THE ACTUAL INFINITE

THE ACTUAL INFINITE:

A LEIBNIZIAN PERSPECTIVE

ON

CANTOR'S PARADISE

By

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ABSTRACT

This thesis is first and foremost an investigation of the actual infinite. It draws on the work of Richard T. W. Arthur in defense of G. W. Leibniz's view that the infinite, while actual, should be understood syncategorematically. The actual infinite has now, due to the work of Georg Cantor (along with Bernard Bolzano and Richard Dedekind), found a permanent home within the foundations of mathematics. This was made possible by the stipulation that the part-whole axiom does not apply to infinite collections in the way it applies to finite ones: an actual infinite set is defined as a collection that can be placed in a one-to-one correspondence with a proper subset of itself. In my view, however, something more than a stipulation is required to guarantee the coherence of an infinite set. It has not been sufficiently demonstrated that an actual infinite multiplicity can be one and whole, fixed and definite—that is, can be categorematic—but this is being assumed. Satisfactory justification is required, I believe, if the actual infinite is to play such a fundamental role in the discipline of mathematics. Georg Cantor does attempt to provide such justification in the form of three philosophical arguments, which I have called the argument from irrationals, the divine intellect argument, and the domain argument. His arguments, however, rely on an equation of the terms "actual" and "potential" with the terms "categorematic" and "syncategorematic" respectively. But based on the work of G. W. Leibniz, such an equation is faulty. It is completely legitimate to maintain that the infinite is both actual and syncategorematic, a possibility not considered by Cantor. Once such a position is on the table—that is, once it is no longer necessary that the actual implies the categorematic—Cantor's arguments are no longer sound and the actual (and categorematic) infinite stands in need of further justification.

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INTRODUCTION

The concept of number has become so refined and subjected to such a detailed analysis within contemporary mathematics that its current characterization seems to have been granted the status of necessary truth. The way that mathematics is conceived, and perhaps more importantly, the way mathematics is taught, takes number to be a genus with a multitude of species falling under it. Moreover, this list of species is treated as though it follows immediately from the genus itself, i.e. from the concept of a number. But the list of species is long and, to the layperson, perhaps even fantastical, containing types of numbers that seem to be nothing more than the concoction of a theoretical scientist. The pinnacle of the entire system is the "natural" number. This species of number accords most readily with the common sense idea of what a number looks like and how it behaves; that is, these numbers are what one "uses" to count, which is perhaps the first experience anyone has with numbers. But the list goes on from there: integers, rational numbers, real numbers, complex numbers, imaginary numbers, and rounding off the list there are the enigmatic transfinite numbers. With each species of number beyond the natural numbers (and even perhaps within the naturals in the form of "0" and "1") there is a departure from the definition of "number" strictly understood. By "strictly understood" I mean understood as "number" was originally conceived, i.e. as a finite plurality.[1] This clearly rules out numbers such as "0" and "1", which are not pluralities. Also ruled out are rational numbers, negative integers, irrational numbers, and imaginary numbers: how can one have a plurality composed of 4/5, -6, √2, or 4*i* elements?

[1] Lavine (1994) identifies the Greek notion of *arithmos* as a finite plurality and cites it as the original notion of number, one which is more akin to the modern notion of set than to the modern "natural" numbers.

It seems, however, that the most troubling extension of the concept of number is the final one listed above, the move to transfinite (or infinite) numbers, although this is certainly debatable.[2] It appears stunningly obvious that, so far as finite numbers are concerned, Euclid's fifth common notion—i.e. the whole is greater than the part—is uncontestable. The obviousness of this axiom with respect to finite numbers implies, for many, that it should apply with equal obviousness to the case of infinite numbers, if such numbers are to be coherent. The history of the discussion of infinite numbers serves as a proof of this claim. Take, for example, Galileo's paradox:

> If I inquire how many roots there are, it cannot be denied that there are as many as there are numbers because every number is the root of some square. This being granted we must say that there are as many squares as there are numbers because they are just as numerous as their roots, and all the numbers are roots. Yet at the outset we said there are many more numbers than squares, since the larger portion of them are not squares. (Galileo 1999, §78)

In this reductio concerning the series of natural numbers and the series of squares, the assumption that the whole is larger than the part was not even something considered for rejection. Neither Galileo (nor Leibniz when he dealt with it) thought it possible to relinquish this metaphysical principle even though they came to differing conclusions as a result of this argument: Galileo concluded that the relations of "greater than", "less than", and "equal to" do not apply when dealing with the infinite while Leibniz concluded that the notion of an infinite number is absurd.[3]

[2] See, for example, Benardete (1964). He endorses transfinite numbers, while rejecting what are considered by many to be more innocuous extensions of the number concept.

[3] Galileo discusses this problem in *Two New Sciences* (EN 79; TLC 56). Leibniz's actual conclusion is that the number of numbers cannot form a whole, and that number is the sort of thing that does not permit a greatest of its kind (TLC 179).

But certain mathematicians thought that perhaps something else ought to be relinquished as a result of Galileo's paradox (and other similar paradoxes). The something else to be relinquished, naturally, was the part-whole axiom itself. This insight, which originated with Bolzano and Dedekind, and which was later taken up by Cantor, provided the basis for a tidal wave of development both in numerical analysis and in the foundations of mathematics. Instead of being paradoxical, an infinite set became defined as a collection which can be set up in a one-to-one correspondence with a proper part of itself; thus, in the case of an infinite collection, the whole is not greater than the part. The fact that the sequence of natural numbers appears to be both equal to and greater than the sequence of squares—the outcome of Galileo's paradox—is avoided by claiming that one-to-one correspondence is the only criterion of equality with respect to infinite sets and so these sequences (or rather the sets they compose) have equal cardinality even though their members differ, and even if one is a proper part of the other.

Once the coherence of infinite sets was established in this way, a wealth of results quickly followed. Dedekind was able to define irrational numbers with a rigour that was not previously attainable: each irrational number is defined by means of dividing the sequence of rational numbers into two infinite sets, essentially "cutting" the line at the spot where the relevant irrational number would sit. Cantor was able to argue that the entire discipline of mathematics can be founded on a theory of sets and went on to describe an elaborate theory of transfinite mathematics including rules analogous to operations such as addition, subtraction, and multiplication as they apply to finite numbers.

Aside from these practical results, however, it is difficult to see the justification for relinquishing an axiom as seemingly obvious as the part-whole axiom. What justification could there be? Ultimately, the soundness of dropping the part-whole axiom can only be evaluated by the results to which it gives rise. At least, this appears to be the common position. But there is a difficulty when it comes to the selection of the standard by which the results themselves will be evaluated. For clearly it is counterintuitive to relinquish the part-whole axiom. And if the results that follow from this are equally counterintuitive, are we even in a position to evaluate them let alone claim that they justify the initial supposition?

Perhaps foreseeing such an impediment, or perhaps for some other reason, Georg Cantor provided arguments for his position that reach beyond the mere coherence of the system he is able to construct. He attempted to provide philosophical justification for the actual infinite. In particular, he gave three substantive arguments for this position: not mathematical arguments, not arguments about what could be accomplished mathematically by embracing the actual infinite, and not mathematical proofs, rather arguments in defense of the concept that he wished to utilize, the concept of the actual infinite.

The notion of the actual infinite, and the attempt made by Cantor to defend it, is precisely what I will address in this thesis. The reason I have introduced this problem in terms of the concept of number is twofold. First of all, if a number is a unified plurality, the question as to whether there can be an infinite number and the question as to whether an infinite collection can be one and whole are essentially the same question. Secondly, by framing my project in terms of the number-concept, the arguments surrounding the philosophy of the infinite, which are often obscure and esoteric, are situated within a context that shows up their relevance to ongoing work in mathematics. What could be more important to mathematics than the number-concept? That a theory of sets, which subscribes to a neo-Cantorian philosophy of the infinite, currently sits at the heart of the foundations of mathematics demonstrates that a certain position on the infinite is presupposed by the number-concept, and in fact, is being presupposed by all work being done in mathematics, even if this is not apparent. If the infinite is to be utilized in this way, it is important to have a sensible theory from which to work.

My aim, therefore, is to evaluate Cantor's philosophical arguments in favour of the actual infinite. It is my view that his arguments cannot stand up to scrutiny. They are philosophically unsatisfactory and so his conclusions must be abandoned or some other arguments must be provided. But this is not merely a critical project. I intend to provide an alternative position, one which is able to avoid the shortcomings of Cantor's arguments, yet maintain some of his insight into the nature of the infinite and how it should be treated mathematically.

The alternative that I will provide originates with Gottfried Leibniz. By providing this alternative, I am not abandoning Cantor's position in favour of a more primitive position on the infinite. That is, Leibniz's position was not once appreciated, but relinquished in favour of a more satisfactory position. It was not unable to deal with certain problems or objections and so abandoned: Leibniz's position was never seriously taken up.[4] Part of my goal, then, is to provide an account of Leibniz's philosophy of the infinite that demonstrates its appeal. This project, then, will have three central components: (1) an exposition and defense of Leibniz's position on the infinite, (2) an exposition and analysis of Cantor's position on the infinite, and (3) an argument for why Cantor's position is unsatisfactory and Leibniz's position is promising. The need for these three components leads to the organization of my thesis into three chapters, which will be organized as follows.

In chapter one I will present Leibniz's position. My focus will be on the way in which he diverges from traditional positions regarding the infinite. Historically, all positions have been framed by the Aristotelian distinction between potential and actual infinities. Leibniz breaks out of this mold by espousing an infinite which is at once actual and syncategorematic. Because of the novelty of his view, it at times seems like Leibniz is contradicting himself. In certain places, he writes that he is very much in favour of the actual infinite, while at other locations he derides the idea of an infinite number as contradictory and absurd. It is this feature of Leibniz's philosophy that elicits the following claim from Dauben (1979): "Leibniz was a particularly difficult case because his opinions concerning the infinite seemed different depending upon occasion and context" (124). My primary goal in the first chapter will be to defend Leibniz against the charge of inconsistency. The apparent discrepancies within Leibniz's treatment of the infinite are attributable to the inability to look beyond the traditional distinctions within

[4] It is not my aim to provide an explanation of why this is the case, but some possible reasons will likely become apparent throughout the course of my exposition.

the philosophy of the infinite and see that Leibniz does have a perfectly consistent position, which nevertheless does not fit within the traditional mold.

In chapter two I will present Cantor's position on the infinite. My focus in this chapter will be the way in which Cantor, unlike Leibniz, is working precisely within the traditional distinction between potential and actual. Although Cantor is clearly not falling in line with the accepted position regarding this distinction—i.e. that the potential infinite is acceptable while the actual infinite is not—he nevertheless remains within it in terms of the inferences that he makes. I will give a brief treatment of Cantor's transfinite mathematics, but I will not focus on the technical aspects of his position. I am interested in his mathematics only insofar as they demonstrate his philosophical commitments and relate to his justification of his acceptance of the actual infinite. I am primarily interested in three philosophical arguments that Cantor provides in defense of his position. I have called these three arguments the argument from irrationals, the divine intellect argument, and the domain argument. In each of these arguments, Cantor believes that he is providing an argument in favour of the actual infinite beyond the mere consistency of the system he is able to construct once the actual infinite is accepted. I will treat these arguments at some length, providing a detailed analysis of precisely what each argument establishes as well as what Cantor believes it to establish. One cannot expect Cantor to take seriously a position on the infinite the coherence of which was not seriously considered until the twentieth century. Thus, I will not be criticizing Cantor for lack of insight. I will, however, be arguing that the arguments he provides for the coherence of the actual infinite (as he understands it) are not satisfactory. In fact, once Leibniz's position is on the table, it seems to be the more acceptable of the two.

In chapter three I will make an explicit argument for this claim, that Leibniz provides us with a more desirable position on the infinite than does Cantor. I will begin with a discussion of two of the known paradoxes that arise from Cantorian set theory—the Burali-Forti paradox and Cantor's paradox—as well as the standard responses. In particular I will discuss the way in which axiomatic set theory follows from the discovery of these paradoxes, attempting to provide a way in which they can be avoided. While it certainly has its appeal (and is widely accepted), I do not believe that axiomatic set theory satisfactorily addresses the problems encountered by Cantor. Thus, although I acknowledge the merits of this approach, I will be taking a different line. To be more precise, I will be providing the foundation for a different line. In my view, once it is acknowledged that Leibniz has a coherent position on the infinite according to which the infinite is both actual and syncategorematic, then Cantor's arguments are no longer sound. I will also consider an argument in favour of the actual infinite not given by Cantor. Rucker (1995) attempts to argue for the coherence of the actual infinite based on the Reflection Principle. While initially compelling, I do not believe that this argument is any more satisfactory than Cantor's own arguments. In the end, I believe that all such arguments can be answered, and that Leibniz's position shows significant promise for providing an alternative theory of the infinite to be utilized by the foundations of mathematics. While incorporating my results into the foundations of mathematics is clearly beyond the scope of the present endeavour, I hope that by the end of the final chapter the possibility for such a project will be seen not as an unnecessary clouding of the matter, but as relevant to achieving a satisfactory foundation for the discipline of mathematics.

Chapter 1: Leibniz and the Infinite

1.1 Introduction

When one begins to enumerate the historical positions on the infinite, one almost inevitably does not get very far before the list is exhausted. All viable positions on the infinite seem to be described in terms of the distinction between potential and actual. One can accept one while denying the other, one can accept both, or one can reject both; there do not seem to be any other options. This distinction between the potential infinite and the actual infinite—as most everyone is aware—originates with Aristotle and can be found within his treatment of one of Zeno's paradoxes of motion. According to Aristotle, the potential infinite is unproblematic; so long as we can avoid the actual infinite we are on firm philosophical ground. And this attitude seems to have endured for quite some time. Generally speaking, it was not until Dedekind and Bolzano (followed by Cantor) proposed that what had previously been considered paradoxical features of the actual infinite could be taken as its defining characteristics (namely that an actually infinite multitude can be placed in a one-to-one correspondence with a proper subset of itself) that the actual infinite was even considered to be philosophically coherent.[5]

Presently, the actual infinite so thoroughly permeates the discipline of mathematics (insofar as mathematics is supposed to have set theory as its fundamental basis) that denying it seems to be tantamount to rejecting the entire foundation upon which mathematics is built. Yet there are basic paradoxes that have pestered the actual

[5] See Bolzano (1851).

infinite and cast doubt upon its coherence, paradoxes that relate to the concept itself and paradoxes that relate to its application in disciplines such as set theory. To consider the case of set theory for a moment, it is clear that these paradoxes can be "dissolved" through the careful construction of axiomatic systems which do not allow certain problematic sets to be constructed. But from a philosophical or metamathematical standpoint, I would argue, these problems or paradoxes have not been satisfactorily dealt with; they have merely been avoided.

There has been a long history of opposition to the actual infinite. Thinkers such as Brouwer and Hilbert have presented alternative positions on the infinite (Intuitionism and Finitism respectively) and attempted to demonstrate how these positions provide a sufficient foundation for mathematics. But both of these responses are still treating the infinite from within the Aristotelian distinction between potential and actual. And while this is not problematic in itself, remaining within this paradigm has led to a seemingly intractable disagreement. Many mathematicians are content with the suspected problems inherent within the actual infinite because of the valuable work that can be accomplished once it is accepted. Many philosophers are unhappy with the amount of work being done by a concept whose status remains dubious. The prospect of reconciliation certainly looks bleak.

The situation as I have described it naturally leads one to hope for a third option in this problematic dilemma, something outside of the traditional Aristotelian distinction.

As I will argue, Leibniz presents us with just such a position.[6] It is a bit of a historical curiosity that Leibniz's position on the infinite has been so casually overlooked for such a long time. Although my aim is not to provide an historical account of why Leibniz's position was not taken up, it is worth noting in a cursory way some possible reasons for this. One possible reason is the divergent and often obscure locations at which his arguments are found: many of his writings on the continuum, for instance, were only published recently.[7] In my view, however, the overriding reason is the pure novelty of Leibniz's position.[8] His view combines two ways of characterizing the infinite which would likely be considered contradictory; in fact, some modern commentators still maintain that Leibniz's position is fundamentally incoherent.[9] Their view is no doubt due to the fact that the actual/potential distinction has come to be considered both mutually exclusive and jointly exhaustive. As a result, any position that breaks away from this taxonomy of the infinite is viewed as nonsensical. Thus, the main hurdle that I have to overcome before I can even begin to rank Leibniz's position against the well-established view of Cantor is to demonstrate that Leibniz's position is self-consistent, i.e. it makes sense.

The present chapter, then, will be dedicated to this goal. I will present a basic sketch of Leibniz's position while defending it against the accusation that it fails to meet even the most basic criterion, namely internal consistency. The second goal will be left

[6] In this view I follow Richard Arthur, who is the original proponent of the coherence of Leibniz's position on the infinite.
[7] I am referring here to the writings contained in *The Labyrinth of the Continuum*.
[8] As I will mention below, Leibniz did have predecessors—namely, Spinoza—who pointed toward this position. For a discussion of Spinoza on this matter see Riesterer (2006). Leibniz was, however, the first thinker to develop it in such detail.
[9] I have in mind Gregory Brown and Samuel Levey, whose objections will be discussed below.

aside for now, to be returned to only in the third chapter once Cantor's position on the infinite is on the table. Something to bear in mind throughout the following exposition is that Leibniz's presentation of his position on the infinite is carried out in large part through examples of "infinity in the small", i.e. through the consideration of parts whose size is ever diminishing. While this is a crucial part of Leibniz's position, it is my intention to apply the position arrived at through these discussions to the question of infinite number, i.e. "infinity in the large". There is an important distinction to be made between an infinite magnitude (great or small) and an infinite multiplicity. It is the latter which is relevant to a discussion of infinite number in the sense that there are infinitely many parts (great or small) that may or may not have a cardinality, that is, a number. Thus, what is important in the following exposition is not the question of infinitely small magnitudes, i.e. infinitesimals, but the multiplicity that they constitute. It is in this sense that the position held by Leibniz is comparable to Cantor's position.

