A Numerical Study of Thermal Buoyancy in Axisymmetric Laminar Vertical Jets

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ABSTRACT

A numerical study of thermal buoyancy in a laminar axisymmetric jet issuing into similar stagnant receiving media is presented. The boundary layer equations governing steady incompressible laminar flow are solved using a finite difference technique developed by Tomich. The results show the predominant effect of positive thermal buoyancy is to increase axial velocity. This effect increases for increasing Prandtl and Grashof numbers. Comparison between the reported numerical solution and the perturbation solution of Mollendorf and Gerhardt shows the latter solution does not adequately describe buoyant flow. Morton's entrainment formulation, on the other hand, is shown to predict the correct scaling factor relating two buoyant flows.

ACKNOWLEDGEMENTS

The author would like to thank his supervisor, Dr. J. Vlachopoulos, for his patience and guidance during the course of this work.

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A NUMERICAL STUDY OF THERMAL BUOYANCY IN AXISYMMETRIC LAMINAR VERTICAL JETS.

I - INTRODUCTION

"Flows commonly referred to as jets may be regarded as part of a more general category of flows characterised by the absence of rigid boundaries. These free boundary flows include jets, wakes, plumes and thermals. It is convenient to further subdivide these flows on the basis of whether they are laminar or turbulent and/or buoyant or non-buoyant. There are a nearly unlimited number of important physical examples of these types of flows. Such flows are common in meteorology, oceanography, air and water pollution. More specific examples are associated with the mechanics of cloud formation, buoyancy driven ocean circulation, and thermal circulation in lakes resulting from water discharges"[7]. In this work thermal buoyancy in laminar vertical jets is investigated.

Attempts have been made to solve the boundary layer equations which describe the effect of thermal buoyancy on round laminar vertical jets. In all instances investigators employed approximate analytic solution techniques. The validity and predictive power of these solutions has never been critically examined. The object of this investigation is, therefore:

- to determine, using numerical techniques, the effect of thermal buoyancy on velocity and temperature fields of an axisymmetric laminar vertical jet, and

- to examine the range of validity and predictive power of the recorded approximate solutions.

II - BACKGROUND

II- 1 ANALYTICAL STUDIES

Using boundary layer approximations Schlichting (1) found similarity solutions for the non-buoyant axisymmetric jet. Andrade and Tsien (2) verified his results experimentally. They photographically observed traces made by particles suspended in a laminar jet flow. Landau (3) used similarity to derive an equivalent solution. His solution to the non-buoyant problem represents one of the few exact solutions to the full Navier-Stokes equations. Squire (4) presented an identical result extended to include a solution of the energy equation.

Thermal buoyancy in laminar jets was analysed by Morton (5). He used entrainment models developed using order of magnitude arguments. His analysis predicts an increase in axial velocity for positive thermal buoyancy. Brand and Lahey (6) performed a similarity analysis. Their solution relies not only on the validity of similarity but also on a particular set of flow and energy matching conditions. The usefulness of their result is not apparent. Mollendorf and Gebhardt (7) extended Schlichting's similarity solution to the buoyant jet. In their analysis they point out that similarity solutions do not apply because the momentum and energy equations are coupled by a position-dependent buoyancy term. For this

reason they used perturbation analysis to extend Schlichting's similarity solution. Their general conclusions are that positive thermal buoyancy increases the axial velocity and that the magnitude of the effect increases for decreasing Prandtl number.

II- 2 NUMERICAL STUDIES

For the non-buoyant jet, Pai's recent numerical study (8) shows that similarity solutions are good only at large distances from the nozzle exit. Vlachopoulos (9) reports a similar result for the magnetohydrodynamic free jet. From the analytical studies it is agreed that positive thermal buoyancy increases the axial velocity resulting in a longer jet, and corresponding rearrangement of the velocity and temperature profiles, and the transition zone (Figure 1). Accordingly approximate solution techniques may not accurately describe the characteristics of the buoyant jet.

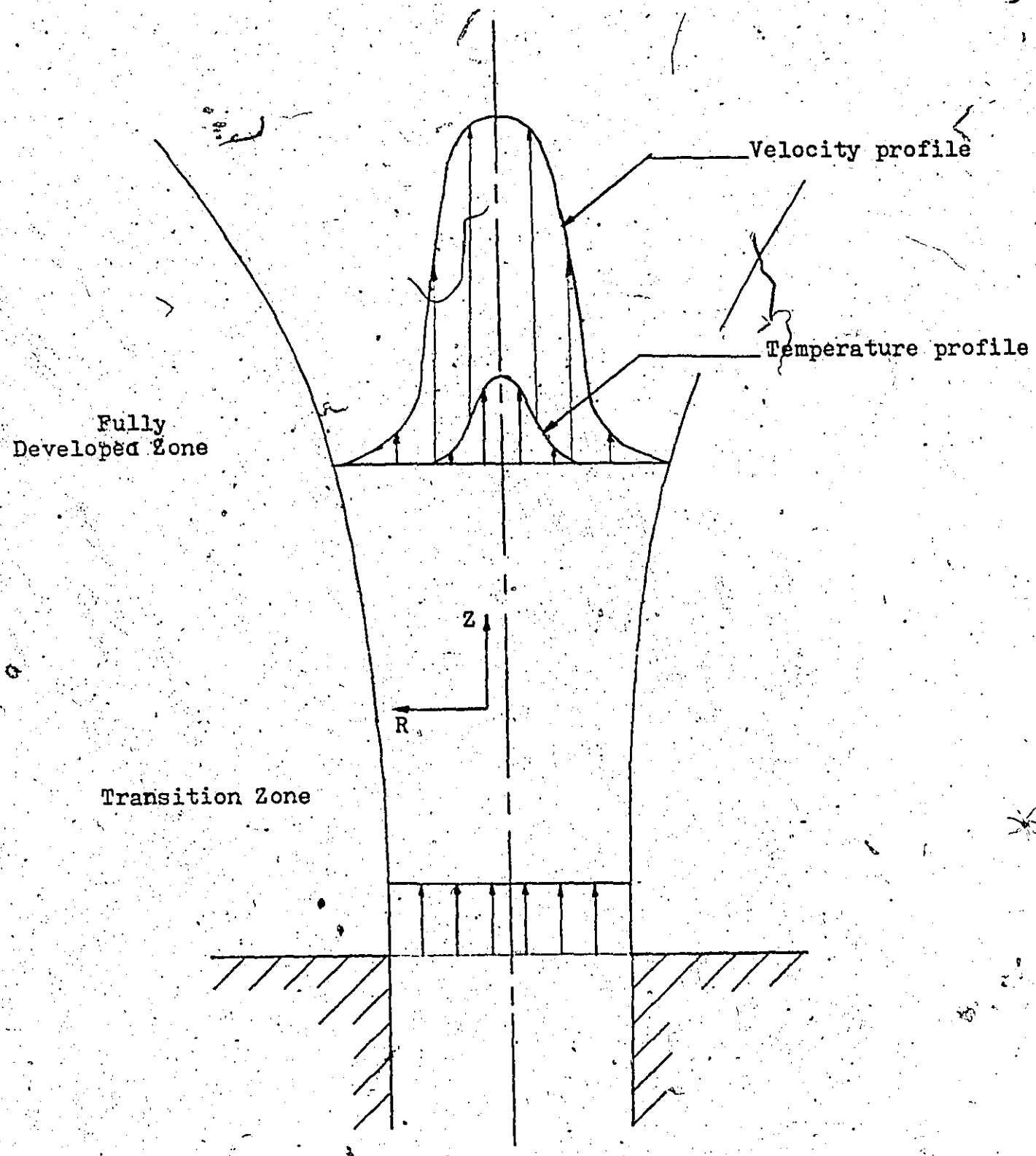

Figure 1 - Schematic of a Thermally Buoyant Axisymmetric Laminar Jet.

III - ANALYSIS

III - 1 MATHEMATICAL FORMULATION

The physical system drawn in Figure 1 is made of a simple jet whose exit temperature is significantly greater than ambient.

The complete formulation of the buoyant jet problem is given in Appendix A. Using the following assumptions:

- the flow is steady, irrotational and axisymmetric
- all fluid properties are constant
- the effect of gravitational forces is negligible
- the flow is laminar
- Prandtl's boundary layer assumptions apply
- the Boussinesq approximation applies ($\Delta\rho/\rho \ll 1$)
- viscous dissipation is negligible

the conservation equations reduce to:

- Continuity equation

$$\frac{\partial V_Z}{\partial Z} + \frac{1}{r}\frac{\partial V_r}{\partial r} = 0 \qquad (1)$$

- Momentum equation

$$\rho V_r \frac{\partial V_Z}{\partial r} + \rho V_Z \frac{\partial V_Z}{\partial Z} = \frac{\mu}{r}\frac{\partial}{\partial r} r \frac{\partial V_Z}{\partial r} + \rho g \beta (t - t_a) \qquad (2)$$

- Energy equation

$$V_Z \frac{\partial t}{\partial Z} + V_r \frac{\partial t}{\partial r} = \frac{\alpha}{r}\frac{\partial}{\partial r} r \frac{\partial t}{\partial r} \qquad (3)$$

The following variable transformations are then introduced:

$$Z = (Z/D)/Re$$

$$R = R/D$$

$$V_Z = V_Z/V_o$$

$$V_R = (V_r/V_o)Re = \frac{\rho V_r D}{\mu}$$

$$T = \frac{t - t_a}{t_o - t_a} \qquad (4)$$

yielding the dimensionless equations of conservation.

- Continuity equation

$$\frac{\partial V_Z}{\partial Z} + \frac{1}{R}\frac{\partial V_R}{\partial R} = 0 \qquad (5)$$

- Momentum equation

$$V_Z \frac{\partial V_Z}{\partial Z} + V_R \frac{\partial V_Z}{\partial R} = \frac{1}{R}\frac{\partial}{\partial R}\left(R \frac{\partial V_Z}{\partial R}\right) + \frac{Gr}{Re} T \qquad (6)$$

- Energy equation

$$V_Z \frac{\partial T}{\partial Z} + V_R \frac{\partial T}{\partial R} = \frac{1}{Pr}\frac{1}{R}\frac{\partial}{\partial R} R \frac{\partial T}{\partial R} \qquad (7)$$

where

$$Re = \frac{\rho V_o D}{\mu}$$

$$Gr = \frac{\rho^2 \beta g D^3 (t_o - t_a)}{\mu^2}$$

$$Pr = \frac{C_p \mu}{k} \qquad (8)$$

The boundary conditions are:

- at R=0 and Z > 0 (centreline)

$$\frac{\partial V_Z}{\partial R} = 0$$

$$\frac{\partial T}{\partial R} = 0$$

$$V_R = 0$$

$$
\begin{aligned}
&\text{as } R \to \infty \\
&\quad V_Z \to 0 \\
&\quad T \to 0 \\
&\text{at } X = 0 \\
&V_Z = V_i(O,R) \\
&V_R = 0 \\
&T = T_i\ (O,R)
\end{aligned}
\tag{9}
$$

The above equations are solved for rectangular initial velocity and temperature profiles:

$$
\begin{aligned}
V_i &= 1, \quad R \leq 1 \\
V_i &= 0, \quad R > 1 \\
T_i &= 1, \quad R \leq 1 \\
T_i &= 0, \quad R > 1
\end{aligned}
\tag{10}
$$

III- 2 NUMERICAL PROCEDURE

A finite difference method developed by Tomich and Weger (10) for axisymmetric, compressible, turbulent free jets was modified and adapted to the buoyant jet case. The following finite difference approximations for temperature, radial velocity and axial velocity were introduced into the equations of energy, momentum and continuity.

Temperature

$$\frac{\partial T}{\partial Z} = \frac{T(Z + \Delta Z, R) - T(Z,R)}{\Delta Z}$$

$$\frac{\partial T}{\partial R} = \frac{T(Z + \Delta Z, R+\Delta R) - T(Z+\Delta Z, R-\Delta R)}{2\Delta R}$$

$$\frac{\partial^2 T}{\partial R^2} = \frac{T(Z+\Delta Z, R+\Delta R) - 2T(Z+\Delta Z, R) + T(Z+\Delta Z, R-\Delta R)}{(\Delta R)^2} \qquad (11)$$

Axial velocity

$$\frac{\partial V_Z}{\partial Z} = \frac{V_Z(Z+\Delta Z, R) - V_Z(Z,R)}{\Delta Z}$$

$$\frac{\partial V_Z}{\partial R} = \frac{V_Z(Z+\Delta Z, R+\Delta R) - V_Z(Z+\Delta Z, R-\Delta R)}{2\Delta R} \qquad (12)$$

$$\frac{\partial^2 V_Z}{\partial R^2} = \frac{V_Z(Z+\Delta Z, R+\Delta R) - 2V_Z(Z+\Delta Z, R) + V_Z(Z+\Delta Z, R-\Delta R)}{(\Delta R)^2}$$

Radial velocity

$$\frac{\partial V_R}{\partial R} = \frac{V_R(Z+\Delta Z, R) - V_R(Z+\Delta Z, R-\Delta R)}{\Delta R} \qquad (13)$$

Upon substitution of the above expressions into the conservation equations (energy, momentum, mass) a system of non-linear equations of the following form was obtained.

Energy

$$A_1 T_1 + B_1 T_Z = D_1$$

$$C_m T_{m-1} + A_m T_m + B_{m+1} T_{m+1} = D_m \text{ for } m = 2, 3, 4, \ldots K-1 \qquad (14)$$

$$C_K T_{K-1} + A_K T_K = D_K$$

Momentum

$$E_1 V_{Z_1} + F_1 V_{Z_2} = H_1$$

$$G_m V_{Z_{m-1}} + E_m V_{Z_m} + F_{m+1} V_{Z_{m+1}} = H_m \text{ for } m = 2, 3, 4 \ldots K-1 \qquad (15)$$

$$G_K V_{Z_{K-1}} + E_K V_{ZK} = H_K$$

Continuity

$$V_{R\,m} = Q_{1\,m}(Q_{2\,m} + Q_{3\,m}) \qquad (16)$$

The derivation of this system of equations is outlined in Appendix B.

By using Wegstein's method (11) for acceleration of convergence, an iterative procedure was used to linearize these algebraic equations. Using this approach the velocities, and temperatures just calculated were replaced with a weighted average of the newly calculated values and those from the previous iteration.

The solution of the tridiagonal system of algebraic equations was performed by Thomas's method (12). This method has the advantage that the round-off error is small in comparison to the truncation error. (13)

The truncation error is a measure of the departure of the finite-difference approximation from the solution of a

partial differential equation at any grid point. If this error tends to zero as the grid spacings tend to zero the finite difference solution is said to converge. If this error grows without bound the solution is unstable.

Richtmeyr (14) presents a theorem by Lax (15) which demonstrates that if the consistency criterion is satisfied stability is both a necessary and sufficient condition for convergence. The consistency criterion requires that the finite difference procedure being used, approximate the solution of the partial differential equation under study.

So far consistency analysis has been limited to the study of linear differential equations. Since the techniques used are not directly applicable to non-linear equations, no analysis of this kind was attempted. Instead to ensure convergence to the true solution various step-sizes, step-size ratios and locations of the outer jet boundary ($R \to \infty$) were tried. The grid sizes used in this study are recorded in Table I.

TABLE I

GRID SIZES

Up to X	m	ΔR	ΔX
.7000	501	0.005	0.1000
.3200	501	0.0100	0.1000
1.000	501	0.0500	0.2500

The results reported are independent of the grid sizes to a small tolerance.

IV - RESULTS AND DISCUSSION

To simplify the study, the existence of two buoyant jets is postulated. The first is a gas jet; the second, a liquid jet. The characteristics of these jets are defined in Table II. In both instances the jet conditions are chosen so that the nozzle exit Reynolds number is 100. The exit Grashof number varies according to the exit temperature difference.

TABLE II

Characteristics of Postulated Jets

Type	Pr	Re	D (cm)	Vo (cm/s)	$\frac{\mu}{\rho}$ (cm^2/s)	t_o (C)	$\frac{g\rho^2\beta D^3}{\mu^2}$ (1/K)
gas	1	100	3.0	7.0	0.21	60.0	2600
liquid	10	100	0.5	1.0 3.0	5.0×10^{-3}	60.0	2600

In the following discussion, the numerical results for these jets are compared to analytical predictions.

The centreline velocities for the two jets are recorded in Tables III and IV, and Figures 2 and 3. As Gr/Re increases from zero the velocity drop becomes a rise indicating a shift from a momentum driven jet to a buoyancy driven jet. When Gr/Re=0 (the non-buoyant case) the numerical solution asymptotically approaches Schlichting's solution, which is given by

$$V_M = \frac{3}{32} z^{-1} \qquad (17)$$

TABLE III

Centreline Velocity for Gas Jet (Pr=1)

Z (=z/DRe)	Gr/Re = 0	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}
0.0500	0.7751	0.7799	0.8216	1.1552	2.7946	8.3285
0.1000	0.5432	0.5521	0.6270	1.1216	3.1483	9.7221
0.1500	0.4120	0.4249	0.5259	1.1024	3.2784	10.2612
0.2000	0.3309	0.3473	0.4690	1.0921	3.3408	10.5238
0.2500	0.2762	0.2962	0.4342	1.0859	3.3758	10.6708
0.3000	0.2368	0.2602	0.4114	1.0819	3.3975	10.7611
0.3500	0.2072	0.2338	0.3957	1.0791	3.4119	10.8203
0.4000	0.1846	0.2141	0.3844	1.0772	3.4219	10.8611
0.4500	0.1663	0.1986	0.3760	1.0757	3.4292	10.8902
0.5000	0.1512	0.1862	0.3696	1.0746	3.4347	10.9118

TABLE IV

Centreline Velocity for Liquid Jet (Pr=10)

Z (=z/DRe)	Gr/Re = 0	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}
0.0500	0.7751	0.7806	0.8295	1.2113	3.0211	9.0212
0.1000	0.5432	0.5552	0.6546	1.2744	3.6893	11.2788
0.1500	0.4120	0.4306	0.5725	1.3249	4.0338	12.4347
0.2000	0.3309	0.3558	0.5316	1.3624	4.2423	13.1284
0.2500	0.2762	0.3072	0.5094	1.3902	4.3808	13.5850
0.3000	0.2368	0.2739	0.4966	1.4112	4.4786	13.9044
0.3500	0.2072	0.2498	0.4889	1.4274	4.5507	14.1560
0.4000	0.1846	0.2321	0.4839	1.4401	4.6040	14.3252
0.4500	0.1663	0.2185	0.4807	1.4503	4.6437	14.4576
0.5000	0.1512	0.2080	0.4785	1.4591	4.6811	14.5621

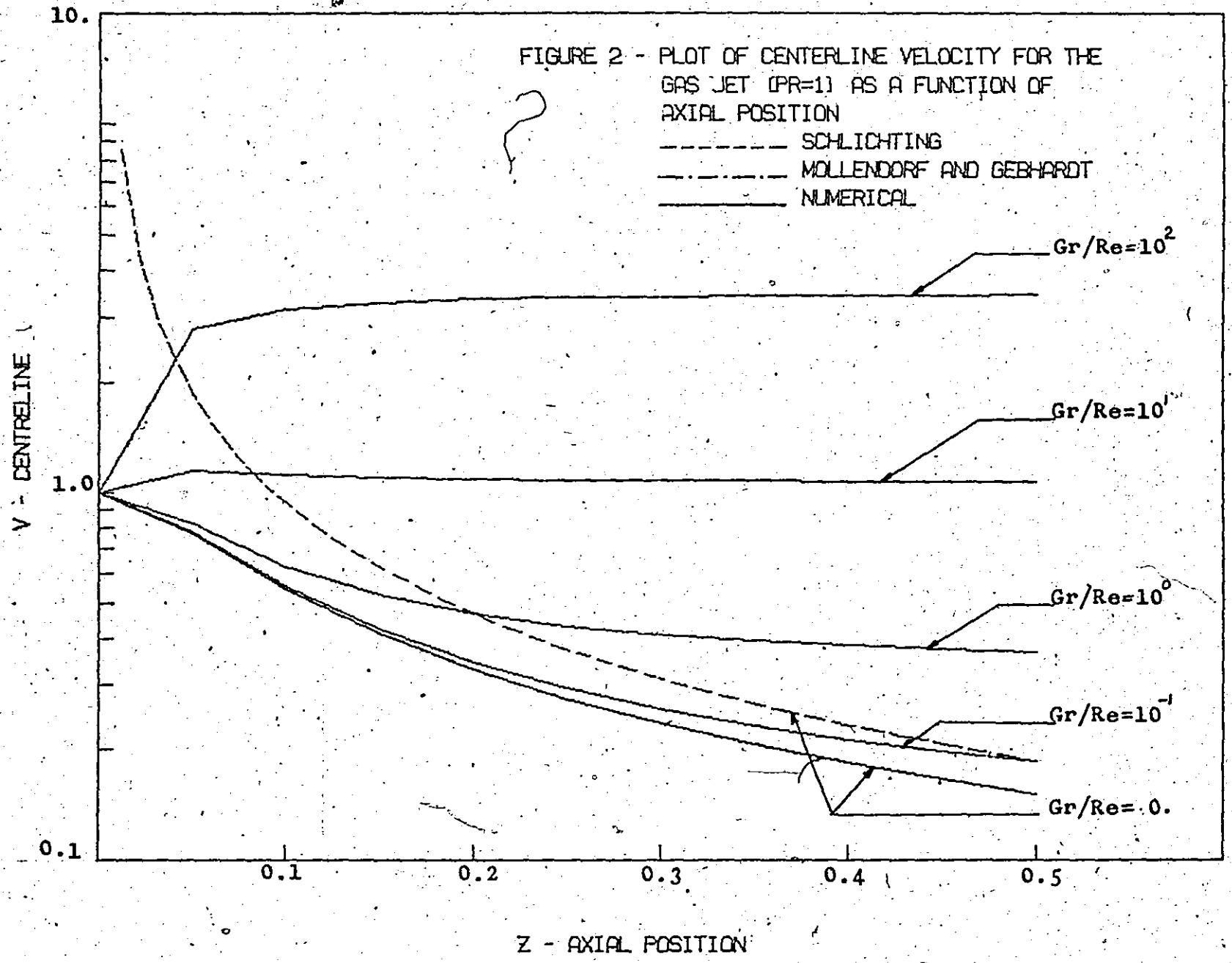

FIGURE 2 - PLOT OF CENTERLINE VELOCITY FOR THE GAS JET (PR=1) AS A FUNCTION OF AXIAL POSITION

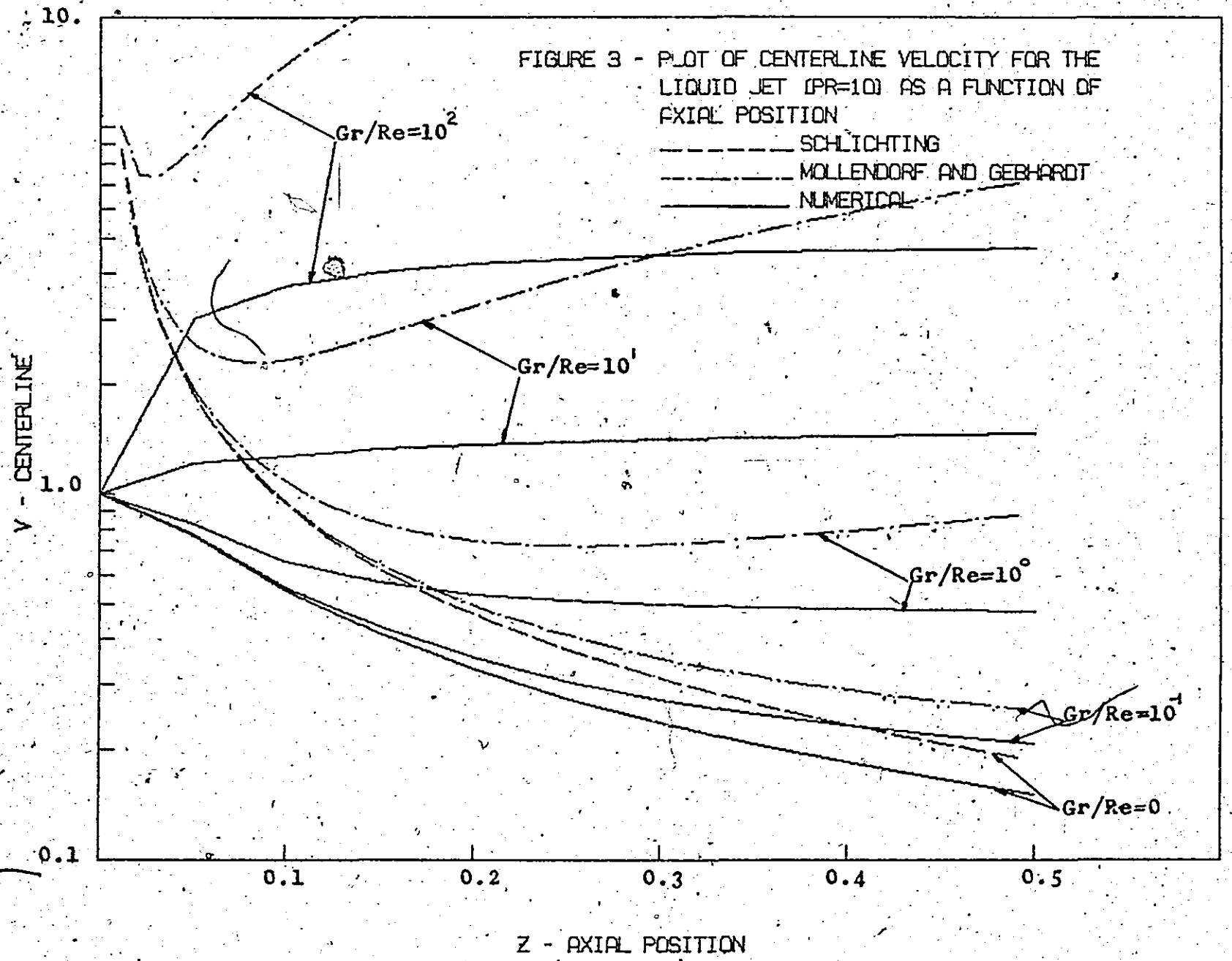

FIGURE 3 - PLOT OF CENTERLINE VELOCITY FOR THE LIQUID JET (PR=10) AS A FUNCTION OF AXIAL POSITION

Furthermore, it is interesting to note that our numerical result for this case is identical to that of Pai (8). The perturbation solution for the buoyant jets as derived by Mollendorf and Gebhardt (7) yields the following equation for centreline velocity.

