A Numerical Study of
Thermal Buoyancy in Axisymmetric
Laminar Vertical Jets

by
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ABSTRACT

A numerical study of thermal buoyancy in a laminar axisymmetric jet issuing into similar stagnant receiving media is presented. The boundary layer equations governing steady incompressible laminar flow are solved using a finite difference technique developed by Tomich. The results show the predominant effect of positive thermal buoyancy is to increase axial velocity. This effect increases for increasing Prandtl and Grashof numbers. Comparison between the reported numerical solution and the perturbation solution of Mollendorf and Gerhardt shows the latter solution does not adequately describe buoyant flow. Morton's entrainment formulation, on the other hand, is shown to predict the correct scaling factor relating two buoyant flows.
ACKNOWLEDGEMENTS

The author would like to thank his supervisor, Dr. J. Vlachopoulos, for his patience and guidance during the course of this work.

Gratitude is expressed to the Chemical Engineering Branch for their assistantship.

Finally the author would like to thank the members of his family for the sacrifices they made in order to make completion of this work possible.
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</tbody>
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A NUMERICAL STUDY OF THERMAL BUOYANCY IN
AXISYMMETRIC LAMINAR VERTICAL JETS.

I - INTRODUCTION

"Flows commonly referred to as jets may be regarded as part of a more general category of flows characterised by the absence of rigid boundaries. These free boundary flows include jets, wakes, plumes, and thermals. It is convenient to further subdivide these flows on the basis of whether they are laminar or turbulent and/or buoyant or non-buoyant. There are nearly unlimited number of important physical examples of these types of flows. Such flows are common in meteorology, oceanography, air and water pollution. More specific examples are associated with the mechanics of cloud formation, buoyancy driven ocean circulation, and thermal circulation in lakes resulting from water discharges" (7). In this work thermal buoyancy in laminar vertical jets is investigated.
Attempts have been made to solve the boundary layer equations which describe the effect of thermal buoyancy on round laminar vertical jets. In all instances investigators employed approximate analytic solution techniques. The validity and predictive power of these solutions has never been critically examined. The object of this investigation is, therefore:

- to determine, using numerical techniques, the effect of thermal buoyancy on velocity and temperature fields of an axisymmetric laminar vertical jet, and

- to examine the range of validity and predictive power of the recorded approximate solutions.
II - BACKGROUND

II-1 ANALYTICAL STUDIES

Using boundary layer approximations Schlichting (1) found similarity solutions for the non-buoyant axisymmetric jet. Andrade and Tsien (2) verified his results experimentally. They photographically observed traces made by particles suspended in a laminar jet flow. Landau (3) used similarity to derive an equivalent solution. His solution to the non-buoyant problem represents one of the few exact solutions to the full Navier-Stokes equations. Squire (4) presented an identical result extended to include a solution of the energy equation.

Thermal buoyancy in laminar jets was analysed by Morton (5). He used entrainment models developed using order of magnitude arguments. His analysis predicts an increase in axial velocity for positive thermal buoyancy. Brand and Lahey (6) performed a similarity analysis. Their solution relies not only on the validity of similarity but also on a particular set of flow and energy matching conditions. The usefulness of their result is not apparent. Mollendorf and Gebhardt (7) extended Schlichting's similarity solution to the buoyant jet. In their analysis they point out that similarity solutions do not apply because the momentum and energy equations are coupled by a position-dependent buoyancy term. For this...
reason they used perturbation analysis to extend Schlichting's similarity solution. Their general conclusions are that positive thermal buoyancy increases the axial velocity and that the magnitude of the effect increases for decreasing Prandtl number.