1.2 Identifying the Problem

Right at the outset there is a curiosity within Leibniz's characterization of the infinite. In fact, at first glance it would appear as though Leibniz is making contradictory claims. In certain locations within his philosophical writings, he clearly advocates the actual infinite, an attitude for which he receives Cantor's praise.[10] However at other locations Leibniz explicitly denies that the notion of infinite number is coherent, which

[10] In fact, in his "Grundlagen einer allgemeinenen Mannigfaltigkeitslehre" Cantor quotes a passage from Leibniz's correspondence with Foucher: "Indeed, I believe there is no part of matter which is not, I do not say divisible, but actually divided; and that consequently the least particle ought to be considered as a world full of an infinity of different creatures" (Arthur, unpublished). Cantor goes on to comment that "already in Leibniz we find in many places essentially the correct point of view" (Cantor 1932, 180).

clearly distinguishes his from a straightforward Cantorian position. The way in which the former attitude is displayed is that at various and divergent locations within his writings Leibniz makes the claim that body—i.e. matter, material stuff—is actually infinitely divided. Leibniz's rationale behind these claims will be explained below. In *A Specimen of Discoveries of the Admirable Secrets of Nature in General* he writes:

> Moreover, there are no atoms, but every part again has parts actually divided from each other and excited by different motions, or what follows from this, *every body however small has actually infinite parts*, and in every grain of powder there is a world of innumerable creatures. (TLC 317; emphasis added)

He reiterates this view in a short work entitled *Created Things are Actually Infinite*:

> Created things are actually infinite. For any body whatever is actually divided into several parts since any body whatever is acted upon by other bodies. And any part whatever of a body is a body by the very definition of body. *So bodies are actually infinite . . . i.e. more bodies can be found than there are unities in any given number*. (TLC 235; emphasis added)

As Russell (1937) observes, "an actual infinite has been generally regarded as inadmissible, and Leibniz, in admitting it, is face to face with the problem of the continuum" (108). Aside from the possible problems with the continuum, of which Leibniz was certainly well aware, this attitude towards the actual infinite was novel to say the least. At the time Leibniz was writing, considerations of the infinite were strongly guided by the Aristotelian distinction between potential and actual infinities, the latter being considered inadmissible by Aristotle and thus generally so. Hence the motivation for Russell's claim. So in this regard, Leibniz is stepping into dangerous philosophical territory by viewing the actual infinite as something coherent.[11] To put it differently, Leibniz is already opening himself up to attack by espousing something the incoherence of which was thought to be well established. But the problems do not end here.

By espousing actually infinite division, it would appear that Leibniz is also espousing the possibility of an actually infinite number. For if an actually infinite multiplicity exists—i.e. as a result of actually infinite division—it seems that we can ask about its cardinality; in other words, we can ask *how many*? At first glance, it seems that giving this multiplicity a cardinality is a straightforward implication of the doctrine of actually infinite division. Thus, if one were only to read this aspect of Leibniz's position on the infinite, one might think that he had anticipated Cantor's acceptance of the actual infinite and was in a position to proceed to the theory of infinite cardinals to be developed by Cantor some two hundred years later. However, there is more to be said about Leibniz's position; and this is where the curiosity lies. For in his dialogue *Pacidius to Philalethes* Leibniz rejects the notion of infinite number as contradictory based on the very concept of number. He writes: "I believe it to be the nature of certain notions that they are incapable of perfection and completion, and also of having a greatest of their kind. Number is such a thing" (TLC 179).[12] Thus, Leibniz cannot accept that the infinite multiplicity implied by his doctrine of actually infinite division has a cardinality. In doing

[11] There are precedents for the espousal of the actual infinite going right back to Ancient Greece. For example, Anaxagoras writes "All things were together, infinite both in number and in smallness; for the small too was infinite. And of all things together none was evident on account of smallness" (DK32B1).
[12] As I will discuss, the literal implication of this statement is not at odds with Cantor's own position regarding infinite number. However, one cannot read too much of a similarity here; for Leibniz had no conception of an infinite number in the sense of a Cantorian transfinite number, a middle ground between the finite and the absolutely infinite. For Leibniz, "infinite number" could mean nothing other than a number that is quantitatively not increasable.

so he would be committed to the acceptance of infinite number. But since he denies that number is the sort of thing that can have a "greatest of [its] kind" his position would very quickly fall into inconsistency.[13] So it would seem that Leibniz is in a rather difficult position. Nevertheless, I do not believe that he can be charged with any inconsistency. It is certainly tenable to espouse the existence of an infinite multiplicity while denying the existence of an infinite cardinal that numbers this multiplicity. The way in which this position takes shape can be found within Leibniz's own writings. That is, the following is not an account of the way in which Leibniz *could have* avoided this apparent inconsistency; it is an account of the way he *did in fact* avoid it. The first step is to see the way in which Leibniz commits himself to the espousal of the actual infinite. An explanation of his doctrine of the actually infinite division of matter will serve to demonstrate this commitment.

1.3 The Infinite as Actual

It seems that the doctrine of actually infinite division has its origin in the Cartesian method of explaining the motion of matter through uneven spaces in a plenum (Arthur 1989). Descartes' account of motion in a plenum is as follows: "a body entering a given place expels another, and the expelled body moves on and expels another, and so on, until the body at the end of the sequence enters the place left by the first body at the precise moment when the first body is leaving it" (CSM I 237-238). In order for this

[13] This is not a completely novel position. See Riesterer (2006) for an argument that Spinoza has a conception of the infinite as syncategorematic. However, it is with Leibniz that the application of such a position within mathematics is discussed.

doctrine to make sense, Descartes requires that all motion be circular, i.e., that all bodies move in a circle. If everything moved in concentric circles, then perhaps no further explanation would need to be given. But then it seems that everything would need to move at a constant rate. So Descartes considered the case of non-concentric circles. In order to understand the scenario Descartes is describing, simply imagine a small circle inscribed within a larger circle, the centers of which do not align with one another. Further imagine corpuscles moving through the corridor created by the exterior edge of the smaller circle and the interior edge of the larger circle. At certain locations during the traversal of this corridor the corpuscles will be moving through a widening space while at other locations the corridor will be narrowing. Descartes argues that bodies passing through a smaller space must move faster in order for enough bodies to move through to fill up the space of the corpuscles on the other side of the corridor. In particular the bodies moving though the smaller space move *n* times as fast as bodies passing through a larger space, where the larger space is *n* times as large as the smaller. It is the need for all of these corpuscles to fit through the small part of the corridor that leads Descartes to espouse what he describes as "indefinite division". Descartes writes,

> for what happens is an infinite, or indefinite, division of the various particles of matter; and the resulting subdivisions are so numerous that however small we make a particle in our thought, we always understand that it is in fact divided into other still smaller particles. (CSM I, 239)

It is only through this indefinite division that a sufficient mass of corpuscles will be able to move through the small space quickly enough to fill the space of the widening corridor, which is clearly required in order to maintain the doctrine of the plenum.

When we compare Descartes' articulation of this doctrine with Leibniz's articulation above, there are two striking similarities: the motion of the bodies is the cause of their division and there is no smallest body (however small a body is, it is *in fact* still divided into smaller particles). The crucial difference between their positions is that while Descartes wants to maintain that the division is merely "indefinite", Leibniz does not hesitate to assert that this division must be actually infinite.[14]

Descartes' explanation of why he distinguishes the indefinite from the infinite (and from the finite as well) can be found in his treatment of whether the world is infinite:

> I do not say that the world is *infinite* but only that it is indefinite. There is quite a notable difference between the two: for we cannot say that something is infinite without a reason to prove this such as we can give only in the case of God; but we can say that a thing is indefinite simply if we have no reason which proves that it has bounds....Having then no argument to prove, and not even being able to conceive, that the world has bounds, I call it *indefinite*. But I cannot deny on that account that there may be some reasons which are known to God though incomprehensible to me; that is why I do not say outright that it is *infinite*. (CSM III, 319-320; Bassler 852-853)

Something is indefinite, then, if there is no proof that it is bounded. To be infinite, on the other hand, it is necessary to construct a proof, but such a proof is beyond the capacity of a finite intellect. Based on this account, the indefinite can be thought of as a kind of middle ground between the finite and the infinite.

[14] This is not to say that Descartes' account is without its difficulties. By stopping at "indefinite" Descartes believes that he can avoid tackling the problems of the continuum. Leibniz, however, attempts to address what he believes to be the actual implications of claiming that any part we can imagine is *in fact* divided into yet smaller parts.

In his early writings, Leibniz also upheld this distinction between the indefinite and the infinite, describing, for example, the number of numbers as indefinite but not infinite (Bassler 850). Later in his career, however, Leibniz moves to the position that the indefinite is infinite, and this is what leads him to a different conclusion than Descartes concerning what indefinite division actually entails.[15] The rejection of this distinction is very much tied up with Leibniz's position that the infinite should be understood syncategorematically as opposed to categorematically. This will be discussed at length in the following section.

As I pointed out, according to Leibniz division is carried out through the motions of bodies. Since all of the parts of a body are "excited by different motions" the division is actual, that is, it has already taken place (TLC 317). Furthermore, since "any part whatever of a body is a body", the same thing can be said of the parts—i.e. they are actually infinitely divided. Although by now it should be apparent, it is important to notice the contrast between Leibniz's view and a doctrine of infinite division in which the division is not necessarily carried out, i.e. is merely potential. For it is in this respect that Leibniz is shown to espouse the actual as opposed to the potential infinite.

Contrast this with Aristotle's position. To say that matter is infinitely *divisible* is to say only that it is capable of being divided. That is, any part can be further divided into smaller parts, which can be further divided, and so on. But the division is only *potential*—it is not necessarily carried out. The import of this is that the parts need not be

[15] Bassler (1998) traces the development of Leibniz's position on this matter. I will not repeat it here, however. I am only interested in Leibniz's mature position as that is what leads to his doctrine of actually infinite division.

further divided (since they may not be actual parts), and thus need not be infinitely small or infinite in number. If this were Leibniz's position, he would not be at odds with his peripatetic ancestors, who maintained that the potential infinite was the only consistent notion of the infinite. For if the parts were, so to speak, waiting to be created, then they would not be present *actually* but only *potentially*, and thus it would not be case that body would seem to have an infinite number of infinitely small parts.[16]

However, when a body is said to be actually *divided*, this problem does arise. And this is precisely what Leibniz does say. In fact, in his letter to Foucher (1692) he makes an explicit effort to distinguish his position from one according to which the divisions are only potential: "Indeed, I believe there is no part of matter which is not, *I do not say divisible, but actually divided*; and that consequently the least particle ought to be considered as a world full of an infinity of different creatures" (G.I.416; Arthur unpublished, 1; emphasis added). For it is not the case that the division *could* be carried out, rather that it is carried out. If a body is actually divided, then it seems to have an infinite number of infinitely small parts. For using the term "divided" gives a connotation of completeness, which leads to the conclusion that Leibniz is in fact espousing a categorematic infinity—i.e. an infinity present all at once. This espousal of the actual infinite is exactly what brings Leibniz face to face with the problem of the composition of

[16] As I will discuss, certain commentators have argued that Leibniz's doctrine of infinite division should have been one of potential division. See, for example, Brown (2000).

the continuum and, in turn, leads to the apparent necessity that Leibniz accept the notion of infinite number.[17]

1.4 The Infinite as Syncategorematic[18]

As clearly as Leibniz accepts that bodies are actually infinitely divided, he denies that there can be such a thing as an infinite number. In *Pacidius to Philalethes*, Leibniz argues that the notion of an infinite number is inconsistent because the whole becomes equal to the part. He uses various examples to draw this conclusion, but the example of the number of all squares (Galileo's paradox) is the easiest to deal with here. It runs thus:

> The number of all squares is less than the number of all numbers, since there are some numbers which are non-square. On the other hand, the number of all squares is equal to the number of all numbers, which I show as follows: there is no number which does not have its own corresponding square, therefore the number of numbers is not greater than the number of squares; on the other hand, every square number has a number as its side: therefore the number of squares is not greater than the number of numbers. Therefore the number of all numbers (square and non-square) will be neither greater than nor less than, but equal to the number of all squares: the whole will be equal to the part, which is absurd. (TLC 177)

As Arthur (2001) points out, Leibniz identifies two statements that may be rejected as a result of this *reductio*. "(W) that in the infinite the whole is greater than the part, and (C) that an infinite collection (such as the set of all numbers) is a whole or unity" (103). From the fact that one set of numbers (i.e. the natural numbers) can be set up in a one-to-one correspondence with one of its proper sub-sets (i.e. the squares of the natural numbers),

[17] There is another aspect of this problem that will not be dealt with here; i.e. the relationship between actually infinite division and the continuity of motion. For a treatment of continuity in Leibniz, see, for example, Crockett (1998) and Levey (1998).

[18] This reading of Leibniz's philosophy of the infinite is presented by R. T. W. Arthur in a series of papers from 1998 through 2001.

Leibniz denies (C). Cantor has the opposite intuition and denies (W). Since (W) and (C) are incompatible with one another, given the other commitments of this argument, Cantor's theory and Leibniz's theory are found to be fundamentally opposed to one another. And thus, as I will argue in a further chapter, this argument cannot be the basis for deciding between the two theories. That is, we cannot argue from a Cantorian perspective that Leibniz should have simply accepted infinite number (Arthur 2001, 104).

It has been objected that this argument is based on an equivocation on the phrase "number of" (Benardete 46). As a result, it has been argued, Leibniz's denial of infinite number is fallacious: it is incorrect to claim that either one maintains the part-whole axiom or one accepts infinite number. For on Benardete's account, the contradiction arises from an equivocation rather than from the part-whole axiom and thus can be avoided by clarifying the various senses of the terms being used. And since the part-whole axiom is not involved in the contradiction, the supposition that the number of numbers forms a whole need not be given up in order to maintain it.

> The three criteria of equality that Benardete distinguishes are as follows:
>
> (1) If A is a proper subset of B, then there is a greater number of elements in B than in A.
>
> (2) If A and B can be placed in 1-1 correspondence, then A and B contain the same number of elements.
>
> (3) If all the elements of A can be placed in 1-1 correspondence with a proper subset of B, then B contains a greater number of elements than A. (47)

The upshot of Benardete's analysis is summarized in the following passage:

> There is no contradiction, as Leibniz supposed, if judiciously we note that in one sense (according to the first criterion) the number of integers is

> greater than the number of even numbers; in another sense (according to the second criterion) the number of elements in the one class is equal to the number of elements in the other; and in still another sense (according to the third criterion) the number of the one is both greater and less than the number of the other. (47)

Even if this avoids the contradiction, it does so at the cost of univocality, which is odd considering that this is the very charge Benardete is bringing to bear against Leibniz. The equivocation that Benardete believes he has identified is rather the contradiction itself: maintaining a univocal sense of "number of" is not possible on the supposition that the number of numbers forms a whole. Benardete has admitted this much. I suppose it is possible if one rejects the part-whole axiom altogether and relies solely on one-to-one correspondence to determine the equality of sets. But since this axiom is certainly legitimate in the case of finite collections, this move is rather counterintuitive. It cannot be argued that one must relinquish the part-whole axiom as a result of this argument, which is ultimately what Benardete is arguing. There ought to be some overriding consideration in favour of this maneuver, if it is to be made. Without such a consideration we are right back where we started: one may reject one of two statements as a result of this reductio. Leibniz rejects that the number of numbers forms a whole, Cantor (pace Bolzano and Dedekind) rejects that the part-whole axiom applies to infinite sets. To argue on the basis of this argument alone that one of them is wrong is at best unproductive.

Even if this objection can be answered, based on the previous section, Leibniz's theory seems to imply the existence of an actually infinite number (as Cantor espouses), but in denying (C)—that an infinite collection is a whole or unity—Leibniz is by implication denying infinite number. There are really only two options open to Leibniz at this point. Either he must acknowledge that his doctrine does commit him to infinite number, a notion which he rejects on separate grounds, and so his theory is inconsistent, or he must provide an explanation of how his theory of actually infinite division does not commit him to infinite number, in which case his theory is (for the moment) coherent. Fortunately, Leibniz does provide such an explanation. It is to be found within his treatment of infinite convergent numerical series (Arthur 1999, 109).[19] Take, for example, the series "$1/2 + 1/4 + 1/8 + 1/16 + \ldots + 1/2^n$". This series is said to have a sum of 1 even though there is an infinity of terms. This, in turn, seems to imply that an infinity of terms combines to make a whole and that there is an infinite number of them. But Leibniz construes the sum of an infinite convergent series in such a way as to avoid this conclusion. In *Infinite Numbers* he describes this construal:

> Whenever it is said that a certain infinite series of numbers has a sum, I am of the opinion that all that is being said is that any finite series with the same rule has a sum, and that the error always diminishes as the series increases, so that it becomes as small as we would like. For numbers do not *in themselves* go absolutely to infinity, since then there would be a greatest number. But they go to infinity when applied to a certain space or to an unbounded line divided into parts. (TLC 99; Arthur 1999, 109)

Since there is no *infinitieth* term of the series it is not the case that the series can be taken as a single collection of terms—i.e. the series is not a completed whole.[20]

To understand how this works, it is necessary to introduce the distinction between "categorematic" and "syncategorematic". This distinction is medieval in origin and was

[19] This point, of course, brings up some interesting questions about the relation, in Leibniz's view, between mathematics and ontology. Unfortunately, I cannot give a detailed treatment of this question here. See, however, Levey (1999) for a discussion of this topic.

[20] As I pointed out above, Leibniz's notion of the sum of a convergent infinite numerical series is essentially the modern notion of a limit. That is, "any finite series with the same rule has a sum, and…the error always diminishes as the series increases, so that it becomes as small as we would like" (TLC 99).

initially intended to be applied to terms; i.e. it is a grammatical distinction. It comes from a famous passage in Priscian's work *Institutiones grammatice* in which he identifies two types of word classes: those that have a definite meaning on their own and those—syncategorema or consignificantia—that must be combined with words from the first class in order to acquire a definite meaning (Spruyt 4). It is then given a more detailed treatment by Peter of Spain in his work *Syncategoreumata* and later taken up and refined by Jean Buridan and Gregory of Rimini (A. W. Moore 51). William of Ockham characterizes the distinction as follows:

> Categorematic terms have a definite and fixed signification, as for instance the word 'man' (since it signifies all men) and the word 'animal' (since it signifies all animals), and the word 'whiteness' (since it signifies all occurrences of whiteness). Syncategorematic terms, on the other hand, as 'every', 'none', 'some', 'whole', 'besides', 'only', 'in so far as', and the like, do not have a fixed and definite meaning, nor do they signify things distinct from the things signified by categorematic terms. (51)

The main feature to note in his discussion is that categorematic terms refer to something fixed and definite, while syncategorematic terms do not. With that in mind let me present the distinction as it is applied to the infinite. As A.W. Moore puts it, "to use 'infinite' categorematically is to say that there is something which has a property that surpasses any finite measure; to use it syncategorematically is to say that, given any finite measure, there is something which has a property that surpasses it" (2002, 51). In the case of an infinite multiplicity, this distinction amounts to the following: to say that for every finite number x there is a finite number y which is greater than it—i.e. $\forall x \exists y(y > x)$—is to use the term 'infinite' syncategorematically; to say that there is a number x which is greater than any number y—i.e. $\exists x \forall y(x \neq y \rightarrow x > y)$—is to use the term 'infinite' categorematically. It is clear from these definitions that there is nothing about 'syncategorematic' that necessitates its equation with 'potential'.

