to physics. This notion—that there can be no separation of pure from applied mathematics—is a view generally held by Cohen regarding mathematics. In the PIM, Cohen discusses the notion of developing foundations of “pure” mathematics distinguished from its application. He claims it reflects an “amazing deficiency of understanding to speak of a pure mathematics as though it may stand in opposition to applied mathematics” (§ 91, 194). Instead, Cohen holds that to be “pure,” mathematics must bear the characteristic of applicability, because “only that is pure which can become applicable under given conditions” (§ 91, 194).⁵⁴ “Given conditions” could refer to given formal conditions, such as the spatial and temporal structures dictated by Newtonian mechanics. But Cohen’s definition of purity does not imply that all mathematical entities are created with the view towards

⁵⁴ Russell underlined and put a question mark beside this comment. This is likely due to the reference to “given conditions.” Cohen does not make explicit in the PIM that experience is constructed according to “formal conditions,” which I am taking to be synonymous here with “given conditions.”

application. To the contrary, Cohen argues that some pure mathematical forms, such as the conic section, are created without any view to their application, and that it may be centuries before one is found. He argues that this is the only possible state of affairs, since nature is not something that exists in itself, such that our mathematical entities need only be directly applied to it (PIM, § 91, 195). Here Cohen gives the example of the planetary orbits, observing that without these mathematical entities, the natural processes that they make comprehensible would not be objectively existent. Thus, mathematical formations are not modeled after things in nature, but the application of mathematical forms within scientific systems brings law-based comprehension and objectivity to phenomena. Furthermore, the more deeply and diversely these mathematical formations can be combined and intertwined, the more available to application (*anwendungsfrei*) and rich they become (PIM, § 91, 195). Cohen asserts that this principle is powerfully demonstrated by the history of the concept of the differential. I imagine he argues this because of the calculus' general applicability to an endless variety of continuous curves, which can be used to analyze any changes in nature. For Cohen, this is the sense in which differential calculus can be considered "pure," and thus it is within the context of its applicability in physics that the grounding and justification of the concept of the infinitesimal must be carried out. Cohen argues that to find evidence for this grounding one must only look to the driving thought (*treibende Gedanke*) behind his discovery of the calculus: to render the things in nature into objects of mathematical science.

A justification of the calculus in terms of its history and applicability to physics necessarily implies a reference to geometry and motion. Frege and Russell both single out this implication as one of the PIM's primary flaws. Frege writes:

The author does not sufficiently distinguish between arithmetic proper and its applications to geometry and mechanics. The infinitesimal calculus is arithmetical in nature, and even though its historical starting-point lay in geometrical and mechanical problems, one must not go back to geometry or mechanics in defining or justifying its fundamental concepts. (“*Rezension*” 327)

In this passage, Frege argues that the infinitesimal calculus is primarily arithmetical in nature. This argument is indicative of his, and Russell's, position that the concept of number should be reduced to an arithmetic foundation. Frege asserts that Cohen's definition of the differential as intensive magnitude is nonsensical from the perspective of pure arithmetic, and hence has no meaning in mathematics, because “the number 3 for example can serve as the measure of a distance in relation to a unit of length; but it can also serve as the measure of an intensive magnitude, e.g., of the intensity of a light measured in terms of brightness” (“*Rezension*” 328). Frege's example is intended to illustrate that although the number 3 can serve different purposes within the context of varied applications, the concept of “the number 3” itself is not thereby changed. The concept of number is defined by Frege prior to its varied applications. One should not, Frege claims, decide the properties of the differential based on an appeal to spatial and temporal properties; rather, one must ask “what properties must spatial and temporal quantities be assumed to have if the differential calculus is to be applicable to geometry and mechanics” (“*Rezension*” 327)?

One may object here that Frege's and Russell's project of securing the purely logico-arithmetic foundation of mathematics is itself problematic, and that although Cohen's grounding of the infinitesimal through an appeal to Kantian notions of thought and intuition may ultimately fail to provide a secure ground for the calculus, Russell and Frege do no better. But any such objection would miss the central thrust of Russell's and Frege's efforts. Russell illustrates this central thrust when he states that Cohen's approach is "vitiating [...] by an undue mysticism, inherited from Kant" (326). Russell is objecting to the larger ontological significance Cohen ascribes to the infinitesimal within the schema of *Erkenntniskritik*, but this objection should not be viewed in isolation. Russell writes in response to Cohen's assertion that the infinitesimal must be justified through transcendental logic that "[t]his Kantian opinion is wholly opposed to the philosophy which underlies this present work [i.e. *The Principles of Mathematics*]" (339).⁵⁵ What this second statement from Russell implies is that aside from his objections to Cohen, Russell seeks to combat an entire philosophical orientation premised on the notion that mathematics is to be grounded in Kantian intuition. Russell singles out Cohen primarily because, in his view, the PIM just happens to be an extremely illustrative example of the negative consequences arising from this viewpoint.

In his book *The Semantic Tradition from Kant to Carnap*, Alberto Coffa documents the development of the semantic tradition leading up to the logical positivism of the Vienna Circle out of the perceived necessity to clarify ambiguities inherent in

⁵⁵ In §18 of the PIM, Cohen writes about the need for *Erkenntniskritik* (transcendental logic) to supplement logic when it comes to the grounding of the differential: "daß jene vermißte logische Begründung des Differentialbegriffs in einem erkenntniskritischen Grundsatz, und zwar in dem der Kategorie der Realität entsprechenden, mithin in dem Grundsatz der intensiven Größe oder der Anticipationen enthalten sei" (57). Russell's underlined the passage and commented: "Absolute rubbish."