II-2 NUMERICAL STUDIES

For the non-buoyant jet, Pai's recent numerical study (8) shows that similarity solutions are good only at large distances from the nozzle exit. Vlachopoulos (9) reports a similar result for the magnetohydrodynamic free jet. From the analytical studies it is agreed that positive thermal buoyancy increases the axial velocity resulting in a longer jet, and corresponding rearrangement of the velocity and temperature profiles, and the transition zone (Figure 1). Accordingly, approximate solution techniques may not accurately describe the characteristics of the buoyant jet.
Figure 1 - Schematic of a Thermally Buoyant Axisymmetric Laminar Jet.
III - ANALYSIS

III - 1 MATHEMATICAL FORMULATION

The physical system drawn in Figure 1 is made of a simple jet whose exit temperature is significantly greater than ambient.

The complete formulation of the buoyant jet problem is given in Appendix A. Using the following assumptions:
- the flow is steady, irrotational, and axisymmetric
- all fluid properties are constant
- the effect of gravitational forces is negligible
- the flow is laminar
- Prandtl's boundary layer assumptions apply
- the Boussinesq approximation applies ($\Delta \rho/\rho < 1$)
- viscous dissipation is negligible

the conservation equations reduce to:

- Continuity equation
  \[ \frac{3V_z}{3z} + \frac{1}{r} \frac{3V_r}{3r} = 0 \]  \(1\)

- Momentum equation
  \[ \rho V_r \frac{3V_z}{3r} + \rho \frac{3V_z}{3z} = \frac{\mu}{r} \frac{3V_z}{3r} + \rho g \beta (t - t_a) \]  \(2\)

- Energy equation
  \[ V_z \frac{3t}{3z} + V_r \frac{3t}{3r} = \frac{\alpha}{r} \frac{3t}{3r} \frac{3t}{3r} \]  \(3\)
The following variable transformations are then introduced:

\[ Z = \frac{(Z/D)}{Re} \]

\[ R = \frac{R}{D} \]

\[ V_Z = \frac{V_Z}{V_o} \]

\[ V_R = \left( \frac{V_R}{V_o} \right) Re = \frac{\rho V_R D}{\mu} \]

\[ T = \frac{t - t_a}{t_o - t_a} \]  

yielding the dimensionless equations of conservation.

- **Continuity equation**
  \[ \frac{3}{2} V_Z + \frac{1}{R} \left( \frac{3}{2} V_R \right) = 0 \]  

- **Momentum equation**
  \[ V_Z \frac{3}{2} V_Z + V_R \frac{3}{2} V_Z = \frac{1}{R} \frac{3}{2} R \left( R \frac{3}{2} V_R \right) + \frac{C_{Gr}}{Re} T \]

- **Energy equation**
  \[ V_Z \frac{3}{2} T + V_R \frac{3}{2} T = \frac{1}{Pr} \frac{3}{2} \frac{3}{2} R \frac{3}{2} \]

where

\[ Re = \frac{\rho V_o D}{\mu} \]

\[ Gr = \frac{\rho^2 \beta g D^3 (t_o - t_a)}{\mu^2} \]

\[ Pr = \frac{C_p \mu}{k} \]

The boundary conditions are:

- at \( R = 0 \) and \( Z > 0 \) (centreline)
  \[ \frac{3}{2} V_Z = 0 \]
  \[ \frac{3}{2} T = 0 \]
  \[ V_R = 0 \]
as \( R \to \infty \)

\[
\begin{align*}
    v_Z & \to 0 \\
    T & \to 0
\end{align*}
\]

at \( x = 0 \)

\[
\begin{align*}
    v_Z &= v_i (0, R) \\
    v_R &= 0 \\
    T &= T_i (0, R)
\end{align*}
\]  

(9)

The above equations are solved for rectangular initial velocity and temperature profiles:

\[
\begin{align*}
    v_i &= 1, \quad R \leq 1 \\
    v_i &= 0, \quad R > 1 \\
    t_i &= 1, \quad R \leq 1 \\
    t_i &= 0, \quad R > 1
\end{align*}
\]  

(10)
III- 2 NUMERICAL PROCEDURE

A finite difference method developed by Tomich and Wager (10) for axisymmetric, compressible, turbulent free jets was modified and adapted to the buoyant jet case. The following finite difference approximations for temperature, radial velocity and axial velocity were introduced into the equations of energy, momentum and continuity.