The distinctions between the potential and actual, categorematic and syncategorematic are no more equivalent than the analogous distinctions between the *a priori* and *a posteriori*, analytic and synthetic were for Immanuel Kant—the distinctions are cross-cutting. This does not necessarily mean that every combination of these terms results in a possible conception of the infinite; however, it does mean that there are more than simply two options—i.e. actual ($=_{df}$ categorematic) or potential ($=_{df}$ syncategorematic). Thus, it is no more necessary that an actual infinity be categorematic than it is (according to Kant) that an *a priori* statement be analytic.

It has been argued that by claiming that body is actually infinitely divided, Leibniz is not in a position to reject that the series has an infinite number of terms. Brown (1998) has claimed that "[Leibniz's] failure to embrace infinite number was due to an uncharacteristic failure of mathematical imagination on his part" (121). I believe, however, that this reading of Leibniz is not taking into account the subtle distinction between the actual infinite and the categorematic infinite.

If Leibniz's rejection of the categorematic infinite is treated as a rejection of the actual infinite, then a contradiction may not be avoidable. But all that Leibniz claims is that the infinite number of numbers is not one and whole. This claim denies the possibility of a *categorematic* infinity; it denies that an infinity of terms can be present as a unity. So long as we equate the categorematic with the actual, then Leibniz's rejection of infinite number entails a denial of the actually infinite, from which follows a contradiction in Leibniz's position. But this equation need not be carried out. For it is certainly possible for there to be an actual, yet syncategorematic conception of the infinite. And if Leibniz's view is understood in this light, then his position does not fall into inconsistency.

1.5 Some Objections Considered

Arguments against Leibniz have generally not given due weight to the distinction between actual and categorematic. According to Brown (2000), the existence of an infinite cardinal number is implied by the fact "that *all* the divisions in such a body [i.e. an infinitely divided body] are conceived by Leibniz to be *actually given*" (27). He continues as follows:

> Given Leibniz's operational treatment of infinite series, it is natural to suppose that the series is not *actually* completed, that it is not, to use Arthur's expression, 'a completed whole.' But the same cannot be said for actual bodies. For Leibniz is quite unambiguous, throughout the whole of his philosophical career, in stating that the divisions in actual bodies, as opposed to those imagined in what he considered to be *ideal* mathematical continua, are *determinate*, and thus *actually*, and not just *potentially*, infinite. (27-28)

From the claim that bodies are *actually infinitely divided*, Brown is inferring that all the divisions are *actually given*. Whereas the former is an articulation of the actual infinite, the latter is the categorematic infinite. Thus, he has reasoned from the actual to the categorematic. While the categorematic clearly implies the actual, the other direction of implication cannot be taken for granted, which is precisely what Brown is doing. While "every division is actual, in that each part is actually subdivided, that does not give you all the divisions" (Arthur 2005). It is one thing to say that all divisions are actual; it is another thing to say that the collection of all divisions can be given as a whole.

Brown makes another similar claim just below: "Given that the body is actually divided to infinity, as Leibniz contends, *all* of the parts, or divisions, of the body are *actually given*, and in *that* sense it *is* a 'completed whole'" (28). It is obvious from the locution "actually given" that the notion of infinity being wielded in these passages is that of a *categorematic* infinite. For a categorematic infinite is precisely an infinite which is present *all at once*. But this is not Leibniz's conception of the infinite. In other words, Leibniz can legitimately claim that for any number of divisions specified, there are *already* (not possibly) more, while still claiming that there is no number that captures *all* of the divisions. This is because the body whose divisions are being considered is not a true whole. As Arthur (1989) puts it, "at each instant, any given body is the sum of all its parts. But this sum is an infinite sum which, like an infinite series, is never completed. It is not a true whole, or even a collection of them, but a distributive whole" (188). It is in this sense, i.e. a distributive sense, that the word "all" is being used in the phrase "all the divisions". As in an infinite sum, all the terms cannot be given all at once, but they are actual, and they can be given in this distributive sense.

The discrepancy identified within these passages on the basis of an equation of actual with categorematic is articulated by Brown as follows: "[Leibniz's] constructivist stance in mathematics led him to treat the mathematical infinite as merely *potentially* infinite, whereas his metaphysics of divided matter led him to treat the divisions in bodies as *actually* infinite" (2000, 36). This has been described as "a dualistic view of the infinite" and an instance of "[Leibniz's] constructivism [spilling] over disastrously into his philosophy of matter" (Levey 1999, 155). As a result, Brown believes that "just as talk of an actual infinity of terms in an infinite series gives way to talk of a merely potential infinity of partial sums, each of which is itself actually finite," "talk of an actual infinity of divisions" should give way "to talk of a merely potential infinity of divisions" (37). But this only follows if actual is equated with categorematic.

If body were infinitely *divisible*, we would have a clear-cut case of a syncategorematic, potential infinity—at any step of the division it would be possible to carry out the division one step further, the infinite number of divisions being thus potentially present. However, for Leibniz, body is infinitely *divided*, for according to this doctrine, all divisions have already been carried out through the different motions of the different parts of the body. This means that at any step along the way, the division has already occurred—in this sense it is *actual*. But any number whatsoever of identified divisions will be less than the actual number of divisions present—in this sense it is *syncategorematic*. It is not possible to identify the number of divisions; there is no completed totality to be identified (this would be a categorematic conception). As Leibniz says, "more bodies can be found than there are unities in any given number" (TLC 235). Thus, all that Leibniz means when he says that there is an actually infinite division is that for *any* number of divisions proposed, there are *already* more divisions than that number. As a result, to claim that this position commits Leibniz to the existence of infinite number commits the "quantifier shift fallacy", that is, to reason from $(\forall x)(\exists y)y > x$ to $(\exists y)(\forall x)y > x$ (Arthur 2001, 107). Therefore, if the infinite division of bodies is (ontologically) actual, yet (numerically) syncategorematic, the problems identified by Brown can be avoided. As Rescher (1955) puts it, "the vast storehouse of nature does indeed encompass an infinite multitude of existents, but this *actual infinity* cannot be contained by any numerical limit whatever" (113).

Even if, as I believe, Leibniz's rejection of infinite number is consistent with his doctrine of actually infinite division, there is a further problem that needs to be addressed. It is clear that Leibniz denies the existence of infinitesimals as anything more than a *façon de parler*, a useful fiction. As Arthur (1999) puts it: "infinitesimals are infinitely small fictional parts into which a continuous whole can be resolved, but not infinitely small actuals or elements that compose into the whole" (109). The following passage from *Pacidius to Philalethes* clearly indicates that Leibniz wants to avoid the conclusion that body is resolved into points:

> Accordingly the division of the continuum must not be considered to be like the division of sand into grains, but like that of a sheet of paper or tunic into folds. And so although there occur some folds smaller than others infinite in number, a body is never thereby dissolved into points or minima. (TLC 185)

The analogy of the tunic is important and I will return to it later. For now, it presents the conclusion that Leibniz wants to avoid—i.e. that body is resolved all the way down into infinitesimals, and thus that points become parts. Nonetheless, it does seem that he is at risk of running into such a problem by claiming that body is actually infinitely divided, for (ontologically speaking) there is an actual infinity of divisions. Further, any part, no matter how small, is itself a body and is therefore itself actually infinitely divided. Thus, there is a problem as to how these parts can be composed of anything but infinitesimals.

There are two paths open to Leibniz if he is to escape this labyrinth. The first is to claim that according to his doctrine of actually infinite division, it is not the case that body is resolved into points or minima. For if the constituent bodies have some finite, determinate magnitude, it is not a problem that they compose into a whole. This argument can be made based on the folds analogy mentioned above. Slightly below the previously quoted passage from *Pacidius to Philalethes* Leibniz continues the analogy of the folded tunic:

> It is just as if we suppose a tunic to be scored with folds multiplied to infinity in such a way that there is no fold so small that it is not subdivided by a new fold: and yet in this way no point in the tunic will be assignable without its being moved in different directions by its neighbors, although it will not be torn apart by them. And the tunic cannot be said to be resolved all the way down into points; instead, although some folds are smaller than others to infinity, bodies are always extended and points never become parts, but always remain mere extrema. (TLC 185)

So based on this analogy, Leibniz believes that he has avoided the conclusion that points become parts of bodies. For all the parts are not of uniform size ("some folds are smaller than others to infinity") and thus cannot be considered minima or infinitesimals. If this were the case, i.e. the parts were of a uniform size, then points would become parts and the labyrinth would get the better of Leibniz. But he thinks that the model of the folds is able to avoid this problem. Initially, this seems plausible: if the divisions result in parts of differing sizes, then even if these divisions are carried out to infinity, body does not become a powder of infinitesimals.

But Levey (1999) sees a problem lurking here. In an attempt to clarify exactly what has gone wrong, he presents two "ontological" models of Leibniz's division analogous to the infinite convergent numerical series. The first is called the "diminishing pennies" model.

> **DP**: Imagine that there is a glass jar halfway full of pennies. The pennies are organized as follows: there is one penny (the largest) one-half inch thick. There is another penny which is half as thick, one-quarter inch thick. There is a third penny one-eighth inch thick, and so on *ad infinitum*. All pennies in the jar are unique and, save their thickness, share the same dimensions.

It is important to note that there are infinitely many pennies, "but while the pennies grow successively thinner and thinner without end, each one has yet some finite thickness to it and stands in some specifiable finite ratio to the thickness of the [largest] penny" (143). The second model is called the "divided block" model:

> **DB**: Imagine a block of one cubic foot in volume. The block is neatly divided down the middle by a hairline fissure. Each of the halves is further divided down the middle. Each quarter is divided down the middle, and so on *ad infinitum*.

In this case it is important to note that "unlike the diminishing pennies model in which…every penny…is finite and undivided, in the divided block model there are no finite and undivided parts of matter to be found" (144). It is Levey's contention that the divided block model more accurately captures Leibniz's folded tunic model, while the diminishing pennies model would have been the sounder choice.

However, the diminishing pennies model is ruled out by Leibniz's claim that the "folds [are] multiplied to infinity in such a way that there is no fold so small that it is not subdivided by a new fold" (TLC 185). Based on this claim, the parts cannot have finite and determinate dimensions. Since *every part* (every body) is divided to infinity, it is hard to avoid the conclusion that body is resolved into infinitesimals. But this is explicitly denied by Leibniz. Is the claim that every fold is subdivided by a new fold simply an instance of over-zealousness on the part of Leibniz? In other words, if we ignore Leibniz's claim that *every part* of matter is infinitely divided, then can we accept the diminishing pennies model and avoid the problem altogether? As Brown (2000) puts it,

> Now Arthur seems to think that infinitesimals can only be avoided by denying that the infinity of parts in a body constitute a completed whole, but that is *not* the only way that infinitesimals might be avoided. For if a body is conceived to be infinitely divided after the manner of Levey's diminishing pennies model, then the parts may be conceived as constituting a completed whole, in the sense that an infinity of parts are actually given, without thereby supposing that there are any infinitesimal parts. (37)

But given the doctrine of actual infinite division as presented by Leibniz, this is not an option, for "there is no fold so small that it is not subdivided by a new fold"—i.e. "all things are subdivided" (TLC 211; Levey 1999, 145). Therefore, it ought to be determined whether Leibniz's position is coherent as it stands before proceeding to suggest what Leibniz should have or could have done instead.

In an attempt to remain within Leibniz's position, I will explore the second option open to Leibniz, which Brown attributes to Arthur, namely that infinitesimals can be avoided by denying that the infinity of parts in a body constitute a completed whole. The reason that Leibniz seems committed to infinitesimals based on his doctrine of actually infinite division is that we expect the parts that exist as a consequence of these divisions to recombine, so to speak, into a unified whole. But does Leibniz expect this to be the case? Brown (2000) has made the claim that

> it is not altogether clear that *Leibniz*, at any rate, actually realized that his commitment to the doctrine that every part of matter is actually divided to infinity entails that bodies are not wholes, despite the fact that such would

> seem to follow rather directly from the former doctrine in conjunction with the doctrine that an infinity of things cannot make a whole. (37-38)

I contend, however, that it *is* altogether clear that Leibniz was aware of this consequence.

Leibniz maintains unambiguously that bodies are merely aggregations. This can be seen in the following excerpts from his correspondence with Arnauld:

> Thus you will never find a body of which we can say that it is truly a substance: it will always be an aggregation of many substances. Or rather, it will never be a real being, since the parts which make it up face just the same difficulty, and so we never arrive at real being, because beings by aggregation can have only as much reality as there is in their ingredients. (116)[21]

In the following instance, he even speaks directly to the doctrine of actually infinite division:

> But not only is a continuum divisible to infinity, but every part of matter is actually divided into other parts which are as different from each other as the two diamonds mentioned above.[22] And since that goes on and on in the same way, you will never arrive at something of which you can say it is a true being until you find animated machines, the substantial form of which produces a substantial unity which is independent of the external union of contact. And if there are none, it follows that except for man there is nothing substantial in the visible world (WF 118).

[21] A similar comment is made a few lines later: "I answer that in my opinion our body in itself, or the *corpse*, considered in isolation from the soul, can only improperly be called *a substance*, like a machine, or a heap of stones, which are only beings by aggregation" (WF 117).

[22] The passage in which Leibniz discusses the diamonds is worth quoting in full: "For imagine there were two stones, for example the diamond of the Grand Duke and that of the Great Mogul. We can use a single collective noun to do service for both of them, and say that they are a pair of diamonds, although hey are a long way apart from one another; but we would not say that they constitute a substance. Now, matters of degree play no part here. If we gradually bring them closer together, therefore, and even bring them into contact, they will not be any more substantially united. And if when they were in contact we joined them to some other body which prevented them from separating—for example, if we mounted them in a single ring—the whole thing would make up only what is called *unum per accidens.* Because it is as if by accident that they are forced to move in unison" (WF 118).

Thus, it is clear that Leibniz does not equate "substantial unity" with "aggregate", and further, that body is unequivocally *not* a substantial unity. Interestingly, in these passages, Leibniz provides a response to the following criticism by Levey:

> Since *any* part of matter we specify would be subject to the precisely same infinite division into parts, it follows that *no* part of matter can truly be one or a whole. But to say that something is not truly one is to say that it does not truly *exist.* Thus in the folds model of matter's infinite division, since no part of matter can truly be one, *there can't be any matter.* (1999, 146)

It seems that within his correspondence with Arnauld, Leibniz agrees with these very claims. That is why there is a "need for something substantial and non-material to act as a principle of unity for body" (Arthur 2001, 110). For according to Leibniz, matter *qua* matter is not real; it is phenomenal. Thus, a body is explicitly not a unity.

But Leibniz does not equate "substantial unity" with "whole" either (Carlin 1997, 8-9). So what we have is this: "substantial unity" ≠ "aggregate" and "substantial unity" ≠ "whole". But where does this leave the relationship between "aggregate" and "whole"? Leibniz writes that "an infinite aggregate is [not] one whole" (G. II. 304; Russell 110). So at least when the infinite is involved we may also say that "aggregate" ≠ "whole", as long as "aggregate" is understood as "infinite aggregate". For to be one whole is to be given *all at once*—i.e. to be categorematic. But as I argued above, Leibniz espouses a syncategorematic conception of the infinite, and thus in an infinite aggregate the parts never combine to form a true whole (or "one whole").

To sum up, we have the following: a body is clearly identified to be an aggregate. Since an aggregate is not a unity (save a *unum per accidens*), a body is not a unity. Furthermore, since an infinite aggregate is clearly not a whole, and a body is an infinite aggregate (based on the doctrine of actually infinite division), a body is explicitly not a whole. Finally, if a body is not a whole, then it is not the case that infinitesimals combine to form a whole. And if infinitely many infinitely small parts are not combining to form a whole, then the problem of the composition of the continuum does not apply. Thus, Leibniz is able to escape the labyrinth.

1.6 CONCLUSION

I would like to conclude with a final word about the diminishing pennies versus the divided block as possible ontological models for the infinite converging numerical series. For it could be objected that my argument in the above section still relies on an ontological interpretation of a mathematical model to which Leibniz is not entitled. That is, the notion of a distributive whole, or even a mechanical aggregation (*unum per accidens*) still rests on the fact that each term of the infinite series is finite and indivisible. In other words, I need the series to do the ontological work that the diminishing pennies model does. I do not believe, however, that an objection along these lines can be successful.

Suppose that we think of the number 1 as a body and we think of the actually infinite divisions as the series 1/2 + 1/4 + 1/8 + 1/16 + . . . and so on.[23] My claim is that the terms of the series combine into an infinite aggregate, which is not a whole and not a unity. Therefore, if 1 (insofar as it represents a body) is a unity, it is not because the terms

[23] It must be remembered that the mathematical series is only an analogy. For insofar as each term is represented by a number, it appears to be a unity. But this is not Leibniz's position. Each term should be thought of as an infinite aggregate, not a whole and not a unity.

combine in such a way as to make it so; rather it is because of some non-material principle of unity. But Leibniz claims that *every part* of a body is a body, and is thus actually infinitely divided itself. So we must also think of 1/2 as a body and 1/4 as a body and so on, which are in turn infinite aggregates as well. For if a body *qua* body is not a whole (as is the case even in the diminishing pennies model), then the parts are not wholes either. Thus, the parts need the same ontological model as the original body. In other words, the diminishing pennies model must be "multiplied to infinity". But we are able to deal with this. For 1/2 may be divided as follows: 1/4 + 1/8 + 1/16 + . . . and so on. What we get in the end is a system of parallel convergent series:

1 = 1/2 + 1/4 + 1/16 + 1/32 + 1/64 + 1/128 + 1/256 + 1/512 + 1/1024 + . . .

1/2 = 1/4 + 1/16 + 1/32 + 1/64 + 1/128 + 1/256 + 1/512 + 1/1024 + . . .

1/4 = 1/16 + 1/32 + 1/64 + 1/128 + 1/256 + 1/512 + 1/1024 + . . .

1/16 = 1/32 + 1/64 + 1/128 + 1/256 + 1/512 + 1/1024 + . . .

1/32 = 1/64 + 1/128 + 1/256 + 1/512 + 1/1024 + . . .

1/64 = 1/128 + 1/256 + 1/512 + 1/1024 + . . .

1/128 = 1/256 + 1/512 + 1/1024 + . . .

1/256 = 1/512 + 1/1024 + . . .