Kant's philosophy. Coffa highlights the grounding of the calculus as a problem historically plagued by ambiguities resulting from basing it on an appeal to motion—and hence on an appeal to space and time. Coffa notes that objections to such an appeal were already voiced before Kant's time. The first objection came from Bishop Berkeley, who in his 1734 work, *The Analyst*, argued against both Newton's and Leibniz's grounding for the calculus, famously calling infinitesimals "the ghosts of departed quantities" (Coffa, 25). I do not wish to enter into a lengthy discussion of the Berkeley's and others' objections to the existence of infinitesimals. What I would like to illustrate is only that Russell's and Frege's project of reducing mathematics to arithmetic and arithmetic to logic must be understood as a response to a long history of debate and confusion. The nature of this response is, I think, expressed admirably here by Coffa:

It is widely thought that the principle inspiring such [reductionist] efforts was epistemological, that they were basically a search for certainty. This is a serious error. It is true, of course, that most of those engaging in these projects believed in the possibility of achieving something in the neighborhood of Cartesian certainty for those principles of logic or arithmetic on which a priori knowledge was to be based. But it would be a gross misunderstanding to see in this belief the basic aim of the enterprise. A no less important purpose was the clarification of what was being said. (26)

It is the second aim Coffa identifies—endeavoring towards clarity of discussion—that I think sets Russell's and Frege's approach to pure mathematics apart from Cohen's. The two approaches do not even compare in terms of terminological precision and clarity of argument. One may argue that this is purely a flaw in Cohen's execution, not the flaw in his project. I would, however, reply that Cohen's approach saddles the concept of infinitesimal with too much systematic baggage and that his view that mathematics

cannot be separated from its applications is ultimately untenable. That view does not reflect the current state of mathematics as a field of knowledge that subsists and is conducted according to its own methodology and in many instances without view to its direct application.

In the PIM, Cohen argues that pure mathematical entities are invented without a view to their application, but this view is ultimately unsatisfactory insofar as the criterion for true purity of a mathematical method is its “availability for application.” Even if we set the term “purity” aside, mathematics as a field of knowledge must have its own methods for determining the validity and significance of its objects without necessary reference to other fields. Hence, I prefer Frege’s formulation presented in *The Foundations of Arithmetic*, that when trying to define a concept such as the infinitesimal, the primary task should be to understand the meaning of statements in which the infinitesimal appears:

The problem here is not, as might be thought, to produce a segment bounded by two distinct points whose length is dx , but rather to define the sense of the identity of the type: $df(x) = g(x) dx$
(*Foundations* 72)

I list this example not to enter into a discussion of Frege’s theories concerning identity relations, but to illustrate his point that mathematical concepts should be defined in terms of their functional meaning within mathematical statements. In this case, infinitesimals have meaning only insofar as they appear in statements involving differentials. The appeal for meaning is one internal to mathematics, and the mathematical concept should derive its meaning and validity within the context of the larger field of mathematical study. This requirement may not preclude the geometrical meaning and interpretation of

some concepts. The important thing is to recognize mathematics as an independent system of knowledge, and Cohen's failure to make this distinction seems to follow directly from the nature of *Erkenntniskritik* itself. *Erkenntniskritik* is focused on the investigation of the conditions for the possibility of natural science, and everything that falls within the scope of this transcendental analysis attains its critical meaning only insofar as it is a contributing component to scientific systems.

4.3 *The Concept of the Limit*

An important aspect of Cohen's argument in the PIM is his critique of the concept of the limit. In section 68, entitled "The Presuppositions and Deficiencies of the Limit Method," Cohen outlines his reasons for why the limit represents a purely "negative" concept, to the extent that it covers over the "positive" critical meaning of the infinitesimal. Frege simply dismisses this "belittlement" of the limit on Cohen's part as stemming from Cohen's misunderstanding of the concept ("*Rezension*" 329). By contrast, Russell makes Cohen's treatment of the limit central to his critique. In particular, he takes exception to Cohen's argument that the concept of the limit relies on the notion of equality (340).⁵⁶ Assuming for now that Russell is correct in his assessment of Cohen's position, I would like to quickly assess his primary arguments against it.

In his first argument against the notion that the concept of the limit relies on a notion of equality, Russell's evokes the example of the series, $2 - (1/n)$, where n is

⁵⁶ In § 2 of the PIM, Cohen writes: "Indessen wird in dem vorausgesetzten Größenbegriffe zugleich die Voraussetzung der Grenzgröße gemacht. Die in der elementaren Größenlehre definierte Gleichheit berücksichtige jene Grenzgrößen nicht" (44). Russell wrote, "Nonsense," in the margin.

capable of assuming all positive integral finite values, and the limit of this series, 2.

Russell argues that Cohen is probably misled by the fact that “the difference between 2 and the successive terms in the series becomes less than any assigned magnitude, and thus we seem to have a sort of extended quality between 2 and the late terms of the series $2 - (1/n)$. However, Russell claims that upon closer examination this is not the case because:

[This notion] depends upon the fact that rationals are in a series in which we have distances which are again rationals. But we know that distances are unnecessary to limits, and that stretches are equally effective.⁵⁷ Now considering stretches, 2 is the limit of $2 - (1/n)$ because no rational comes between 2 and all terms of the series $2 - (1/n)$ —precisely the sense in which ω is the limit of the finite integers. (340)

Here, Russell’s intends to demonstrate that there is no equality between 2 and the series $2 - (1/n)$ in the arithmetic sense. The limit only implies is that there is no rational that comes between 2 and all the terms of the series. This somewhat difficult notion is based on Russell’s appropriation of Cantor’s transfinite mathematics, and one may argue that Russell is speaking at cross-purposes to Cohen. This is not the case. To avoid the inherent difficulties of transfinite mathematics and Russell’s complicated terminology, I would like to consider an earlier statement Russell makes regarding Newton that will allow us to ground the pending discussion in terms with which we are already familiar. After a discussion of Leibniz’s “extremely crude” notion of infinitesimals, which Russell claims mysteriously leave the calculus in a simultaneous state of being approximate and exact, he goes on to say:

⁵⁷ Earlier in the book, Russell defines a “stretch” in the following way: “In all series there are terms intermediate between any two whose distance is not the minimum. These terms are determinate when the two distant terms are specified. The intermediate terms may be called the *stretch* from a_0 to a_n ” (Russell, 181).