$$V_M = \frac{3}{32} Z^{-1} + \frac{2}{3} (2Pr + 1)\left(\frac{Gr}{Re}\right) Z\, f(Pr) \tag{18}$$

The first term is identical to Schlichting's solution for the non-buoyant jet. The second term is the perturbation term. In Figure 3 this solution is compared with the numerical solution. The comparison shows significant disagreement over all Z. This is not surprising as for small Z the first term is dominant and suffers from the same inadequacy as Schlichting's solution. For large Z, the second term dominates and increases without bound. Mollendorf and Gebhardt's solution, therefore, does not accurately describe the centreline velocity for the buoyant jet. Morton's entrainment formulation (5) predicts a constant centreline velocity given by

$$V_M = \left(\left(\frac{1}{4E}\right)\left(\frac{1+Pr}{Pr}\right)\left(\frac{Gr}{Re}\right)\right)^{0.5} \tag{19}$$

This result is consistent with the numerical results only for large Z. Because E, the entrainment parameter, is an unknown function of Pr, it is impossible to predict absolute values

of the centreline velocity. Nevertheless, it is possible to compare the effect of Gr/Re as predicted by Morton with the numerical solution by comparing relative centreline velocities.* As seen in Tables V and VI, Morton's entrainment model predicts the correct relative centreline velocity. Brand and Lahey (6) report a similarity solution. Their solution predicts a constant centreline velocity for all Z. Its dependence on Pr, Re and Gr is not clear. Since they predict a constant velocity their solution is at best only good for large Z.

Through their experimental studies McNaughton and Sinclair (16) observed spreading in a laminar jet. For small density differences between the jet and the surrounding fluid (media) their jet appeared to thin. Figures 4 and 5 record the velocity half-widths for the liquid and gas jets. Thinning is present for large Gr/Re(>10). The degree of thinning increases as Gr/Re and Pr increase. Since thinning is a function of Prandtl number spreading of the velocity profile is dependent on spreading of the temperature profile. The effect of buoyancy on spreading can also be seen in plots of axial velocity profiles for the liquid jet, Figure 6a-e. Similar results for the gas jet are recorded in Appendix D.

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* For a given Gr/Re, the relative centreline velocity is defined as the ratio between the absolute centreline velocity and the absolute centreline velocity for Gr/Re=1.

TABLE V

Relative Centreline Velocity
for Gas Jet (Pr=1)
at Z (=z/DRe) = 0.50

Gr/Re	V_{rel} Morton	V_{rel} Numerical
10^{-1}	0.3162	0.5040
10^{0}	1.0000	1.0000
10^{1}	3.1622	2.9072
10^{2}	10.0000	9.2934
10^{3}	31.6224	29.5233

TABLE VI

Relative Centreline Velocities
for Liquid Jet (Pr=10)
at Z (=z/DRe) = 0.50

Gr/Re	V_{rel} Morton	V_{rel} Numerical
10^{-1}	0.3162	0.4353
10^{0}	1.0000	1.0000
10^{1}	3.1622	3.0493
10^{2}	10.0000	9.7834
10^{3}	31.6223	30.4332