1/512 = 1/1024 + . . .

1/1024 = . . .

If the series applies in the first instance, as Levey and Brown believe that it does in the diminishing pennies model, I believe that it works in all other instances as well. But the extension of the diminishing pennies model that I have presented above is manifestly not equivalent to the divided block model. To jump from the diminishing pennies model to the divided block model simply because Leibniz says that every part (or body) is actually infinitely divided is a mistake. For if Levey's analysis is granted, the divided block leads to infinitesimals, a conclusion which Leibniz was consciously trying to avoid as he explicitly acknowledged it to be unacceptable. By simply extending the diminishing pennies model to infinity, as Leibniz extends the folded tunic model, we can reach a much more accurate picture of what Leibniz had in mind, *without being committed to infinitesimals*. And since, in the context of the present objection, Leibniz is committed to a categorematic infinite (and by extension infinite number) if he is committed to infinitesimals, the inference does not go through.

Based on this, then, I believe it is possible and even quite *plausible* to remove the apparent contradictions from Leibniz's view while remaining within the textual evidence. Thus, the doctrine of actually infinite division, far from being inconsistent, is actually quite coherent. Having thus established the internal consistency of Leibniz's position, I will now proceed to an exposition of Georg Cantor's position on the infinite. My focus will be conceptual; that is, I will investigate the philosophical arguments that Cantor proposes to establish that a categorematic infinite is coherent. As we will see, Cantor is operating with the presupposition that "potential" is synonymous with "syncategorematic" and that "actual" is synonymous with "categorematic". This oversight will play a central role in the evaluation of his argument.

CHAPTER 2: CANTOR AND THE INFINITE

2.1 INTRODUCTION

In the previous chapter I argued that Leibniz's position on the infinite is in fact coherent. In particular I argued for the following three claims: Leibniz describes the infinite as actual; he maintains that this actuality is to be syncategorematically understood; and finally, the combination of these views is not contradictory. The third claim is by no means obvious and various commentators have argued the contrary. Both Brown and Levey have taken this position. I believe, however, that their arguments do not satisfactorily take into account the distinction between actual and categorematic or the distinction between potential and syncategorematic. While these terms are generally treated as synonymous, they are not synonyms and treating them as such leads to the rejection of a perfectly coherent position on the infinite: i.e. that the infinite is actual, yet syncategorematically understood.

In the present chapter, I will move from a discussion of Leibniz to a discussion of Georg Cantor. Like chapter one, this chapter will be mainly expository. I will present the basic components of Cantor's theory of transfinite numbers. In particular, I will focus on Cantor's philosophical justification for his mathematical conclusions. Thus, I am not interested in an analysis of the mathematics involved, except perhaps indirectly. My goal is to present a thorough account of Cantor's philosophical arguments as well as the principles behind his mathematical ones. Mirroring chapter one, in this chapter I plan to establish the following three claims: Cantor describes the infinite as actual; he maintains that this actual infinite is to be categorematically understood; and finally, he takes "actual" and "categorematic" to be synonymous. Once these arguments have been presented, chapter three will begin what I call a Leibnizian response to Cantor's view. That is to say, I will apply the conclusions of chapter one to the content of chapter two.

2.2 The Formalism of Transfinite Mathematics

In metaphysical discourse, the part-whole axiom enjoyed the status of an intuitively obvious principle for some time. As discussed in chapter one, the likes of Galileo and Leibniz would not give it up in the face of clear reductio arguments. Instead they chose to give up other statements. Galileo concluded that relations such as "larger than", "equal to" and so on simply do not apply when dealing with the infinite.[24] Leibniz concluded, as we saw, that the notion of an infinite totality, i.e. an infinite number, is incoherent. And these views persisted for some time. It was not until Bolzano and Dedekind decided that what seemed like a contradiction, i.e. that a part is equal to the whole, could actually be taken as the defining characteristic of an infinite set that the status of the part-whole axiom was reduced to that of a postulate.[25] This move opened the door for Cantor, who proceeded to construct a comprehensive system of transfinite arithmetic. Ultimately, Cantor was motivated by his conviction that any set has a determinate power or cardinality; in his mind it was obvious that this would apply to

[24] The passage from *Two New Sciences* runs as follows: "And in final conclusion, the attributes of equal, greater, and less have no place in infinities, but only in bounded quantities" (EN 79; TLC 356).
[25] See Bolzano (1851), §20.

infinite sets in the same way it does to finite ones (Hallett 32 ff.).[26] But there is a problem here that needs to be overcome, for in the case of finite sets Cantor supposed (not without some reasonableness) that the power or cardinality is a natural number. For example, the power of the set {4, 657, 9} is clearly "3". But what is the power of the set {1, 2, 3, …, n, …}? It is by no means clear (Hallett 2). Cantor's initial account of transfinite numbers attempts to provide an answer to this question in terms of what he calls "transfinite ordinals".

2.2.1 Generation Rules, Number Classes, and Transfinite Arithmetic

Transfinite ordinal numbers come to life by means of two generation principles. However, as Hallett notes, the principles actually only postulate the existence of such numbers; the term "generation principle" seems to imbue these statements with a power they do not possess. Cantor first presents these principles in his essay *Ueber unendliche lineare Punktmannigfaltigkeiten* published in 1883. Here are the principles as stated by Hallett:

> (1) if α is an ordinal number (whether finite or transfinite) then there is a new ordinal number $\alpha+1$ which is the immediate successor of α;
>
> (2) given any unending sequence of increasing ordinal numbers there is a new ordinal number following them all as their 'limit' (that is to say, no ordinal number smaller than this limit can be strictly greater than all ordinals in the given sequence). (49)

[26] This position is referred to as Cantor's finitism. Certain commentators have seen Cantor as extending the definition of finite rather than attributing finite characteristics to the infinite. See, for example, Mayberry (2000).

Two important observations about these principles must be made. First of all, it is the second of these two principles that is responsible for "generating" the first transfinite ordinal number. The first taken on its own only gives rise to an indefinite series of finite ordinals; something else is needed to give rise to the transfinite. The crucial difference between the role of each principle can be illuminated by appealing to the quantifier shift fallacy discussed in chapter one.[27] In logical symbolism, if the domain under consideration is the ordinal numbers and "Sxy" represents "x is the successor of y", the first principle can be captured by "$(\forall x)(\exists y)(Sxy \,\&\, x \neq y)$"—i.e. for all ordinal numbers there exists at least one ordinal that is its successor. By shifting the quantifiers we arrive at the second principle: "$(\exists x)(\forall y)(x \neq y \rightarrow Sxy)$". That is, there exists an ordinal number that is the successor of all other ordinal numbers within a given sequence. As we saw in chapter one, it is the second formula that captures the notion of a categorematic infinite. The first transfinite ordinal generated by this principle is denoted by "ω". Based on the generating principles, the ordinals do not come to an end here. Every ordinal has a successor, which means that ω has a successor as well. Moore (2001) gives the following representation of the hierarchical series of ordinals to which these two principles give rise:

$$
\begin{array}{l}
0, 1, 2, \ldots \\
\omega, \omega + 1, \omega + 2, \omega + 3, \ldots \\
\omega \times 2, (\omega \times 2) + 1, (\omega \times 2) + 2, (\omega \times 2) + 3, \ldots \\
\omega \times 3, (\omega \times 3) + 1, \ldots \\
\omega \times 4, \ldots \\
\omega \times 5, \ldots \\
.
\end{array}
$$

[27] See chapter one.

$\omega^2, \omega^2 + 1, \omega^2 + 2, \ldots$
$\omega^2 + \omega, \omega^2 + \omega + 1, \omega^2 + \omega + 2, \ldots$
$\omega^2 + (\omega \times 2), \ldots$
.
$\omega^2 \times 2,$
.
$\omega^2 \times 3, \ldots, \omega^2 \times 4, \ldots,$
$\omega^3, \ldots$
$\omega^4, \ldots$
$\omega^5, \ldots$
.
$\omega^\omega,$
. (A. W. Moore 127)

It is clear that this can be continued indefinitely and that there is apparently no limit to the application of the generation principles. Since the only thing that the principles can generate is more ordinal numbers, it is certain that they can always be applied.[28]

The second important observation is that the second principle is an extension of the concept of a limit. The standard definition of a limit is what is called the ε-δ definition. It runs as follows:

> Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the limit of f(x) as x approaches a is L, and we write lim [x → a] f(x) = L if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that $| f(x) - L | < \varepsilon$ whenever $0 < | x - a | < \delta$. (Stewart 115)

The crucial aspect of this definition is that for any value one chooses, there is always one *closer* to the limit value within the open interval. That is to say, as x approaches a, L – f(x) approaches 0. In the case of ω, however, each successive finite ordinal is no closer to ω than the previous one. That is, as x increases to infinity ω – f(x) does not decrease. This means that although in an intuitive sense ω may look like the limit of the series of natural

[28] In chapter three it will be shown that these principles give rise to certain paradoxes.

numbers, in a technical sense, the concept of a limit has been extended. Cantor is no longer using "limit" in a univocal way. He was certainly aware of this, however. So much so in fact that he felt the need to clarify his use of the term: "he added that by this he meant only to emphasize the character of ω taken as the first whole number following next after *all* the numbers n ∈ N. The idea of ω as limit served to satisfy its role as an ordinal, the smallest integer larger than any integer n ∈ N" (Dauben 98). So the aspect of the concept of limit that Cantor wanted to stress was the idea that the limit follows every member of, say, a given series. Thus, Cantor's infinite is not categorematic in the sense that there is an infinitieth term in the series of natural numbers within the series itself. In this respect his position coincides with Leibniz's comment noted in chapter one that number is not the sort of thing to have a greatest of its kind. But unlike Leibniz, Cantor holds that for any infinite series of numbers, convergent (akin to the limit of a function) and divergent (akin to the series of natural numbers) alike, there will be a number standing outside of the series that is the next greatest number than all the numbers in the series. Cantor's contention is that every series has such a "limiting" ordinal.

The second step in the explication of the transfinite is the idea of a number class. This idea is important because it begins to connect the theory of ordinals with the question posed at the beginning of this section; i.e. how can the notion of power (or cardinality) be understood in the case of infinite sets? Before this question can be answered, some details need to be presented. In what could be called the first application of the second generation principle ω results; this application also gives rise to the first class of transfinite ordinals (I). Further applications of the first principle give rise to the series ω, $\omega + 1$, $\omega + 2$, … All of these ordinals are still within the first number class; i.e. the first number class is closed under transfinite addition. Once the second principle is applied to this series, however, the result is a new class of transfinite ordinals, the second class (II). This process can be continued indefinitely giving rise to more and more classes of ordinals—i.e. ordinals that cannot be reached from the series "below" them by successive addition. Cantor needed a way to distinguish in a definite way each number class from those below it and those above it. What he came up with was a third principle, which he called the principle of limitation:

> *Definition:* We define therefore the second number class (II) as the collection of all numbers α (increasing in definite succession) which can be formed by means of the two principles of generation:
>
> $$\omega, \omega + 1, \ldots, \nu_0\, \omega^{\mu} + \nu_1\, \omega^{\mu-1} + \ldots + \nu_{\mu}, \ldots, \omega^{\omega}, \ldots, \alpha, \ldots,$$
>
> which are subject to the condition that all numbers preceding α (from 1 on) constitute a set of power equivalent to the first number class (I). (Cantor 1932, 197; Dauben 98)[29]

The importance of this principle is that it gave Cantor a way to compare the size of different transfinite number classes, which is a large step towards understanding the notion of power as it applies to infinite sets. Instead of just having an indeterminate mess of transfinite ordinals, Cantor now had a completely determinate succession of numbers whose comparative sizes could be considered in detail (Dauben 99). Questions such as is (I) > (II)? Is (I) = (II)? and so on now had some sense.

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[29] In the original: "Wir definieren daher die zwiete Zahlenklasse (II) als den Inbegriff aller mit Hilfe der beiden Erzeugungsprinzipe bildbaren, in bestimmter Sukzession fortschrietenden Zahlen α
$\omega, \omega + 1, \ldots, \nu_0\, \omega^{\mu} + \nu_1\, \omega^{\mu-1} + \ldots + \nu_{\mu}, \ldots, \omega^{\omega}, \ldots, \alpha, \ldots,$
welche der Bedingung unterworfen sind, dass alle der Zahl α voraufgehenden Zahlen, von 1 an, eine Menge von der Maechtigkeit der Zahlenklasse (I) bilden."

At this point I can begin explicitly to tie Cantor's theory of ordinals to the notion of power. An ordinal number, in a figurative or intuitive sense, measures a well-ordering of a given set. Moore (2001) defines "well-ordering" as follows:

> A well-ordering of a set X (finite or infinite) is an imposition of order on the members of X satisfying the following three conditions: it singles out one of the members of X as the first, unless, of course, X has no members (that is, unless X is the empty set); it singles out another member of X as the second, unless X has only one member; it singles out another as the third, unless X has only two members; and quite generally, it singles out, for each member of X that has already been singled out, another as its immediate successor, unless there are none left; more generally still, it singles out, for each set of members of X that have already been singled out (finite or infinite), a first to succeed them all, again unless there are none left. (123)

A well-ordering, then, is a relation, which imposes an order on a given set. For example, a well-ordering of the set N of natural numbers would be the relation "is the immediate successor of". Sets whose well-orderings have the same "length" or "shape", then, have the same ordinal number. Some examples of well-ordered sets along with their ordinal number may help to elucidate this definition.

Set	Ordinal Number
$\{0, 1, 2, \ldots\}$	ω
$\{2, 4, 6, \ldots\}$	ω
$\{1, 2, 3, \ldots, 0\}$	$\omega + 1$
$\{3, 5, 7, \ldots, 1\}$	$\omega + 1$
$\{1, 2, 4, 5, \ldots, 0, 3\}$	$\omega + 2$
$\{0, 2, 4, \ldots, 1, 3, 5, \ldots\}$	$\omega \times 2$
$\{2, 4, 6, \ldots, 1, 3, 5, \ldots, 0\}$	$(\omega \times 2) + 1$

{2, 4, 6, …, 3, 5, 7, …, 0, 1} $(\omega \times 2) + 2$. (A. W. Moore 124-126)

It is obvious that the sets with different ordinal numbers have different "shapes" in an intuitive sense, but how does this tell us anything about their size? There is a clear way in which the ordinal number of a given set is tied to its power or cardinality, which is further tied to the specification of number classes given above.

Cantor was convinced that just as in the case of finite sets, infinite sets must have a determinate cardinality. His demarcation of the number classes of transfinite ordinals supplied him with a straightforward way to identify those cardinalities. It is relatively simple: every set that has an ordinal within a given number class has the same cardinality. That is, the elements of a set with a given ordinal number can be placed in a one-to-one correspondence with the elements of a set whose ordinal number is the same number class. Thus, just as there is an increasing hierarchy of transfinite ordinal numbers, there is also an increasing hierarchy of transfinite cardinal numbers—one for each ordinal number class. Cantor chose the first letter of the Hebrew alphabet, $\aleph$, to represent the transfinite cardinals. $\aleph_0$ is the first transfinite cardinal and represents the cardinality of sets whose ordinal number falls within the first number class—for example, {0, 1, 2, …} or {2, 4, 6, …}. The series of cardinals continues as follows: $\aleph_1$, $\aleph_2$, $\aleph_3$, …, $\aleph_\omega$, … $\aleph_\nu$, … As in the case of the ordinals, there is no end to this series.[30]

Cantor also provided rules of arithmetic for his transfinite numbers, some of which have been tacitly utilized in the foregoing exposition. I will focus here on two

[30] That the series of cardinals has no end is stated explicitly by Cantor in his letter to Dedekind (1899). In this letter (among other things) Cantor introduces a problem that would later be known as Cantor's paradox. This will be discussed in chapter three.

simple examples: addition and multiplication.[31] The addition of transfinite numbers is defined in terms of well-ordered sets such that the numbers being added represent the numbering of the respective sets. Addition is carried out, then, by constructing a set made up of the sequence of elements from one set followed by the sequence of elements from another set. As a result, addition of transfinite numbers is not always commutative. For example, $1 + \omega = 1, 1, 2, 3, \ldots \neq 1, 2, 3, \ldots, 1 = \omega + 1$ (Dauben 104). For while the sequence corresponding to $1 + \omega$ has the same "shape" as the sequence of natural numbers (given the specified ordering), the sequence corresponding to $\omega + 1$ does not:

$$\mathbf{N} = 1, 2, 3, 4, \ldots$$

$$1 + \omega = 1, 1, 2, 3, \ldots$$

$$\omega + 1 = 1, 2, 3, 4, \ldots, 1$$

It is clear that the "shape" of $\mathbf{N}$ differs from $\omega + 1$ but not from $1 + \omega$. Despite the differing ordinal numbers, however, the cardinality of the sets formed by these sequences is the same. Ultimately it is one-to-one correspondence that serves as the measure of cardinality and thus as the foundation for transfinite arithmetic.

Multiplication was also defined in terms of well-ordered sets: "Given a well-ordered set of numbering β, by replacing each of its elements by elements of numbering α,...if $\beta = \omega$, $\alpha = 2$, then: $\beta \cdot \alpha = \omega$. On the other hand: $\alpha \cdot \beta = 2\omega$" (Dauben 104). Thus, multiplication fails to be commutative as well. Many more operations on transfinite numbers were defined, but these two examples are sufficient to illustrate that Cantor worked out a comprehensive and self-consistent system of transfinite numbers. The

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[31] These operations are described in "Contributions to the founding of the theory of transfinite numbers" §3 and §8 (Cantor 1955, 91-94 and 119-122).

question that remains is whether or not this is enough to guarantee the existence of such numbers. Before I address this question, however, I would like to consider the extension of the idea of number that has occurred here. That is to say, it does not seems as though the transfinite numbers can be numbers in the same sense of "number" as, for example, the natural numbers.

2.2.2 Cantor's Idea of Number

Aristotle provides us with objections to the existence of transfinite numbers. Although he is clearly not commenting directly on Cantor's conception of transfinite numbers, his objection is so fundamental that it must be considered. In the *Physics* he writes, "nor, for that matter, can there be a separated infinite number: for number, or what has number, is countable, and so, if it is possible to count what is countable, it would then be possible to traverse the infinite" (III.5 204b7-9). An obvious example of a "separated infinite number" would be a Cantorian transfinite number. The reason Aristotle makes such an objection is that he conceives of number as something that is by definition countable. As a consequence, an infinite number could not exist, for the counting of such a thing would continue indefinitely. To count to infinity, so to speak, would not be possible, for one cannot move through an infinite number of steps in a finite time (i.e. "traverse the infinite"). Therefore, Aristotle concludes that such numbers cannot exist; or perhaps better, they cannot be numbers at all.