In this respect, Newton is preferable to Leibniz: his Lemmas give the true foundation of the Calculus in the doctrine of limits, and, assuming the continuity of space and time in Cantor's sense, they give valid proofs of its rules so far as spatio-temporal magnitudes are concerned. But Newton was, of course, entirely ignorant of the fact that his Lemmas depend upon the modern theory of continuity; moreover, the appeal to time and change, which appears in the word fluxion, and to space, which appears in the Lemmas, was wholly unnecessary, and served merely to hide that no definition of continuity had been given.
(326)

The modern theory of continuity to which Russell refers is based on the notion of a compact series. Specifically he says, "*Continuity* applies to series (and only to series) whenever these are such that there is a term between any two given terms" (Russell, 193). An example of a compact series is the series up from the real numbers, since between any two real numbers, say 1 and 1.2, there is another term, 1.1.⁵⁸ Now, our forthcoming discussion of Newton will not rely on the Russell's modern theory of continuity, but I give this passage in full for two reasons: 1) to show that Cohen is not the only participant in this discussion who, rightly or wrongly, is prone to read his own ideas into the work of historical figures;⁵⁹ and 2) that we can suspend Russell's arithmetical terminology and consider the issues of contention involving limits with reference to Newton's more intuitive (no reference to Kant intended) geometrical arguments.

Once again I would like to consider Newton's Lemma 1, from the *Principia*:

⁵⁸ In the case of the real numbers, there are an infinite number of terms between any two given terms. Hence the real number continuum corresponds to a higher order of infinity than the integers in Cantor's transfinite hierarchy.

⁵⁹ Just as Russell tries to argue that Newton's ideas accord with his own, he attempts to associate Cohen with Leibniz's "extremely crude" notion of infinitesimals. Russell says, "Leibniz employed the form dy/dx because he believed in infinitesimals" (Russell, 338), which implies that Leibniz believed in actual infinitesimal components of the continuum. Shortly thereafter Russell writes, "But when we turn to works such as Cohen's, we find the dx and the dy treated as separate entities, as real infinitesimals, as the intensively real elements of which the continuum is composed" (Russell, 339). In these cases we see that Russell is foisting the notion that infinitesimals are the real constituent elements of the continuum on Leibniz and Cohen. As was shown in chapter 3, this assertion is arguably unfair to both.

Quantities, and also ratios of quantities, which in any “finite time” constantly tend to equality, and which before the end of that time approach so close to one another that their difference is less than any given quantity, become ultimately equal. If you deny this, let them become ultimately unequal, and let their difference be D. Then they cannot approach so close to equality that their difference is less than the given difference D, contrary to hypothesis. (433)

First, it is important to note here that Newton does say, seemingly in contradiction with Russell, that the quantities ultimately become equal. Russell would argue that this apparent discrepancy is the result of a lack of precision that comes with geometrical arguments. For our purposes, the essential aspect of the Lemma is the *reductio* argument Newton employs at the end. The argument shows that no finite difference can be assigned between the quantities by supposing that there is one and deriving a contradiction. Put another way, this means that there is no finite quantity that can be said to be between the quantities and their limit. Russell would argue that his arithmetic formulation of the doctrine of limits—that there is no term between all the terms in the series $2 - (1/n)$ and 2—expresses the same meaning that Newton’s formulation does, but with greater rigor. Russell argues that analogies with motion and of quantities “approaching” the limit are misleading, because quantities are thought of as though they “pass through” intermediate values (344). Certainly this is a difficult notion and there seems to be something missing from Newton’s proof above. For how can the quantities, or their ratios, reach their limit if they must first traverse an infinite number of intermediate positions? Russell’s solution is to argue that there is no “approach,” only the progression of terms in a given arithmetic series, whose terminus is a limit. This notion of progression is entirely static.

Cohen's solution is markedly different, and I believe it unfair to claim that it stems from "misunderstanding" the concept of limits, as Russell and Frege assert. Rather it stems from Cohen not accepting the relationship between the infinite and finite inherent in the concept of the limit. Russell and Frege simply take their position to be so demonstrably and self-evidently true that "rejection" is taken to be synonymous with "misunderstanding" when it comes to the acceptance of limits and the arithmetic basis for mathematics.⁶⁰ As Russell claims, Cohen begins his critique of the limit method in the PIM by arguing that it presupposes the indifference between "limit" and "equality" (§ 68, 144).⁶¹ However, Cohen argues, in accordance with Leibniz, that equality is known to be infinitely small inequality. With this recognition, the concept of the limit corrects the concept of equality through the supposition of the infinitesimal (§ 68, 144). Cohen's apparent misunderstanding comes into play in the notion that the infinitesimal is somehow required to bridge the ultimate difference that must exist between the limit and approaching terms. Arithmetic foundations aside, one may argue that this is purely a misunderstanding of Newton's Lemma 1, where he challenges an opponent to find the ultimate difference and shows that it leads to a contradiction. The evidence clearly supports the interpretation, however, that Cohen did not misinterpret the argument. For example, I noted in chapter 3 that Cohen quotes Leibniz saying basically the same thing

⁶⁰ As Patton observes, Cohen does not indicate in the PIM how familiar he was with the work that had been done in finding an arithmetic grounding for calculus ("Hermann" 111).

⁶¹ An important aspect of Russell's argument against Cohen is that Cohen fails to make a distinction between "quantity" and "magnitude," and he is quite correct in arguing that Cohen does consider all finite numbers to be magnitudes. Russell differentiates between quantities and magnitudes based on the distinction between terms that can be equal to each other and terms that can be greater or less. Quantities are defined as the former, whereas magnitudes are the latter. For example, he says, "an actual foot-rule is a quantity: its length is a magnitude" (159).

about the foundation of the differential: “The error is less than the adversary wanted to assign, and by consequence we could not assign any” (§ 54, 110). Here Cohen dismisses Leibniz’s argument, but he does not claim that Leibniz is wrong, only that the grounding of the infinitesimal requires more than purely logical considerations. I believe Cohen holds the same view of Newton’s argument. Newton’s Lemma 1 shows that presupposing a *finite* difference between the limit and the approaching quantities leads to a contradiction; however, no contradiction is necessarily implied if another type of quantity is supposed to come between the two.⁶²