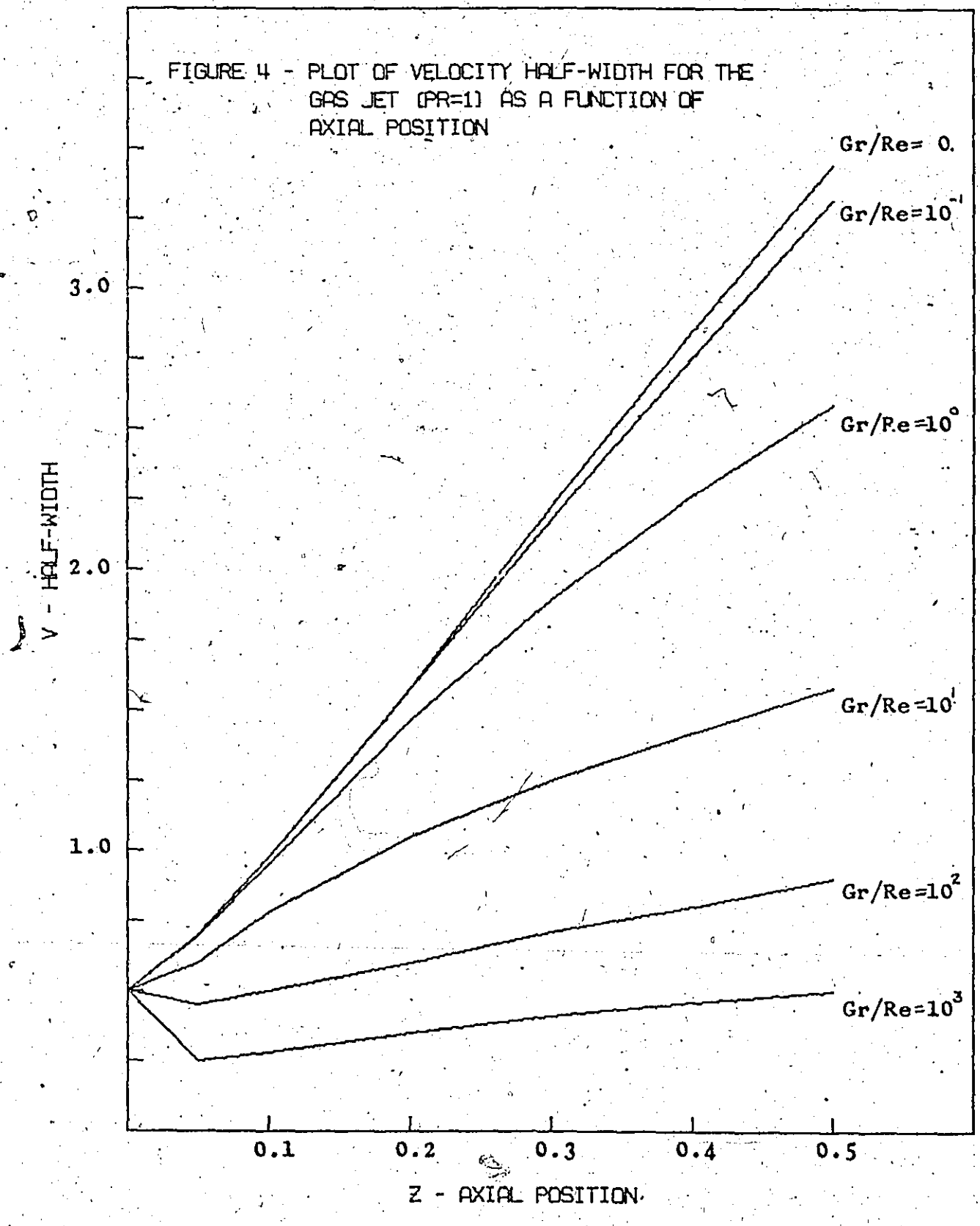

FIGURE 4 - PLOT OF VELOCITY HALF-WIDTH FOR THE GAS JET (PR=1) AS A FUNCTION OF AXIAL POSITION

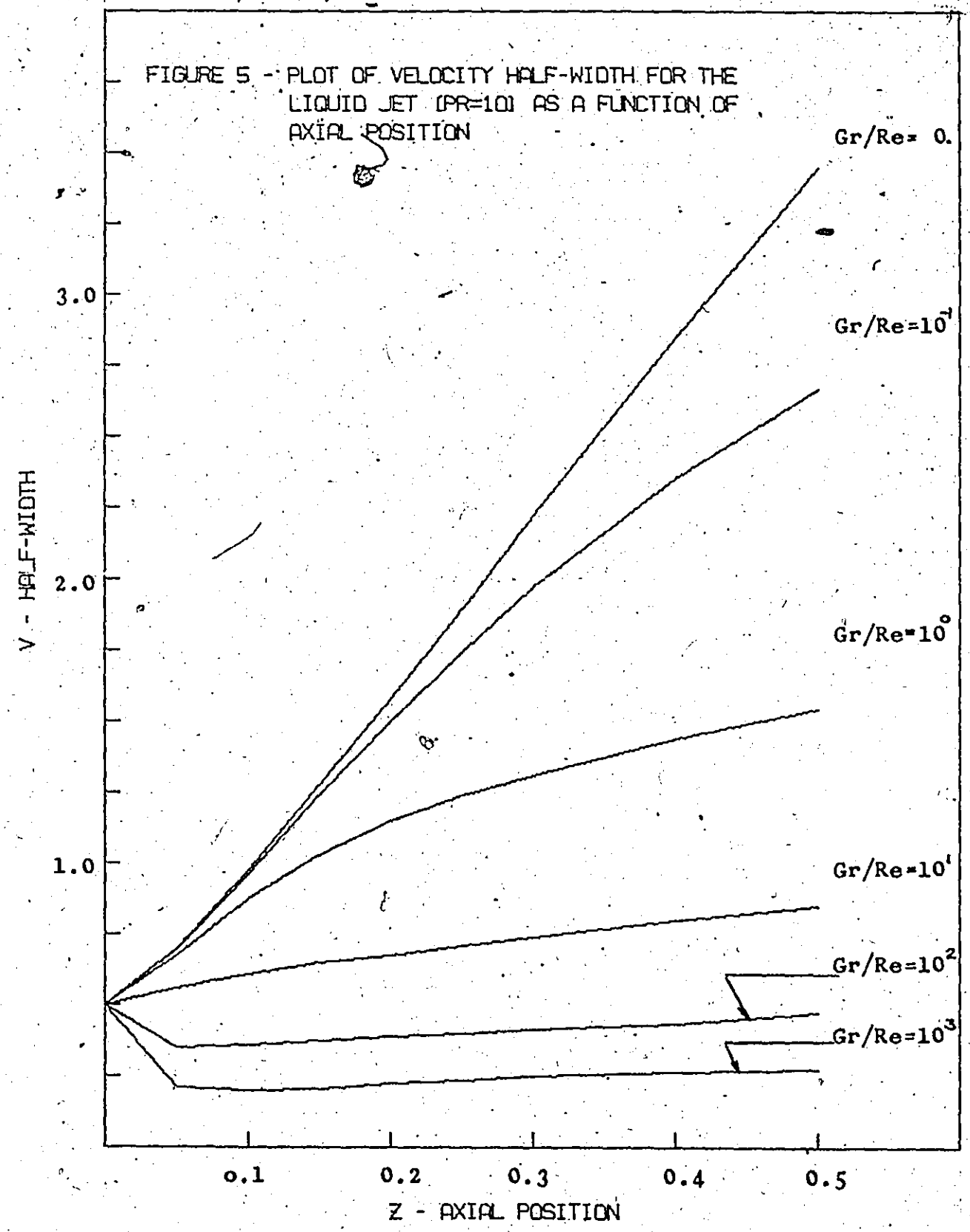

FIGURE 5 - PLOT OF VELOCITY HALF-WIDTH FOR THE LIQUID JET (PR=10) AS A FUNCTION OF AXIAL POSITION

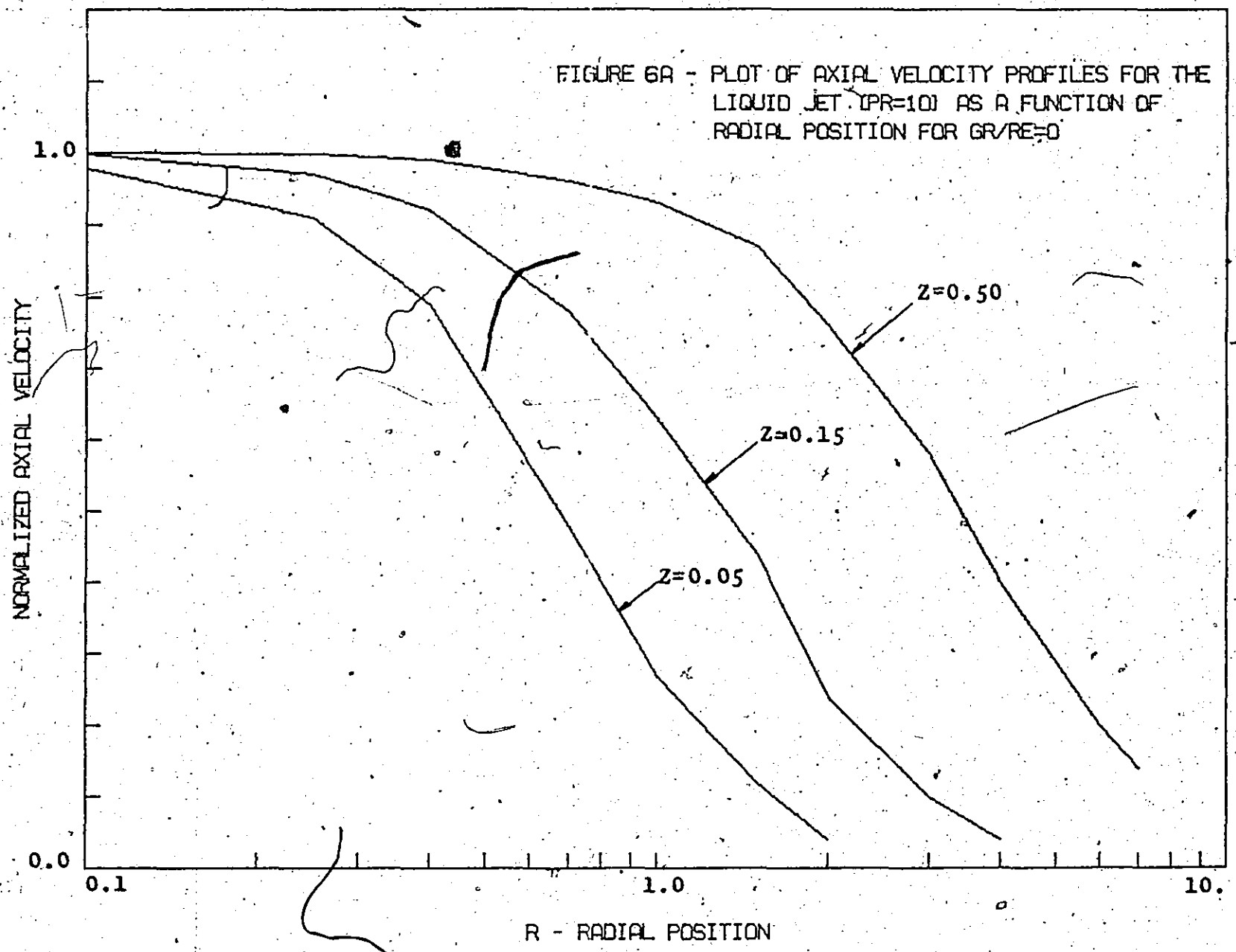

FIGURE 6A - PLOT OF AXIAL VELOCITY PROFILES FOR THE LIQUID JET (PR=10) AS A FUNCTION OF RADIAL POSITION FOR GR/RE=0

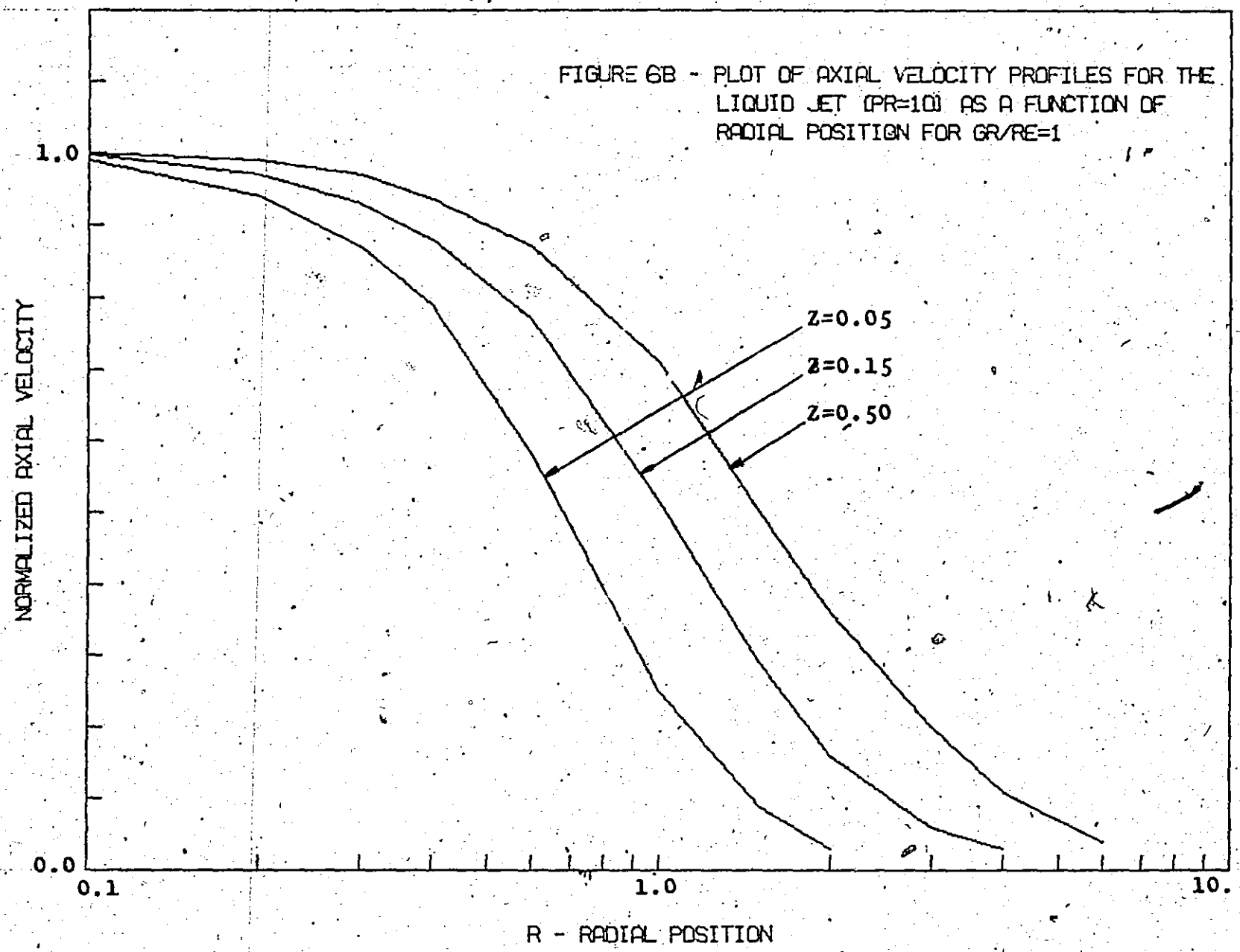

FIGURE 6B - PLOT OF AXIAL VELOCITY PROFILES FOR THE LIQUID JET (PR=10) AS A FUNCTION OF RADIAL POSITION FOR GR/RE=1

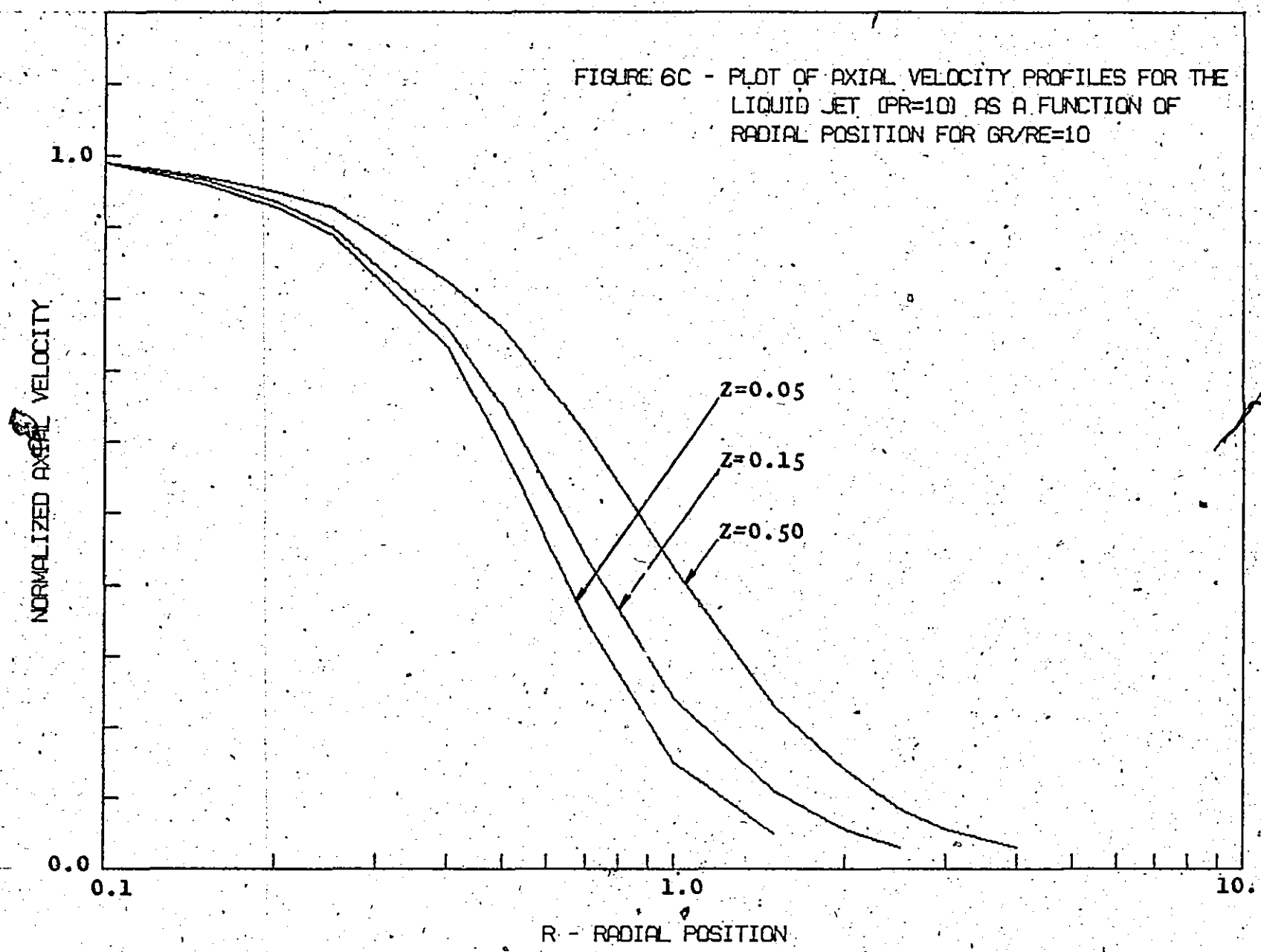

FIGURE 6C - PLOT OF AXIAL VELOCITY PROFILES FOR THE LIQUID JET (PR=10) AS A FUNCTION OF RADIAL POSITION FOR GR/RE=10

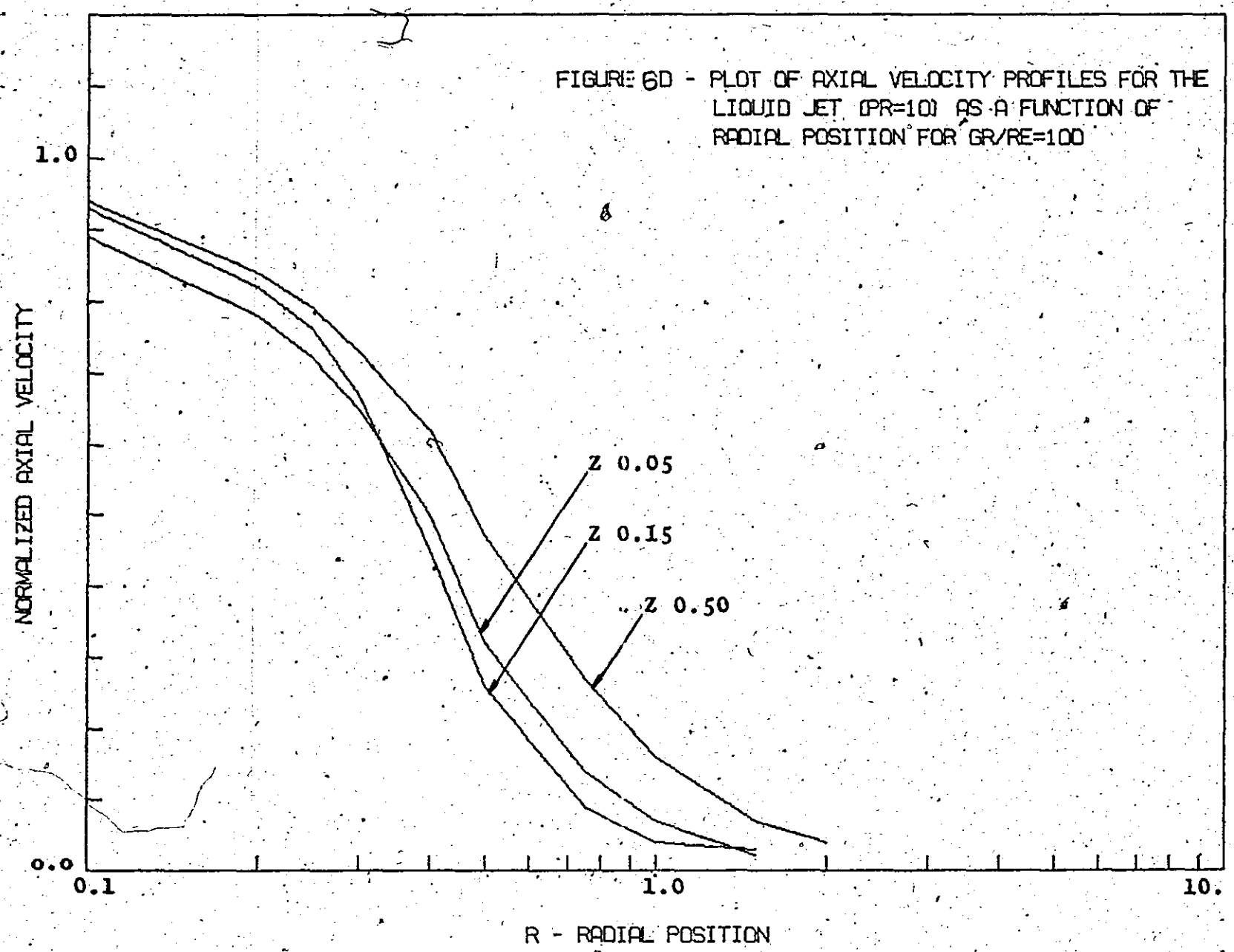

FIGURE 6D - PLOT OF AXIAL VELOCITY PROFILES FOR THE LIQUID JET (PR=10) AS A FUNCTION OF RADIAL POSITION FOR GR/RE=100

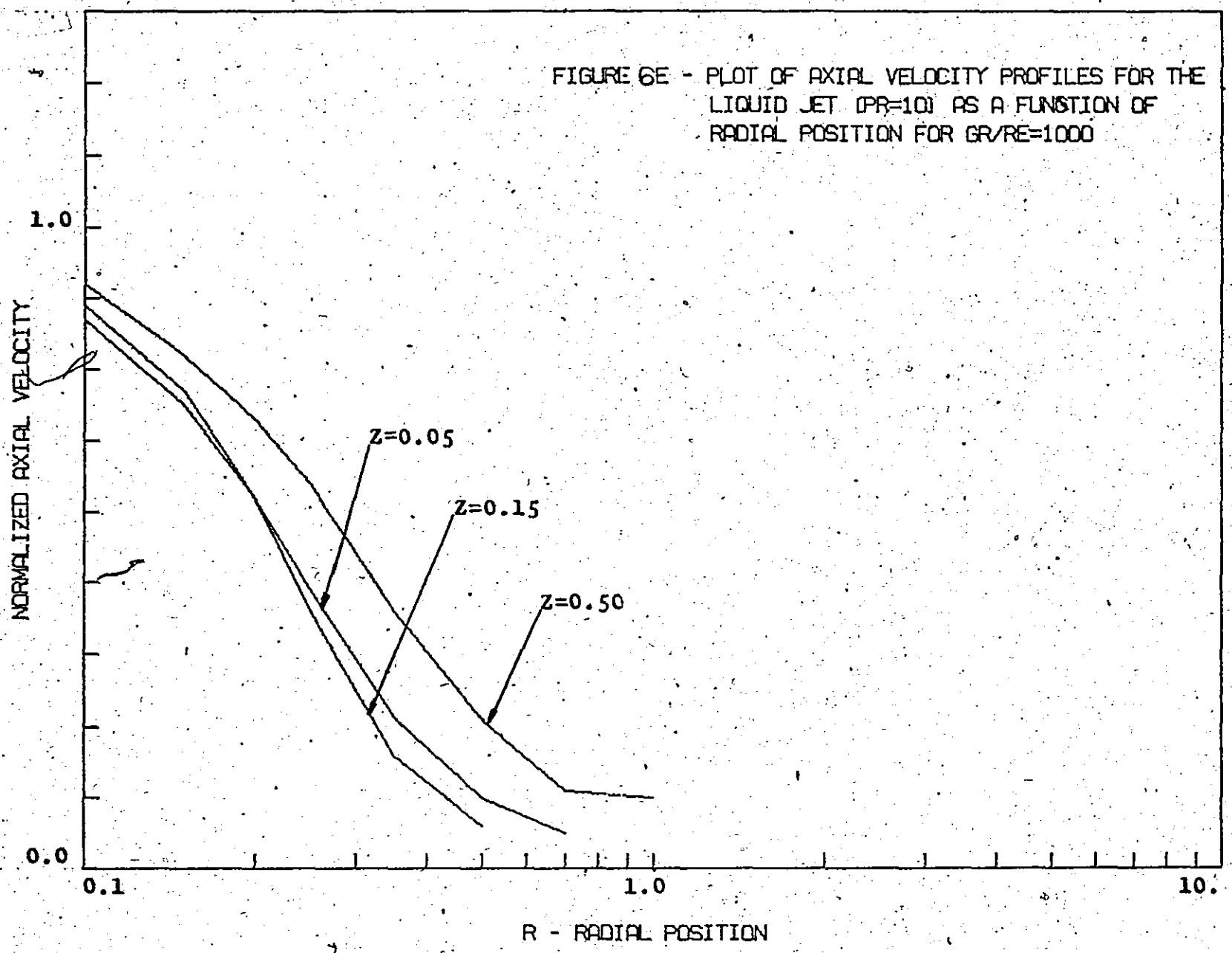

FIGURE 6E - PLOT OF AXIAL VELOCITY PROFILES FOR THE LIQUID JET (PR=10) AS A FUNCTION OF RADIAL POSITION FOR GR/RE=1000

The radial velocity profiles for the liquid jet are reported in Figures 7a-e. As Gr/Re increases the positive portion of this profile decreases and becomes increasingly negative. For large values of Gr/Re (> 10) and small Z, the radial velocity is always negative. This means the jet is contracting or thinning. This is consistent with McNaughton and Sinclair's experimental observations.

The centreline temperatures for the two jets are recorded in Tables VII and VIII and Figures 8 and 9. For Gr/Re = 0, the similarity solution of Yih (17) and Mollendorf and Gebhardt (7) applies. This solution is represented by

$$T_M = \frac{(2Pr + 1)}{32} Z^{-1} \qquad (20)$$

The numerical solution for this case is shown to asymptotically approach the similarity solution (Figures 9 and 10). These plots also show that agreement between numerical and similarity solutions is better for the gas jet (Pr = 1) case. This result implies that the smaller the Prandtl number the more accurate the similarity prediction of centreline temperature.