Temperature

\[ \frac{\partial T}{\partial Z} = \frac{T(Z + \Delta Z, R) - T(Z, R)}{\Delta Z} \]

\[ \frac{\partial T}{\partial R} = \frac{T(Z + \Delta Z, R + \Delta R) - T(Z + \Delta Z, R - \Delta R)}{2\Delta R} \]

\[ \frac{\partial^2 T}{\partial R^2} = \frac{T(Z + \Delta Z, R + \Delta R) - 2T(Z + \Delta Z, R) + T(Z + \Delta Z, R - \Delta R)}{(\Delta R)^2} \]  \( \text{(11)} \)

Axial velocity

\[ \frac{\partial \bar{V}_Z}{\partial Z} = \frac{\bar{V}_Z(Z + \Delta Z, R) - \bar{V}_Z(Z, R)}{\Delta Z} \]

\[ \frac{\partial \bar{V}_Z}{\partial R} = \frac{\bar{V}_Z(Z + \Delta Z, R + \Delta R) - \bar{V}_Z(Z + \Delta Z, R - \Delta R)}{2\Delta R} \]

\[ \frac{\partial^2 \bar{V}_Z}{\partial R^2} = \frac{\bar{V}_Z(Z + \Delta Z, R + \Delta R) - 2\bar{V}_Z(Z + \Delta Z, R) + \bar{V}_Z(Z + \Delta Z, R - \Delta R)}{(\Delta R)^2} \]  \( \text{(12)} \)

Radial velocity

\[ \frac{\partial \bar{V}_R}{\partial R} = \frac{\bar{V}_R(Z + \Delta Z, R) - \bar{V}_R(Z + \Delta Z, R - \Delta R)}{\Delta R} \]  \( \text{(13)} \)

Upon substitution of the above expressions into the conservation equations (energy, momentum, mass) a system of non-linear equations of the following form was obtained.
Energy

\[ A_1 T_1 + B_1 T_2 = D_1 \]
\[ C_m T_{m-1} + A_m T_m + B_{m+1} T_{m+1} = D_m \quad \text{for} \ m = 2, 3, 4, \ldots, K-1 \]
\[ C_K T_{K-1} + A_K T_K = D_K \] \hfill (14)

Momentum

\[ E_1 V_{Z1} + F_1 V_{Z2} = H_1 \]
\[ G_m V_{Zm} + E_m V_{Zm+1} + F_{m+1} V_{Zm+1} = H_m \quad \text{for} \ m = 2, 3, 4, \ldots, K-1 \]
\[ G_K V_{ZK-1} + E_K V_{ZK} = H_K \] \hfill (15)

Continuity

\[ V_{Rm} = Q_{1m} (Q_{2m} + Q_{3m}) \] \hfill (16)

The derivation of this system of equations is outlined in Appendix B.

By using Wegstein's method (11) for acceleration of convergence, an iterative procedure was used to linearize these algebraic equations. Using this approach the velocities and temperatures just calculated were replaced with a weighted average of the newly calculated values and those from the previous iteration.

The solution of the tridiagonal system of algebraic equations was performed by Thomas's method (12). This method has the advantage that the round-off error is small in comparison to the truncation error. (13)

The truncation error is a measure of the departure of the finite-difference approximation from the solution of a
partial differential equation at any grid point. If this error tends to zero as the grid spacings tend to zero the finite difference solution is said to converge. If this error grows without bound the solution is unstable.

Richtmeyr (14) presents a theorem by Lax (15) which demonstrates that if the consistency criterion is satisfied stability is both a necessary and sufficient condition for convergence. The consistency criterion requires that the finite difference procedure being used, approximate the solution of the partial differential equation under study.