Cantor's reply, however, is that the numbers ω and $\aleph_0$ are different kinds of numbers than the finite numbers (Rioux 116). Aristotle is assuming that "number" is a species, and therefore, that a given number must have the same characteristics as any other. However, Cantor is making a more fundamentally conceptual move than a simple introduction of a quantitatively larger number (in Aristotle's sense of "number"). Cantor is, essentially, making the claim that number is not a species but a genus (Rioux 166). Thus, the transfinite numbers are an entirely separate species of number than the finite ones, both of which reside within a larger genus. Cantor provides a reason for this difference of species; it comes from the way in which ω is obtained. It is discovered by thinking of "the limit to which numbers ν [i.e. the series of whole numbers] approach, ω is the first integer which succeeds all numbers ν, that is, it is to be regarded as greater than every one of the numbers ν" (Cantor 1932, 195; Rioux 104).[32] It is not obtained by the successive addition of finite numbers, but by an entirely conceptual method. Thus, by definition, it is not of the same kind as any finite number. To put it differently, ω is not the infinitieth term in the sequence of natural numbers; rather it stands outside of the series. Since Aristotle's objection is based on the supposition that "number" is a single species, it is not able to hold in light of Cantor's claim.

Interestingly, many of the "numbers" that make up the common currency of mathematics are anomalous, if, that is, the term "number" must be applied univocally to each and every case.[33] Take, for example, the numbers 0 and 1. 1 was thought by the Greeks to be the unit out of which numbers are constructed and not a number itself. 0 was introduced as a place holder for the absence of a number and, again, not a number itself.

[32] In the original: "welcher die Zahlen ν zustreben, wenn darunter nichts anderes verstanden wird, als dass ω die *erste* ganze Zahl sein soll, welche auf alle Zahlen ν folgt, d. h. grosser zu nennen ist als jede der Zahlen ν."
[33] See Benardete (1964).

The same claims can be made regarding the negative integers, imaginary numbers, and so on. Thus, if an objection to Cantor's transfinite numbers is to be made on this basis, an account must be provided as to why the status of, for example, the irrational numbers or the negative integers is any different. And in fact, Cantor exploits just such a consideration in order to demonstrate that his transfinite numbers are actually coherent.

2.2.3 "Free" Mathematics and Ontology

At this point I would like to return to the question of what exactly follows from the fact that Cantor has created a comprehensive and self-consistent theory of transfinite numbers. Or more precisely, I would like to investigate Cantor's view about what follows from this.

Cantor believes that mathematics is inherently free. By this he means that the only limitation on mathematical advances is contradiction: if a purported theory leads to a contradiction then it is not an accurate theory and needs to be corrected. However, if no contradiction can be found, then the theory is actually self-justifying. This in itself is not peculiar. It is a common practice in mathematics to judge the merits of a supposition by the results to which it gives rise. And if this alone were Cantor's position, then he may or may not be susceptible to criticism on this basis. But Cantor makes a further inference beyond the mere technical viability of the mathematical structure whose consistency he believes to have demonstrated. He believes that ontological conclusions follow from pure logical or formal consistency (Hallett 16). This is the peculiar nature of Cantor's methodology, his Platonism.

From this it follows that the pure coherence of transfinite mathematics is sufficient to guarantee the existence of its content. Nevertheless, Cantor does provide philosophical arguments in justification of his mathematics. Whether he thought these arguments necessary because his ontological conclusions move beyond what is justifiably concluded based on arguments from coherence or whether he provided them for those of us simply unconvinced by his argument from coherence is another matter altogether. In either case, I would like to consider these philosophical arguments in some detail. First, however, I would like to present Cantor's philosophy of the infinite in general, beginning with what he thought the actual/potential distinction amounts to.

2.3 Cantor's Philosophy of the Infinite

Despite Cantor's position that a coherent mathematical system is somehow self-justifying, he did not therefore shun philosophical inquiry into the content of his mathematical theories. In fact, Cantor seems to have believed that the coherence of a mathematical system is determined through philosophical investigation. That is, he seems to have believed that pure logical consistency is not sufficient; one also needs to provide positive arguments for the coherence of the concepts involved. At least, this is one way of understanding Cantor's motivation for providing the arguments that will occupy us throughout this section.

It seems clear that Cantor was not only a historian of mathematics, but of philosophy as well. Cantor's philosophy of the infinite obviously originates with the Aristotelian distinction between actual and potential infinities. But it does not end there.

Cantor seems to be aware of developments in metaphysics from the medieval period right through to the time of Leibniz and further on into his own century. Cantor believed that many arguments against the actual infinite in fact beg the question. That is, the arguments attempt to attribute properties of finite numbers to infinite numbers; when the application leads to contradiction it is concluded that the actual infinite is nonsensical. Take, for example, Galileo's argument from chapter one. The reductio leads Leibniz to the conclusion that infinite number is problematic; this is because Leibniz believes it more plausible to maintain the part-whole axiom in the face of this contradiction. Cantor, however, believes that while the part-whole axiom may be fine when finite quantities are involved, to assume that it must be maintained when dealing with infinite quantities is a mistake. Thus, while Cantor fits his theory into the traditional jargon, he very much alters what must be accepted or rejected based on the traditional distinctions. I will now move on to a consideration of how Cantor understood the traditional distinctions within the philosophy of the infinite.

2.3.1 The Actual and Potential, the Categorematic and Syncategorematic

For Cantor, the potential infinite is more accurately described as a variable finite (Jané 378). "Potentially infinite" essentially means that for any given value, a larger value can be conceived. This larger value, in turn, will be nothing but another finite value, with respect to which a larger value can again be conceived. Thus, inherent within Cantor's conception of the potential infinite is the notion of incompleteness. For example, if one only had the first generating rule at one's disposal, it could be applied successively without approaching a definite totality. As Cantor puts it, "a variable magnitude x successively taking the different finite whole number values 1, 2, 3, ..., v, ... represents a potential infinite" (Cantor 1932, 409; Jané 379).[34] Mathematical induction, then, would be an example of a process that would lead to a potential infinity: each step along the way is finite but the operation that generated it can be repeated an indefinite amount of times.

This conception is in fact identical to the Aristotelian conception of the potential infinite. As Aristotle says, "in general, the infinite exists in this way: by one thing's always being taken after another—each thing taken is finite, but it is always one followed by another" (Physics III.6 206a27-b). Thus, Aristotle asserts (along with Cantor) that any step along the way of a potential infinity is itself finite. The only thing that makes it infinite is the fact that another step can always be taken. It is what could be described as a successive actualization. All of the parts will never be actualized at once, and therefore, the infinity can only be potential, not actual.

Cantor approaches an explanation of the actual infinite through the notion of a set. As a continuation of the quotation above, Cantor says: "the set (v) of all whole finite numbers, conceptually determined in full by a conceptual law, offers the simplest example of an actual infinite quantum" (Cantor 1932, 409; Jané 379).[35] Thus, an actual infinity is an infinite multiplicity of things, which comprise a totality or a completed whole. Even though there is no largest whole number, it is nonetheless possible to

[34] In the original: "So stellt uns beispielsweise eine veraenderliche Groesse x, die nacheinander die verschiedenen endlichen ganzen Zahlwerte 1, 2, 3, ..., v, ... anzunehmen hat, ein petentiales Unendliches vor..."

[35] In the original, continuing from the above footnote: "...wogegen die durch ein Gesetz begrifflich durchaus bestimmte Menge (v) aller ganzen endlichen Zahlen v das einfachste Beispiel eines actual-unendlichen Quantums darbietet."

conceive of the set of whole numbers as something limited (although infinite). An infinite set, therefore, is an actual infinity. It is something determinate, bounded, and complete, and in this way actual, yet it is made up of infinitely many members, and in this way infinite.

As mentioned in chapter one, the distinction between categorematic and syncategorematic originates with Peter of Spain, although the exposition I have utilized is from William of Ockham. As previously stated, the distinction is this:

> Categorematic terms have a definite and fixed signification, as for instance the word 'man' (since it signifies all men) and the word 'animal' (since it signifies all animals), and the word 'whiteness' (since it signifies all occurrences of whiteness). Syncategorematic terms, on the other hand, as 'every', 'none', 'some', 'whole', 'besides', 'only', 'in so far as', and the like, do not have a fixed and definite meaning, nor do they signify things distinct from the things signified by categorematic terms. (51)

Although this distinction seems to be analogous to the distinction between actual and potential, it is not (as I spent a great deal of time arguing in chapter one). I will not repeat those arguments here; rather I will consider the way in which Cantor seems to have understood this distinction. One thing is worth noting, however: the syncategorematic seems to encompass the potential infinite. In other words, if someone holds that the infinite is potential, then it follows that it is syncategorematic. The difference comes in when the converse is considered. Simply because the infinite is described as syncategorematic, one cannot infer that it is potential (as chapter one serves to demonstrate). It is questionable whether or not Cantor realized this.

It is clear that he found the distinction between categorematic and syncategorematic to be both important and useful. In fact, it seems as though he may have found it preferable, in some sense, to the actual/potential distinction made by Aristotle: among Cantor's immediate predecessors, Bolzano utilized this distinction in a way that impressed Cantor (Dauben 124). With respect to this distinction, however, Cantor does not seem to realize that syncategorematic and potential are not synonyms. He seems to have taken syncategorematic as a more precise (or perhaps more accurate) way of describing what had traditionally been described as a potential infinite. In the *Grundlagen*, Cantor writes: "This infinite (called by some scholastics the 'syncategorematic infinite') is a mere helping-concept, a relation-concept of our thought; in its definition it includes variability, and so 'datur' can never be said of it in the proper sense" (Cantor 1932, 180).[36] But to think of the syncategorematic infinite merely as a refinement of the potential infinite is to overlook something rather important. This in itself is not problematic, for as my argument in chapter one indicates, the fact that the terms "potential" and "syncategorematic" are not synonymous has not been widely acknowledged. The problem is that, based on Cantor's arguments for the coherence of the transfinite, i.e. the coherence of the actual infinite, this distinction, or at least his failure to identify it, adversely effects the conclusions that he believes he can draw. With this in mind, I will now move on to a discussion of these arguments themselves.

[36] In the original: "Dieses Unendliche (von einigen Scholastikern "synkategorematisches Unendliches" genannt) ist ein blosser Hilfs- und Beziehungsbegriff unseres Denken, welcher seiner Definition nach die Veraenderlichkeit einschliesst und von dem somit das "datur" niemals im eigentlichen Sinne ausgesagt warden kann."

2.3.2 Three Philosophical Arguments

It is important to note that the following arguments are not intended to establish the same conclusion, although the conclusions of each one are intended to support the position that the transfinite is coherent, more precisely that the actual/categorematic[37] infinite is coherent. As a result, an objection to, or refutation of, one is not a refutation of them all. This must be kept in mind. Each argument is presented as the basis for the acceptance of the actual/categorematic infinite, but there is a slightly different intermediary goal in mind in each case. There are three arguments that I will discuss here: the argument from irrationals, the divine intellect argument, and the domain argument. My aim here is not to accept or refute these arguments, merely to present them. They will serve as the basis for the presentation in chapter three of a possible Leibnizian response to Cantor.

One of the reasons that Cantor felt justified in extending the concept of number in the way indicated above[38] is that he felt the basis for such numbers was already presupposed by more mundane and uncontroversial types of numbers. In particular, Cantor believed that the irrational numbers provided the basis for his transfinite numbers. Essentially, he argued that there is a slippery slope. If one accepts the irrational numbers as coherent, one must also accept the transfinite numbers. If one denies the transfinite numbers one must also deny the irrationals, something no serious mathematician is likely to do. In Cantor's words:

[37] Since, as I have argued, Cantor took "actual" to be synonymous with "categorematic" and "potential" to be synonymous with "syncategorematic", when referring to Cantor's view I will use the compound terms "actual/categorematic" and "potential/syncategorematic" respectively to indicate the applicability of both terms.
[38] See section 2.2.2.

> The transfinite numbers themselves are in a certain sense *new irrationals*, and in fact I think the best way to define the *finite* irrational numbers is entirely similar; I might even say in principle it is the same as my method…for introducing transfinite numbers. One can absolutely assert: the transfinite numbers *stand* or *fall* with the finite irrational numbers; they are alike in their most intrinsic nature [*inneresten Wesen*]; for the former like these latter (numbers) are definitely, delineated [*abgegrenzte*] forms or modifications…of the actual infinite. (Cantor 1932, 395-396; Dauben 128)[39]

As Dauben goes on to say, the reason that Cantor believed that the transfinite numbers followed so directly from the existence of the irrationals is that "to define the irrational numbers, infinite collections of rational numbers had been required" (128). Thus, when the curtain is pulled back, all that this argument amounts to is the claim that transfinite numbers follow directly from the existence of actual/categorematic infinite sets. The irrationals just happen to give us an example of infinite sets being used in mathematics. Beyond this, there is no significant similarity between the irrationals and the transfinites.[40] This is not so much a criticism of Cantor's argument as an identification of its actual content. I mention it because it will be important in further sections to understand exactly what this argument establishes.

In fact, this argument does not establish very much. It does not establish the connection between infinite sets and transfinite numbers, nor does it establish the legitimacy of infinite sets; it only shows that their legitimacy is assumed in areas of

[39] In the original: "Die transfiniten Zahlen sind in gewissem Sinne selbst *neue Irrationalitaeten* und in der Tat ist die in meinen Augen beste Methode, die *endlichen* Irrationalzahlen zu definieren, ganz aehnlich, ja ich moechte sogar sagen im Prinzip dieselbe wie meine oben beschriebene Methode der Einfuehrung transfiniter Zahlen. Man kann ungedingt sagen: die transfiniten Zahlen *stehen oder fallen* mit den endlichen Irrationalzahlen; sie gleichen einander ihrem innersten Wesen nach; denn jene wie diese sind bestimmt abgegrenzte Gestaltungen oder Modifikationen…des aktualen Unendlichen."

[40] I suppose one could argue that a similarity between them still exists on the grounds that both involve an extension of the number concept beyond the original notion of counting numbers. But one extension does not justify any extension one pleases and so another argument would nonetheless be required to establish the legitimacy of extending the number concept to include transfinites.

mathematics other than transfinite arithmetic. What it does establish is merely the following hypothetical syllogism:

(P1) If one accepts irrational numbers (as defined by Dedekind), one accepts infinite sets.

(P2) If one accepts infinite sets, one accepts transfinite numbers.

(C) If one accepts irrational numbers (as defined by Dedekind), one accepts transfinite numbers.[41]

By way of this formalization of Cantor's argument, it is clear that by a simple Modus Tollens argument, the rejection of transfinite numbers leads one to a rejection of irrational numbers. The component of this argument most susceptible to refutation is P1. P2 is fairly innocuous; as we will see in the final chapter, even Leibniz would grant this conditional. But one man's Modus Ponens is another man's Modus Tollens: this conditional itself does not do the work Cantor needs it to do, not, that is, without P1. Thus, what is required is a justification of P1.

The obvious response to this analysis is that P1 has already been justified. Through the use of the actual infinite, Dedekind has finally set irrational numbers on a firm foundation. If P1 is rejected, then the irrationals need to be grounded in some other way. This is clearly not a welcome task. There are two things that I would say in response to such a challenge. In the first place, this is not an objection to my analysis of Cantor's argument; rather it is an objection to the rejection of infinite sets. Such an objection is based purely on pragmatic considerations. It has nothing to do with the concept of an

[41] In fact, from Cantor's comments above, the conclusion he is after is actually the following biconditional statement: One accepts irrational numbers if and only if one accepts transfinite numbers. I have elected to focus on Cantor's argument for this direction of implication, since it is more relevant to the present discussion.

actual/categorematic infinite set; it merely points out the fruitfulness of their acceptance. I do not disagree with this claim: it seems clear that the acceptance of actual/categorematic infinite sets is useful, but this does not provide sufficient justification for their coherence and by extension for their use in defining irrational numbers. Furthermore, and as I will discuss in the next chapter, the rejection of actual/categorematic infinite sets does not necessarily undermine Dedekind's characterization of irrational numbers. If this is the case, then P1 is left without any ground to stand on. And since Dedekind's characterization only legitimates Cantor's transfinites if actual/categorematic infinite sets are employed, if Dedekind's characterization can be modified, one need not accept Cantor's transfinites on the basis of their analogy to the irrationals.

This is not Cantor's only attempt to justify his transfinites, however. In the second argument that I will consider, the divine intellect argument, Cantor attempts to justify the acceptance of transfinite numbers without appealing to Dedekind's characterization of the irrationals; in fact, he does not appeal to any pragmatic considerations whatsoever. The basic starting point of this argument is Cantor's definition of set: "When…the totality of elements of a multiplicity can be thought without contradiction as 'being together', so that their collection into '*one* thing' is possible, I call it a…*set*" (Cantor 1932, 443; Hallett 34).[42] Alternatively, and more succinctly: "By a 'manifold' or 'set' I understand in general any many [*Viele*] which can be thought of as a one [*Eines*]" (Cantor 1932, 204;

[42] In the original: "Wenn hingegen die Gesamtheit der Elemente einer Vielheit ohne Widerspruch als "zusammenseiend" gedacht warden kann, so dass ihr Zusammengefasstwerden zu "*einem* Ding" moeglich ist, nenne ich sie eine…"Menge"."

Hallett 33).[43] The method by which the elements are unified into a single whole is somewhat vague, but it is nonetheless a relatively clear (if not technical) characterization of what a set is. In the case of sets with a finite number of elements it seems obvious that a unification of these elements by a human intellect is possible. That is, I can "hold together" the set of, say, even whole numbers less than ten—{2, 4, 6, 8}. But in the case of, say, all even whole numbers, there is something about the set that eludes completion—{2, 4, 6, 8, …}. I am forced to resort to "…" in order to diagrammatically unify this set and in terms of some more abstract or mental form of unification, I am at a loss. Since there is no last element of the set, I have nothing to occupy the final slot of my mental characterization of it in the way that "8" occupies the final slot of the finite set mentioned above. While this may be a counterargument to Cantor's claim that an infinite set can in fact be complete, i.e. be actual/categorematic, he has a response.

The most obvious case is that of the natural numbers themselves. Since there is always a number larger than the one chosen, it seems impossible to call the entire collection a "one" in the sense required for the natural numbers to be a set as defined by Cantor. He writes:

> Each individual *finite* cardinal number is in God's intellect both a representative idea and a unified form for the knowledge of innumerably many compound things, that is, those which possess the cardinal number in question. All *finite* cardinal numbers are thus distinct and simultaneously present in God's intellect. They form in their *totality a manifold, unified thing for itself* [*Ding fuer sich*], *delimited* from the remaining content of God's intellect, and this thing is itself again an object [*Gegenstand*] of God's knowledge. (Cantor *Nachlass* VI, 170; Hallett 36).