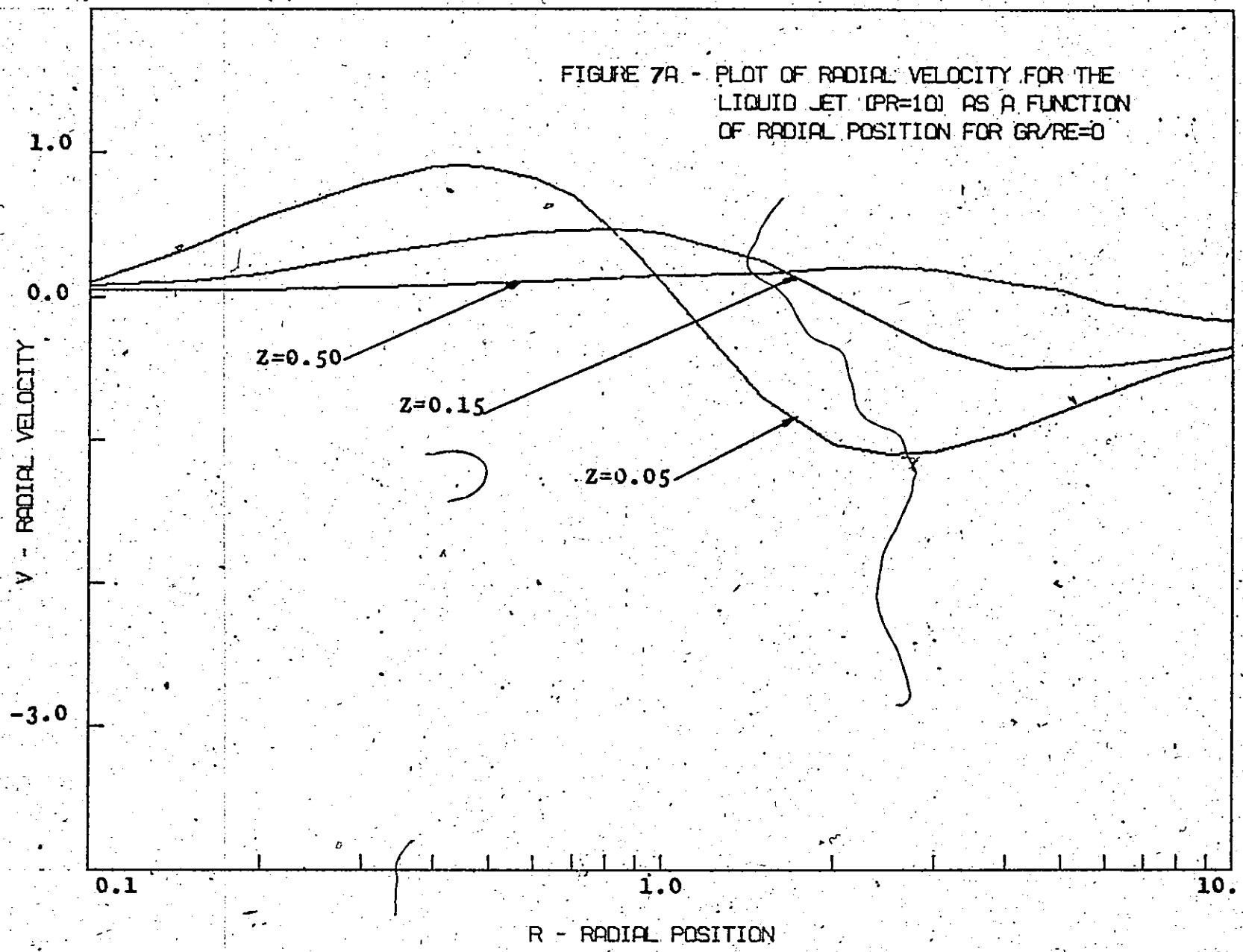

FIGURE 7A - PLOT OF RADIAL VELOCITY FOR THE LIQUID JET (PR=10) AS A FUNCTION OF RADIAL POSITION FOR GR/RE=0

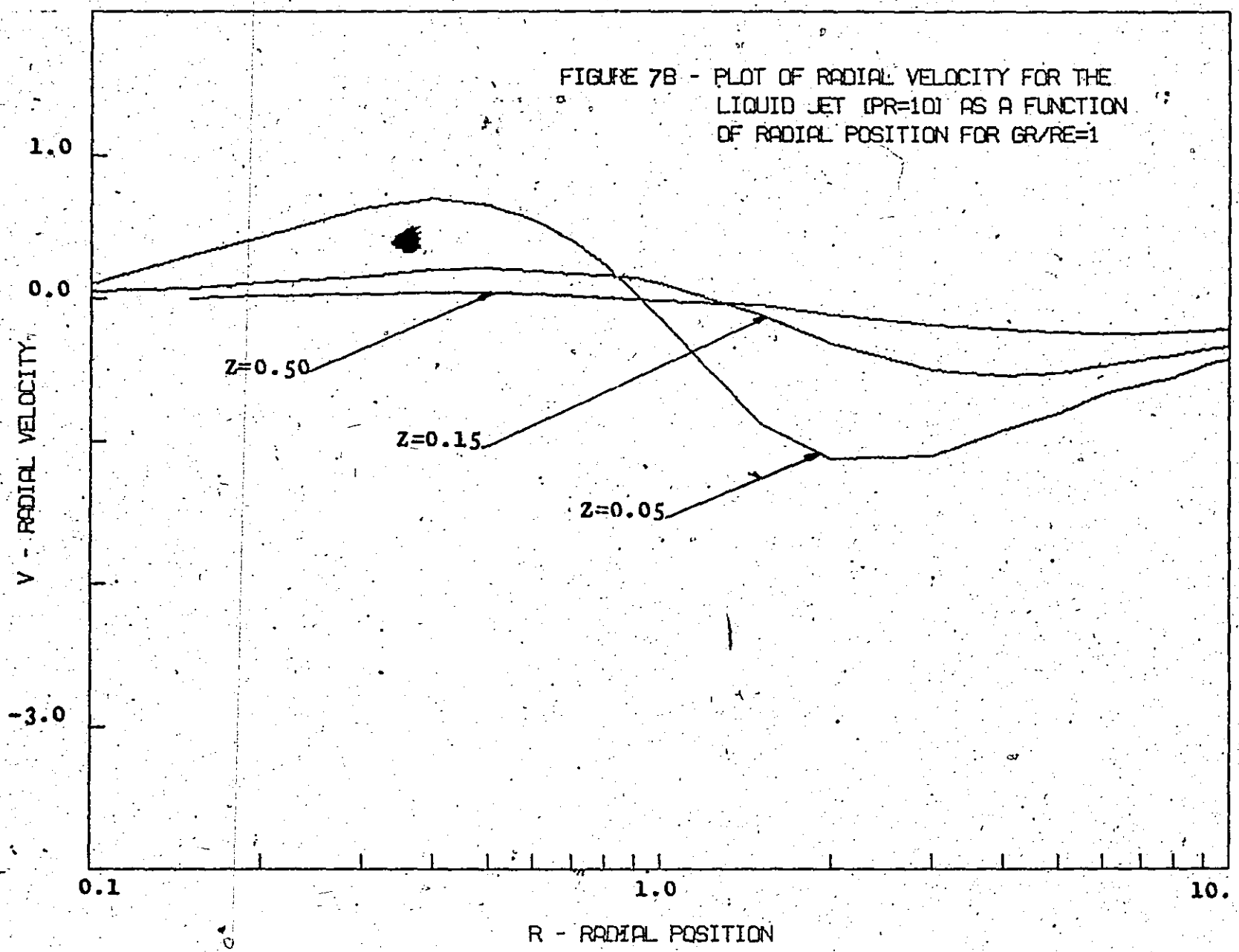

FIGURE 7B - PLOT OF RADIAL VELOCITY FOR THE LIQUID JET (PR=10) AS A FUNCTION OF RADIAL POSITION FOR GR/RE=1

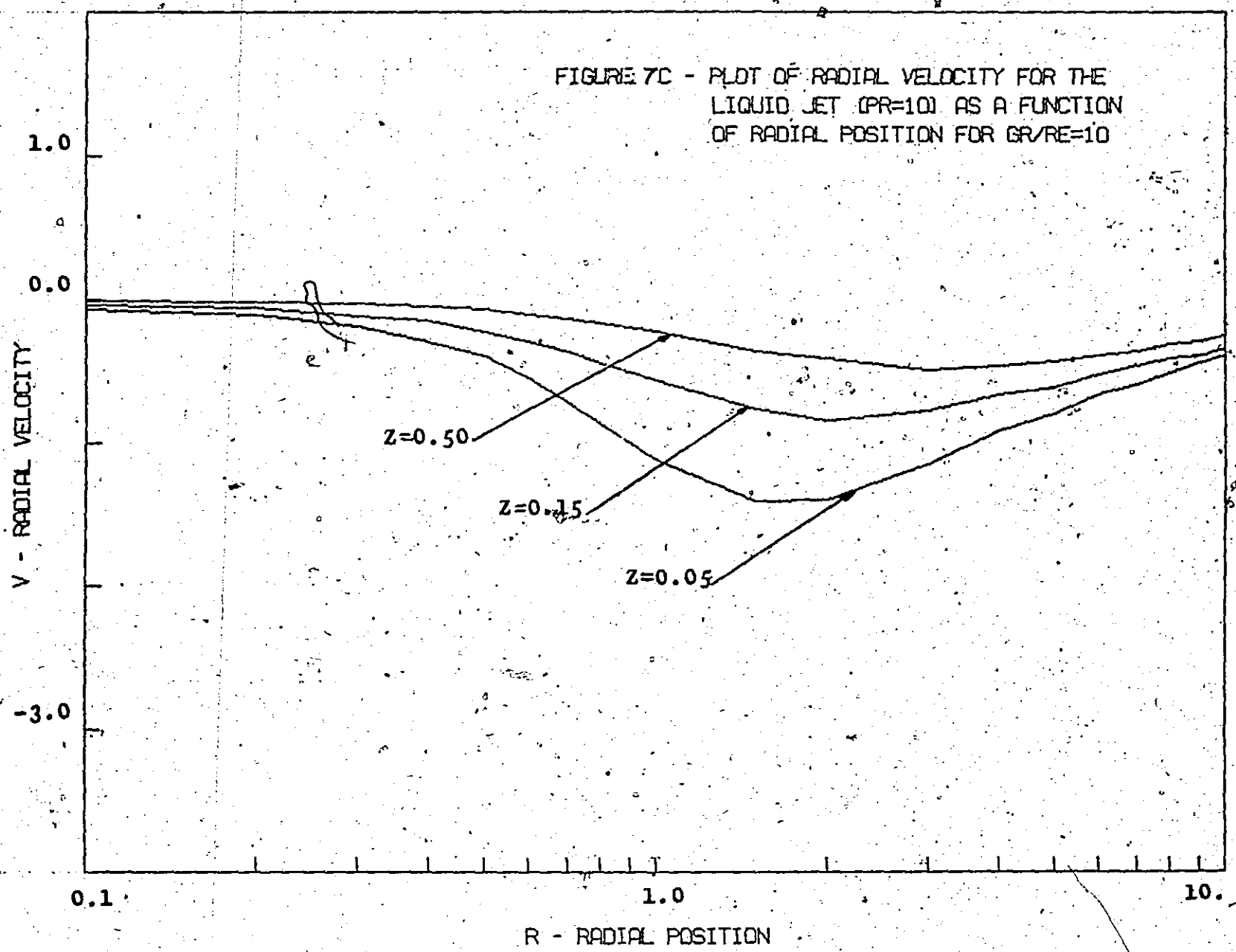

FIGURE 7C - PLOT OF RADIAL VELOCITY FOR THE LIQUID JET (PR=10) AS A FUNCTION OF RADIAL POSITION FOR GR/RE=10

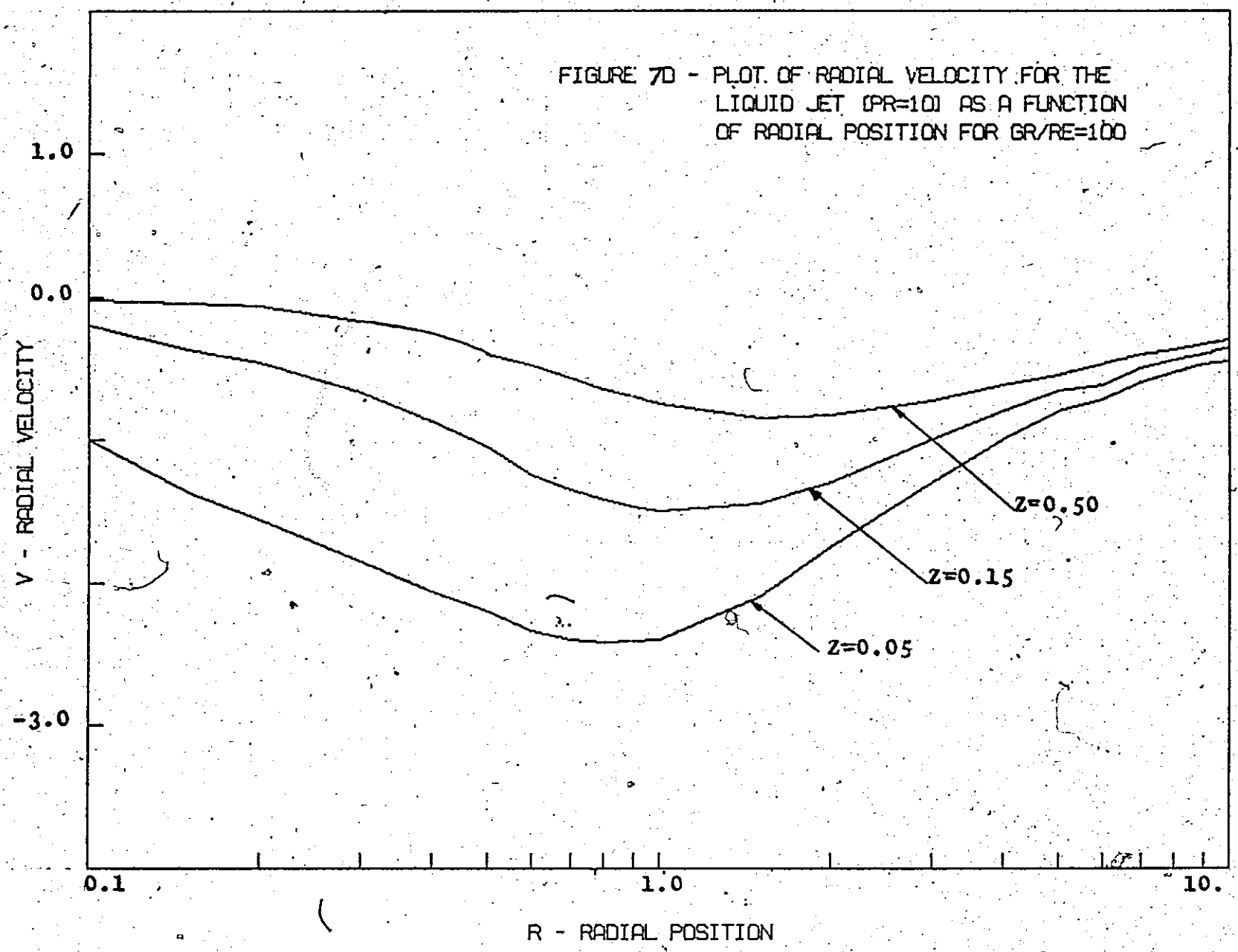

FIGURE 7D - PLOT OF RADIAL VELOCITY FOR THE LIQUID JET (PR=10) AS A FUNCTION OF RADIAL POSITION FOR GR/RE=100

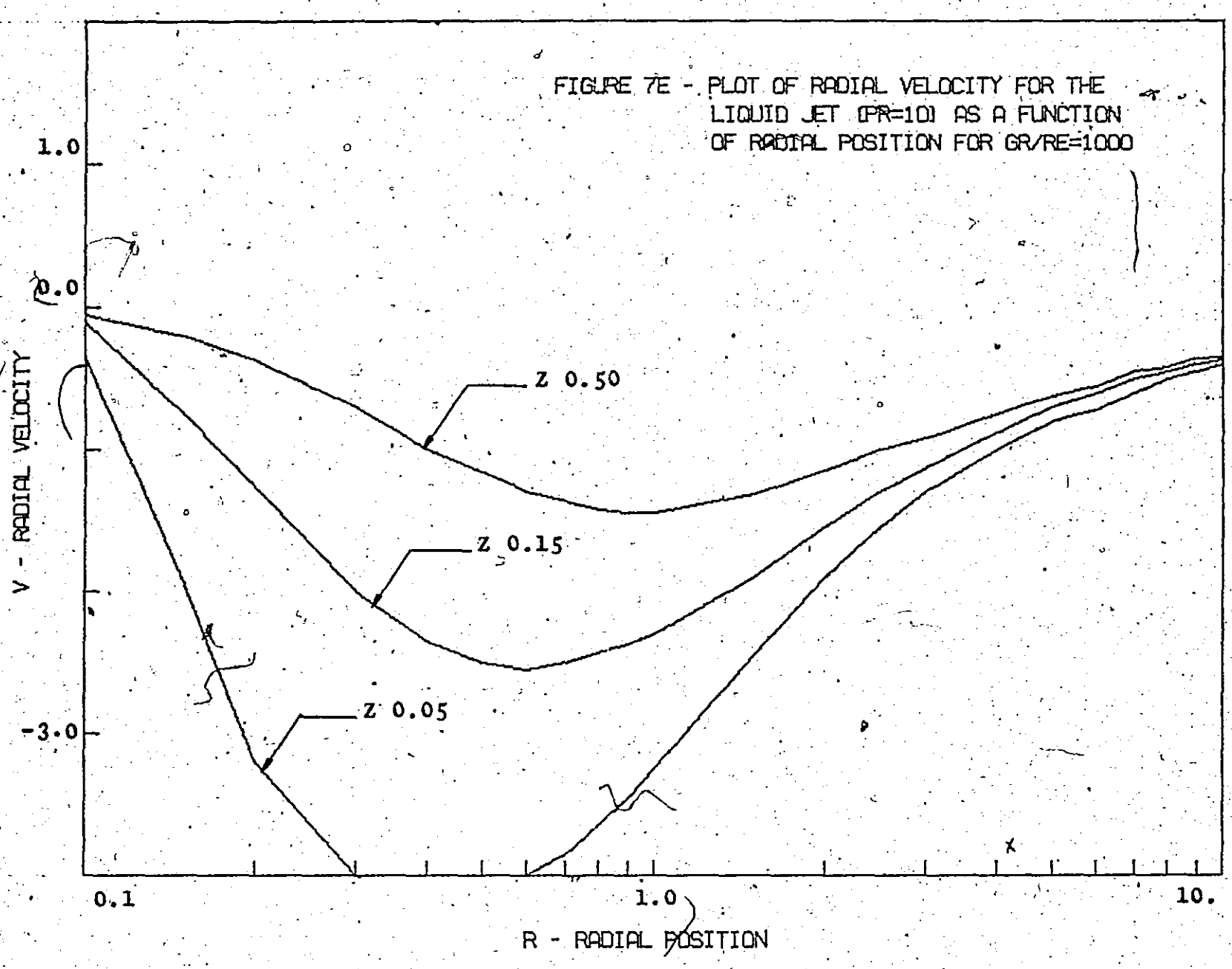

FIGURE 7E - PLOT OF RADIAL VELOCITY FOR THE LIQUID JET (PR=10) AS A FUNCTION OF RADIAL POSITION FOR GR/RE=1000

TABLE VII

Centreline Temperature for Gas Jet (Pr=1)

Z (=z/DRe)	Gr/Re					
	0	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}
0.0500	0.7751	0.7750	0.7741	0.7698	0.7651	0.7376
0.1000	0.5432	0.5427	0.5400	0.5331	0.5350	0.5241
0.1500	0.4120	0.4113	0.4072	0.4004	0.4051	0.4008
0.2000	0.3309	0.3297	0.3248	0.3190	0.3247	0.3231
0.2500	0.2762	0.2748	0.2693	0.2647	0.2705	0.2701
0.3000	0.2368	0.2353	0.2296	0.2260	0.2317	0.2319
0.3500	0.2072	0.2056	0.1999	0.1970	0.2025	0.2030
0.4000	0.1846	0.1828	0.1769	0.1746	0.1799	0.1805
0.4500	0.1663	0.1644	0.1585	0.1568	0.1617	0.1624
0.5000	0.1512	0.1493	0.1436	0.1422	0.1469	0.1476

TABLE VIII

Centreline Temperature for Liquid Jet (Pr=10)

Z (=z/DRe)	Gr/Re					
	0	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}
0.0500	1.0000	1.0000	1.0000	1.0000	0.9998	0.9950
0.1000	0.9985	0.9985	0.9984	0.9978	0.9953	0.9769
0.1500	0.9879	0.9878	0.9876	0.9864	0.9815	0.9513
0.2000	0.9632	0.9632	0.9634	0.9632	0.9586	0.9211
0.2500	0.9275	0.9277	0.9288	0.9312	0.9291	0.8884
0.3000	0.8858	0.8864	0.8889	0.8941	0.8955	0.8546
0.3500	0.8424	0.8432	0.8474	0.8551	0.8602	0.8206
0.4000	0.7994	0.8007	0.8066	0.8163	0.8242	0.7875
0.4500	0.7582	0.7602	0.7675	0.7786	0.7897	0.7552
0.5000	0.7206	0.7221	0.7307	0.7426	0.7561	0.7322

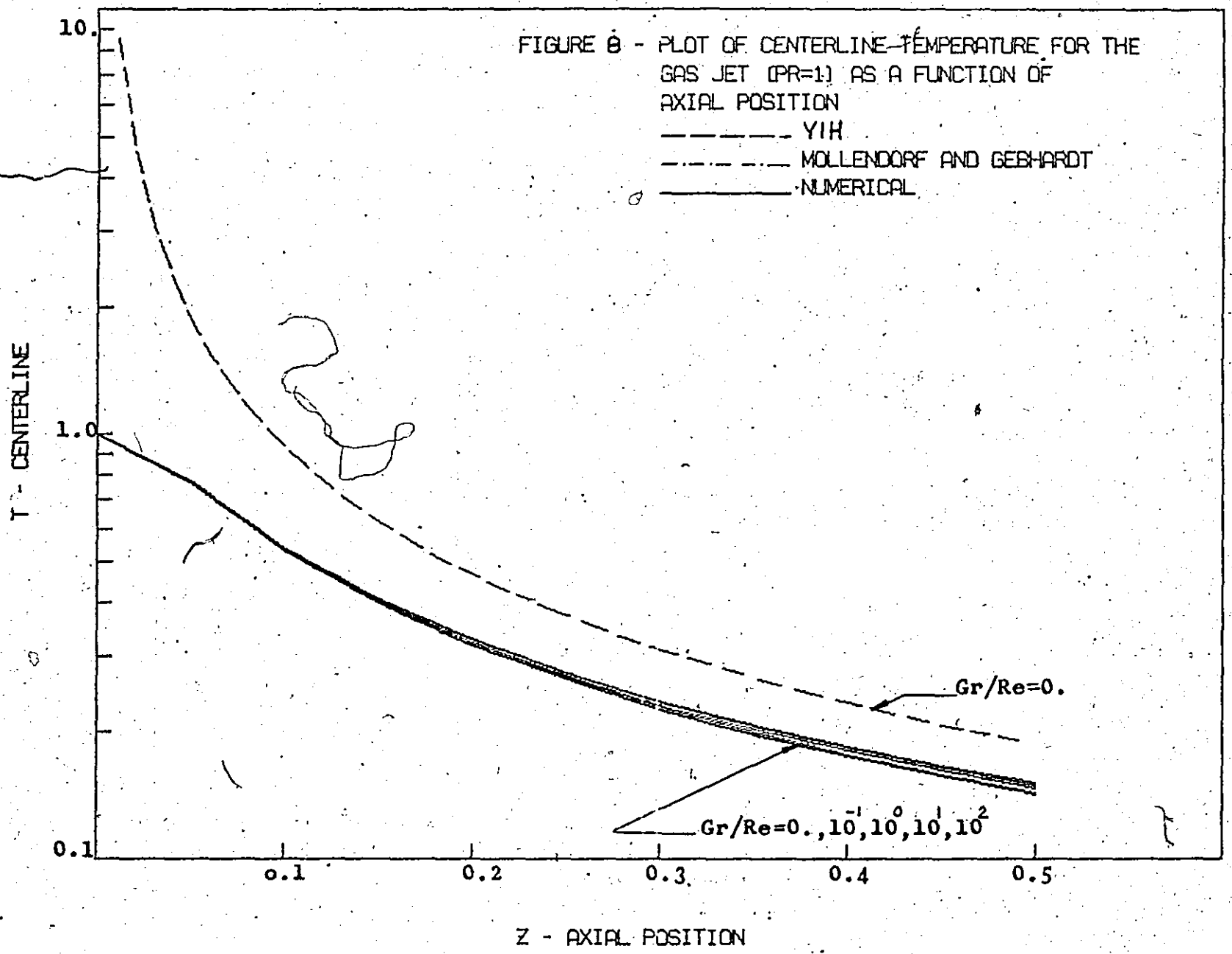

FIGURE 8 - PLOT OF CENTERLINE TEMPERATURE FOR THE GAS JET (PR=1) AS A FUNCTION OF AXIAL POSITION

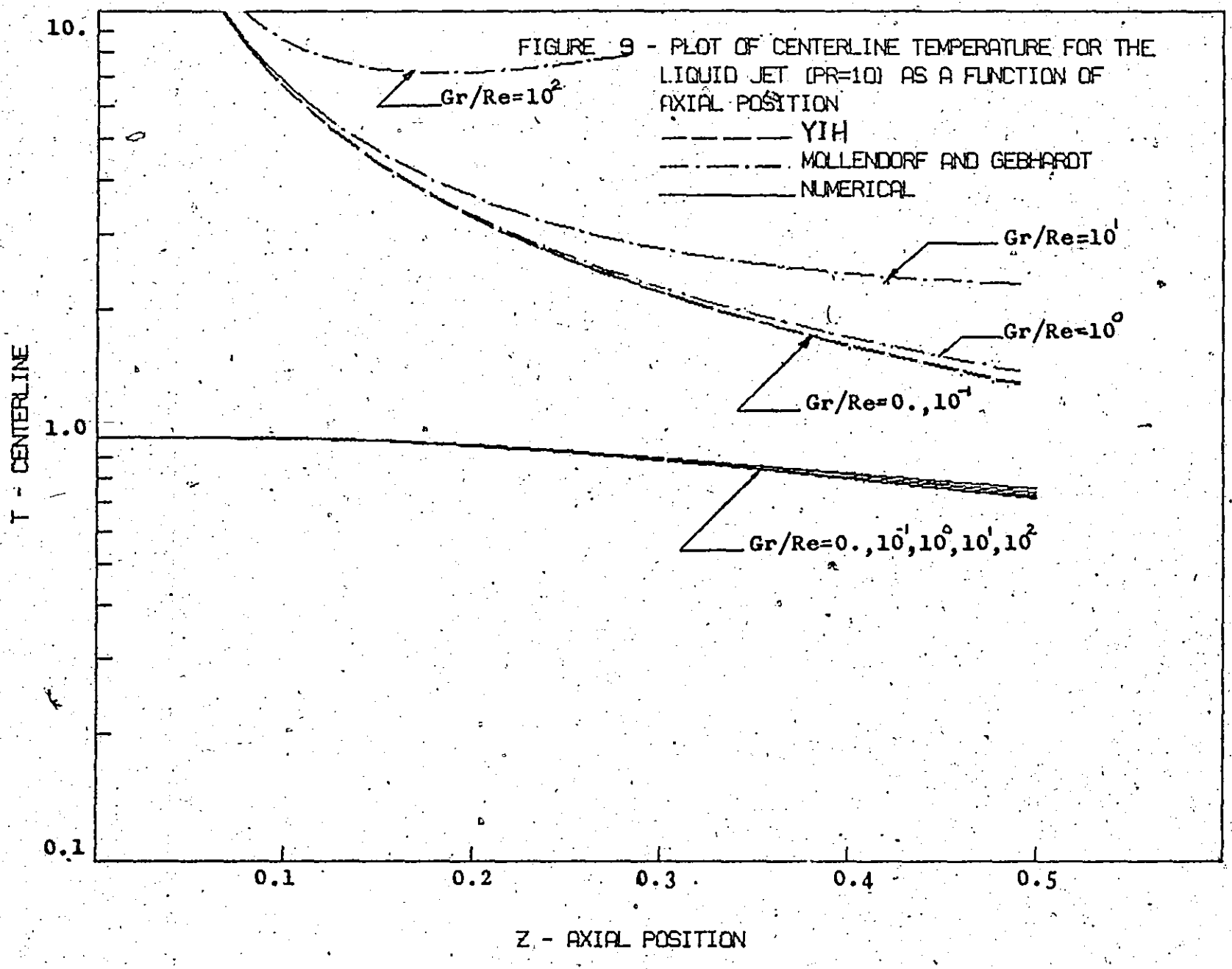

FIGURE 9 - PLOT OF CENTERLINE TEMPERATURE FOR THE LIQUID JET (PR=10) AS A FUNCTION OF AXIAL POSITION

From Tables VII and VIII it is observed that buoyancy has little effect on the centreline temperature. Mollendorf and Gebhardt (7) derive the following equation for the centreline temperature.

$$T_M = \left(\frac{2Pr+1}{32}\right) Z^{-1} + \frac{4}{9}(2Pr+1)^2\left(\frac{Gr}{Re}\right) Z\, h(Pr) \tag{21}$$

Figure 9 shows that the temperature predicted by this relationship is inadequate. For small Z the first term in equation (2) dominates and the resultant solution suffers from the same inadequacy as the similarity solution. For large Z the second term dominates and increases without bound. Morton (5) predicts the following relationship for centreline temperatures

$$T_M = \left(\frac{1}{8E}\right)\left(\frac{1+Pr}{Pr^2}\right) Z^{-1} \tag{22}$$

It suggests that the centreline temperature is inversely proportional to the axial position. Their relationship also predicts that T is independent of Gr/Re. Comparison of the numerical values for relative centreline temperatures* as recorded in Tables IX and X agrees with this observation.

The temperature half-widths are recorded in Figures 10 and 11. As in the case of the axial velocity the temperature profile thins. The degree of thinning increases for increasing Gr/Re and Pr.

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* For a given Gr/Re, the relative centreline temperature is defined as the ratio between the absolute centreline temperature and the corresponding centreline temperature for Gr/Re=1.

TABLE IX

Relative Centreline Temperatures
for Gas Jet (Pr=1)
at Z (=z/DRe) = 0.50

Gr/Re	T_{rel} Morton	T_{rel} Numerical
10^{-1}	1.0000	1.0400
10^{0}	1.0000	1.0000
10^{1}	1.0000	0.9914
10^{2}	1.0000	1.0231
10^{3}	1.0000	1.0283

TABLE X

Relative Centreline Temperatures
for Liquid Jet (Pr=10)
at Z (=z/DRe) = 0.50

Gr/Re	T_{rel} Morton	T_{rel} Numerical
10^{-1}	1.0000	0.9884
10^{0}	1.0000	1.0000
10^{1}	1.0000	1.0162
10^{2}	1.0000	1.0353
10^{3}	1.0000	1.0022

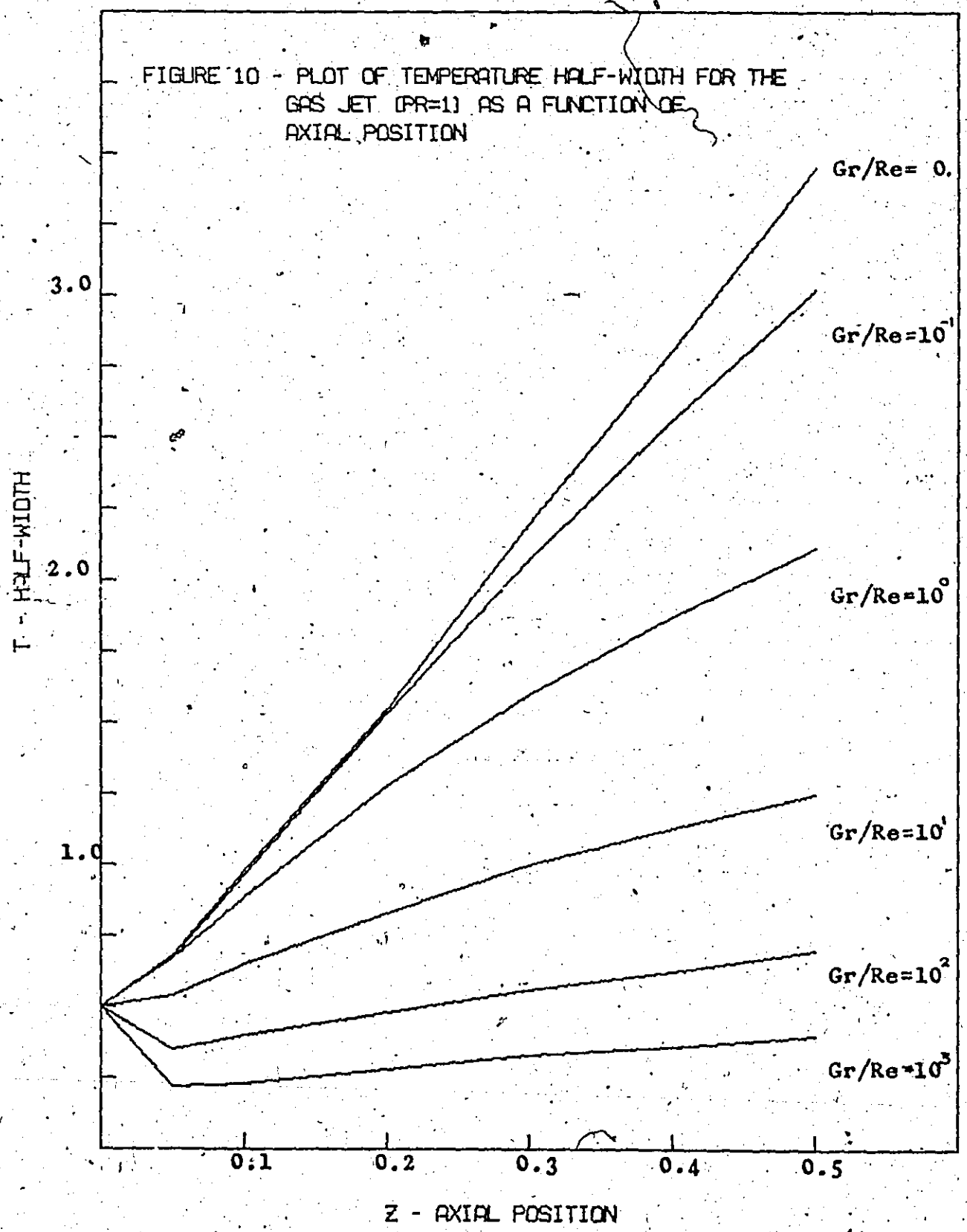
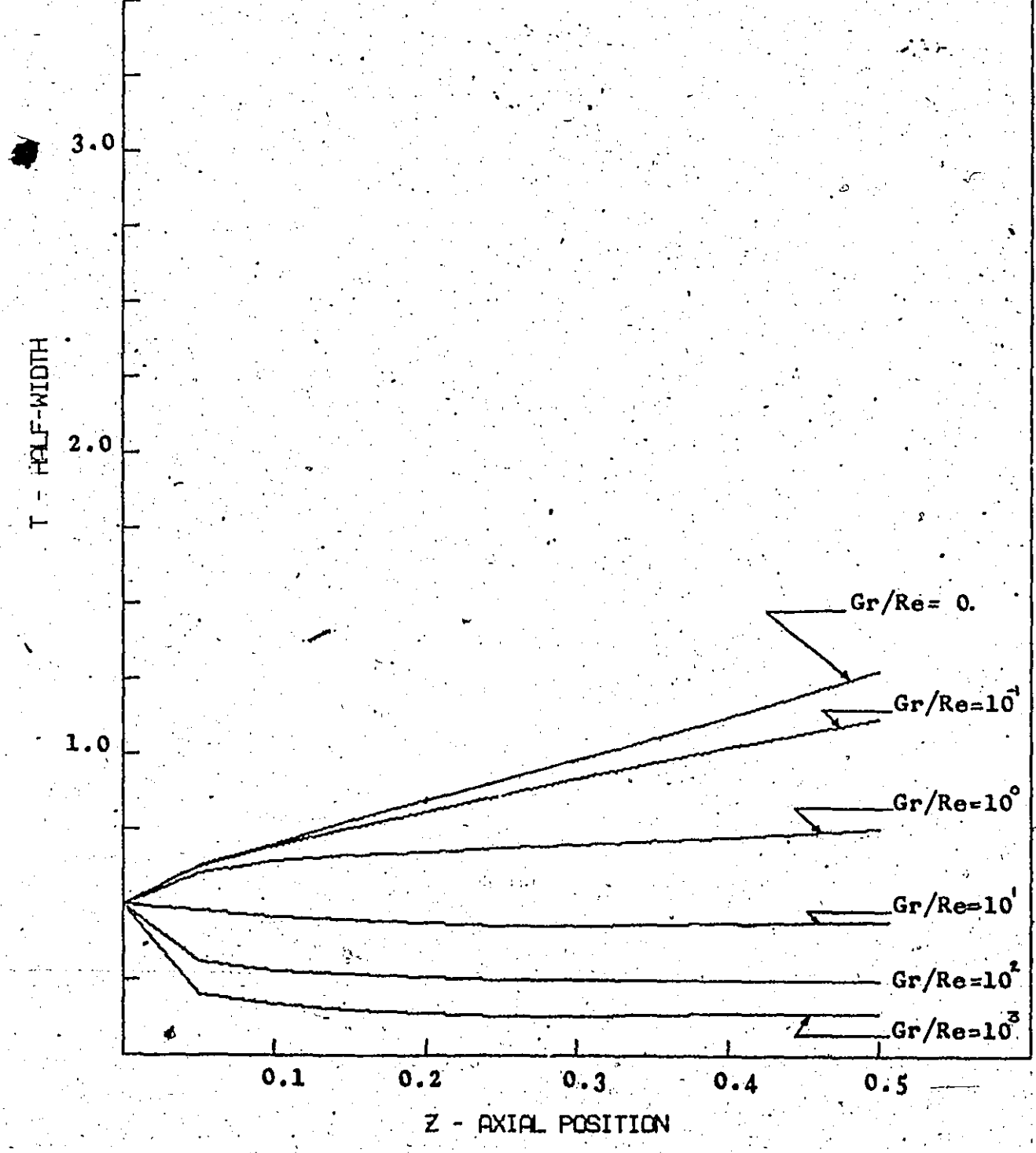