So far consistency analysis has been limited to the study of linear differential equations. Since the techniques used are not directly applicable to non-linear equations, no analysis of this kind was attempted. Instead to ensure convergence to the true solution various step-sizes, step-size ratios and locations of the outer jet boundary ($R \rightarrow \infty$) were tried. The grid sizes used in this study are recorded in Table I.
### TABLE I
GRID SIZES

<table>
<thead>
<tr>
<th>Up to X</th>
<th>m</th>
<th>$\Delta R$</th>
<th>$\Delta X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7000</td>
<td>501</td>
<td>0.005</td>
<td>0.1000</td>
</tr>
<tr>
<td>0.3200</td>
<td>501</td>
<td>0.0100</td>
<td>0.1000</td>
</tr>
<tr>
<td>1.000</td>
<td>501</td>
<td>0.0500</td>
<td>0.2500</td>
</tr>
</tbody>
</table>

The results reported are independent of the grid sizes to a small tolerance.
IV - RESULTS AND DISCUSSION

To simplify the study, the existence of two buoyant jets is postulated. The first is a gas jet, the second, a liquid jet. The characteristics of these jets are defined in Table II. In both instances the jet conditions are chosen so that the nozzle exit Reynolds number is 100. The exit Grashof number varies according to the exit temperature difference.

<table>
<thead>
<tr>
<th>Type</th>
<th>Fr</th>
<th>Re</th>
<th>D</th>
<th>Vo</th>
<th>( \mu )</th>
<th>( \frac{\rho}{\mu^2} )</th>
<th>( \frac{g \alpha b}{\mu^2} )</th>
<th>( D^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>gas</td>
<td>1</td>
<td>100</td>
<td>3.0</td>
<td>7.0</td>
<td>0.21</td>
<td>60.0</td>
<td>2600</td>
<td></td>
</tr>
<tr>
<td>liquid</td>
<td>10</td>
<td>100</td>
<td>0.5</td>
<td>1.0</td>
<td>5.0x10^{-3}</td>
<td>60.0</td>
<td>2600</td>
<td></td>
</tr>
</tbody>
</table>

In the following discussion, the numerical results for these jets are compared to analytical predictions.

The centreline velocities for the two jets are recorded in Tables III and IV, and Figures 2 and 3. As \( \text{Gr/Re} \) increases from zero the velocity drop becomes a rise indicating a shift from a momentum driven jet to a buoyancy driven jet. When \( \text{Gr/Re}=0 \) (the non-buoyant case) the numerical solution asymptotically approaches Schlichting's solution, which is given by

\[
V_M = \frac{3}{32} z^{-1}
\]
## TABLE III

Centreline Velocity for
Gas Jet (Pr=1)

<table>
<thead>
<tr>
<th>Z (=z/DRe)</th>
<th>0</th>
<th>10⁻¹</th>
<th>10⁰</th>
<th>10¹</th>
<th>10²</th>
<th>10³</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0500</td>
<td>0.7751</td>
<td>0.7799</td>
<td>0.8216</td>
<td>1.1552</td>
<td>2.7946</td>
<td>8.3285</td>
</tr>
<tr>
<td>0.1000</td>
<td>0.5432</td>
<td>0.5521</td>
<td>0.6270</td>
<td>1.1216</td>
<td>3.1483</td>
<td>9.7221</td>
</tr>
<tr>
<td>0.1500</td>
<td>0.4120</td>
<td>0.4249</td>
<td>0.5259</td>
<td>1.1024</td>
<td>3.2784</td>
<td>10.2612</td>
</tr>
<tr>
<td>0.2000</td>
<td>0.3309</td>
<td>0.3473</td>
<td>0.4690</td>
<td>1.0921</td>
<td>3.3408</td>
<td>10.5238</td>
</tr>
<tr>
<td>0.2500</td>
<td>0.2762</td>
<td>0.2962</td>
<td>0.4342</td>
<td>1.0859</td>
<td>3.3758</td>
<td>10.6708</td>
</tr>
<tr>
<td>0.3000</td>
<td>0.2368</td>
<td>0.2602</td>
<td>0.4114</td>
<td>1.0819</td>
<td>3.3975</td>
<td>10.7611</td>
</tr>
<tr>
<td>0.3500</td>
<td>0.2072</td>
<td>0.2338</td>
<td>0.3957</td>
<td>1.0791</td>
<td>3.4119</td>
<td>10.8203</td>
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<tr>
<td>0.4000</td>
<td>0.1846</td>
<td>0.2141</td>
<td>0.3844</td>
<td>1.0772</td>
<td>3.4219</td>
<td>10.8611</td>
</tr>
<tr>
<td>0.4500</td>
<td>0.1663</td>
<td>0.1986</td>
<td>0.3760</td>
<td>1.0757</td>
<td>3.4292</td>
<td>10.8902</td>
</tr>
<tr>
<td>0.5000</td>
<td>0.1512</td>
<td>0.1862</td>
<td>0.3696</td>
<td>1.0746</td>
<td>3.4347</td>
<td>10.9118</td>
</tr>
</tbody>
</table>