[43] In the original: "Unter einer "Mannigfaltigkeit" oder "Menge" verstehe ich naemlich allgemein jedes Viele, welches sich als Eines denken laesst…"

What Cantor takes this argument to establish is just that the natural numbers do in fact exist as a set (a "unified thing for itself") "and that it is God, not us, who has conceived the collection with the necessary unity" (Dauben 36). Since this collection exists as a unified whole in the divine intellect, it is not necessary that it exist as a unified whole for the finite, human intellect. The fact that God can "hold it together" guarantees its status as a set.

This argument certainly comes with its theological commitments. However, this is not what I am concerned with. I will not provide an objection to this argument on its own terms. Such an approach would no doubt take me much too far afield. I have included this argument for one reason only: it demonstrates that Cantor felt it necessary to argue for the view that an infinite collection can be a unified whole. In other words, he believed that something was needed to justify his assumption that an actual/categorematic infinite set is a coherent thing. Whether or not this particular argument is sufficient to establish this position is not important at present. What is important is that, even in Cantor's mind, some such argument is needed. If the argument that Cantor gives is judged to be unsatisfactory, it is unclear what argument could take its place. I will return to this point in the next chapter.

The final argument that I will consider, the domain argument, argues that the actual/categorematic infinite must be presupposed as a domain for the potential/syncategorematic infinite. In fact, Cantor believed that in order to utilize the potential/syncategorematic infinite, as one is free to do, the actual/categorematic infinite must be accepted. In Cantor's words,

> There is no doubt that we cannot do without variable quantities in the sense of the potential infinite; and from this the necessity of the actual infinite can also be proven, as follows: In order for there to be a variable quantity in some mathematical inquiry, the 'domain' of its variability must strictly speaking be known beforehand through a definition. However, this domain cannot itself be something variable, since otherwise each fixed support for the inquiry would collapse. Thus, this 'domain' is a definite, actually infinite set of values. Thus, each potential infinite, if it is rigorously applicable mathematically, presupposes an actual infinite. (Cantor 1932, 410-411; Jané 385)[44]

Cantor believes that the existence of a variable (or potential) infinite presupposes an actual infinite over which it varies. To show how much within the history Cantor remains with this argument, it is worth noting that a surprisingly similar claim appears in Aristotle's Physics:

> To be [infinite] so as to exceed every [definite quantity] by addition is not possible even potentially unless there is something which is actually infinite, accidentally, as the natural philosophers say that the body outside the world-system, of which the substance is air or some other such thing, is infinite. But if it is not possible for there to be a perceptible body which is actually infinite in this sense, it is manifest that there cannot be one even potentially infinite by addition. (Physics III.6 206b20-27)

In this passage Aristotle is considering bodies, and therefore he does not arrive at the same conclusion as Cantor. However, it is interesting that the relation between the actual and potential infinite held by each thinker is the same in this case. In an admittedly basic formalization, these can be seen as competing Modus Ponens and Modus Tollens arguments:

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[44] In the original: "Unterliegt es naemlich keinem Zweifel, dass wir die *veraenderlichen* Groessen im Sinne des petentialen unendlichen nicht missen koennen, so laesst sich daraus auch die Notwendigkeit des Aktual-Unendlichen folgendermassen beweisen: Damit eine solche veraenderliche Groesse in einer mathematischen Betrachtung verwertbar sei, muss strenggenommen das "Gebiet" ihrer Veraenderlichkeit durch eine Definition vorher bekannt sein; dieses "Gebiet" kann aber nicht selbst wieder etwas Veraenderliches sein, da sonst jede feste Unterlage der Betrachtung fehlen wuerde; also ist dieses "Gebiet" eine bestimmte aktual-unendliche Wertmenge. So setzt jedes potentiale Unendliche, soll es streng mathematisch verwendbar sein, ein Aktual-Unendliches voraus."

<u>Cantor</u>: 1) potential infinite → actual infinite
2) potential infinite
Thus, 3) actual infinite.

<u>Aristotle</u>: 1) potential infinite → actual infinite
2) ~ actual infinite
Thus, 3) ~ potential infinite

Both Aristotle and Cantor, therefore, believe that the potential infinite somehow depends on the actual infinite. It is important to note that in this argument Aristotle is only referring to the infinite by addition, not to the infinite by division. In the case of the infinite by division Aristotle credits the potential infinite without being committed (in his mind) to the actual infinite; simply because a line can be divided at any point does not mean that it is divided at every point (in fact this was the central consideration in Aristotle's response to Zeno's Dichotomy paradox). Perhaps, the differing conclusions drawn are indicative of the objects under consideration in each case. In any case, what is interesting here is that Aristotle clearly did not have at his disposal the distinction between categorematic and syncategorematic infinities. That he arrived at the same conclusion as Cantor regarding the relationship between the syncategorematic/potential and the categorematic/actual is further (although admittedly not conclusive) evidence that Cantor had no conception of the difference between potential and syncategorematic. For, as I will show in chapter three, when this distinction is taken seriously, Cantor's argument does not establish what he takes it to establish.

Although these three arguments provide justification for three different claims, it is easy to see that they all serve as justification for the larger position that the actual/categorematic infinite is coherent. The first argument, the argument from irrationals, attempts to establish that if one accepts the irrational numbers one must also accept the transfinite numbers (implicit in this argument is that one must accept the irrationals). The second argument, the divine intellect argument, attempts to show that an actual/categorematic infinite collection is a coherent notion even if it seems problematic for a finite intellect. This argument is very much on the mark as to what Cantor needs to establish if his transfinite numbers are to have any ground to stand on. The final argument tries to demonstrate that the potential/syncategorematic infinite, something everyone no doubt accepts, actually presupposes or necessarily relies upon the actual/categorematic infinite. This is perhaps the strongest of the three and so it will receive the most attention in the next chapter.

2.3.3 The Absolute

One final point of interest regarding Cantor's philosophy of the infinite is that Cantor's transfinite numbers, both the cardinals and ordinals, end up becoming a syncategorematic infinity themselves. It is interesting because Cantor's account of the hierarchy of transfinite numbers sounds very much like Leibniz's position regarding any infinite collection whatever. As Cantor says,

> To every transfinite cardinal number… there is a next greater proceeding out of it according to a unitary law, and also to every unlimitedly ascending well-ordered aggregate of transfinite cardinal numbers… there is a next greater proceeding out of that aggregate in a unitary way. (Cantor 1932, 296; Cantor 1955, 109)[45]

[45] In the original: "Zu *jeder transfiniten Kardinalzahl*…gibt es eine nach einheitlichem Gesetz aus ihr hervorhehende *naechstgroessere*; aber auch zu jeder unbegrenzt aursteigenden wohlgeordneten Menge…von transfiniten Kardinalzahlen…gibt es eine *naechstgroessere*, einheitlich daraus hervorgehende."

Cantor calls infinities of this sort, i.e. those analogous to the unending series of transfinites, an "absolute" infinity: in this case it appears to be a potential infinity composed of actual infinities. Therefore, when number is "increased" as much as possible, we do not end up with a completed, actual infinity. In fact, this hierarchy of transfinite cardinals very much resembles a syncategorematically understood actual infinite. For there is a cardinal greater than any cardinal one may choose. However, this hierarchy cannot merely be a potential infinity, for the so-called "generation" principles do not actually bring these numbers into existence; i.e. the numbers already exist before they are reached by the generating principles. It is not only the series of transfinite cardinals that has this characteristic, i.e. that it comprises an absolute infinity. Any set that cannot be coherently thought of as a unity falls into this category (Cantor 1932, 443).[46] Such sets are called "inconsistent multiplicities" in contrast to "consistent multiplicities", the unity of which presents no problems. For example, Russell's set—the set of all sets that are not members of themselves—is an inconsistent multiplicity as is the set of all sets. For one reason or another these sets result in contradiction when thought of as a unity and thus they are not strictly speaking sets, i.e. they are not consistent multiplicities. So Cantor did acknowledge that not all infinite collections form coherent unities. The sets that do not are deemed Absolutely Infinite.

Interestingly, Cantor's domain argument should apply equally well in this case, making it impossible to have a variable range without a fixed domain over which it

[46] In the original: "Eine Vielheit kann naemlich so beschaffen sein, dass die Annahme eines "Zusammenseins" *aller* ihrer Elemente auf einen Widerspruch fuehrt, so dass es unmoeglich ist, die Vielheit als eine Einheit, als "ein fertiges Ding" aufzufassen. Solche Vielheiten nenne ich *absolute unendliche* oder *inkonsistente Vielheiten*."

ranges. But unlike the case of the natural numbers, Cantor now has an independent consideration against the unity of this domain; namely, his distinction between consistent and inconsistent multiplicities. The fact that Cantor seems to have a coherent model of an actual infinite, syncategorematically understood, at work within his characterization of the Absolute serves to further bolster the coherence of Leibniz's position.

2.4 CONCLUSION

In this chapter, I have provided a basic account of Cantor's theory of the transfinite as well as three main arguments put forth by Cantor in support of his position. The three main claims that I hope to have established in this chapter are that Cantor views the infinite as actual, that he understands this actual infinite as categorematic, and that these terms turn out to be synonyms on his account. Moreover, the terms "potential" and "syncategorematic" turn out to be synonymous for Cantor as well. These equations of terms will cause significant difficulty for Cantor. Bearing these claims in mind, I will now move on to the third and final chapter, in which I attempt to compare the characterization of Leibniz provided in chapter one with the characterization of Cantor in the present chapter.

CHAPTER 3: A LEIBNIZIAN RESPONSE

3.1 INTRODUCTION

In the first two chapters I have presented two very different positions concerning the infinite. According to Leibniz, the infinite is actual yet syncategorematically understood. That is to say, although Leibniz espouses the actual infinite (as indicated by his doctrine of the actual infinite division of matter) he denies that an infinite multiplicity (even if it is actually infinite) is one and whole. While it has been objected that such a position is inconsistent, I have maintained that these objections blur the distinction between actual and categorematic and between potential and syncategorematic. Although this distinction is subtle and at times tricky to maintain without ambiguity, I believe that it is a justifiable distinction to make; in fact, I believe that Cantor needs this distinction in order to make sense of his position on the Absolute. Thus, I have argued, Leibniz does in fact have a coherent position on the infinite. However, coherence is only the most basic of steps towards a plausible theory and so I will attempt to take further steps in the present chapter.

In chapter two I presented Georg Cantor's theory of transfinite mathematics. This theory is certainly of interest in itself—i.e. as a purely mathematical theory—since it rejects certain axioms (in particular the part-whole axiom) that seem to be so deeply entrenched in our way of thinking as to be considered irrefutable, yet nonetheless produces a (mathematically) coherent and self-consistent system. Of more interest to my present inquiry, however, is whether or not the philosophy behind the mathematics is coherent and self-consistent. As I demonstrated, Cantor presents three main philosophical arguments for the view that the actual infinite is a coherent notion, despite the treatment that it received from his predecessors. Thus, Cantor was interested in the philosophical as well as the technical viability of his position. But if this viability is to be taken seriously, Cantor's arguments need to be able to stand up to scrutiny. For just as in the case of Leibniz's position, simple coherence is not sufficient to justify the acceptance of this theory.[47]

In this chapter I intend to take more substantive steps towards the position that not only is Leibniz's position coherent, it is a plausible and viable alternative to the Cantorian theory of the infinite. In fact, it is possibly a more desirable position to maintain. In my view Leibniz is able to provide adequate responses to Cantor's arguments. Furthermore, I believe that while Cantor's conclusions do not validly follow from his arguments, these arguments do support (in a certain sense) Leibniz's position. In other words, Leibniz's position follows from Cantor's arguments. What I mean by this will become clear throughout the course of this chapter. I will argue that the Leibnizian position on the infinite is in many respects philosophically superior to that of Cantor.

I will begin this chapter with a consideration of two central paradoxes that have led mathematicians and philosophers to question Cantor's theory of the infinite (or at least question the way in which Cantor presents it). These paradoxes have been known for some time and so I will simply be repeating them. What is original is the response to

[47] Although this may seem to contradict Cantor's methodological commitments (see section 2.2.3), as I have argued the presentation of the arguments in section 2.3.2 above indicates the possibility that Cantor believed that technical consistency was not enough to justify ontological claims insofar as he provided the philosophical arguments I have discussed.

these paradoxes that I intend to give. Modern set theorists have responded by taking an axiomatic approach; and while this may have its merits, it is not the line that I intend to take. I will, however, spend some time describing the line that modern set theorists have taken and attempting to justify my motivation for choosing a different path. Ultimately, it comes down to a question of priorities. That is, the sort of theory one wishes to end up with will determine what is prioritized, what is essential and inessential to the theory. If one wants a theory to be able to do a great deal of mathematical work, then one may sacrifice certain philosophical considerations in favour of technical prowess. If, however, one is willing to sacrifice a certain amount of mathematical fruitfulness, or at least the ease with which developments are made as a result of the theory, then one may prioritize different things. Finally, I will provide a Leibnizian response to Cantor's three philosophical arguments. The basis for these responses will in large part be the distinction between potential and syncategorematic, actual and categorematic, which is why I spent so much time in chapter one attempting to establish that this is in fact a tenable distinction.

3.2 Problems with Cantorian Set Theory

Right from the start Cantor is confronted with problems that need to be overcome. First of all, his definition of "set" is rather vague. To say that a set is simply a many that can be thought of as a one, i.e. a multiplicity that can be unified, may have a certain intuitive appeal, but it buys this appeal at the cost of substantive content. How does this definition allow one to determine whether a given multiplicity is a set? What exactly does unification amount to? Who needs to do the unifying? These questions, as well as many others that I have not included, do not seem to have a clear answer. Cantor certainly tries to answer them, but his responses do not seem satisfactory. When confronted with certain sets that seem to lead to contradiction, Cantor introduces the distinction between consistent and inconsistent multiplicities, the former being sets proper and the latter being described as absolutely infinite and therefore not capable of being unified. But what does this distinction amount to? An inconsistent multiplicity is defined as a collection that cannot be thought of as a unity. But this definition does not even attempt to explain why it is that the unification is impossible, it only claims that it is. Until a contradiction is discovered, there is no way to differentiate between sets and non-sets. One needs to have the ingenuity to discover the contradiction in order to classify the collection correctly. At best, then, this distinction is simply not very helpful; at worst, it is vacuous.

This inability to define "set" satisfactorily led many mathematicians to abandon the project altogether. Instead, they decided to put together a system of axioms for set construction that simply did not allow one to construct sets that lead to contradictions. Before I describe such an axiomatic system, I will present the paradoxes that gave rise to the need (in the minds of some mathematicians) to take this approach.

3.2.1 Two Paradoxes

The two paradoxes that I will be treating here are analogous. The first is the paradox of the largest cardinal number and the other the paradox of the largest ordinal number. The schema of the paradoxes is identical; different terms can be substituted into this schema to create each paradox. Thus I will spend more time describing the first paradox, the details of which can be applied to the second paradox as well.

Cantor himself is the originator of the paradox of the largest cardinal. He describes it in a letter to Richard Dedekind dated 1899. While the actual intention of the letter is to establish that all infinite sets (i.e. consistent multiplicities) have an aleph as their cardinal number and furthermore that for any two cardinal numbers a and b, either a = b or a < b or a > b—what Dauben calls the comparability hypothesis[48]—Cantor also addresses the problem inherent in the notion of a largest cardinal number or, alternatively, of the set of all cardinal numbers. Cantor already has a response in mind when he presents the paradox, however, so he does not see it as harmful to his system in any way.

The paradox itself relies on certain prior results concerning the transfinite numbers. In an 1891 paper, Cantor introduces what is now called the Power Set Axiom; namely, that from any set A a new set with a larger cardinality can be formed by taking the set of all subsets of A. This result ensures that there is no end to the sequence of transfinite cardinals, since larger and larger cardinalities can always be generated by forming the power set. The second result on which this paradox relies is the fact that all sets can be well-ordered.[49]

[48] Interestingly, this is a direct refutation of Galileo's position with respect to the infinite (see footnote 24). This demonstrates Cantor's full acceptance of the actual infinite and furthermore what Hallett calls Cantor's "finitism", i.e. the infinite numbers must be treated as far as possible just like the finite ones.
[49] This statement was not satisfactorily established until Zermelo's 1904 paper; however, Cantor proceeds as though it were a proven result. Interestingly, one reaction to this paradox was simply to deny that all sets can be well-ordered, Ω being the obvious example.

Consider the set Ω of all numbers: "since it is a well-ordered set, there would correspond to it a number δ greater than all numbers of the system Ω; but the number δ also occurs in the system Ω, because this system contains *all* numbers; δ would thus be greater than δ, which is a contradiction" (Cantor 1899, 115). As indicated, this result follows directly from the assumption that Ω is a set and thus well-orderable and that every set has a power set, which guarantees that there is no end to the series of cardinals, i.e. that δ exists.

Burali-Forti discovers an analogous paradox with respect to the transfinite ordinal numbers (1897). By means of the second generation principle (which plays the role of the power set axiom above), it is clear that there is no end to the series of ordinal numbers; for any series of ordinals there always exists an ordinal greater than every member of the series. Just as with Cantor's paradox, then, we can consider the set Ω of all ordinal numbers. By the second generation principle, there exists an ordinal number δ greater than every member of Ω. But since, Ω contains all ordinal numbers, we have a contradiction. Once again δ must be greater than δ. This is unacceptable.