FIGURE 10 - PLOT OF TEMPERATURE HALF-WIDTH FOR THE GAS JET (PR=1) AS A FUNCTION OF AXIAL POSITION

FIGURE 11 - PLOT OF TEMPERATURE HALF-WIDTH FOR THE LIQUID JET (PR=10) AS A FUNCTION OF AXIAL POSITION



V - CONCLUSIONS AND RECOMMENDATIONS

The effect of thermal buoyancy on laminar vertical axisymmetric jets has been analysed. The results show that the effect is a function of Prandtl, Reynolds and Grashof numbers. The ratio, Gr/Re, is of particular importance because it is a measure of the relative strength of buoyancy and inertial forces. As Gr/Re increases from zero the predominant effect is to increase the axial velocity.

Agreement between the numerical and analytical solutions to the buoyant jet problem is not good. Mollendorf and Gebhardt's perturbation solution approaches and then diverges from the numerical solution. Brand and Lahey's solution may at best agree for large Z. Morton's entrainment model gives the best agreement. But, even this agreement is only good in a relative sense. In general, due to the coupling of momentum and energy equations when buoyancy is present, the only suitable method for studying the resultant flow is a numerical technique such as the one used in this study.

Recommendations are in two directions, experimental and theoretical. Since the author has found little experimental work in this field, it is recommended that the experimental side of this work be done. On the theoretical side we recommend that the extension of this project include turbulent

jets, the effect of negative bouyancy, (i.e. cold jet issuing into a hot medium) and the effect of buoyancy caused by density differences. In the latter the momentum equation would no longer be coupled to the energy equations. Instead it would be coupled to a conservation of species equation. The study of thermal buoyancy in turbulent jets is important because most buoyant flows in nature occur in the turbulent regime.

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NOTATION

A, Am	-	Coefficients defined in Appendix B
B, Bm	-	Coefficients defined in Appendix B
C, Cm	-	Coefficients defined in Appendix B
D, Dm	-	Coefficients defined in Appendix B
D	-	Nozzle diameter
E, Em	-	Coefficients defined in Appendix B
E	-	Morton;s entrainment parameter
F, Fm	-	Coefficients defined in Appendix B
Gr	-	Grashof number
H, Hm	-	Coefficients defined in Appendix B
Pr	-	Prandtl number
Q_1, Q_2, Q_3	-	Coefficients defined in Appendix B
Re	-	Reynolds number
R	-	Dimensionless radial coordinate, r/D
T	-	Dimensionless temperature, (t-ta)/(to-ta)
V_R	-	Dimensionless radial velocity, (Vr/Vo)Re
V_Z	-	Dimensionless axial velocity, (Vz/Vo)
Z	-	Dimensionless axial coordinate, Z/DRe
g	-	Gravitational constant
f, h	-	Represent similarity function defined by Mollendorf and Gebhardt (7). At the centreline f=0.10 and h=0.001 for Pr=10.
p	-	Absolute pressure
r	-	Radial coordinate
t	-	Temperature
k	-	Thermal conductivity
C_p	-	Specific heat

v - Velocity

z - Axial Coordinate

Greek letters

ΔR - Step size in R-direction

ΔZ - Step size in Z direction

β - Expansion coefficient

δ - Boundary layer thickness

λ - Iteration counter

μ - Viscosity

ρ - Density

α - Thermal diffusivity, $k/\rho c_p$

Subscripts

a - Denotes ambient conditions

k - Denotes conditions along the outer boundary

M - Denotes free jet centreline properties

m - Denotes grid points in a direction normal to the direction of flow

n - Denotes grid points in the direction of flow

o - Denotes nozzle exit properties

R,r - Denotes radial component

rel - Denotes relative velocity or temperature

t - Refers to energy

v - Refers to momentum

Z,z - Denotes axial component in free jet

APPENDIX A

THE DERIVATION OF THE BOUNDARY LAYER EQUATIONS FOR AN INCOMPRESSIBLE THERMALLY BUOYANT VERTICAL JET.

Assuming the flow is steady and incompressible, and radial pressure gradients and viscous dissipation are negligible, the general equations for the conservation of mass, momentum and energy are;

Continuity

$$\frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{\partial}{\partial Z}V_Z = 0 \qquad \text{A-1}$$

Momentum

-Z direction

$$\rho V_r \frac{\partial V_Z}{\partial r} + \rho V_Z \frac{\partial V_Z}{\partial Z} = -\frac{\partial p}{\partial Z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial V_Z}{\partial r} + \frac{\partial V_r}{\partial Z}\right)\right] + 2\mu\frac{\partial}{\partial Z}\left(\frac{\partial V_Z}{\partial Z}\right) + \rho g_Z \qquad \text{A-2}$$

-r direction

$$\rho V_r \frac{\partial V_r}{\partial r} + \rho V_Z \frac{\partial V_r}{\partial Z} = \frac{2\mu}{r}\frac{\partial}{\partial r}\left(r\frac{dV_r}{\partial r}\right) - 2\mu\frac{V_r}{r^2} + \mu\frac{d}{dZ}\left(\frac{\partial V_Z}{\partial r} + \frac{\partial V_r}{\partial Z}\right) \qquad \text{A-3}$$

Energy

$$\rho Cp\left(V_r\frac{\partial T}{\partial r} + V_Z\frac{\partial T}{\partial Z}\right) = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial Z^2}\right] \qquad \text{A-4}$$

Note also since the jet is vertical the radial gravitational term, ρg_r, is zero.

If the fluid is at rest (moving slow relative the speed of sound in the media) and has some mean temperature, the vertical pressure gradient, $\partial p/\partial z$ is given by

$$\frac{\partial p}{\partial z} = \bar{\rho} g z \qquad \text{A-5}$$

Furthermore, assuming that the Boussinesq approximation ($\Delta\rho/\rho \ll 1$) applies, the relationship between the density at any point, ρ, and the mean density, $\bar{\rho}$, is

$$\rho = \bar{\rho} - \bar{\rho}\beta(T-\bar{T}) \qquad \text{A-6}$$

Substituting equations (A-5) and (A-6) into equation (A-2) yields

$$\rho V_r \frac{\partial V_z}{\partial r} + \rho V_z \frac{\partial V_z}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left[v\mu\left(\frac{\partial V_z}{\partial r} + \frac{\partial V_r}{\partial z}\right)\right]$$

$$+ \frac{\partial}{\partial z}\left(2\mu \frac{\partial V_z}{\partial z}\right) - \bar{\rho}\bar{\beta}(t-t_a) \qquad \text{A-7}$$

Making the boundary layer assumptions that

- the thickness of the boundary layer is small compared to the length, and
- the velocity along the boundary layer is much larger than the velocity across the boundary layer

the following order of magnitude approximations can be postulated.

$$r = (\delta)$$
$$z = (1)$$
$$v_r = (\delta)$$
$$v_z = (1) \qquad \text{A-8}$$

These equations are more easily solved if they are rearranged in dimensionless form. By introducing the following dimensionless variables

$$Z = (Z/D)/Re$$

$$R = R/D$$

$$V_Z = V_Z/Vo$$

$$V_R = (V_r/Vo)Re = \frac{\rho V_r D}{\mu}$$

$$T = \frac{t - t_a}{t_o - t_a}$$

$$Re = \frac{\rho VoD}{\mu}$$

$$Gr = \frac{g\rho^2\beta D^3(t_o - t_a)}{\mu^2}$$

$$Pr = \frac{C_p\mu}{k} \qquad \text{A-9}$$

the conservation equations become

Continuity equation -

$$\frac{\partial V_Z}{\partial Z} + \frac{1}{R}\frac{\partial V_R}{\partial R} = 0 \qquad \text{A-10}$$

Momentum equation -

$$V_Z\frac{\partial V_Z}{\partial Z} + V_R\frac{\partial V_Z}{\partial R} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial V_Z}{\partial R}\right) + \frac{Gr}{Re}T \qquad \text{A-11}$$

Energy equation -

$$V_Z\frac{\partial T}{\partial Z} + V_R\frac{\partial T}{\partial R} = \frac{1}{Pr}\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial T}{\partial R} \qquad \text{A-12}$$

where δ is the boundary layer thickness.

Introducing these order of magnitude approximations in the general conservation equations makes the entire r-direction momentum equation and certain terms of the z-direction momentum and energy equations insignificant and therefore negligible if

$$\mu = (\delta^2)$$
$$g\beta = (1)$$
$$k/C_p = (\delta^2) \qquad \text{A-13}$$

Thus, the boundary layer equations are

- Continuity equation

$$\frac{\partial V_z}{\partial z} + \frac{1}{r}\frac{\partial V_r}{\partial r} = 0 \qquad \text{A-14}$$

- Momentum equation

$$\rho V_r \frac{\partial V_z}{\partial r} + \rho V_z \frac{\partial V_z}{\partial z} = \frac{\mu}{r}\frac{\partial}{\partial r} r \frac{\partial V_z}{\partial r} + \rho g \beta (t - t_a) \qquad \text{A-15}$$

- Energy equation

$$V_z \frac{\partial t}{\partial z} + V_r \frac{\partial t}{\partial r} = \frac{\alpha}{r}\frac{\partial}{\partial r} r \frac{\partial t}{\partial r} \qquad \text{A-16}$$

APPENDIX B

THE DERIVATION OF THE FREE JET DIFFERENCE EQUATIONS

Tomich (10), in developing a free jet finite difference technique, derived the difference equations and used them in writing the computer program. Their derivation is outlined below.

The dimensionless equations for the conservation of mass, momentum and energy in a thermally buoyant, incompressible free jet are

$$\frac{\partial}{\partial Z}(V_Z) + \frac{1}{R}\frac{\partial V_R}{\partial R} = 0 \qquad \text{B-1}$$

$$V_Z \frac{\partial V_Z}{\partial Z} + V_R \frac{\partial V_Z}{\partial R} = \frac{1}{R}\frac{\partial}{\partial R} R \frac{V_Z}{R} + Gr/Re\ T \qquad \text{B-2}$$

$$V_Z \frac{\partial T}{\partial Z} + V_R \frac{\partial T}{\partial R} = \frac{1}{Pr}\frac{1}{R}\frac{\partial}{\partial R} R \frac{\partial T}{\partial R} \qquad \text{B-3}$$

By expanding the above equations B-1, B-2, and B-3, we obtain, respectively

$$\frac{\partial V_Z}{\partial Z} + \frac{1}{R}\underline{V_R} + \frac{\partial V_R}{\partial R} = 0 \qquad \text{V-4}$$

$$\underline{V_Z}\frac{\partial V_Z}{\partial Z} + \underline{V_R}\frac{\partial V_Z}{\partial R} = \frac{1}{R}\frac{\partial V_Z}{\partial R} + \frac{\partial^2 V_Z}{\partial R^2} + \frac{Gr}{Re}\underline{T} \qquad \text{B-5}$$

$$\underline{V_Z}\frac{\partial T}{\partial Z} + \underline{V_R}\frac{\partial T}{\partial R} = \frac{1}{Pr}\frac{1}{R}\frac{\partial T}{\partial R} + \frac{1}{Pr}\frac{\partial^2 T}{\partial R^2} \qquad \text{B-6}$$

Now, to obtain the difference equations for the finite difference network shown in Figure B-1, the underlined variables in Equations(B-4), (B-5) and (B-6) are replaced by their respective values at a point $(Z+\Delta Z,R)$. Then, the finite difference approximations for derivatives at a point $(Z+\Delta Z,R)$ are substituted into these equations.

The following finite difference approximations were used for temperature:

$$\frac{\partial T}{\partial Z} = \frac{T(Z+\Delta R,R)-T(Z,R)}{\Delta Z} \qquad \text{B-7}$$

$$\frac{\partial T}{\partial R} = \frac{T(Z+\Delta Z,R+\ R)-T(Z+\Delta Z,R-\Delta R)}{2\Delta R} \qquad \text{B-8}$$

$$\frac{\partial^2 T}{\partial R^2} = \frac{T(Z+\Delta Z,R+\Delta R)-2T(Z+\Delta Z,R)+T(Z+\Delta Z,R-\Delta R)}{(\Delta R)^2} \qquad \text{B-9}$$

for velocity:

$$\frac{\partial V_Z}{\partial Z} = \frac{V_Z(Z+\Delta Z,R)-V_Z(Z,R)}{\Delta Z} \qquad \text{B-10}$$

$$\frac{\partial V_Z}{\partial R} \quad \frac{V_Z(Z+\Delta Z,R+\Delta R)-V_Z(Z+\Delta Z,R-\Delta R)}{2\Delta R} \qquad \text{B-11}$$

$$\frac{\partial^2 V_Z}{\partial R^2} \quad \frac{V_Z(Z+\Delta Z,R+\Delta R)-2V_Z(Z+\Delta Z,R)+V_Z(Z+\Delta Z,R-\Delta R)}{(\Delta R)^2} \qquad \text{B-12}$$

$$\frac{\partial V_R}{\partial R} = \frac{V_R(Z+\Delta Z,R)-V_R(Z+\Delta Z,R-\Delta R)}{\Delta R} \qquad \text{B-13}$$

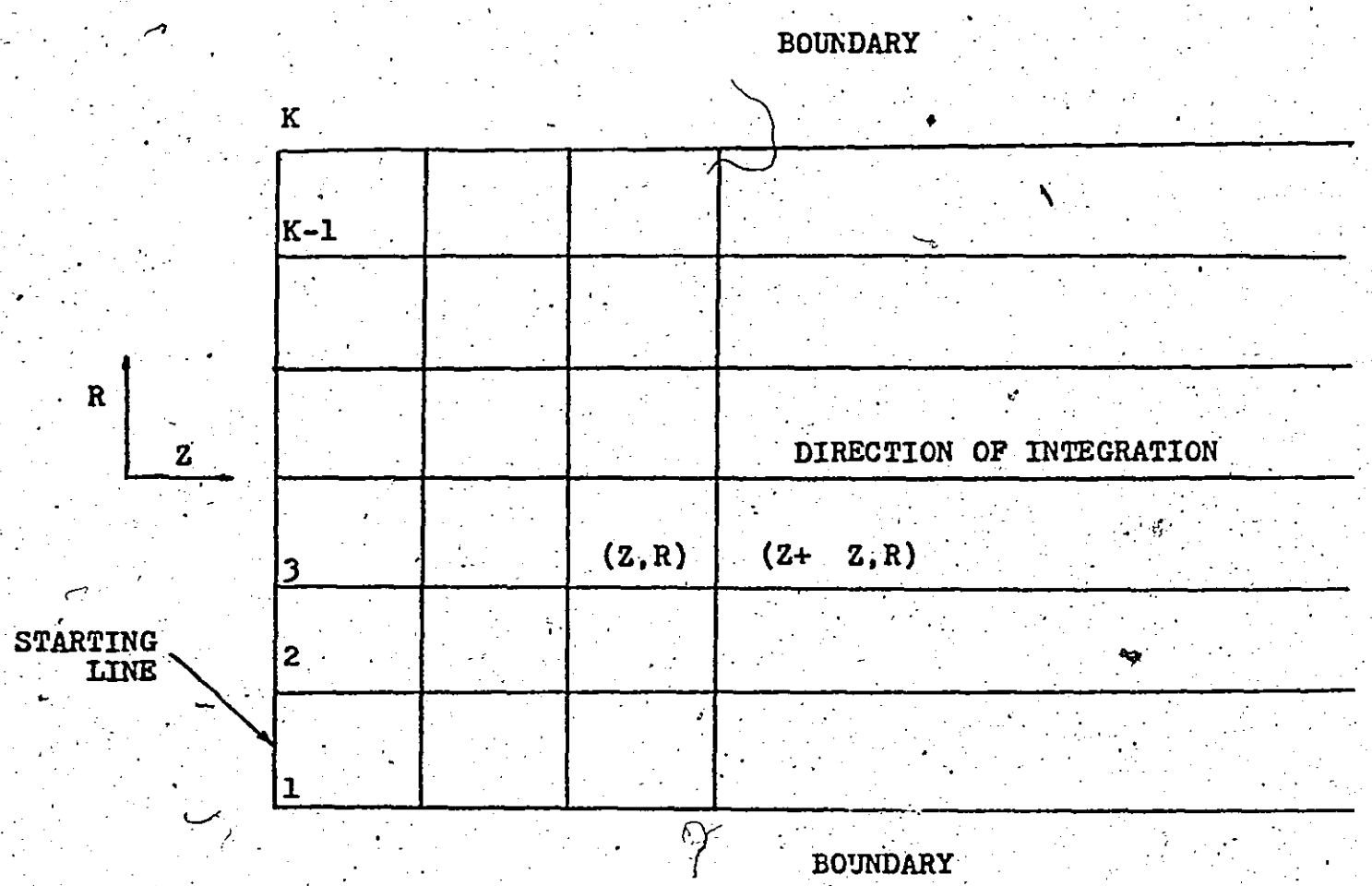


FIGURE B-1
THE FREE JET FINITE DIFFERENCE NETWORK

After making these substitutions and rearranging, the following difference equations were obtained, for energy.

$$A \cdot T(Z+\Delta Z, R) + B \cdot T(Z+\Delta Z, R+\Delta R) + C \cdot T(Z+\Delta Z, R-\Delta R) = D \qquad \text{B-14}$$

where:

$$A = \frac{V_Z(Z+\Delta Z, R)}{Z} + \frac{2}{Pr\ (\Delta R)^2} \qquad \text{B-15}$$

$$B = \frac{V_R(Z+\Delta Z, R)}{2\Delta R} - \frac{1}{2\ Pr\ R\Delta R} - \frac{1}{Pr\ (\Delta R)^2} \qquad \text{B-16}$$

$$C = \frac{V_R(Z-\Delta Z, R)}{2\Delta R} + \frac{1}{2\ Pr\ R\Delta R} - \frac{1}{Pr\ (\Delta R)^2} \qquad \text{B-17}$$

$$D = \frac{V_Z(Z+\Delta Z, R)}{\Delta R}\ T(Z, R) \qquad \text{B-18}$$

for momentum:

$$E \cdot V_Z(Z+\Delta Z, R) + F \cdot V_Z(Z+\Delta Z, R+\Delta R) + G \cdot V_Z(Z+\Delta Z, R-\Delta R) = H \qquad \text{B-19}$$

where:

$$E = \frac{V_Z(Z+\Delta Z, R)}{\Delta Z} + \frac{2}{(\Delta R)^2} \qquad \text{B-20}$$

$$F = \frac{V_R(Z+\Delta Z, R)}{2\Delta R} - \frac{1}{2\ \Delta RR} - \frac{1}{(\Delta R)^2} \qquad \text{B-21}$$

$$G = \frac{V_R\ (Z+\Delta Z, R)}{2\,\Delta R} + \frac{1}{2\ \Delta\ RR} - \frac{1}{(\Delta R)^2} \qquad \text{B-22}$$

$$H = \frac{V_Z(Z+\Delta Z, R)}{\Delta Z}\ V_Z(Z,R) + \frac{Gr}{Re}\ T\ (Z+\Delta Z\ R) \qquad \text{B-23}$$

Then, the continuity equation was written in an explicit form

$$V_R(Z+\Delta Z, R) = Q_1(Q_2+Q_3) \qquad \text{B-24}$$

where:

$$Q_1 = 1/\left(\frac{1}{R} + \frac{1}{\Delta R}\right) \qquad \text{B-25}$$

$$Q_2 = \frac{V_Z(Z,R) - V_Z(Z+\Delta Z, R)}{\Delta Z} \qquad \text{B-26}$$

$$Q_3 = V_R(Z+\Delta Z, R-\Delta R)/\Delta R \qquad \text{B-27}$$

Now, in order to write the equations for the points 1,2,3,...k of the finite difference network of Figure B-1 we start from the axis of symmetry along which the points are labelled with the subscript 1. First we apply the boundary conditions, Equations (9), to the differential equations B-1, B-2, and B-3, then we make the finite difference substitutions. The following difference equations are obtained.