## TABLE IV

Centreline Velocity for
Liquid Jet (Pr=10)

<table>
<thead>
<tr>
<th>Z (=z/DRe)</th>
<th>0</th>
<th>10⁻¹</th>
<th>10⁰</th>
<th>10¹</th>
<th>10²</th>
<th>10³</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0500</td>
<td>0.7751</td>
<td>0.7806</td>
<td>0.8295</td>
<td>1.2113</td>
<td>3.0211</td>
<td>9.0212</td>
</tr>
<tr>
<td>0.1000</td>
<td>0.5432</td>
<td>0.5552</td>
<td>0.6546</td>
<td>1.2744</td>
<td>3.6893</td>
<td>11.2788</td>
</tr>
<tr>
<td>0.1500</td>
<td>0.4120</td>
<td>0.4306</td>
<td>0.5725</td>
<td>1.3249</td>
<td>4.0338</td>
<td>12.4347</td>
</tr>
<tr>
<td>0.2000</td>
<td>0.3309</td>
<td>0.3558</td>
<td>0.5316</td>
<td>1.3624</td>
<td>4.2423</td>
<td>13.1284</td>
</tr>
<tr>
<td>0.2500</td>
<td>0.2762</td>
<td>0.3072</td>
<td>0.5094</td>
<td>1.3902</td>
<td>4.3808</td>
<td>13.5850</td>
</tr>
<tr>
<td>0.3000</td>
<td>0.2368</td>
<td>0.2739</td>
<td>0.4966</td>
<td>1.4112</td>
<td>4.4786</td>
<td>13.9044</td>
</tr>
<tr>
<td>0.3500</td>
<td>0.2072</td>
<td>0.2498</td>
<td>0.4889</td>
<td>1.4274</td>
<td>4.5507</td>
<td>14.1560</td>
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<tr>
<td>0.4000</td>
<td>0.1846</td>
<td>0.2321</td>
<td>0.4839</td>
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<td>4.6040</td>
<td>14.3252</td>
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<tr>
<td>0.4500</td>
<td>0.1663</td>
<td>0.2185</td>
<td>0.4807</td>
<td>1.4503</td>
<td>4.6437</td>
<td>14.4576</td>
</tr>
<tr>
<td>0.5000</td>
<td>0.1512</td>
<td>0.2080</td>
<td>0.4785</td>
<td>1.4591</td>
<td>4.6811</td>
<td>14.5621</td>
</tr>
</tbody>
</table>
FIGURE 2 - PLOT OF CENTERLINE VELOCITY FOR THE GAS JET (PR=1) AS A FUNCTION OF AXIAL POSITION

- SCHLICHTING
- MOLLENDORF AND GEBHARDT
- NUMERICAL

Gr/Re = 10^2
Gr/Re = 10^1
Gr/Re = 10^0
Gr/Re = 10^{-1}
Gr/Re = 0.