Hence, while Russell or Frege may argue it is *illogical*, Cohen’s argument that intensive magnitudes intervene in the method of first and ultimate ratios does not lead to a direct logical contradiction. He holds that the grounding of the limit method, properly understood, “lies beyond logic” (PIM, § 2, 44). For Cohen, this grounding requires that one make the critical distinction between equality and identity. “Equality,” he says, “means a relation between assumed extensive unities, whose creation is carried out in intuition” (PIM, § 68, 144). For Cohen, however, extensive finite unities taken on their own have only a fictional and arbitrary grounding and therefore the relation is unresolved. Cohen argues that to arrive at a more certain ground, one must look to the origin of extensive magnitudes, and hence the origin of the objects of intuition, in the operation of thought. Locating this origin, which resides in the proper balance of thought and intuition, requires the concept infinitesimal, thought of as intensive magnitude. This

⁶² Thanks to Richard Arthur for pointing out this implication of Newton’s Lemma to me, and for drawing my attention to the basic equivalence between it and Leibniz’s syncategorematic definition of the infinitesimal.

in turn requires that the concept of the limit be placed within the broader context of *Erkenntniskritik*, where the limit represents that creative moment, or instant, from which extensive unities spring. In this sense, the limit is thought to be truly limitative inasmuch as extensive magnitudes arise through the functioning of the limiting judgment.

Cohen regards the limiting judgment to be a function of the continuity of consciousness.⁶³ He claims that it is the continuity of consciousness that allows for certainty that “what is not ‘table’, therefore need not be ‘melancholy’” (PIM, § 68, 146). This rather bizarre statement is intended by Cohen to be illustrative of a deficient form of limiting judgment. For him, it is the continuity of consciousness is what allows for the securing of those judgments that use negative predicates (non – A) to arrive at positive ones. Cohen holds that the very possibility of reality itself depends on such limiting judgments, and this is the critical meaning of the limit. His view is that the critical meaning of continuity—fundamentally the continuity of consciousness—allows for the possibility of extensive unities to lose their fictional character. He says that the limit, properly understood represents, in a limitative sense, this deeper point of origin from which the line goes forth (§ 68, 146). That is why Cohen argues that the absolute nascent and evanescent points in Newton’s method of first and ultimate ratios cannot be understood merely negatively as points of convergence. By his account they must be recognized as the beginning and ending through which thought determines its object as real. For Cohen, this is the critical meaning of “identity”, and it is closely tied to the

⁶³ The references Cohen makes throughout the PIM to the “continuity of consciousness” and the “unity of consciousness” are likely a major reason for Russell accusing him of relying on undue mysticism. In § 27, Russell underlined the words “unity of consciousness” in a passage where Cohen argues that this unity is the origin of number. Russell also commented in the margin that he felt this conclusion was “unspeakable nonsense.”

mechanical motive, and hence to the critical grounding of the differential, because it is the differential as intensive magnitude that mediates the application of mathematical form to the things of nature.

Cohen argues that modern “English logic” has its source in the neglect of the distinction between equality and identity. He writes that “it is false from the outset to say: all A are = B; just as it is false to say: A = A” (§ 68, 146). The explanation that follows this statement is quite terse. He says that with this formulation the requirement falls away that the predicate “B” be quantified (*quantifiziert*).⁶⁴ From the context of the overall argument one can infer that Cohen is saying that these kinds of judgments overlook the source of objects in an act of consciousness, and hence are not proper logical formulations of the judgment of identity. I cannot say “A is equal to A” unless I first have an “A” to compare to itself. Hence “A” cannot be grounded in an identity relation with itself, anymore than it can be grounded through equality with “B.” In both cases, this is a “fictional” relation. As such, one must walk the critical path. Before the object can be presupposed, a “unified consciousness” must be there to ground it. Object “A” is the thought determination of the active consciousness. Consciousness, by virtue its “continuity,” determines its objects through limiting judgments. I imagine this condition is what Cohen means when he says that the predicate “B” above has not been properly quantified. “B” must first be made determinate through the predicate “non—B.” This is

⁶⁴ My thanks to Nicolas Griffin for pointing out that Cohen’s references to “English logic” and the quantification of the predicate are actually references to the work of the Scottish metaphysician William Hamilton, (1788-1856).

the function of identity. Certainly, this is a challenging notion and is obviously intricately tied to the whole of Cohen's *Erkenntniskritik*, and his notions of objectivity and reality.

These ideas will be the topic of the next and final section of this chapter; however, there is one remaining topic outstanding that I first wish to address before moving on. I have discussed Cohen's critique of the limit primarily through geometrical arguments while leaving Russell's arithmetic considerations aside. As we have seen, Russell formulates the doctrine of limits strictly by arithmetic means involving the progression of terms of an arithmetic series. But what would Cohen say about ω being the limit of the series of integers? Russell is quite correct in asserting that there is no relation of equality at play here. He says only that there is no integer between all the terms in the series and the limit, ω . It is here that we come to the crux of Cohen's and Russell's differing views. For what does it mean to say that there is no term between all the terms of the series and its limit? If the series is infinite, then how can we speak of all the terms? And how can we speak of ω as being greater than all the natural numbers? Certainly this notion is not something that can be represented in intuition, whereby we would imagine the terms proceeding inexhaustibly one after another, as if in time. If we think of numbers in this way, infinity can never be reached inductively, because no matter how high one counts an infinite number of terms remain. Cohen's understanding of number is based on intuition, thus he holds the view that one cannot go from the finite to the infinite without a sense of "wonder" (§ 44, 86). Cohen argues that one must always proceed from the infinite to the finite, as though every finite number is a determination whose possibility rests on a more primordial absolute unity. Since number, for Cohen, represents a unity, one must return

to the unity of consciousness as the ultimate grounds for its possibility. Of course Russell would disagree with this line of reasoning and perhaps rightly so. As I have discussed, there are good reasons for his desire to banish Kantian intuition from the realm of mathematics. I think, however, it is safe to say that Cohen's views of the limit stem from genuine disagreement with Russell and Frege as opposed to mere naïve misunderstanding.