for energy:

$$A_1 \cdot T(Z+\Delta Z,1) + B_1 \cdot T(Z+\Delta Z,2) = D_1 \qquad \text{B-28}$$

where:

$$A_1 = \frac{V_Z(Z+\Delta Z,1)}{\Delta Z} + Pr \frac{4}{(\Delta R)^2} \qquad \text{B-29}$$

$$B_1 = - Pr \frac{4}{(\Delta R)^2} \qquad \text{B-30}$$

$$D_1 = \frac{V_Z(Z-\Delta Z,1)}{\Delta Z} T(Z,1) \qquad \text{B-31}$$

for momentum:

$$E_1 \cdot V_Z(Z+\Delta Z,1) + F_1 \cdot V_Z(Z+\Delta Z,2) = H_1 \qquad \text{B-32}$$

where:

$$E_1 = \frac{V(Z+\Delta Z,1)}{\Delta Z} + \frac{4}{(\Delta R)^2} \qquad \text{B-33}$$

$$F_1 = - \frac{4}{(\Delta R)^2} \qquad \text{B-34}$$

$$H_1 = \frac{V_Z(Z+\Delta Z,1)}{\Delta Z} V(Z,1) + \frac{Gr}{Re} T(Z+ Z,1) \qquad \text{B-35}$$

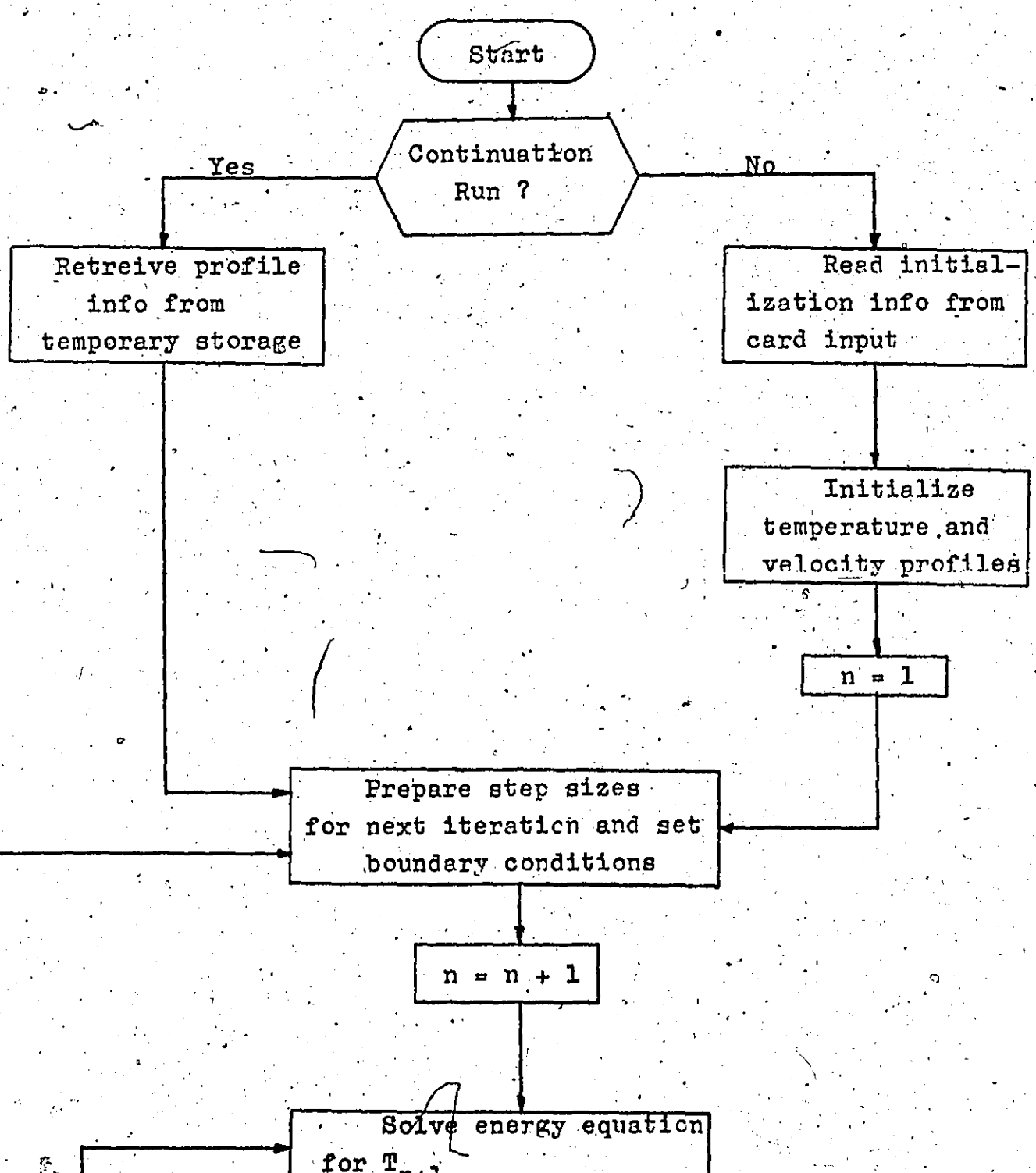
The difference equations for the rest of the points are directly obtained from the general form, Equations B-14, B-19, and B-24.

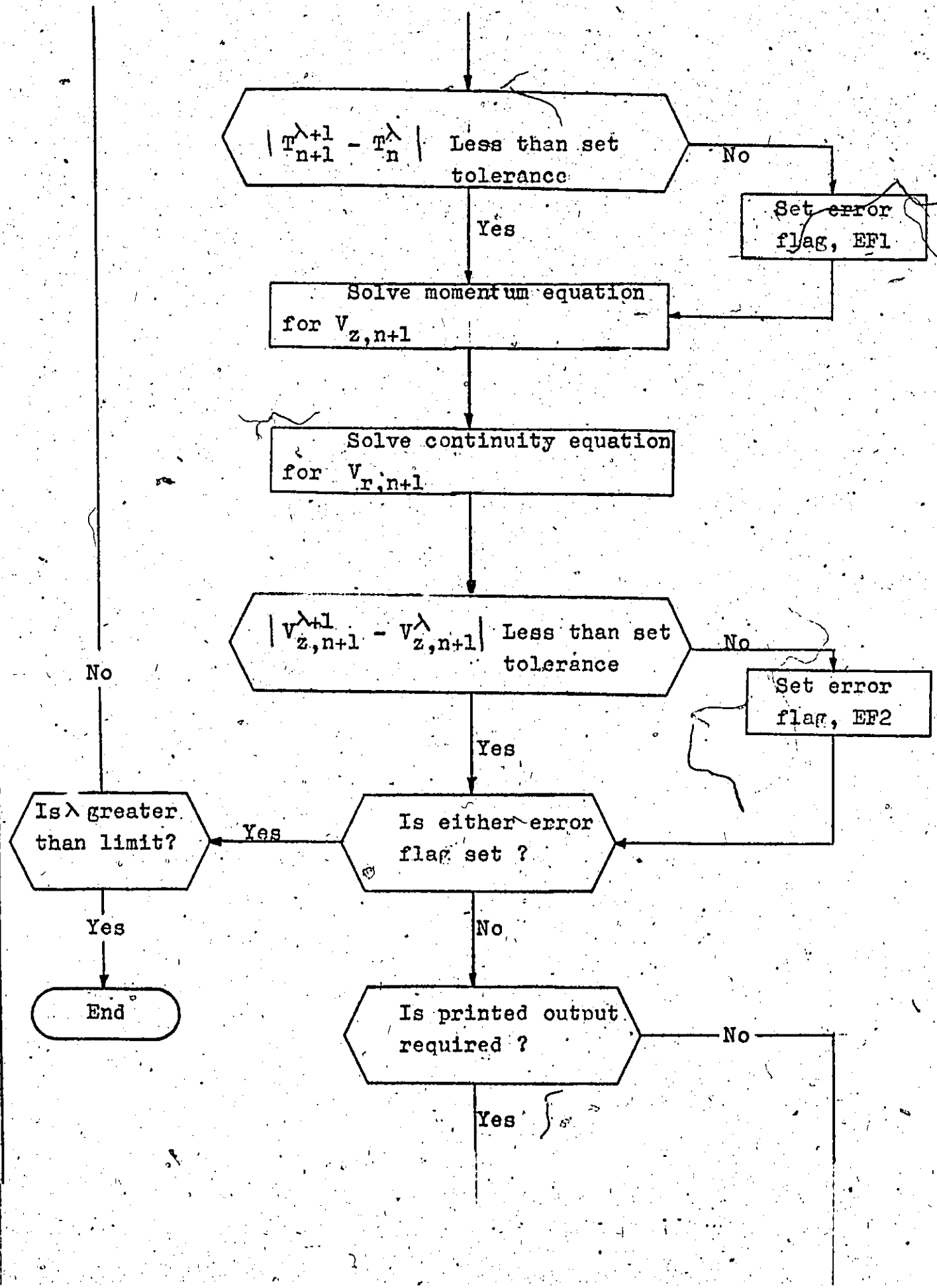
APPENDIX C

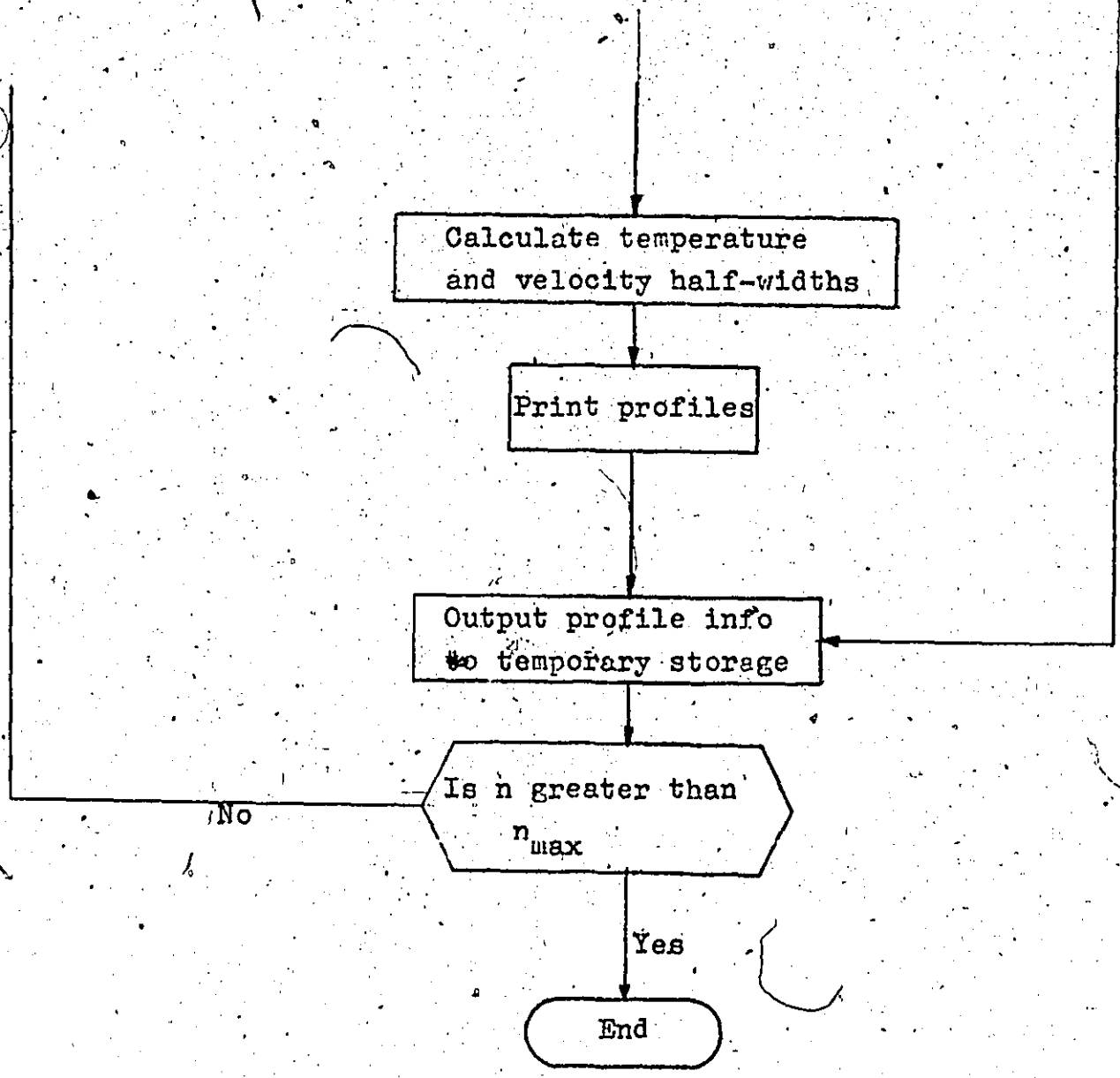
Algorithm and Computer Programme

The object of the numerical calculation was to determine both temperature and velocity profiles in a thermally buoyant jet. This object was achieved by using the following algorithm and computer programme.

Algorithm


Start
Continuation Run ?
Yes
No
Retreive profile info from temporary storage
Read initialization info from card input
Initialize temperature and velocity profiles
n = 1
Prepare step sizes for next iteration and set boundary conditions
n = n + 1
Solve energy equation for T_{n+1}


$|T_{n+1}^{\lambda+1} - T_n^{\lambda}|$ Less than set tolerance
No
Set error flag, EF1
Yes
Solve momentum equation for $V_{z,n+1}$
Solve continuity equation for $V_{r,n+1}$
$|V_{z,n+1}^{\lambda+1} - V_{z,n+1}^{\lambda}|$ Less than set tolerance
No
Set error flag, EF2
Yes
Is λ greater than limit?
Yes
Is either error flag set ?
No
Yes
End
No
Is printed output required ?
No
Yes


Calculate temperature and velocity half-widths
Print profiles
Output profile info to temporary storage
Is n greater than n_{max}
No
Yes
End