V - CENTRELINE

Z - AXIAL POSITION
FIGURE 3 - PLOT OF CENTERLINE VELOCITY FOR THE LIQUID JET (Pr=10) AS A FUNCTION OF AXIAL POSITION

- - - - - - - - SCHLICHTING
- - - - - - - - MOLLENDORF AND GEBHARDT
- - - - - - - - NUMERICAL

Gr/Re=10²
Gr/Re=10¹
Gr/Re=10⁰
Gr/Re=0

Z - AXIAL POSITION

V - CENTERLINE
Furthermore, it is interesting to note that our numerical result for this case is identical to that of Pai (8). The perturbation solution for the buoyant jets as derived by Mollendorf and Gebhardt (7) yields the following equation for centreline velocity.

\[ V_M = \frac{3}{32} \frac{1}{Pr} + \frac{2}{3} (2Pr + 1) \frac{Gr}{Re} Z f(Pr) \]  \hspace{1cm} (18)

The first term is identical to Schlichting's solution for the non-buoyant jet. The second term is the perturbation term. In Figure 3 this solution is compared with the numerical solution. The comparison shows significant disagreement over all \( Z \). This is not surprising as for small \( Z \) the first term is dominant and suffers from the same inadequacy as Schlichting's solution. For large \( Z \), the second term dominates and increases without bound. Mollendorf and Gebhardt's solution, therefore, does not accurately describe the centreline velocity for the buoyant jet. Morton's entrainment formulation (5) predicts a constant centreline velocity given by

\[ V_M = \left( \frac{1}{4E} \left( \frac{1+Pr}{Pr} \frac{Gr}{Re} \right) \right)^{0.5} \]  \hspace{1cm} (19)

This result is consistent with the numerical results only for large \( Z \). Because \( E \), the entrainment parameter, is an unknown function of \( Pr \), it is impossible to predict absolute values
of the centreline velocity. Nevertheless, it is possible to compare the effect of Gr/Re as predicted by Morton with the numerical solution by comparing relative centreline velocities. As seen in Tables V and VI, Morton's entrainment model predicts the correct relative centreline velocity. Brand and Lahey (6) report a similarity solution. Their solution predicts a constant centreline velocity for all Z. Its dependence on Pr, Re and Gr is not clear. Since they predict a constant velocity their solution is at best only good for large Z.

Through their experimental studies McNaughton and Sinclair (16) observed spreading in a laminar jet. For small density differences between the jet and the surrounding fluid (media) their jet appeared to thin. Figures 4 and 5 record the velocity half-widths for the liquid and gas jets. Thinning is present for large Gr/Re (>10). The degree of thinning increases as Gr/Re and Pr increase. Since thinning is a function of Prandtl number spreading of the velocity profile is dependent on spreading of the temperature profile. The effect of buoyancy on spreading can also be seen in plots of axial velocity profiles for the liquid jet, Figure 6a–e. Similar results for the gas jet are recorded in Appendix D.

---

For a given Gr/Re, the relative centreline velocity is defined as the ratio between the absolute centreline velocity and the absolute centreline velocity for Gr/Re=1.
TABLE V

Relative Centreline Velocity
for Gas Jet (Pr=1)
at \( z \) \((=z/\text{DRe}) = 0.50 \)

<table>
<thead>
<tr>
<th>( \text{Gr/Re} )</th>
<th>( V_{rel} )</th>
<th>( \text{Morton} )</th>
<th>( \text{Numerical} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^{-1} )</td>
<td>0.3162</td>
<td>0.5040</td>
<td>0.5040</td>
</tr>
<tr>
<td>( 10^0 )</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>( 10^1 )</td>
<td>3.1622</td>
<td>2.9072</td>
<td>2.9072</td>
</tr>
<tr>
<td>( 10^2 )</td>
<td>10.0000</td>
<td>9.2934</td>
<td>9.2934</td>
</tr>
<tr>
<td>( 10^3 )</td>
<td>31.6224</td>
<td>29.5233</td>
<td>29.5233</td>
</tr>
</tbody>
</table>

TABLE VI

Relative Centreline Velocities
for Liquid Jet (Pr=10)
at \( z \) \((=z/\text{DRe}) = 0.50 \)