4.4 *Objectivity and Reality*

Ultimately, the acceptance of Cohen's grounding for the calculus hinges on the acceptance or rejection of the relationship he attempts to establish between the infinitesimal, intensive magnitude, and the category of reality. Both Russell and Frege identify this central notion. Russell states:

As regards the nature of the infinitesimal, we are told that the differential, or the inextensive, is to be identified with the intensive, and the differential is regarded as the embodiment of Kant's category of reality. This view (in so far as it is independent of Kant) is quoted with approval from Leibniz; but to me, I must confess, it seems completely destitute of justification.
(342)

Of course, Cohen does try to provide some justification for this claim—establishing this connection is the main purpose of the PIM. But Cohen's historical justification is not the kind of argument that Russell is willing to accept when it comes to the grounding of a mathematical concept, and he responds to it with an elaborate proof demonstrating the non-existence of infinitesimals.

I do not think Russell can be chastised for not giving Cohen's argument more credence within the context of *The Principles of Mathematics*. Russell's purpose for

mentioning Cohen at all is really only to demonstrate how effectively his system cuts through such theories. By contrast, Frege takes up Cohen's interpretation of Kant in more depth, and his comments are more illustrative for my purposes. Frege's first critical remark is that "the Kantian transition from the quality of a judgment to qualitative reality and intensive magnitude seems already questionable to me" ("*Rezension*" 110). When he says "quality of a judgment," Frege is referring to the fact that in Kant's table of judgments, the limiting, or infinite, judgment is placed in the second class—"Quality of Judgments" (CPR, B95) Furthermore, Kant places the category of reality in the second class—"Of Quality" (B106). It seems hardly coincidental that Cohen would then connect the limiting judgment with the category of reality. He obviously ascribes significance to the connection between the corresponding classes on the table of judgments and the table of categories. It is interesting to observe that in the table of categories, each of Kant's classes is made up from three. In the case of quality, they are: 1) reality, 2) negation, and 3) limitation. Kant claims that in each class, the first and the second form a dichotomy and the third "arises from the combination of the second category and the first" (B110). Hence, limitation arises out of the combination of reality and negation. It is possible that Cohen reads some degree of significance into this, combined with the fact that the infinite judgment belongs to the corresponding class of categories. This possibility is strengthened by the place of the anticipations of perception amongst the "system of the principles of pure understanding." The principles of pure understanding are a priori conditions that govern appearances such that "they yield knowledge of an object corresponding to them" (B198). Amongst these principles of pure understanding, the

anticipations of perception correspond to the qualitative aspect, i.e., that the “real that is an object of sensation has intensive magnitude, that is, degree” (B207). Looking at the correspondence between the table of judgments, categories, and the principles of pure understanding, it would hardly appear coincidental that Cohen emphasizes the qualitative aspect of each. He seems to be trying to ascertain and bring to the fore a deep schematic unity in Kant’s work, and use it, in conjunction with the concept of the infinitesimal, to underpin his notion of objectivity in science. However, if this is the case, Cohen never plainly spells it out and justifies it directly in the PIM. Hence, Frege is certainly justified in questioning this transition across the *Critique*.

It is beyond the scope of this project to undertake such an involved analysis of Kant’s schema, but there is one other key argument made by Frege that I would like to address:

[Cohen] seems to think of the act of realization performed by the infinitesimal in such a way that a distance for example is generated by the force with which it increases; but does this give it a higher degree of reality than geometrical forms possess elsewhere? It seems that objectification is not always clearly distinguished from realization. While geometrical objects may well be granted objectivity, they cannot very well be granted reality proper, which is also what Cohen says himself; but how can we alter this fact by thinking of the forms as being continuously generated? I do not find that the infinitesimal has an intimate connection with reality. Nor do I understand how the principle of the anticipations of perception can be so interpreted, or how one could infer from it, that an intensive magnitude or the differential has the power of realization. (“*Rezension*” 329)

The observation Frege makes—that Cohen fails to make a proper distinction between objectification and realization—is decisive in my view when it comes to judging the overall success of Cohen’s critical grounding of the infinitesimal. That the real in

perception is intensive magnitude is intended by Cohen to allow for the apriorization of sensation and the application of geometry to nature. Cohen argues that the objectivity of a natural process or thing consists in its geometrical ideality. And we may well wish to agree with Cohen that objectivity resides in knowledge and systems of knowledge, such as Newtonian mechanics. But can we say that by applying geometry to nature we thereby grant nature its reality? Furthermore, can we say that if the infinitesimal is the necessary mediator in this process of ideation that it bestows reality upon the things in nature? I think Frege is ultimately correct in answering no to both questions, but just why that is warrants further investigation.

In regard to geometrical forms themselves, Frege also finds difficulty with Cohen's view. He says that geometrical forms are objective, but not necessarily real, and it is unclear how being continuously generated would make them more real than when they are considered to be static. This comment is a reference to Cohen's view that geometrical forms are continuously determined, or generated, out of intensive magnitudes. To decide whether these forms would be "more real" or not if they were static is really a question of whether one is willing to accept Cohen's arguments about consciousness determining its objects. By phrasing the problem in a simple dichotomy between "static" and "generated" forms, Frege trivializes this aspect of Cohen's argument. Frege argues in the above passage that Cohen admits that "while geometrical objects may well be granted objectivity, they cannot very well be granted reality proper" (329). Frege does not back up this claim with a reference to the PIM, and I think that is because this charge is a misrepresentation of Cohen's position. For Cohen, real objects

are the objects of natural science, and mathematical objects have no significant meaning outside of their application within physics. In the PIM, Cohen considers whether all higher-order differentials allow for a physical and geometrical interpretation. His response is that it does not matter if all higher-order differentials can be so interpreted. All that matters is that there are cases in which higher-order differentials have important applications and there is no sense in worrying about others that may not (§ 101, 212). Inapplicable differentials would not be “pure” objects of mathematics, since purity entails applicability.