Reactions to these apparent paradoxes vary. Burali-Forti took this result to be a refutation of the comparability hypothesis, falling in line somewhat with Galileo's response to the series of squares problem, i.e. that the relations "less than", "greater than", and "equal to" do not apply to the infinite in the same way they do to the finite. Cantor, however, had a ready-made response. The "set" Ω is simply not a set; it is an inconsistent multiplicity, absolutely infinite, and therefore not subject to the same restrictions as a set proper. It is not clear that this response is sufficient, however. For there is no way to identify consistent versus inconsistent multiplicities save the discovery of paradox. What sets other than that of all the ordinals or all the cardinals are absolutely infinite? What differentiates an inconsistent multiplicity from a consistent one? What characteristic guarantees that a multiplicity is consistent? The inability to provide satisfactory answers to these questions led many mathematicians to the view that the problem needed to be addressed more thoroughly. This, ultimately, seems to be the motivation for the axiomatization of set theory.[50]

3.2.2 Modern Axiomatics

Current set theory has its basis in the Zermelo-Fraenkel axiom system introduced at the start of the twentieth century. The goal of the theory is to provide a way to characterize sets without running into the traditional paradoxes. This is accomplished by specifying certain axioms and definitions that allow one to characterize only unproblematic sets. Nowhere is "set" defined; nowhere does one find intuitive proofs such as one can find in Cantor. Proofs of theorems are carried out via the apparatus of first order logic, leaving no room for ambiguity. There are seven standard axioms at the

[50] Lavine (1994) does not share this view. He sees the move to axiomatics as a clarification of Cantorian set-theory that is supposed to clear up the misunderstandings responsible for the paradoxes. However, the view he endorses does not, in my opinion, differ in any significant respect. Whether to avoid the paradoxes by setting Cantorian set-theory on a firmer foundation or to clarify Cantorian set-theory, the outcome is the same. For Lavine's discussion of the motivation for the axiomatic system, see especially chapter one to chapter three.

heart of the theory.[51] The Axiom of Extensionality (A1) states that if two sets have the same members then the sets are identical:

$$\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)$$

The Axiom of Replacement (A2) states that if the domain of a function is a set, then its range is also a set:

$$\theta(x,y) \text{ is a function} \rightarrow \forall A \exists B \forall y(y \in B \leftrightarrow \exists x(x \in A \ \& \ \theta(x,y)))$$

The Set Existence Axiom (A3) states that there exists at least one set:

$$\exists A(A = A)$$

The Power Set Axiom (A4) states that for every set there exists a set containing all the subsets of that set:

$$\forall A \exists C \forall x(x \in C \leftrightarrow x \subseteq A)$$

The Union Axiom (A5) states that the union of a set—i.e. the collection of all individuals contained in its subsets—is a set:

$$\forall A \exists w \forall z(z \in w \leftrightarrow \exists y(z \in y \ \& \ y \in A))$$

The Axiom of Infinity (A6) states that there is at least one infinite set:

$$\exists A(A \text{ is inductive})$$

The Axiom of Choice (A7) states that every set has a choice function, which turns out to be equivalent to the claim that for every set, there exists a relation that well-orders that set. One can clearly see that these axioms essentially specify how to legitimately construct sets from other sets, on the necessary assumption that at least one set exists. It is acknowledged that these seven statements are unprovable, which is why they have been

[51] This particular statement of the axioms is taken from Moore (2005).

granted axiom-status. From these axioms along with certain definitions a wealth of theorems can be proven, none of which fall into the contradictions that are found in Cantor's theory. For example, it is now a provable result that there is no set of all sets, i.e. $\sim\exists A \forall x(x \in A)$, where the variables range over sets.

In an alternate approach—von Neumann-Bernays-Gödel set theory—Cantor's notion of inconsistent multiplicities has been refined. In its place stands a distinction between sets and classes. Classes that are not sets are called proper classes and they are defined by their inability to become members of other classes without contradiction (Suppes 12). Thus, all sets are classes but not all classes are sets. As a result the Burali-Forti paradox and Cantor's paradox cannot be constructed, since they require proper classes to be members of other classes, which is not permitted (Suppes 12).

The paradoxes have certainly been avoided, but at what cost? This axiomatized system, while it may coincide with the results of Cantor's set theory, has the air of arbitrariness. While A1 through A5 may be intuitively obvious and so undisputable, A6 and A7 are not. There was significant controversy surrounding the introduction of these axioms and, while they have come to be accepted, this acceptance is not due to their intuitive obviousness, rather to their mathematical efficacy: they allow one to prove the theorems and to obtain the results that one would like to prove and to obtain. This is perhaps a somewhat bold statement to make but I will not spend much time trying to defend it. It seems to me that whatever the status of this assertion as to the motivation for adopting these two axioms, it is clear that the only justification that they enjoy is their usefulness; the absence of any other form of justification is implicitly granted by their status as axioms.

I do not intend to dispute the usefulness of the axiom system. It is coherent, elegant, and able to avoid the problems faced by Cantor. However, there is something lacking: it has failed to define the key concept with which it is working—namely, that of a set. As Dauben notes, Cantor's notion of set underwent significant revisions, all of which failed to solve the problems that it faced (241). This failure prompted the move to an axiomatic approach, one that was supposed to reclaim certainty in mathematics.[52] But a clear definition of set is conspicuously absent. This may be acceptable to some, but I feel that a clear definition of set is a crucial aspect of any complete theory of sets.[53] Since all of the paradoxes seem to originate with Cantor's definition of set, it seems that another way to resolve them would be to refine this definition in such a way as to avoid the paradoxes. While this approach was abandoned when it became clear that no simple revision would suffice, it is the approach that I favour. For to make the move to an axiomitized system avoids rather than resolves the paradoxes. And if these problems are to be satisfactorily dealt with, I believe that resolution is what is required.

These paradoxes seem to point to some important considerations regarding the infinite. To ignore them is to sacrifice understanding for the sake of practicality or usefulness. The philosophical commitments implied by such a method are far-reaching.

[52] Once again, Lavine (1994) has a slightly different take on the motivation for axiomatic set-theory. But I do not believe that his account differs from my own in any respect relevant to the present argument.
[53] Mayberry (2000) argues, and I completely agree, that axiomatic set-theory actually presupposes a theory of sets as its foundation insofar as it uses the tools of first-order logic. For the logic, rather than being the foundation for a theory of sets, actually requires a theory of sets to get itself off the ground. If this is the case, then anything done in axiomatic set-theory, rather than providing an argument for a particular view of sets or of the infinite, is actually just a statement of that position.

As Cantor himself noted, mathematics is self-justifying only insofar as it avoids contradiction. One problem with this is that what counts as a contradiction to one person counts as something else to another. In my view, while the formalism of the transfinite may indeed be free from contradiction as a result of the axiomatization, its conceptual basis is not. This needs to be sorted out if the theory is to be anything more than a clever game of symbol manipulation. Thus, I intend to provide not a refutation of axiomatic set theory, but an alternative to it (or at least the foundation for one). I will argue that Leibniz has a response to each of Cantor's conceptual arguments for the coherence of the actual infinite and thus if one proceeds with a Leibnizian notion of set at the heart of one's theory, perhaps the paradoxes can be avoided in a way that provides a true resolution of the problems that lay at their centre.

As mentioned in chapter one, Leibniz cannot simply respond to Cantor's arguments by appealing to the part-whole axiom. While this would clearly be an adequate response to Cantor, it would even more clearly be question-begging. From Galileo's paradox, Cantor and Leibniz chose to give up different statements, that the whole is greater than the part and that the number of numbers forms a whole respectively. One statement cannot be wielded against the other with any force. The only possible consideration that could lend Leibniz's position any force over Cantor's is that Cantor does not as a result of Galileo's paradox give up the part-whole axiom per se; it is only given up in the case of infinite sets. Thus there is a tension with Cantor between the desire to treat the infinite as much as possible like the finite[54] and the need to depart so

[54] I.e. Cantorian finitism as discussed by Hallett (1984).

dramatically from this maxim with respect to the part-whole axiom. This type of consideration was dealt with in chapter one when I responded to the charge of equivocation against Leibniz in relation to Galileo's paradox; and so I will not dwell on it here. For as I concluded in chapter one, all that this line of argument serves to demonstrate is that the choice of which statement to abandon cannot provide the basis for a refutation of either position.

3.3 A Leibnizian Response

As indicated, I believe that Leibniz has a response to Cantor's arguments in favour of the actual infinite. That is not to say I believe that within Leibniz one can find a refutation of modern axiomatic set theory. What would such a refutation consist in? I believe that although modern set theory has departed from Cantor's line of argument his conclusions and his foundational assumptions remain. This is an odd state of affairs to say the least. To accept Cantor's conclusions but not his arguments leaves one in the unfortunate position of either constructing new arguments for these conclusions or acknowledging that no such arguments have been made. As the above section shows, a consistent theory—i.e. axiomatic set theory—can be constructed on the assumption that the actual infinite is coherent, but this by no means justifies this assumption. In light of these observations, this section will contain a challenge to the very notion of an actually infinite multiplicity, understood categorematically. I will argue that none of Cantor's arguments establish the coherence of the actual infinite (in the categorematic sense). The basis for my arguments will be the Leibnizian position, namely that there is an important distinction between "potential" and "syncategorematic" on the one hand and between "actual" and "categorematic" on the other, coupled with a tacit appeal to Ockham's razor (which is no doubt favourable to modern set theorists). That is to say, I am assuming that if it seems that two statements can be inferred from a given argument, the more modest statement is the one that should be accepted.

3.3.1 Three Philosophical Arguments Refuted

I will maintain the same order of presentation here that I established in chapter two, beginning with what I called the argument from irrationals and ending with the domain argument. The argument from irrationals claims that transfinite numbers and irrational numbers must stand or fall together. This is how I presented this argument in chapter two:

> (P1) If one accepts irrational numbers (as defined by Dedekind), one accepts infinite sets.
>
> (P2) If one accepts infinite sets, one accepts transfinite numbers.
>
> (C) If one accepts irrational numbers (as defined by Dedekind), one accepts transfinite numbers.

This argument is supposed to appeal to those who believe that irrationals are necessary mathematical content and not subject to controversy. It is supposed to prevail on the efficacy and indubitability in mathematics of the irrationals and claim that their status must be called into question if one doubts the validity of transfinite numbers. More intermediately, it argues that if one does not accept actual/categorematic infinite sets, then one cannot accept irrational numbers. However, this argument rests upon a particular definition of irrational numbers, or perhaps better, a particular way of characterizing them—namely, by means of the Dedekind cut. Lavine (1994) explains this characterization of irrationals as follows:

> Dedekind's theory of cuts defines √2, for example, in terms of the infinite set of all positive rational numbers p such that $p^2 > 2$. (That set and the set of remaining rational numbers—that is, those rational numbers p that are negative or such that $p^2 < 2$—cut the rational numbers into two parts, an initial segment and a final segment, hence the name "cut"). (38)

I do not intend to contest this definition. However, I must be clear about what this actually demonstrates. This argument appeals to the fact that actual/categorematic infinite sets are used in order to define irrational numbers. Insofar as this definition is successful, it is supposed to legitimize the actual/categorematic infinite sets to which it appeals. But I do not believe that this argument is sufficient.

The innocuous part of this argument is P2: this is a statement that even Leibniz would grant. But he would turn Cantor's Modus Ponens into a Modus Tollens: since Leibniz has rejected infinite number, he needs to deny the unity of any infinite multiplicity in order to avoid the consequence that Cantor has rightly identified. For once an actual/categorematic infinite is accepted, it certainly appears necessary that it should have a least upper bound (its limiting ordinal) and a cardinality (its cardinal).

The heart of this argument, however, is the alleged implication of P1's consequent by its antecedent. That is, in order for P1 to hold, actual/categorematic infinite sets must be a necessary condition of Dedekind's characterization of the irrationals (assuming, of course, that this is the only way to characterize the irrationals with the desired rigour). Thus, as an argument for the coherence of the actual/categorematic infinite, the argument from irrationals can get nowhere unless Dedekind's characterization of the irrationals (as it is understood by Cantor) necessitates actual/categorematic infinite sets. This, then, is where I will direct my objection. I believe that there is a way in which one can maintain Dedekind's characterization of the irrationals without accepting actual/categorematic infinite sets.

In fact, the possibility of such a response to Cantor can be found within Leibniz's position on the infinite. And as with many of my criticisms of Cantor's arguments, this objection is based on his failure to distinguish between actual and categorematic.[55] Once the apparatus of the actual/syncategorematic infinite is at one's disposal, the infinite sets referred to in Dedekind's cut need not be actual/categorematic infinite sets. A Leibnizian infinite multiplicity will suffice. In the case of the sum $\frac{1}{2} + \frac{1}{4} + \ldots + 1/2^n + \ldots$, as described in chapter one, even though it is acknowledged that there is no last term, the infinite sum can be thought of as a distributive whole—that is, all the parts are actual since they follow from the law of the series, but they do not compose a unified whole. In the case of the Dedekind cut, each required set can be thought of as a distributive whole, following from what is analogous to the law of the series. For example, in the case of $\sqrt{2}$ the "law of the series" would be $\{p: p^2 < 2\}$ and $\{p: p^2 > 2\}$, just as with the series above the law would be $\{p: 1/2^p, p \in \mathbf{N} \geq 1\}$. As in the case of the infinite series, there is no

[55] I do not mean to imply by this statement that Cantor should have seen the distinction between categorematic and actual that I am attributing to Leibniz via the interpretation given by Richard Arthur. As I said in chapter one regarding Brown's objections to Leibniz's position, one cannot be faulted for not espousing an apparently contradictory position when one feels that one has a perfectly consistent theory of one's own. The lengths to which I have gone to argue that Leibniz's position is merely consistent demonstrates that such a position would not have seemed a viable option to Cantor.

need that the collections specified be categorematic in order for the specification to have sense.

It is important to note that I am not presenting an alternate theory of irrationals. Something so grandiose would be beyond the purview of this project. I am merely identifying a tacit assumption within Cantor's understanding of Dedekind's cut method. The assumption is that the infinite sets being utilized must be categorematic. It is clear that they must be actual—a potential infinite certainly would not do—but nothing necessitates that they be categorematic. The mere possibility that an actual/syncategorematic infinite would suffice is all that I require for a successful objection to P1.[56] And without P1, the hypothetical syllogism that is the argument from irrationals cannot possibly go through.

The next argument to be considered is the divine intellect argument. This argument states that ω is a consistent multiplicity, i.e. a set, because it can be "held together" as a unity in the divine intellect. That is, even if the limited human intellect cannot see how an infinite multiplicity can be a unified whole, this does not prevent ω from being a set. It is sufficient that some intellect can unify the multiplicity, even if that intellect belongs to God. This argument cannot be criticized on the same grounds as the argument from irrationals. It certainly does not assume anything about infinite sets. In fact, it is directed precisely at providing justification for the coherence of an actual/categorematic infinite set. Its main flaw, however, is that it is too strong. Leaving aside for a second the theological commitments inherent in the premises of this argument,

[56] P1 is the following conditional statement: if one accepts irrationals, one accepts actual/categorematic infinite sets.

it proves too much. It seems that no multiplicity whatsoever would be incapable of unification in the divine intellect. Even if one grants the theological presuppositions of this argument, I do not believe that this objection can be avoided. What prevents God from ensuring that every multiplicity is a set? This seems to collapse Cantor's distinction between consistent and inconsistent multiplicities. For how can a multiplicity be inconsistent if one can always appeal to the divine intellect in order to guarantee its unity? The unwelcome answer is that one cannot maintain this distinction. But if Cantor no longer has the distinction between consistent and inconsistent multiplicities at his disposal, then he cannot avoid the paradoxes of the largest cardinal and the largest ordinal.

These considerations have been given in an attempt to refute Cantor's divine intellect argument on its own terms. That is, I have argued that even if one ignores the theological commitments of this argument, it is not able to prove what it is supposed to prove. But one cannot simply ignore the theological commitments of this argument. To appeal to the divine intellect in order to justify the actual/categorematic infinite is simply not a good strategy. It relies on too many metaphysical premises, premises which would certainly be disputed by many philosophers and mathematicians including those who have carried on with Cantor's work on the theory of sets. Ultimately, then, this argument is no more satisfactory that the argument from irrationals: in order to accept it one must also accept significant metaphysical claims, claims whose status is perhaps even more controversial than the statement that they are intended to support.

Interestingly, Leibniz also holds the view that God or the divine intellect serves as the foundation for mathematics. For Leibniz, however, it is the practice of mathematics as a discipline that is grounded by God, not particular mathematical results, insofar as mathematical laws are prescribed by God (Letter to Herman Conring, 1678; L 189). But to appeal to the divine intellect to justify particular results can be nothing other than self-serving. And so Cantor's divine intellect argument is no more compelling that his argument from irrationals.

Finally, I arrive at the domain argument. In many ways this argument can be seen as the strongest of the bunch. And so it ought to receive the most thorough refutation. The domain argument states that any variable finite quantity presupposes an infinite and definite (i.e. actual/categorematic) domain within which it ranges. Therefore, the use of potentially infinite quantities or numbers in mathematics, which is certainly legitimate, leads one necessarily to the postulation of actually infinite quantities or numbers (understood categorematically). The initial problem with this argument is that it relies completely on the equation of "actual" with "categorematic". That is, it is fine to say that the potential infinite commits one to the actual infinite in the way described above; however, why must one understand this categorematically? It seems that this follows only so long as "categorematic" is taken to be synonymous with "actual". The possibility of holding the position that the infinite is actual, understood syncategorematically, implies that Cantor's conclusion is too strong. That is to say, even if I grant Cantor's point that the variable finite presupposes an actual infinite domain, I can still maintain that the domain should be understood syncategorematically. In other words, there is nothing that guarantees the unity of the domain. When one takes "actual" as synonymous with "categorematic", that the domain is categorematic follows directly from the fact that the domain is actually infinite. But since I have established that it is perfectly coherent to maintain that an infinite multiplicity is actually infinite but that it is not one and whole, there is a more modest conclusion that can be drawn from Cantor's argument. This is the appeal to Ockham's razor that I mentioned above. The more modest, and in my view legitimate, conclusion is simply that the domain is actually infinite, but its actuality is to be understood syncategorematically. As I have discussed, this is precisely Leibniz's position regarding the infinite; and thus Leibniz's and not Cantor's position is what validly follows from Cantor's argument.

There is a further consideration regarding this argument. The application of the domain argument to the series of transfinite ordinals lands one right in the middle of the Burali-Forti paradox.[57] That is, as I explained in chapter two, Cantor's transfinite ordinals form an ordered sequence that very much resembles the sequence of finite ordinal numbers. In the case of the finite ordinals, the argument that their variability presupposes a fixed and definite domain within which they range leads one to conclude that ω, the set of all finite ordinals, is a legitimate set. However, when this reasoning is applied in the case of the transfinite ordinals, a paradox results. For to assume that the transfinite ordinals range within a fixed and definite domain ought to give rise to the set Ω, the set of all ordinal numbers. But such a set, as we have seen in section 3.2.1, gives rise to the

[57] An analogous line of argument could be carried out with respect to the series of transfinite cardinals and Cantor's paradox. However, since the two arguments would be virtually identical, I have chosen to focus on the ordinals alone with the understanding that this equally applies to the cardinals.

Burali-Forti paradox and is thus deemed an inconsistent multiplicity by Cantor. What makes the set of finite ordinals special? Why does this argument apply to them but not to the transfinite ordinals? These considerations indicate that this argument is not as strong as it may appear. And they further support my point that while the domain of the variable finite may need to be actual, it need not be categorematic.