COMPUTER PROGRAMME

```
      PROGRAM TST(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT,TAPE3,TAPE2)
C
C
C
C     THERMALLY BUOYANT AXISYMMETRIC LAMINAR JET
C
C     INPUT DATA
C
C     TO-JET EXIT TEMPERATURE,DEG. R
C     TA-AMBIENT TEMPERATURE,DEG. R
C     UO-JET EXIT VELOCITY,FT./SEC.
C     DELZ1-FIRST X-DIRECTION FINITE DIFFERENCE STEP SIZE
C     DELZ2-SECOND X-DIRECTION FINITE DIFFERENCE STEP SIZE
C           THE FREE JET PROGRAM
C     AXISYMMETRIC
C     DELZ3-THIRD X-DIRECTION FINITE DIFFERENCE STEP SIZE
C     DELR1-FIRST R-DIRECTION FINITE DIFFERENCE STEP SIZE
C     DELR2-SECOND R-DIRECTION FINITE DIFFERENCE STEP SIZE
C     DELR3-THIRD R-DIRECTION FINITE DIFFERENCE STEP SIZE
C     PRNO-TURBULENT PRANDTL NUMBER
C     TOL-TOLERANCE FOR ITERATIVE CONVERGENCE
C     II-NUMBER OF LINES OF THE FIRST X-DIRECTION STEP
C     SIZE PLUS ONE.
C     IS-NUMBER OF LINES OF THE SECOND X-DIRECTION STEP
C     SIZE PLUS ONE.
C     IT-NUMBER OF LINES OF THE THIRD X-DIRECTION STEP
C     SIZE PLUS ONE.
C     JI-NUMBER OF LINES OF FIRST R-DIRECTION STEP SIZE
C     JS-NUMBER OF LINES OF SECOND R-DIRECTION STEP SIZE
C     JT-NUMBER OF LINES OF THIRD R-DIRECTION STEP SIZE
C     KNOZ-X-DIRECTION LINE NUMBER OF NOZZLE
C     LIMIT-LIMITING NUMBER OF ITERATIONS
C       OUTPUT INCLUDES
C     VELOCITIES,TEMPERATURES AND DENSITIES
C     IN DIMENSIONLESS FORM
      DIMENSION OLDVZ(501),OLDVR(501),VZ(501),VR(501),T(501),
     1A(501),B(501),C(501),TEST(501),VZNC(501),VREN(501)
     2,D(501)
      DIMENSION VZM(501),TNO(501)
      DIMENSION VRM(501)
      DIMENSION OLDT(501)
      EXTERNAL TIMT
      CALL RECOVR(TIMT,148,0)
      SS=SECOND(CP)
      NPROF=0
      ITAPE=0
      ITAPE=1
C
C     ITAPE = 0 - NEW RUN
C     ITAPE = 1 - CONTINUATION RUN.  DATA STORED ON TAPE2
C
      IF(ITAPE.EQ.0) GO TO 2231
 1021 READ (2)      TO,TA,UO,DELZ1,DELZ2,DELZ3,DELR1,DELR2,DELR3,PRNO,T
 1022 READ (2)      II,IS,IT,JI,JS,JT,KNOZ,LIMIT
      READ (2)      CONST,CPC,GRAS
      READ(2) Z,DELR
      READ (2) VZ
```

```
      READ (2) VR
      READ (2) T
      READ (2) M,NPROF
      WRITE (3)    T0,TA,U0,DELZ1,DELZ2,DELZ3,DELR1,DELR2,DELR3,PRNO,T
      WRITE (3)    II,IS,IT,JI,JS,JT,KNOZ,LIMIT
      WRITE (3)    CONST,CPC,GRAS
      WRITE (3) Z,DELR
      WRITE (3) VZ
      WRITE (3) VR
      WRITE (3) T
      WRITE (3) M,NPROF
      M=M+1
 2231 CONTINUE
      IF(ITAPE.EQ.1) GO TO 2241
      READ (5,100) T0,TA,U0,DELZ1,DELZ2,DELZ3,DELR1,DELR2,DELR3,PRNO,T
  100 FORMAT (8F10.4)
      READ (5,200) II,IS,IT,JI,JS,JT,KNOZ,LIMIT
  200 FORMAT (8I5)
      READ (5,300) CONST,CPC,GRAS
  300 FORMAT (3F10.5)
      J=JI
C
C     NOZZLE EXIT PROFILES ARE GENERATED
C
        DO 440 N=1,JI
       T(N)=0.0
 440     CONTINUE
      KNOZ1=KNOZ-1
      DO 1 N=1,KNOZ1
      RPJ=N-1
      RJ=RPJ*DELR1
      OLDVZ(N)=1.0
      OLDVR(N)=0.0
    1 OLDT(N)=1.0
      DO 2 N=KNOZ,J
      OLDVZ(N)=0.0
      OLDVR(N)=0.0
    2 OLDT(N)=0.0
      DO 8 N=1,J
      VZ(N)=OLDVZ(N)
      VR(N)=OLDVR(N)
    8 T(N)=OLDT(N)
      M=2
 2241 CONTINUE
      WRITE (6,70) T0,TA,U0,DELZ1,DELZ2,DELZ3,DELR1,DELR2,DELR3,PRNO,T
   70 FORMAT (8F10.4)
      WRITE (6,71) II,IS,IT,JI,JS,JT,KNOZ,LIMIT
   71 FORMAT (8I5)
      WRITE (6,73) CONST,CPC,GRAS
   73 FORMAT (3F14.5)
      COUNT=0.0
 1002 CONTINUE
      IF(ITAPE.EQ.0) GO TO 872
      J=JI
      DO 871 N=1,J
      OLDVZ(N)=VZ(N)
      OLDVR(N)=VR(N)
```

```
  871 OLDT(N)=T(N)
  872 CONTINUE
      PAR=0.333
      J=JI
      KAPPA=DELR2/DELR1
      NU=DELR3/DELR2
      PKAP=DELZ2/DELZ1
      OMEGA=DELZ3/DELZ1
      POS=1.0
      SET=51
C
C     PREPARE STEP SIZES FOR NEXT ITERATION
C        AND SET BOUNDARY CONDITIONS
C
 2001 IF (II-M) 1003,2003,3003
 1003 IF (IS-M) 4003,5003,6003
 4003 IF (IT-M) 2000,2000,7003
 3003 ZP1=M-1
      Z=ZP1*DELZ1
      DELZ=DELZ1
      DELR=DELR1
      J=JI
      POS=POS+1.0
      GO TO 1001
 2003 DO 13 N=1,JI,KAPPA
      NEW=(N-1)/KAPPA+1
      OLDVZ(NEW)=OLDVZ(N)
      OLDVR(NEW)=OLDVR(N)
   13 OLDT(NEW)=OLDT(N)
      DO 14 N=NEW,JS
      OLDVZ(N)=0.0
      NM=N-1
      RP=N
      RPM=N-1
      OLDVR(N)=OLDVR(NM)*RPM/RP
   14 OLDT(N)=0.0
 6003 ZP1=(II-2)
      ZP2=M-II+1
      Z=ZP1*DELZ1+ZP2*DELZ2
      DELZ=DELZ2
      DELR=DELR2
      J=JS
      POS=POS+PKAP
      GO TO 1001
 5003 DO 15 N=1,JS,NU
      NEW=(N-1)/NU+1
      OLDVZ(NEW)=OLDVZ(N)
      OLDVR(NEW)=OLDVR(N)
   15 OLDT(NEW)=OLDT(N)
      DO 16 N=NEW,JT
      OLDVZ(N)=0.0
      NM=N-1
      RP=N
      RPM=N-1
      OLDVR(N)=OLDVR(NM)*RPM/RP
   16 OLDT(N)=0.0
 7003 ZP2=IS-II
```

```
      ZP3=M-IS+1
      ZP1=II-2
      Z=ZP1*DELZ1+ZP2*DELZ2+ZP3*DELZ3
      DELZ=DELZ3
      DELR=DELR3
      J=JT
      POS=POS+OMEGA
 1001 INCR=0.0
      J1=J-1
      J2=J-2
      REYK=CONST*CPC
      PRK=CONST*CPC/PRNO
      C1=REYK/(DELR*DELR)
      CC1=PRK/(DELR*DELR)
 3001 WRITE (6,50)
   50 FORMAT(* T                TNO              VZM              VZNO
     1  VRM              ZDRE              RR              M              N*,/)
      VZ(J)=0.0
      T(J)=0.0
C
C     SOLVE ENERGY EQUATIONS FOR TEMPERATURE
C
 8001 A(1)=VZ(1)/DELZ+4.0*CC1
      B(1)=-4.0*CC1
      IF(T0-TA) 75,76,75
   76 D(1)=0.0
      GO TO 86
   75 CONTINUE
      D(1)=VZ(1)/DELZ*OLDT(1)
   86 CONTINUE
      DO 18 N=2,J1
      RP=N-1
      R=RP*DELR
      CC2=PRK/(2.0*R*DELR)
      CC3=VR(N) /(2.0*DELR)
      CC4=VZ(N)/DELZ
      NP=N+1
      NM=N-1
      C(N)=CC2-CC3-CC1
      A(N)=CC4+2.0*CC1
      B(N)=CC3-CC2-CC1
      IF(T0-TA) 77,78,77
   78 THETA=0.0
      GO TO 88
   77 THETA=OLDT(N)
   88 CONTINUE
   18 D(N)=CC4*THETA
      DO 22 N=1,J
   22 TEST(N)=T(N)
      D(1)=D(1)/A(1)
      DO 19 N=2,J1
      N1=N-1
      B(N1)=B(N1)/A(N1)
      A(N)=A(N)-C(N)*B(N1)
   19 D(N)=(D(N)-C(N)*D(N1))/A(N)
      T(J1)=D(J1)
      DO 20 N=1,J2
```

```
      NN=J2-N+1
      NN1=NN+1
   20 T(NN)=D(NN)-B(NN)*T(NN1)
C
C     APPLY WEGSTEIN CONVERGENCE PROMCTICN
C
      DO 66 N=1,J
      T(N)=PAR*TEST(N)+(1.0-PAR)*T(N)
   66 CONTINUE
      DO 23 N=1,J
C
C     TEST TEMPERATURE PROFILE FOR CONVERGENCE
C
      DIFF=ABS(TEST(N)-T(N))
      IF(DIFF-TOL) 23,23,9001
   23 CONTINUE
      ERR1=0.0
      GO TO 24
 9001 ERR1=1.0
C
C     SOLVE MOMENTUM EQUATION FOR AXIAL VELOCITY
C
   24 A(1)=VZ(1)/DELZ+4.0*C1
      B(1)=-4.0*C1
      D(1)=VZ(1)*OLDVZ(1)/DELZ
      D(1)=D(1)+GRAS*(REYK**2)*T(1)
      DO 3 N=2,J1
      RP=N-1
      R=RP*DELR
      C2=REYK/(2.0*R*DELR)
      C3=VR(N)/(2.0*DELR)
      C4=VZ(N)/DELZ
      C(N)=C2-C3-C1
      A(N)=C4+2.0*C1
      B(N)=C3-C2-C1
      D(N)=C4*OLDVZ(N)
      D(N)=D(N)+GRAS*(REYK**2)*T(N)
    3 CONTINUE
      DO 17 N=1,J
   17 TEST(N)=VZ(N)
      D(1)=D(1)/A(1)
      DO 4 N=2,J1
      N1=N-1
      B(N1)=B(N1)/A(N1)
      A(N)=A(N)-C(N)*B(N1)
    4 D(N)=(D(N)-C(N)*D(N1))/A(N)
      VZ(J1)=D(J1)
      DO 5 N=1,J2
      NN=J2-N+1
      NN1=NN+1
    5 VZ(NN)=D(NN)-B(NN)*VZ(NN1)
      DO 666 N=1,J
C
C     APPLY WEGSTEIN CONVERGENCE PROMCTION TO AXIAL VELOCITY
C
  666 VZ(N)=PAR*TEST(N)+(1.0-PAR)*VZ(N)
C
```

```
C     SOLVE CONTINUITY EQUATION EXPLICITLY FOR RADIAL VELOCITY
C
      DO 6 N=2,J
      N1=N-1
      RP=N1
      R=RP*DELR
      CE1=1./(1./R+1./DELR)
      CE2=(OLDVZ(N)-VZ(N))/DELZ
      CE3=VR(N1)/DELR
    6 VR(N)=CE1*(CE2+CE3)
      DO 10 N=1,J1
      ERROR=ABS(TEST(N)-VZ(N))
C
C     TEST AXIAL VELOCITY FOR CONVERGENCE
C
      IF (ERROR-TOL) 10,10,5001
   10 CONTINUE
C
C     PROCECD TO NEXT AXIAL POSITION IF BOTH TEMPERATURE AND AXIAL VELC
C        HAVE CONVERGED
C
      IF (ERR1) 26,26,5001
   26 CONTINUE
      IF(POS-SET) 4001,3002,3002
C
C     IF POS EQUALS SET OUTPUT PROFILES
C
 3002 POS=1.0
      TX=1./T(1)
      VZA=1./VZ(1)
      DO 5553 N=1,J
      XT=T(N)*TX
      XV=VZ(N)*VZA
      IF(XT.GT.0.35) JBT=N
      IF(XV.GT.0.35) JBV=N
      IF(XT.GT.0.75) JBBT=N
      IF(XV.GT.0.75) JBBV=N
 5553 CONTINUE
C
C     CALCULATE VELOCITY HALF WIDTH
C
      NCTRF=1
      JBF=JBV
      IF(JBF.LT.1) JBF=1
      JBBF=JBBV
      IF(JBF.EQ.JBBF) JBBF=JBF+1
      FB=VZA*VZ(JBF)-0.5
      RP=JBF-1
      RF=RP*DELR
 5501 FBB=VZA*VZ(JBBF)-0.5
      RP=JBBF-1
      RFF=RP*DELR
      IF((FB-FBB).EQ.0.0) STOP 261
      BVZ=(RFF*FB-RF*FBB)/(FB-FBB)
      IF(NCTRF.EQ.1) GO TO 5502
      DIF=BVZVZ-BVZ
      IF(ABS(DIF).LE.1.E-02) GO TO 5503
```

```
 5502 BVZVZ=BVZ
      JST=JBBF
      JBBF=BVZ/DELR+1.
      IF(JBBF.GT.501) JBBF=(501+JST)/2
      IF(JBBF.LT.1) JBBF=(1+JST)/2
      IF(JBF.EQ.JBBF) JBBF=JBF+1
      NCTRF=NCTRF+1
      IF(NCTRF.GT.50) STOP 271
      GO TO 5501
 5503 CONTINUE
C
C     CALCULATE TEMPERATURE HALF WIDTH
C
      NCTRF=1
      JBF=JBT
      IF(JBF.LT.1) JBF=1
      JBBF=JBBT
      IF(JBF.EQ.JBBF) JBBF=JBF+1
      FB=TX*T(JBF)-0.5
      RP=JBF-1
      RF=RP*DELR
 5511 FBB=TX*T(JBBF)-0.5
      RP=JBBF-1
      RFF=RP*DELR
      IF((FB-FBB).EQ.0.0) STOP 263
      BT =(RFF*FB-RF*FBB)/(FB-FBB)
      IF(NCTRF.EQ.1) GO TO 5512
      DIF=BTT-BT
      IF(ABS(DIF).LE.1.E-02) GO TO 5513
 5512 BTT=BT
      JST=JBBF
      JBBF=BT /DELR+1.
      IF(JBBF.GT.501) JBBF=(501+JST)/2
      IF(JBBF.LT.1) JBBF=(1+JST)/2
      IF(JBF.EQ.JBBF) JBBF=JBF+1
      NCTRF=NCTRF+1
      IF(NCTRF.GT.50) STOP 273
      GO TO 5511
 5513 CONTINUE
C
C     OUTPUT PROFILES
C
      NPRT=1
      PRINT 1660
 1660 FORMAT(1H1,//)
      PRINT 1661
 1661 FORMAT(39X,55H***   THERMAL  BOUYANCY IN LAMINAR  VERTICAL  JETS
     1***,/)
      NPROF=NPROF+1
      PRINT 1665, NPROF
 1665 FORMAT(105X,*PROFILE NUMBER = *,I5,/)
      RE=1./REYK
      PRINT 1662,RE,BVZ,GRAS,BT,PRNO
 1662 FORMAT(4X,*FOR      RE = *,G10.3,60X,*VELOCITY HALF WIDTH = *,
     1G15.7,/,11X,*GR = *,G15.7,56X,*TEMPERATURE HALF WIDTH = *,G15.7,
     2,11X,*PR = *,F10.2,/)
      PRINT 1663
```

```
 1663 FORMAT(133H**-------------------------------------------------------
     3-------------------------------------------------------------------
     3-----------*,/,2H**,30X,1H*,31X,1H*,18X,1H*,11X,*POSITION*,7X,1H
     3,21X,1H*,/,2H**,12X,*TEMPERATURE*,7X,1H*,10X,*AXIAL VELOCITY*,7X
     3H*,
     3 1X,*RADIAL VELOCITY*,2X,1H*,7X,*DIMENSIONLESS*,6X,1H*,3X,*GRID
     3SITION*,5X,1H*,/,133H**-------------------------------------------
     3-------------------------------------------------------------------
     3---------------------*,/,2H**,1X,*DIMENSIONLESS*,2X,1H+,2X,*NORM
     3IZED*,1X,1H*,1X,*DINENSIONLESS*,1X,1H+,3X,*NORMALIZED*,2X,1H*,1X
     3*DIMENSIONLESS*,4X,1H*,2X,*AXIAL*,6X,1H+,5X,*RADIAL*,1X,1H*,1X,
     3*AXIAL*,2X,1H+,3X,*RADIAL*,3X,1H*,/,133H**------------------------
     3-------------------------------------------------------------------
     3-----------------------------------------*,/)
      DO 7 N=1,J,5
      NPRT=NPRT+1
      IF(NPRT.LT.46) GO TO 1557
      NPRT=1
 1557 IF(NPRT.NE.1) GO TO 1558
      PRINT 1660
      PRINT 1663
 1558 CONTINUE
      RP=N-1
      R=RP*DELR
      ZDRE=Z*REYK
      RR=R
      VZM(N)=VZ(N)
      VZNO(N)=VZM(N)/VZM(1)
      TNO(N)=T(N)/T(1)
      VRM(N)=VR(N)/REYK
      WRITE(6,150) T(N),TNO(N),VZM(N),VZNO(N),VRM(N),ZDRE,RR,M,N
  150 FORMAT(5F15.8,4X,2F15.8,I7,I9)
    7 CONTINUE
C
C     STORE MOST RECENT PROFILES ON TAPE3
C
      PRINT 1660
 4001 REWIND 3
      WRITE (3)    T0,TA,U0,DELZ1,DELZ2,DELZ3,DELR1,DELR2,DELR3,PRNO,T
      WRITE (3)    II,IS,IT,JI,JS,JT,KNOZ,LIMIT
      WRITE (3)    CONST,CPC,GRAS
      WRITE (3) Z,DELR
      WRITE (3) VZ
      WRITE (3) VR
      WRITE (3) T
      WRITE (3) M,NPROF
      ENDFILE 3
      DO 9 N=1,J
      OLDVZ(N)=VZ(N)
      OLDVR(N)=VR(N)
    9 OLDT(N)=T(N)
      WRITE (6,999) INCR
  999 FORMAT (I5)
      M=M+1
C
C     EXIT IF DOWNSTREAM LIMIT IS MET
C
```

```
      IF(IT-M) 3000,3996,3996
 3996 CONTINUE
      GO TO 2001
 5001 INCR=INCR+1
      SF=SECOND(CF)
      SEC=SF-SS
      SS=SF
      TMLF=XTIME(2)-SEC
      IF(TMLF.LT.SEC) STOP541
      IF (INCR-LIMIT) 1009,1009,7001
 1009 GO TO 8001
 7001 DO 12 N=1,J
      RP=N-1
      R=RP*DELR
      WRITE(6,250) VZ(N),VR(N),T(N),M,N,Z,R
  250 FORMAT(3F18.8,2I5,2F15.8)
      WRITE(6,809) TEST(N)
  809 FORMAT(F20.8)
   12 CONTINUE
      WRITE (6,9999) INCR
 9999 FORMAT (I5)
      GO TO 2000
 2000 CONTINUE
 3000 CONTINUE
       END
```

```
RY OF CHANGES MADE BY THE OPTIMIZER
 WORDS OF INVARIANT RLIST REMOVED FROM THE LOOP BEGINNING AT LINE 235
 WORDS OF INVARIANT RLIST REMOVED FROM THE LOOP BEGINNING AT LINE 255
 WORDS OF INVARIANT RLIST REMOVED FROM THE LOOP BEGINNING AT LINE 280
 WORDS OF INVARIANT RLIST REMOVED FROM THE LOOP BEGINNING AT LINE 288

TER ALLOCATION
ISTERS ASSIGNED OVER THE LOOP BEGINNING AT LINE 235
ISTERS ASSIGNED OVER THE LOOP BEGINNING AT LINE 280
```

```
      SUBROUTINE TIMT(NT,IGAIN,NUL)
      DIMENSION NT(17)
      INTEGER B0
      B0=(NT(1).AND.177B)
      IF(B0.NE.8) GO TO 1
      IGAIN=0
      RETURN
    1 IGAIN=1
      STOP333
      END
```