For Cohen, the only philosophically significant sense in which one can speak of reality is within the context of mathematical science. This restriction applies equally to geometrical objects, as well as the things in nature. Cohen’s arguments in the PIM ultimately falls back on the notion that consciousness is the ultimate ground of reality. The principle of intensive magnitude is intended to reflect the qualitative unity of consciousness, and the concept of the infinitesimal (thought of as a differential or as Newton’s absolute moment) is “an implement [*Werkzeug*] and testament [*Zeugnis*] to reality” (§ 105, 219). The infinitesimal is the implement of reality insofar as it is representative of the process through which thought determines the objects of intuition by means of the limiting judgment. But even if one were to accept this process as valid, can Cohen’s notion of reality be accepted? He says that it is in consciousness that all reality is secured, but what kind of “consciousness” is he speaking of? Under Cohen’s schema there is no philosophically significant world, only science—a set of synthetic relations. “Consciousness” then can only be thought of as a dynamic, temporal element that

determines objects in a creative moment. This process is loosely understood through an analogy by which a particular object is generated out of an indefinite series of synthetic relations, as an integral is generated out of an indefinite series of differentials. But this “consciousness” is not subjectivity. Furthermore, science is abstract knowledge of the world, but it is not the world itself. To say that experience is synonymous with natural science involves the reification of both subject and world. Hence, the infinitesimal is not the embodiment of reality.

That said it should be clear that Frege’s and Russell’s criticisms of the PIM are part of a much larger debate in epistemology. Their philosophical viewpoints are put to the test with regard to the infinitesimal just as much as Cohen’s and it is certain that both Frege and Russell have a great stake in showing Cohen’s account to be flawed, even as they dismiss it for being absurd. For example, Frege’s *Foundations of Arithmetic* was published only a year after the PIM, and the two works are diametrically opposed in their arguments about the foundations of mathematics. Furthermore, Cohen’s was a highly public academic figure and head of school of thought whose influence would still have been strong in 1903 when Russell’s book was published. Cohen’s dismissive comments about “English logic” and Frege’s and Russell’s charges of Cohen’s mathematical ineptitude indicate a larger clash of ideologies at play that goes beyond the grounding of the infinitesimal or the dismissal of the concept.⁶⁵ Hence, although neither Cohen nor Frege and Russell can be said to have unproblematically and conclusively succeeded in

⁶⁵ Moynahan goes so far as to argue that Cohen wrote the PIM as a history of the calculus because he needed to hide the overtly controversial political and theoretical views contained within it. In technocratic Willhelmine Germany, Moynahan argues, Cohen’s philosophy of science must be regarded also as a political project (38).

grounding the infinitesimal, or dismissing the concept in Frege's and Russell's case, their attempts are worthwhile as exemplars of two opposing schools of thought on these subjects. This is because the dispute is indicative of a larger clash within German academic philosophy at the time between mainstream philosophers of the cultural elite, such as Cohen, and symbolic logicians.⁶⁶

⁶⁶ For an account of this larger clash, see Jarmo Pulkkinen's, *Thought and Logic*.

Conclusion

This thesis has been a presentation and critical evaluation of Cohen's PIM and of the critical neo-Kantian philosophical perspective put forward therein. I would like to conclude this discussion with some general reflections on the book. As I have endeavored to show, the argument put forward in the PIM involves a complex weaving of historical and systematic concerns. However, it has also been shown that the argument put forward in the PIM is ultimately untenable—the claim that the infinitesimal is an “instrument of reality” cannot be justified. That said, it is my view that, although flawed, the PIM is a work of value for three primary reasons: 1) the high quality of Cohen's historical presentation; 2) the importance of the basic question that Cohen seeking to address; and 3) the book's significance within the larger context of Cohen's work and of the Marburg school generally.

It is telling that upon the first mention of the PIM in *The Principles of Mathematics*, Russell, although criticizing the book, deemed it necessary to add a footnote pointing out that “the historical part of this work, it should be said, is admirable” (326). This comment is reflective of the fact that although the overall argument for critical grounding of the infinitesimal is questionable, there is no question that Cohen's historical presentation is extremely well researched and that many of the connections he identifies between thinkers and the development of key concepts of calculus are accurate. It is also clear upon reading the PIM that Cohen read into the work of Leibniz and Newton deeply, and his attempts to situate and contextualize their discovery of calculus

within the broad context of their larger philosophical projects is admirable. Frege and Russell criticize Cohen for not separating the calculus from its application to physics; however, Cohen's account is based on the recognition that for Leibniz and Newton there was no clear distinction between the pure form of the calculus and its applicability. Indeed, such high value was placed on the method because it worked so well and allowed for a seemingly unlimited variety of applications. Hence Cohen's conclusion that applying geometry to nature was a pressing motivation for Leibniz and Newton is a valid one. Furthermore, Russell's and Frege's notion that mathematics can be clearly separated from its applications and geometrical interpretations is itself highly questionable. It may well be that the meaning of mathematical statements and objects is inherently tied to the ways in which they are used, both within mathematics and in conjunction with other disciplines. That said, where Cohen's history goes awry is in his argument that Leibniz's differential and Newton's fluxions are intensive magnitudes.⁶⁷ It is a consequence of Leibniz's principle of general order and Newton's method of first and ultimate ratios that the differential and fluxion are finite quantities. However, Cohen is correct in his assessment that the dispute over who really discovered the calculus and the argument that Newton's method rests on a more secure foundation than Leibniz's are by and large groundless.