Jané (1995) argues that Cantor's domain argument actually serves to guarantee the existence of the Absolute. Since this is in direct opposition to the claims I have just made, it is worth taking some time to consider his argument. The following passage summarizes Jané's argument:

> Cantor's claim [i.e. the domain argument] implies that a potentially infinite sequence, or, more generally, any sequence of terms (of any length) is, strictly speaking, unintelligible and unacceptable in mathematics if it is not previously determined what all its different terms are, which is equivalent to say, if the totality whose elements are the terms of the sequence has not been previously defined.
>
> Thus generalized, the domain principle will also apply to the sequence of all ordinal numbers. According to it, the whole ordinal sequence, or, what amounts to the same, the unlimited applicability of both generating principles, is unintelligible unless the absolute totality of all ordinal numbers actually exists. (1995, 386).

Jané goes on to cite the following passage from Cantor in support of this position:

> The transfinite with its wealth of arrangements and forms necessarily *points at* an absolute, at the "true infinite", whose magnitude is unable to increase or decrease at all, and thereby must be considered quantitatively as an absolute maximum. The latter exceeds, so to speak, all human comprehension and eludes mathematical determination. (Cantor 1932, 405)

The interesting feature of this passage is Cantor's use of the verb "hinwesen", which means "to point at". Jané notes that Cantor uses the same verb when discussing the way in which the finite natural numbers "point at" ω. This similarity completes the analogy between the two relations, between the finite numbers and ω on one hand and between the transfinite ordinals and Ω on the other.

Despite the valid semantic argument for this analogy, it breaks down upon consideration of what is established in each case. For in the case of the relation between the finite numbers and ω, it must be the case that the resulting domain is categorematic. If the domain is not categorematic, then it will not serve the function required of it. In the latter case, however, the case of the relation between the transfinite ordinals and Ω, it must be the case that the resulting domain is *not* categorematic. If the domain is categorematic, then the Burali-Forti paradox rears its head. As I see it, this gives rise to a dilemma for the analogy Jané is attempting to draw. Since both applications of the domain principle must be analogous, they must both give rise to a categorematic domain or neither can. Since in one case the categorematic domain is required and in the other case it needs to be avoided, there is no viable option to be found here. If the domain argument itself is to be considered valid, there are only two options: either the Absolute must be considered a consistent multiplicity, which Cantor denies, or my initial argument is correct: there is no necessity in inferring an actual/categorematic domain from the variable finite range of the natural numbers; one may, and in my view should, only infer an actual infinite domain from this argument if this domain is understood syncategorematically.

There is one objection to my responses that I should consider. If I am claiming, as I have been, that it is possible to grant that the potential infinite implies the actual infinite (in the sense described above), then I need to consider the possibility that a syncategorematically understood infinite implies a categorematically understood infinite. For initially, these distinctions seem as though they could have an analogical relationship to one another. But this cannot be the case. The reason is simple and I have already discussed it: to infer a categorematically understood infinite from a syncategorematically understood infinite is to commit the quantifier shift fallacy, i.e. to reason from, for example, $(\forall x)(\exists y)y > x$ to $(\exists y)(\forall x)y > x$, which is clearly invalid. Thus, there is not a strict analogy between potential/actual and syncategorematic/categorematic, even though it may appear that there should be. As a result, there is nothing committing one to a categorematically understood infinite simply by accepting a syncategorematically understood infinite.

3.3.2 The Reflection Principle

Although I do not believe that any one of Cantor's three philosophical arguments is sufficient to demonstrate the accuracy of his position, there may be another route to his conclusion. That is to say, there may be an independent argument for the coherence of the actual infinite. I did not include this possibility in the second chapter because I do not believe that it is an argument Cantor himself put forward. However, it does flow naturally out of certain statements that Cantor did accept and so it is important to consider it here. This alternative argument is based on what is called the Reflection Principle. The reflection principle applies to the Absolute, what Cantor called "Ω". It states that for any property x possessed by Ω, there always exists some set s—i.e. some consistent multiplicity—that has property x.

As I discussed in the previous chapter, the Absolute is beyond any formal or mathematical determination; it is neither complete nor one and whole. It falls under the heading of what Cantor calls inconsistent multiplicities. In fact, Cantor goes so far as to identify the Absolute with God: he calls it "the true Absolute, which is God, and which permits no determination" (Cantor 1932, 175; Hallett 44). In this way, the Absolute is seen as utterly unintelligible, i.e. incoherent.

The incoherence of the Absolute is the starting point for the argument from the Reflection Principle in support of the actual infinite. Since the Absolute is incoherent, there cannot be any description peculiar to it, any description that can successfully refer only to the Absolute. As Rucker (1995) puts it:

> The motivation behind the Reflection Principle is that the Absolute should be totally inconceivable. Now, if there is some conceivable property *P* such that the Absolute is the only thing having property *P*, then I can conceive of the Absolute as "the only thing with property *P*". The Reflection Principle prevents this from happening by asserting that whenever I conceive of some very powerful property *P*, then the first thing I come up with that satisfies *P* will *not* be the Absolute, but will instead be some smallish rational thought that just happens to reflect the facet of the Absolute that is expressed by saying it has property *P*. (50)

When considered in this way, the Reflection Principle pre-empts any individuating property of the Absolute by stipulating that there will always be a non-Absolute collection that shares the specified property.

The way in which the Reflection Principle can be used to construct an argument in favour of the actual infinite is as follows. As the Reflection Principle states, "every conceivable property of Ω is shared by some ordinal less than Ω. Thus,...since we know that Ω is greater than all the finite numbers n, we know by Reflection that there must be some existing ordinal, call it ω, that is also greater than all the finite *n*" (Rucker 80). What Rucker believes to follow from this is that "if one accepts the various infinite Absolutes, then one is fairly well committed to accepting the existence of infinite…sets" (50). In other words, Rucker is arguing that if one accepts the Absolute then one must also accept the actual/categorematic infinite. This argument can be thought of as the converse of Cantor's domain argument as it was applied by Jané to the question of the Absolute. In that case Cantor argued that the actual infinite implies the Absolute. Here Rucker is arguing that the Absolute implies the actual infinite.

I do not believe, however, that this argument is sufficient to demonstrate the coherence of the actual/categorematic infinite. Aside from certain required assumptions, such as Rucker's particular characterization of the Absolute as non-characterizable (even negatively) and the coherence of the Reflection Principle itself, the main problem that I see with his argument is his treatment of the Absolute as though it actually were mathematically determinable. Consider the property described above by Rucker: "Ω is greater than all the finite numbers *n*". To set up the absolutely infinite collection Ω in the relation "greater than" to the set of all finite numbers *n* is to make an illicit move. For what could "mathematically undeterminable" mean but (among other things) something along the lines of what Galileo concluded from his paradox of the squares: "the attributes of equal, greater, and less have no place". Rucker has assumed that Ω can be placed in the second of these relations. But that must be incorrect. The Absolute cannot simply be a superlatively large collection, *greater than* any other collection whatsoever. Despite what our intuitions may tell us, it is not correct to say that Ω is *greater than* anything. And if this is the case, it cannot be inferred that there exists a consistent multiplicity, ω, greater than all finite numbers. Ultimately, then, the Reflection Principle cannot provide a basis for the actual/categorematic infinite.

In the end, the perspective given by Leibniz's philosophy of the infinite is able to provide an adequate response to Cantor's arguments. The refutation I have provided clearly depends on the distinction between "potential" and "syncategorematic", the meaningfulness of which I believe to have established in chapter one. While this is by no means a definitive refutation of Cantor and of the notion of an actual/categorematic infinite set, it is certainly the first step. The next step will be to provide a coherent notion of set from the Leibnizian perspective that is not susceptible to the paradoxes that plague Cantor's theory of sets. While such an investigation is much too involved to be carried out here, I believe that the considerations in this section provide the foundation for such a project. The conclusion of this chapter will provide some brief comments as to the motivation for this type of endeavour.

3.4 CONCLUSION

I would be well-advised to pause for a moment and begin to compile exactly what I have argued in this chapter. First of all, I have presented the traditional problems with Cantor's set theory. Cantor's paradox and the Burali-Forti paradox are problematic for Cantor, despite his confidence that they can be explained away by appealing to the distinction between consistent and inconsistent multiplicities. I then presented the most common response to these paradoxes: the axiomatization of set theory. This response is appealing insofar as it is straightforward, elegant, and coherent. It is crucial to note that this approach implicitly accepts Cantor's philosophical position with respect to the infinite, although it does not explicitly espouse it within the axioms A1 through A7. Axiomatic set theory is essentially an attempt to provide adequate foundations as well as adequate rigour for the standard arguments and conclusions that are found in Cantor's writing. It does not seem to be susceptible to the two paradoxes I have discussed, since the construction of the contradictions is no longer possible within the framework of the seven axioms I have presented. Many of Cantor's claims regarding the transfinite are still provable within this system, but their proofs are no longer carried out in the intuitive manner utilized by Cantor; set theorists now have the apparatus of first-order logic to guarantee the validity of their proofs. This theory appears to be airtight.

However, there is still something very unsatisfactory about this approach. First of all, and as I have already mentioned, it tacitly accepts Cantor's philosophical position regarding the infinite. But this position is littered with contradictions, as the paradoxes themselves indicate. Thus, to avoid the paradoxes but maintain the same underlying theory is problematic. This would not be the case if the axiomatic system satisfactorily dealt with the paradoxes. But it does not. The paradoxes are certainly avoided, but they are by no means resolved. For the paradoxes seem to originate with Cantor's definition of set, or at least to involve the identification of some multiplicity as a set. Nowhere in axiomatic set theory, however, is "set" defined; axioms are merely given that allow one to "construct" sets from previously given sets. As Mayberry (2000) notes, to characterize something axiomatically is not so much to define it as to define it away (60). Thus, in one sense the problem has been solved (or rather dis-solved) but in another sense it has not. What exactly is a set? What makes one multiplicity a set and another a proper class? Simply indicating that one can be characterized by the axioms and the other cannot is circular.[58] Something more is required.

This is the motivation for section 3.3.1, in which I provide refutations of Cantor's philosophical arguments. It seems to me that the problems that continually appear all stem from Cantor's assumption that any actual infinite is categorematic. By demonstrating the shortcomings of these arguments, I am attempting to isolate the origin of the paradoxes that are found within Cantor's theory of the transfinite. Based on the content of these arguments, it is clear that Cantor himself found it necessary to justify his acceptance of the actual/categorematic infinite. It seems that the usefulness or fruitfulness of his theory has blinded some to the insufficient foundations on which it is constructed. It is not clear that the distinction between consistent and inconsistent multiplicities and by extension sets and proper classes can be maintained in a univocal way. Ultimately it seems to come down to a pragmatic decision: which sets are going to lead to contradiction and how can we avoid the construction of these contradictions?

Furthermore, how does one maintain the obvious insight that Cantor had into the nature of the infinite and its role in mathematics without falling into the same contradictions that Cantor falls into? This is where Leibniz's position displays its attractiveness. By crediting the actual infinite, Leibniz leaves the door open for the

[58] I am not alone in my diagnosis that the absence of a clear definition of set is a problem for axiomatic set theory. See also Lavine (1994) and Mayberry (2000). Although their approach differs from mine, they are responding to the same need, i.e., a definition of "set" sufficient for the role it needs to play in the foundation of mathematics.

acceptability of, for example, Cantor's domain argument, with one small modification: although the domain is actually (not potentially) infinite this actuality is not to be understood as categorematic but as syncategorematic. What this means, to put it in modern terms, is that the domain is similar although not identical to a proper class. It is actually infinite, but it is not a unified whole. The details of this need to be worked out in more detail, but the crucial point is that many of Cantor's results in the theory of sets could be maintained (in an admittedly modified form) while the paradoxes and problems could be avoided. In the end this possibility is due to the consistency of maintaining both that the infinite is actual and that this actuality should be understood syncategorematically. This insight, which I have credited to Leibniz via an interpretation of his writing first proposed by Richard Arthur, can, I believe, provide the possibility for a rich and fruitful, yet consistent and philosophically justified theory of sets.

CONCLUSION

A lot of ground has been covered in a relatively small space. I began with a presentation of Leibniz's position on the infinite and a defense of it against the most basic charge, i.e. that it is not even self-consistent. I ended by arguing that within Leibniz one can find a viable alternative to the Cantorian position on the infinite, one that has the potential to play a part in the foundations of mathematics. There is a substantial gap between these two components, and so I would like to conclude by attempting to bridge it, at least slightly. I will describe what I believe to have accomplished in the preceding chapters as well as give some considerations as to what remains to be done, i.e. what other possible avenues the above arguments point towards.

Despite some initial apprehension at the time of his first publications, Cantor's mathematical treatment of the infinite is now so widely accepted that to challenge it is to move into difficult territory. Although the technical viability of Cantor's theory is hard to dispute, I believe that conceptually it leaves much to be desired. Nevertheless, the Cantorian take on the infinite has become common currency in contemporary mathematics. The actual infinite plays a large role in the foundations of mathematics, since axiomatic set theory (arguably a refinement of Cantorian set theory) performs the function of grounding the entire discipline. But the question remains: how does one justify the acceptance and the use of the actual/categorematic infinite? This is a crucial question. For, at least in some sense, contemporary mathematics is piggy-backing on Cantor's justification of the actual infinite without accepting many of Cantor's central premises. It seems unlikely that anyone would accept, for example, Cantor's justification of the unity of an infinite collection by appeal to the divine intellect. But they nevertheless want to use the result. There is not necessarily anything wrong with this, but I feel that it is at least important to clarify what this commits one to. It commits one to the position that any stipulation that leads to a consistent set of results is thereby justified, that coherence is the only standard of evaluation in mathematics. It is ultimately because I do not want to be committed to such a position that I believe it important to find an alternative. And I have championed Leibniz's position because of his complex treatment of the actual infinite.

As I discussed in chapter one, Leibniz's position initially looks as though it cannot help but fall into inconsistency. On the one hand, Leibniz espouses the doctrine of actually infinite division. That is to say, he does not claim along with Aristotle that matter is capable of being infinitely divided, that such a division is possible. He claims that matter is already infinitely divided, that the division is actual. On the other hand, Leibniz denies that the notion of an infinite number is coherent. As a result of Galileo's paradox of the squares, Leibniz denies that the number of numbers forms a whole. But it does not seem consistent to endorse actually infinite division, to claim that there is an actually infinite multiplicity, without also granting that the multiplicity has a cardinality. But if the multiplicity has a cardinality, then one must accept infinite number; not to do so would lead to a clear inconsistency in one's theory.

From within the traditional discussion of the philosophy of the infinite, there is no avenue open to Leibniz. Even though the grammatical distinction between categorematic and syncategorematic terms had been known for some time before Leibniz, this distinction was taken to be a refinement of the distinction between actual and potential. That is, the term "infinite" can be said in two ways: syncategorematically, it means "for any number specified there are/can be more"—this is equated with the potential infinite—and categorematically, it means "there are more than any number given"—this is equated with the actual infinite. But when each option is formalized, it is clear that there is nothing within syncategorematic that prevents one from maintaining that the collection of elements is actual. That is why I have included "are/can be" in my description of syncategorematic. To say that the term "infinite" is being used syncategorematically is perfectly consistent with either the actual or the potential infinite. Thus, the relationship between the four terms is quite complex. "Categorematic" implies "actual"; but "actual" does not imply "categorematic". "Potential" implies "syncategorematic", but "syncategorematic" does not imply "potential". The objections I considered all assume that "actual" implies "categorematic" and that "syncategorematic" implies "potential". When these implications are shown to be faulty, all of the objections to Leibniz's point of view become answerable.

With this, the situation is complicated somewhat, for there are now more than simply two options to consider: the actual infinite may be categorematic or syncategorematic. This, ultimately, is what prompts the reconsideration of Cantor's arguments in favour of the actual infinite. Even if these arguments lead one to the actual infinite, it remains to be seen whether that infinite is categorematic or syncategorematic. In chapter two I argued that in Cantor's mind, "actual" and "categorematic" are synonymous. As a result, when Cantor provides an argument that supports the actual infinite, he automatically concludes that the same argument supports that the infinite is to be understood categorematically. Because of this, I believe that he has drawn conclusions that are underdetermined by his premises.

In the case of the argument from irrationals, Cantor's first premise is subject to dispute because he assumes that Dedekind's characterization of the irrationals requires the categorematic infinite. But the collections of rationals used in this characterization, can be understood as actually infinite, yet syncategorematic. Thus, all that is needed is that for any rational less than, for example, $\sqrt{2}$, there is always one closer, not potentially, but actually, yet the collections described by $\{p: p^2 < 2\}$ and $\{p: p^2 > 2\}$ are not unified wholes. In this way the "sets" of rational numbers that compose the Dedekind cut need not be categorematic. So long as they are actual, they will do the work required of them. Clearly the technical details of such an approach would need to be worked out before it could be considered an actual alternative in the sense that it could do any mathematical work. While I am unable to provide such a treatment here, the possibility of doing so casts some doubt on the assumption that the collections specified in Dedekind's characterization of the irrationals need to be categorematic.

In the domain argument similar considerations apply. Even if a potential infinite (in the sense of a variable finite) implies an actual infinite, there is nothing necessitating that it be understood categorematically. In fact, Cantor himself demonstrates this by his utilization of the same argument in the case of the Absolute. The Absolute cannot be categorematic, but based on the domain argument it must be actual. Thus, even within Cantor's own system, there is an example of the syncategorematic actual infinite. And as

I have argued, once this option is on the table, then any defense of the actual infinite that understands it categorematically becomes problematic.

This is essentially where my argument ends. Even if the actual infinite is necessitated by certain other acceptable beliefs, the categorematic infinite stands in need of further justification. In an attempt to be somewhat constructive, rather than wholly destructive, I have attempted to point towards an alternative point of view, namely the one held by Leibniz. It seems to me that by using the actual/syncategorematic infinite as the basis for one's theory, it is possible to construct a theory of sets free from the problems of the Cantorian approach.

I am well aware that Axiomatic set theory claims to have done just that, but with the advantage of being able to utilize the actual/categorematic infinite. What I am doing here, however, is attempting to provide an alternative to that approach. I do not claim to be refuting it. What would such a refutation consist in? It is unclear how one would refute the specification of seven axioms of set construction. But this is, in part, the trouble I have with this approach and why I am looking for an alternative. The selection of the axioms must be based on certain underlying principles. But these principles, whatever they may be, are merely tacit. As Mayberry (2000) argues, a theory of sets is assumed by an axiomatic approach. It is such a theory that I am after, and it is in the specification of such a theory that I believe Leibniz's position has something to offer.

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