APPENDIX D

ADDITIONAL RESULTS

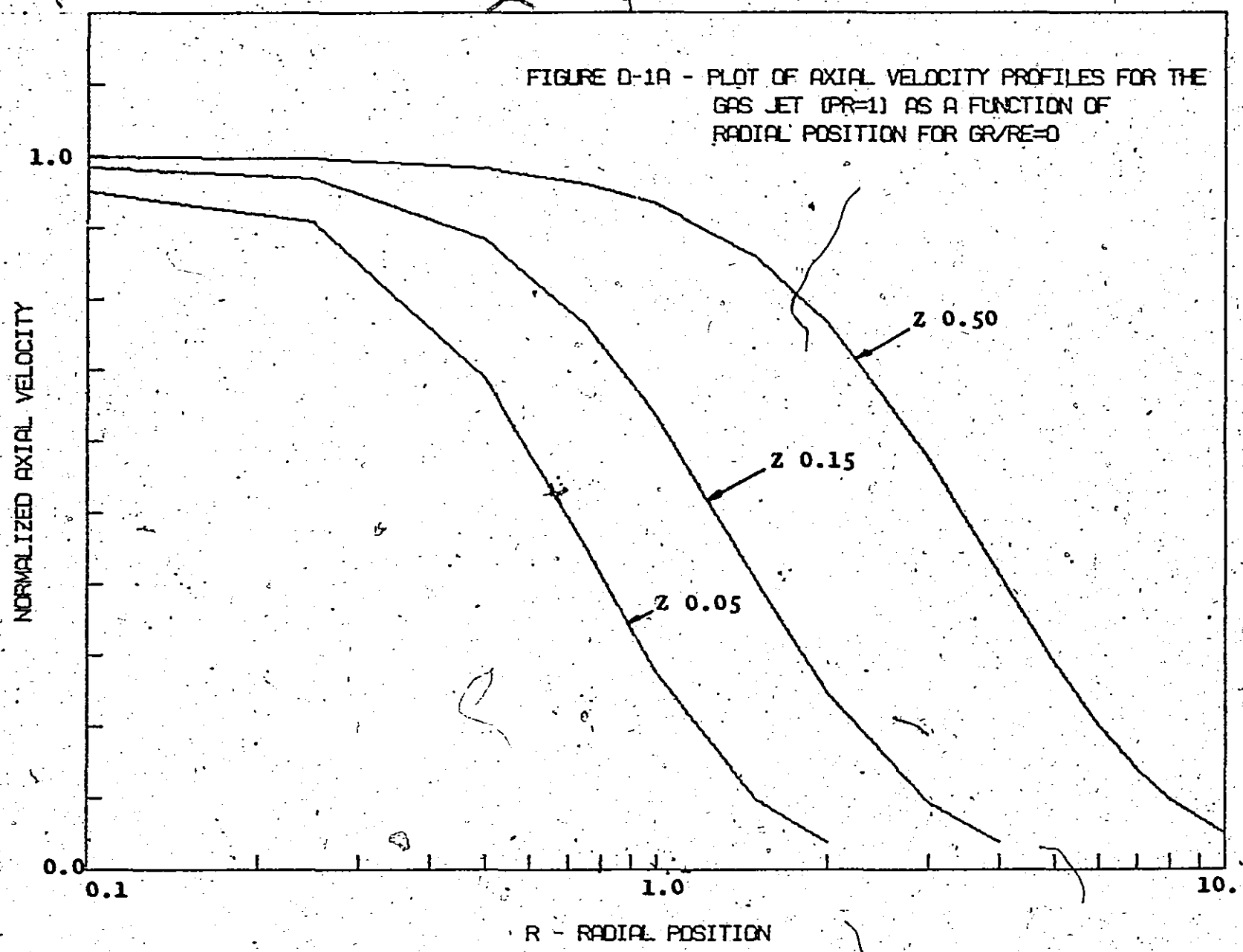


FIGURE D-1A - PLOT OF AXIAL VELOCITY PROFILES FOR THE GAS JET (PR=1) AS A FUNCTION OF RADIAL POSITION FOR GR/RE=0

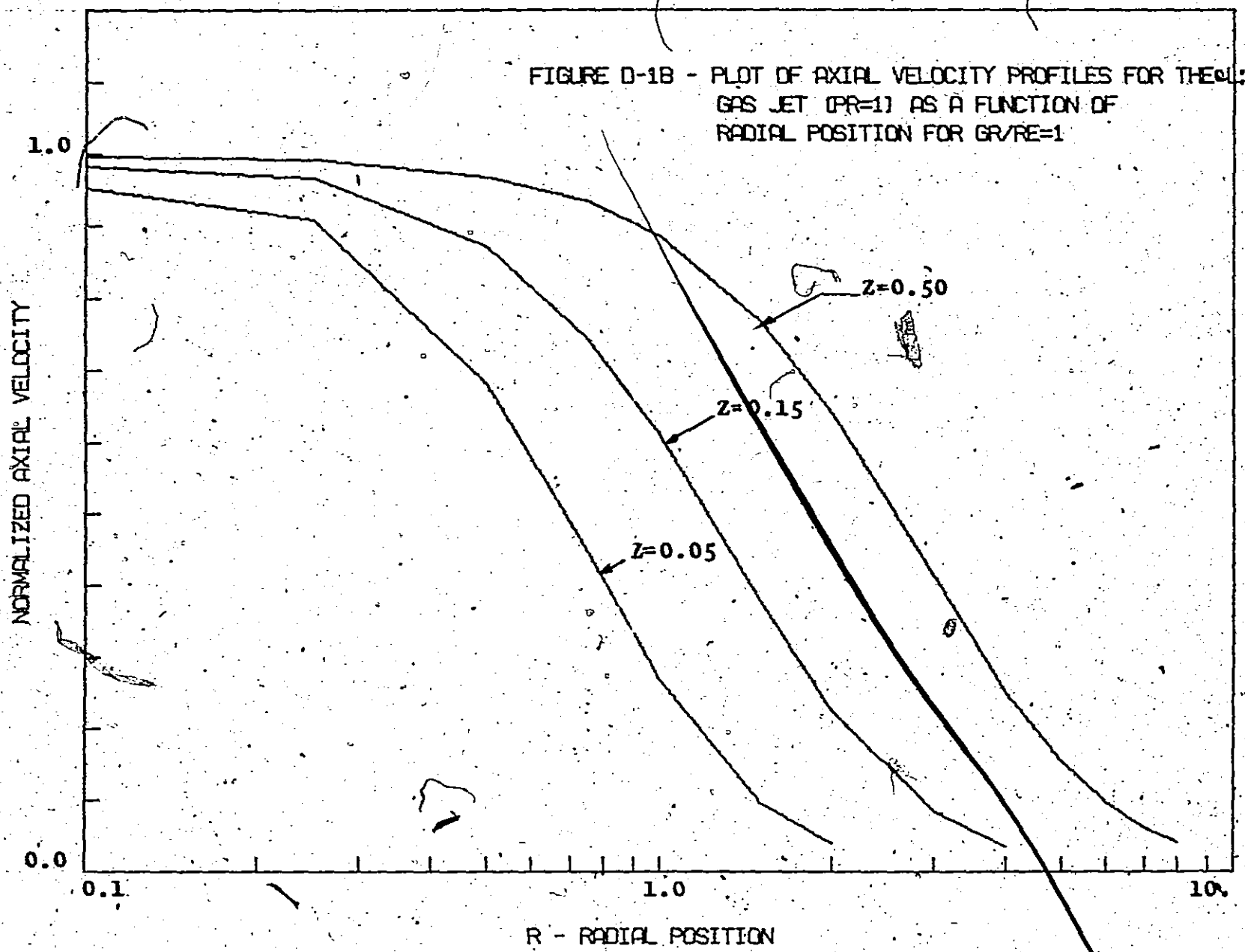


FIGURE D-1B - PLOT OF AXIAL VELOCITY PROFILES FOR THE AL GAS JET (PR=1) AS A FUNCTION OF RADIAL POSITION FOR GR/RE=1

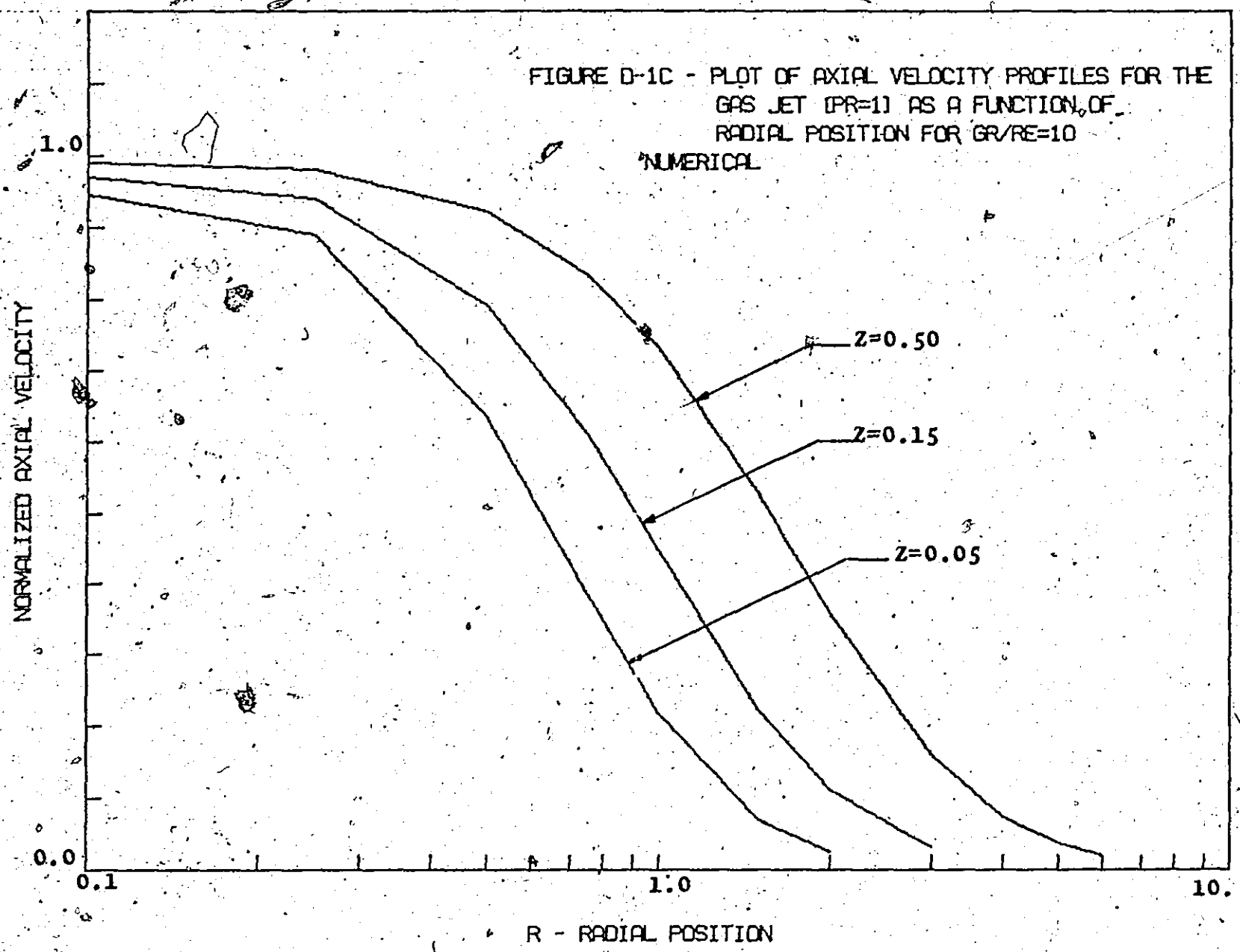


FIGURE D-1C - PLOT OF AXIAL VELOCITY PROFILES FOR THE GAS JET (PR=1) AS A FUNCTION OF RADIAL POSITION FOR GR/RE=10

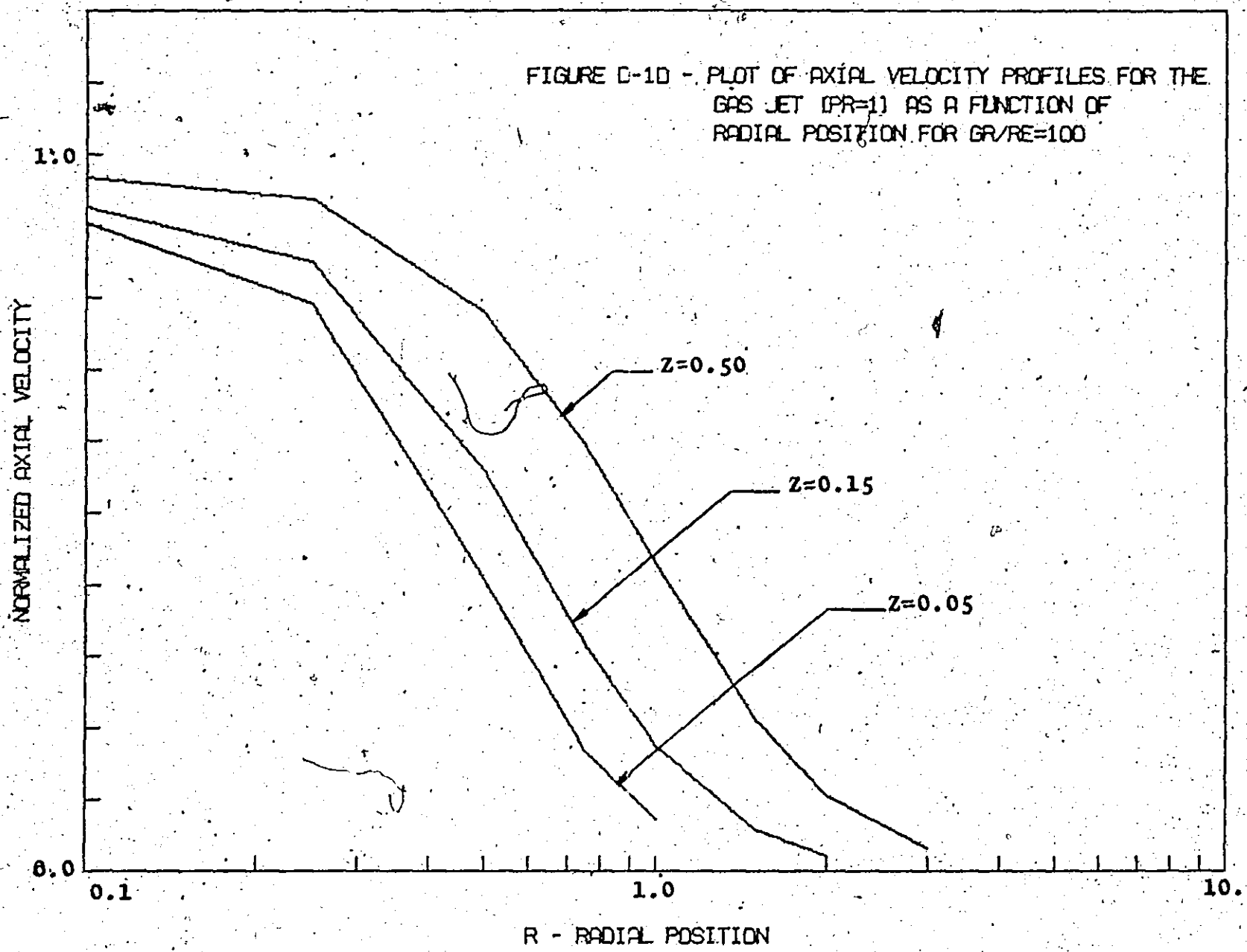


FIGURE D-10 - PLOT OF AXIAL VELOCITY PROFILES FOR THE GAS JET (PR=1) AS A FUNCTION OF RADIAL POSITION FOR GR/RE=100

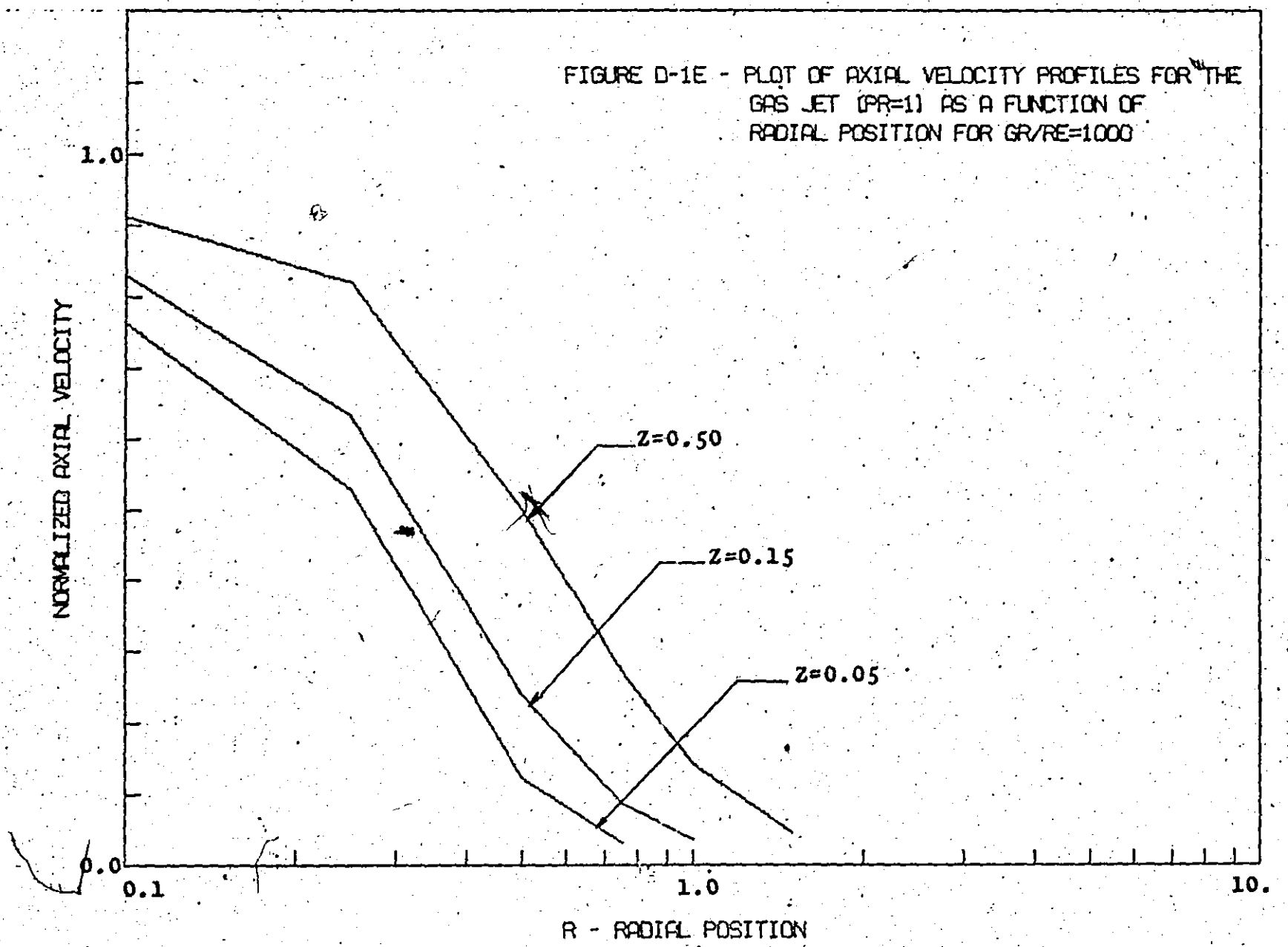


FIGURE D-1E - PLOT OF AXIAL VELOCITY PROFILES FOR THE GAS JET (PR=1) AS A FUNCTION OF RADIAL POSITION FOR GR/RE=1000

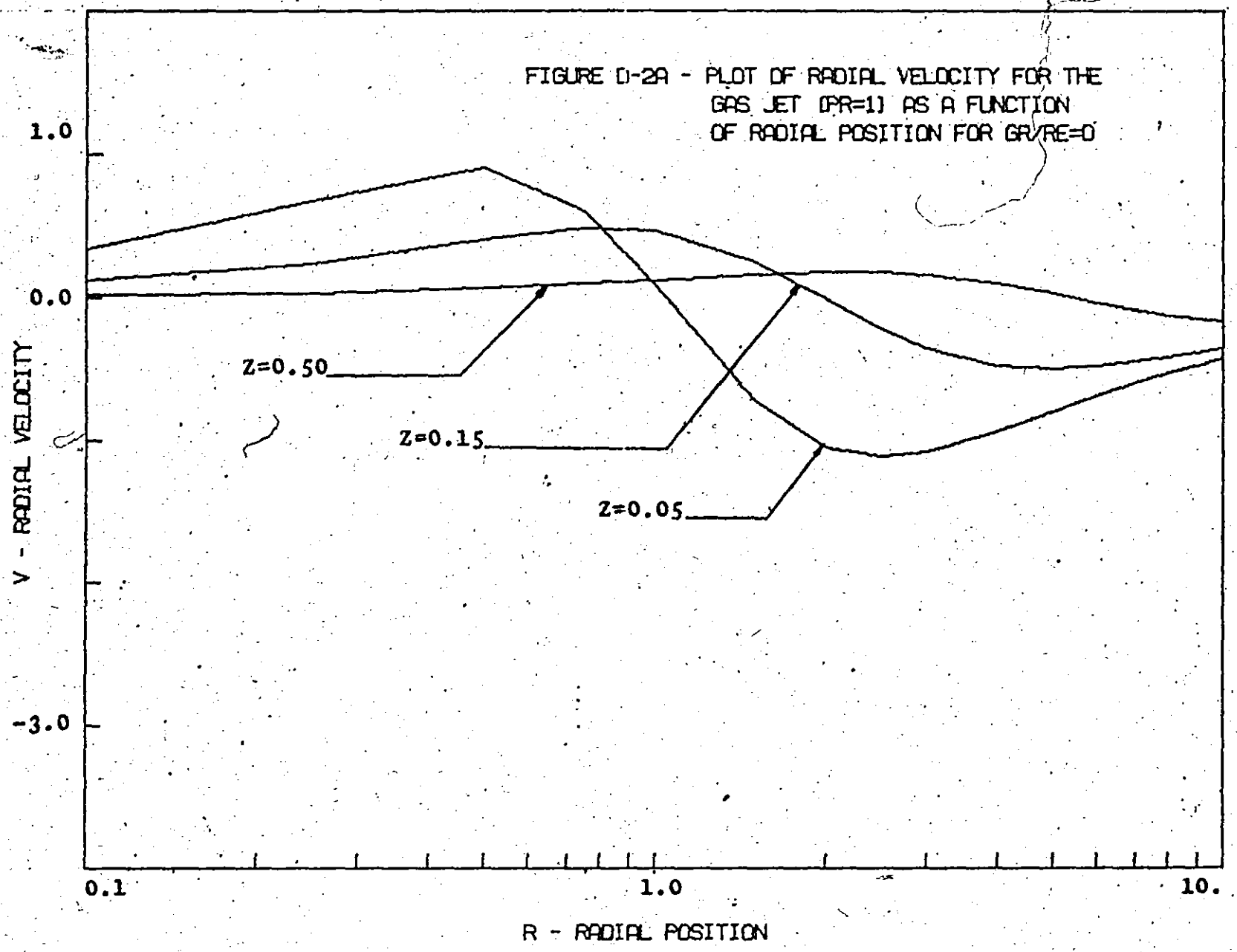


FIGURE D-2A - PLOT OF RADIAL VELOCITY FOR THE GAS JET (PR=1) AS A FUNCTION OF RADIAL POSITION FOR GR/RE=0

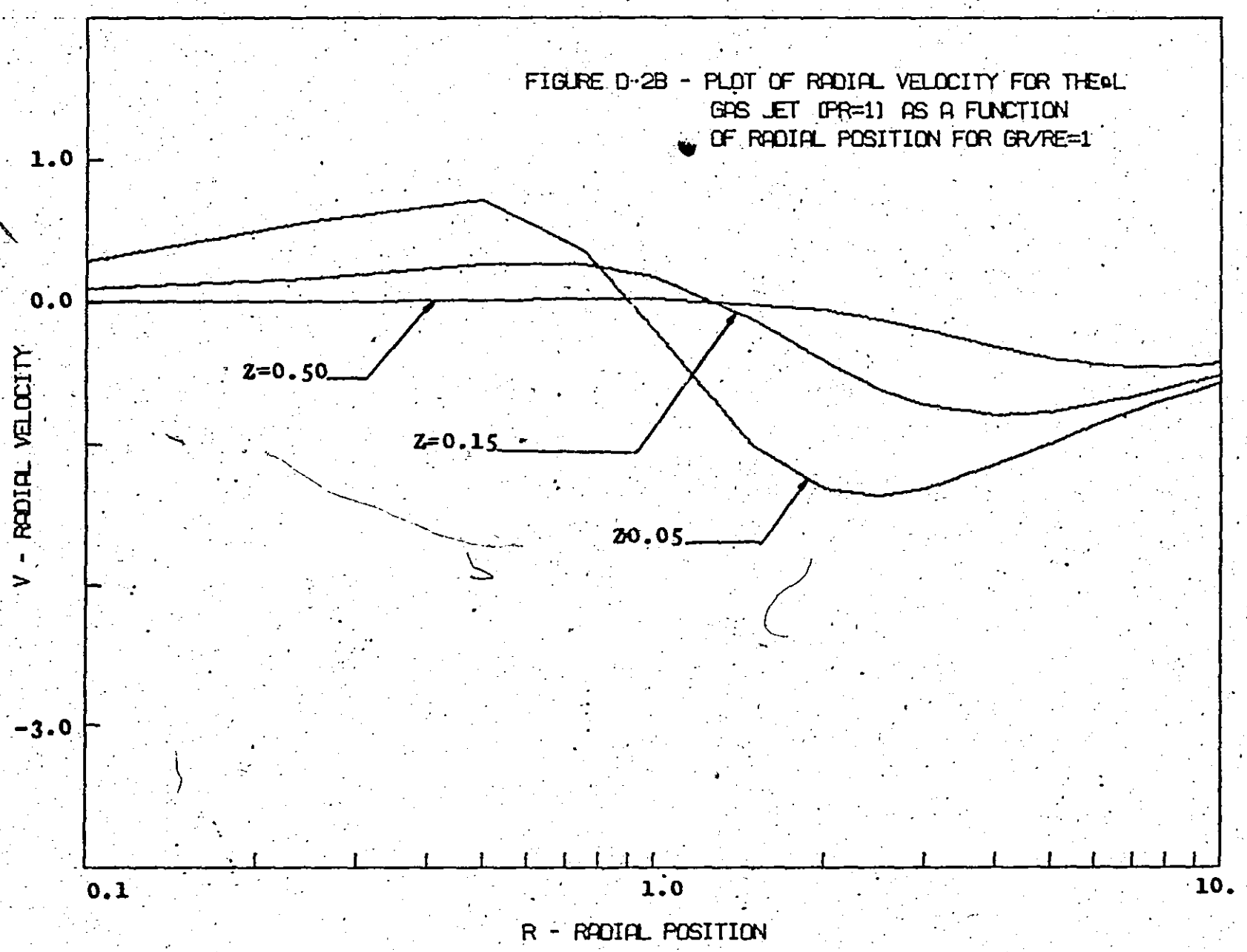


FIGURE D-2B - PLOT OF RADIAL VELOCITY FOR THE GAS JET (PR=1) AS A FUNCTION OF RADIAL POSITION FOR GR/RE=1

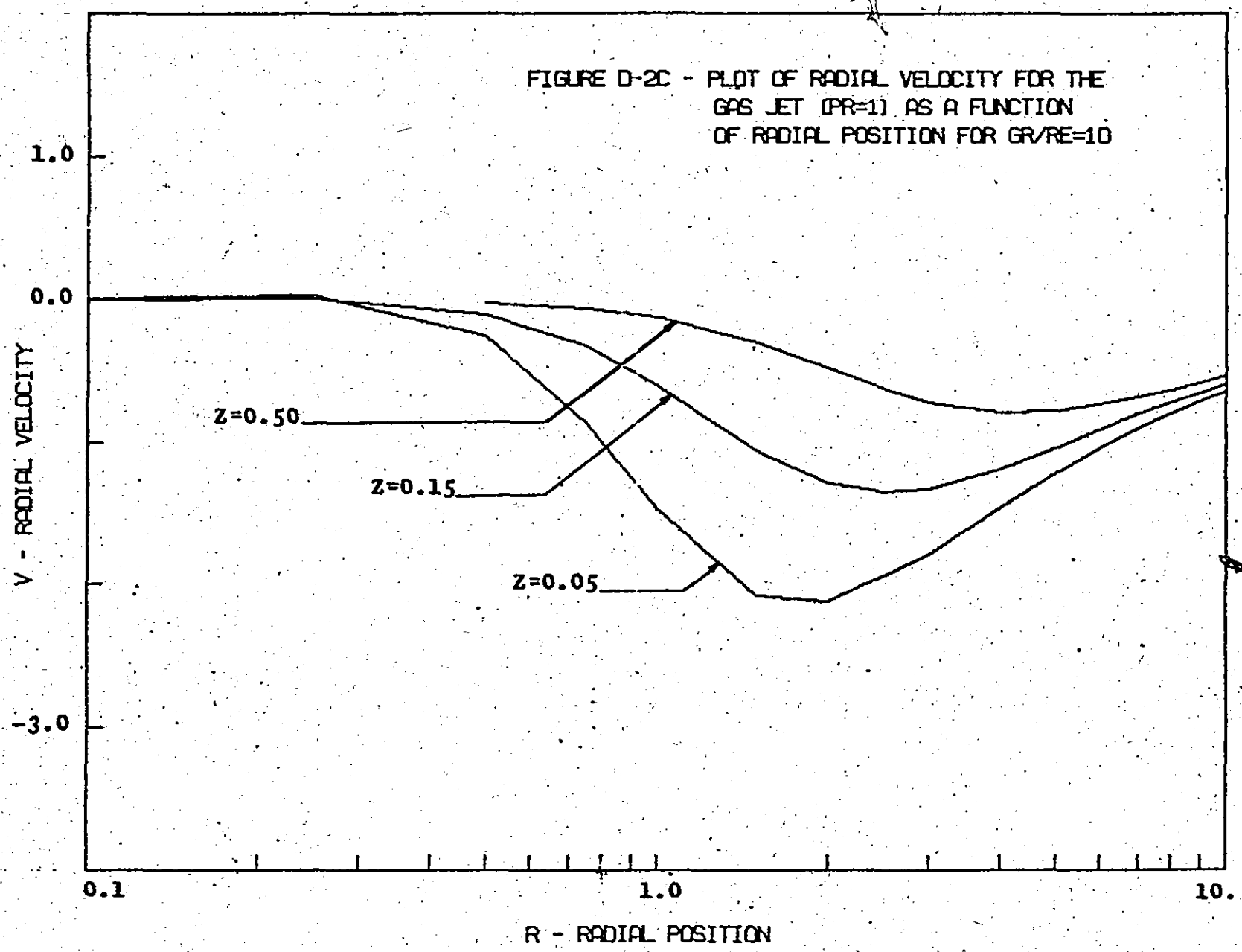


FIGURE D-2C - PLOT OF RADIAL VELOCITY FOR THE GAS JET (PR=1) AS A FUNCTION OF RADIAL POSITION FOR GR/RE=10

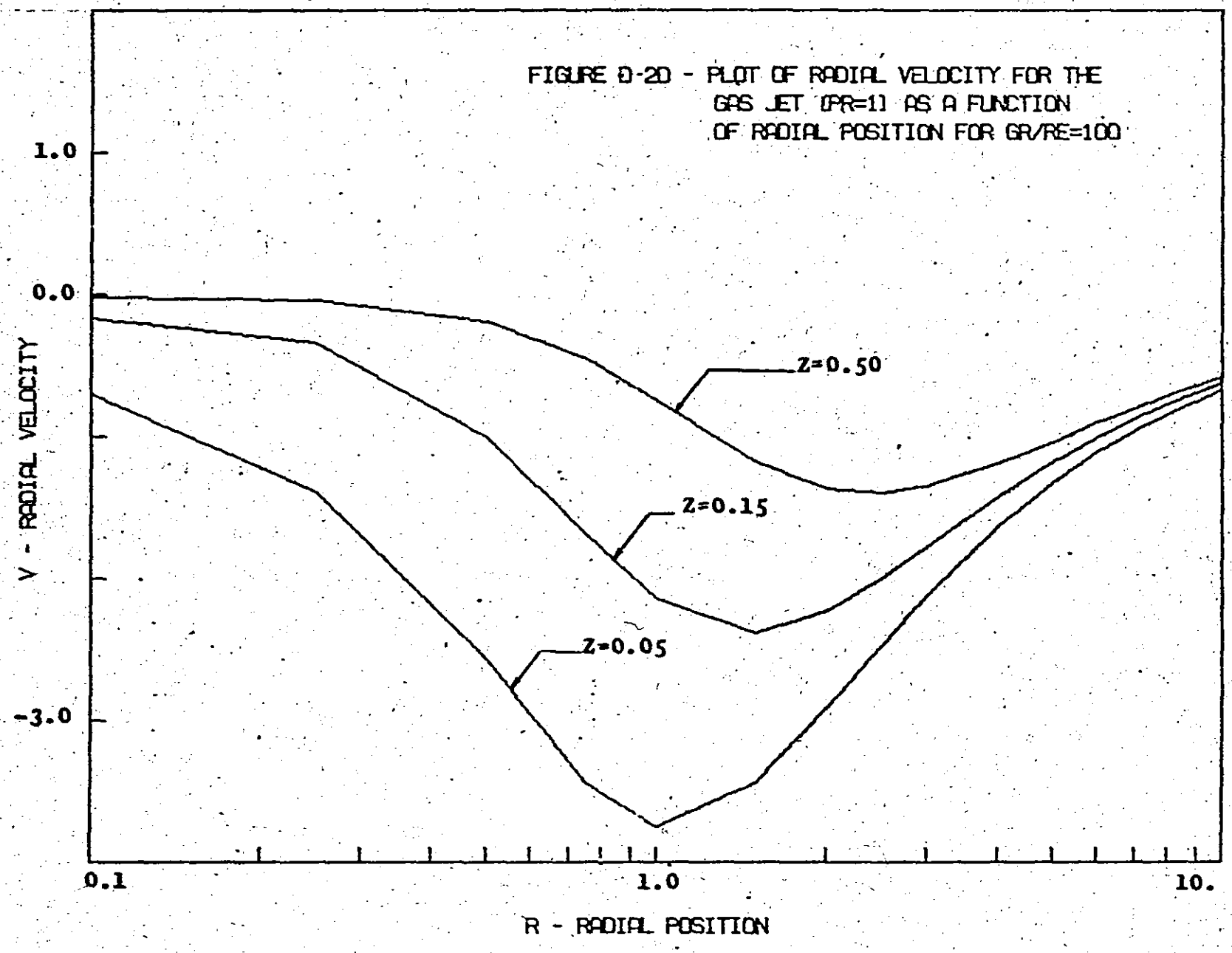


FIGURE D-20 - PLOT OF RADIAL VELOCITY FOR THE GAS JET (PR=1) AS A FUNCTION OF RADIAL POSITION FOR GR/RE=100