While the quality of Cohen's historical account should not be questioned, what he does with the history should. His critical grounding of the calculus and the interpretation of Kant on which it is based are ultimately untenable as they appear in the PIM. This is

⁶⁷ For an account of intensive magnitudes in Leibniz' thought, including a brief mention of Cohen's interpretation of Leibniz, see: De Risi, Vincenzo, *Geometry and Monadology*, 266-270.

due to the conflation of experience with mathematical science, and of reality with objectivity. However, I believe that the PIM still has merit in this respect by virtue of the basic questions that Cohen seeks to answer. What does it mean to have objective mathematical sciences, and to what extent do the objects of these sciences correspond to reality? It is my view that these questions remain pertinent and unresolved, and if anything they have only increased in their importance since Cohen's time. This is due to the reliance of all the natural sciences on increasingly abstract and complex mathematics and mathematical models. For example, the "objects" of quantum mechanics are based upon probability functions and elaborate interpretations of experiments. There is a key difference here between these objects and those of classical mechanics. No one questions the existence of cannonballs or of planets, or wonders to what extent physics creates these objects. In the case of sub-atomic particles, whose existence can only be indirectly confirmed, questions concerning the ontological status of scientific objects attain new force. The same can be said of climate change. The trends in temperature, precipitation, glacial melting, and carbon dioxide levels in the atmosphere can all be quantified and measured. However, the interpretation of this data and its significance is determined largely through complex mathematical models run on computers. How is it that these models tell us something about reality? Do the mathematical relations on which these models are based exist manifestly in nature? To what extent can we grasp nature directly as an object and know things about it with certainty? These questions become all the more pertinent when political policy must be decided on the basis of scientific theories. These are the types of questions that Cohen seeks to address, albeit somewhat indirectly,

in the PIM, and the book should be valued for raising the right questions regardless of whether its answers are satisfactory.

The same can be said of Cohen's larger body of work and that of the Marburg School generally. In respect to this larger movement, the PIM stands as an important work because of its widely recognized status, along with the second edition of *Kant's Theory of Experience*, as signaling a transition in Cohen's thought from the early works on Kant towards his own independent philosophical perspective (Moynahan, 35). In Cohen's arguments about the limiting judgment one can see the beginnings of his later "logic of origin" that would be fully articulated in his 1902 *Logik der Reinen Erkenntnis*.⁶⁸ Beyond Cohen scholarship the PIM is important insofar as the differential remained an important notion within the Marburg School generally. Moynahan explains that for younger members of the school, such as Cohen's close disciple and intellectual heir Cassirer, the "specific importance of the calculus was replaced by a general reading of functional and serial relations" (66). What Moynahan refers to here is the idea prevalent in the Marburg school that the judgment of a particular object of knowledge is the result of a kind of "infinite sum" of general relations and tacit understanding that make up a system of knowledge. There is now a rejuvenated interest in the Marburg school—and in Cassirer in particular—as providing a model for a mediate view of scientific rationality that, in the words of Michael Friedman, could "have the resources for developing a satisfying resolution to the relativist and historicist predicament arising

⁶⁸ It is beyond the scope of my project to discuss Cohen's later "Critical Idealism" and how it differs from *Erkenntniskritik*. Poma gives a detailed overview in chapters 4 and 5. Also, in chapter 4 of her dissertation, Patton gives a detailed account of Cohen's mature philosophy of science and how it relates to ideas he develops in the PIM.

from Kuhn’s theory of scientific revolutions” (121). That means determining how systems of knowledge such as Newtonian mechanics can make claim to objectivity without necessarily having the final say on reality. This is certainly a worthwhile basic question, and if we were to take it up, the PIM would be a good place to begin.

Works Cited

- Albertini, Francesca. Das Verständnis des seins bei Hermann Cohen. Würzburg: Königshausen & Neumann, 2003.
- Arthur, Richard T. W. "Leibniz's Syncategorematic Infinitesimals, Smooth Infinitesimal Analysis, and Newton's Proposition 6." Draft ed.
- . "Newton's Fluxions and Equably Flowing Time." Studies in the History and Philosophy of Science 26.2 (1995): 323-51.
- Cassirer, Ernst. "Hermann Cohen und die Erneuerung der Kantischen Philosophie." Kantstudien 17 (1912): 252-73. English translation: "Hermann Cohen and the Renewal of Kantian Philosophy." Trans: Lydia Patton. Angelaki 10.1 (2005): 95-108.
- Coffa, J. Alberto. The Semantic Tradition from Kant to Carnap: To the Vienna Station. Cambridge: Cambridge University Press, 1991.
- Cohen, Hermann. Das Prinzip der Infinitesimal-Methode und seine Geschichte: Ein Kapitel zur Grundlegung der Erkenntniskritik. (1883) Frankfurt am Main: Suhrkamp Verlag, 1968. Partial English translation: The Principle of the Infinitesimal Method and its History. Trans. Lydia Patton. Draft ed.
- . Kants Theorie der Erfahrung. (1913) Hildesheim: G. Olms, 1987.
- . "Zur Kontroverse zwischen Trendelenburg und Kuno Fischer." (1871) Hermann Cohens Schriften zur Philosophie und Zeitgeschichte. Ed. Albert Görland and Ernst Cassirer. Berlin: Akademie Verlag, 1928. 229-275. English translation: "On the Trendelenburg-Fischer Debate." Trans. Lydia Patton. Draft ed.
- Cohen, I. Bernard. "A Guide to Newton's *Principia*." The Principia. Berkeley: University of California Press, 1999. 1-370.
- De Risi, Vincenzo. Geometry and Monadology: Leibniz's *Analysis Situs* and Philosophy of Space. Berlin: Birkhäuser, 2007.
- Fischer, Kuno. Anti-Trendelenburg. eine Duplik. Jena: H. Dabis, 1870.

- Flach, Werner. "Einleitung." in Hermann Cohen's Das Prinzip der Infinitesimal-Methode und seine Geschichte: Ein Kapitel zur Grundlegung der Erkenntniskritik. Frankfurt am Main: Suhrkamp Verlag, 1968. 11-38.
- Frege, Gottlob. Die Grundlagen der Arithmetik: Eine Logisch-Mathematische Untersuchung über den Begriff der Zahl. Breslau: W. Koenner, 1884. English translation: The Foundations of Arithmetic. Trans. J. L. Austin. Evanston: Northwestern University Press, 1980.
- . "Rezension Von: H. Cohen, *Das Prinzip Der Infinitesimal-Methode Und Seine Geschichte*." (1885) Kleine Schriften. Ed. Ignacio Angelelli. Hildesheim: Georg Olms Verlagsbuchhandlung, 1967. 99-102. English translation: "Review of H. Cohen, *Das Prinzip Der Infinitesimal-Methode Und Seine Geschichte*." in Gottlob Frege, Collected Papers on Mathematics, Logic, and Philosophy. Ed. Brian McGuinness. New York: Basil Blackwell, 1984. 108-111.
- Friedman, Michael. "Ernst Cassirer and Contemporary Philosophy of Science." Angelaki 10.1 (2005): 119-28.
- Funkenstein, Amos. "The Persecution of Absolutes: On the Kantian and Neo-Kantian Theories of Science." The Kaleidoscope of Science. Ed. E. Ullmann-Margalit. Boston: D. Reidel Publishing Company, 1986. 39-63.
- Holzhey, Helmut. "Cohen and the Marburg School School in Context." Hermann Cohen's Critical Idealism. Ed. Reiner Munk. Dordrecht: Springer, 2005. 3-37.
- Kant, Immanuel. Critique of Pure Reason. Trans. Norman Kemp Smith. New York: St. Martin's Press, 1929.
- Leibniz, G. W. "Disseratio exoterica de statu praesenti et incrementis novissimis deque usu geometriae." Mathematische Schriften, Bd. 7. (1859) Ed. Carl Immanuel Gerhardt. Hildesheim: G. Olms, 1962. 316-326.
- . "Leibniz an Wallis." Mathematische Schriften, Bd. 4. (1859) Ed. Carl Immanuel Gerhardt. Hildesheim: G. Olms, 1962. 23-29.
- . "Letter of Mr. Leibniz on a General Principle Useful in Explaining the Laws of Nature through a Consideration of the Divine Wisdom: To Serve as a Reply to the

- Response of the Rev. Father Malebranche, 1687." Philosophical Papers and Letters. Ed. Leroy E. Loemker. 2nd ed. Dordrecht: D. Reidel, 1969. 351-354.
- . "Réplique aux réflexions, contenues dans la seconde edition du Dictionnaire critique de M. Bayle, article Rorarius sur le système de l'harmonie préétablie." (1702) Opera philosophica quae exstant latina gallica germanica omnia. (1840) Ed. J. E. Erdmann. Meisenheim/Glan: Anton Hain K.G., 1959. 183-191.
- . "Responsio ad nonnullas difficultates a Dm. Bernardo Niewentiit circa methodum differentialem seu infinitesimalem motas." Mathematische Schriften, Bd. 5. (1859) Ed. Carl Immanuel Gerhardt. Hildesheim: G. Olms, 1962. 320-328.
- . "Specimen dynamicum." (1695) Philosophical Papers and Letters. Ed. Leroy E. Loemker. 2nd ed. Dordrecht: D. Reidel, 1969. 435-452.
- . "The Theory of Abstract Motion, 1671." The Labyrinth of the Continuum: Writings on the Continuum Problem, 1672-1686. Ed. Richard T. W. Arthur. New Haven: Yale University Press, 2001. 339-343.
- . "Theory of Abstract Motion: Fundamental Principles, 1671." Gottfried Wilhelm Leibniz: Philosophical Papers and Letters. Ed. Leroy E. Loemker. 2nd ed. Dordrecht: D. Reidel, 1969. 139-142.
- . "Zwei Briefe Leibnizens an Joh. Ch. Schulenburg." (1698) Mathematische Schriften, Bd. 7. (1859) Ed. Karl Immanuel Gerhardt. Hildesheim: G. Olms, 1962. 238-243.
- Merrick, Teri. "What Frege Meant When He Said: Kant is Right about Geometry." Philosophia Mathematica 14.1 (2006): 44-75.
- Moynahan, Gregory B. "Hermann Cohen's *Das Prinzip der Infinitesimalmethode*, Ernst Cassirer, and the Politics of Science in Wilhelmine Germany." Perspectives on Science 11.1 (2003): 35-75.
- Natorp, P. "Kant und die Marburger Schule." Kant-Studien 17 (1912): 193-222.
- . Plato's Theory of Ideas: An Introduction to Idealism. (1902) Sankt Augustin: Academia, 2004.

- Newton, Isaac. "De gravitatione et aequipondio fluidorum." Unpublished Scientific Papers of Isaac Newton. Ed. A. Rupert Hall and Marie Boas Hall. Cambridge: Cambridge University Press, 1962. 89-156.
- . The Principia. Trans. I. Bernard Cohen. Berkeley: University of California Press, 1999.
- . "A Treatise of the Method of Fluxions and Infinite Series." (1671) The Mathematical Works of Isaac Newton. Ed. D. T. Whiteside. New York: Johnson, 1964. 29-137.
- Patton, Lydia. "The Critical Philosophy Renewed: The Bridge between Hermann Cohen's Early Work on Kant and Later Philosophy of Science." Angelaki 10.1 (2005): 109-18.
- . "Hermann Cohen's History and Philosophy of Science." Ph.D. McGill University, 2004.
- Plato. The Republic. Trans. Desmond Lee. Harmondsworth: Penguin Classics, 1975.
- Poma, Andrea. The Critical Philosophy of Hermann Cohen. Trans. John Denton. Albany: State University of New York Press, 1997.
- Pulkkinen, Jarmo. Thought and Logic: The Debates Between German-Speaking Philosophers and Symbolic Logicians at the Turn of the 20th Century. New York: Peter Lang, 2005.
- Richardson, Alan W. "Conceiving, Experiencing, and Conceiving Experiencing: Neo-Kantianism and the History of the Concept of Experience." Topoi 22 (2003): 55-67.
- Rucker, Rudy. Infinity and the Mind. Toronto: Bantam Books, 1983.
- Russell, Bertrand. The Principles of Mathematics. (1903) New York: W.W. Norton & Company, 1996.
- Trendelenburg, Adolf. Kuno Fischer Und Sein Kant: Eine Entgegnung. Leipzig: S. Hirzel, 1869.
- . "Über eine Lücke in Kants Beweis von der Ausschliessenden Subjektivität des Raumes und der Zeit." Historische Beiträge zur Philosophie, Bd. 3. Berlin: G. Bethge, 1867. 215-276.

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Veal, Damian. "Continental Philosophy and the Sciences: The German Tradition."

Angelaki 10.1 (2005): 1-31.