<table>
<thead>
<tr>
<th>( \text{Gr/Re} )</th>
<th>( V_{rel} )</th>
<th>( \text{Morton} )</th>
<th>( \text{Numerical} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^{-1} )</td>
<td>0.3162</td>
<td>0.4353</td>
<td>0.4353</td>
</tr>
<tr>
<td>( 10^0 )</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>( 10^1 )</td>
<td>3.1622</td>
<td>3.0493</td>
<td>3.0493</td>
</tr>
<tr>
<td>( 10^2 )</td>
<td>10.0000</td>
<td>9.7834</td>
<td>9.7834</td>
</tr>
<tr>
<td>( 10^3 )</td>
<td>31.6223</td>
<td>30.4332</td>
<td>30.4332</td>
</tr>
</tbody>
</table>
FIGURE 4 - PLOT OF VELOCITY HALF-WIDTH FOR THE GAS JET (PR=1) AS A FUNCTION OF AXIAL POSITION

Gr/Re = 10^0
Gr/Re = 10^1
Gr/Re = 10^2
Gr/Re = 10^3

0.1 0.2 0.3 0.4 0.5
Z - AXIAL POSITION
FIGURE 5 - PLOT OF VELOCITY HALF-WIDTH FOR THE LIQUID JET (Pr=10) AS A FUNCTION OF AXIAL POSITION

-Gr/Re= 0.
-Gr/Re=10
-Gr/Re=10'1
-Gr/Re=10°
-Gr/Re=10°3

Z - AXIAL POSITION
FIGURE 6A: PLOT OF AXIAL VELOCITY PROFILES FOR THE LIQUID JET. Z/TR=0 AS A FUNCTION OF RADIAL POSITION FOR GR/AE=0.
FIGURE 6B - PLOT OF AXIAL VELOCITY PROFILES FOR THE LIQUID JET (PR=10) AS A FUNCTION OF RADIAL POSITION FOR GRAY=1

R - RADIAL POSITION
0.0  0.1  1.0

Z = 0.05  Z = 0.15  Z = 0.50

NORMALIZED AXIAL VELOCITY
FIGURE 80 - PLOT OF AXIAL VELOCITY PROFILES FOR THE LIQUID JET (PR=10) AS A FUNCTION OF RADIAL POSITION FOR GE=100.
FIGURE GE - PLOT OF AXIAL VELOCITY PROFILES FOR THE LIQUID JET (PP=10) AS A FUNCTION OF RADIAL POSITION FOR GVRE=1000.
The radial velocity profiles for the liquid jet are reported in Figures 7a-e. As Gr/Re increases the positive portion of this profile decreases and becomes increasingly negative. For large values of Gr/Re (> 10) and small Z, the radial velocity is always negative. This means the jet is contracting or thinning. This is consistent with McNaughton and Sinclair's experimental observations.

The centreline temperatures for the two jets are recorded in Tables VII and VIII and Figures 8 and 9. For Gr/Re = 0, the similarity solution of Yih (17) and Mollendorf and Gebhardt (7) applies. This solution is represented by

\[ T_M = \frac{(2Pr + 1)}{32} Z^{-1} \]  

(20)

The numerical solution for this case is shown to asymptotically approach the similarity solution (Figures 9 and 10). These plots also show that agreement between numerical and similarity solutions is better for the gas jet (Pr = 1) case. This result implies that the smaller the Prandtl number the more accurate the similarity prediction of centreline temperature.
FIGURE 7A - PLOT OF RADIAL VELOCITY FOR THE LIQUID JET (PR=10) AS A FUNCTION OF RADIAL POSITION FOR GR/RE=0.
FIGURE 7C - PLOT OF RADIAL VELOCITY FOR THE LIQUID JET (Fr=10) AS A FUNCTION OF RADIAL POSITION FOR Gr/Re=10
FIGURE 70 - PLOT OF RADIAL VELOCITY FOR THE LIQUID JET D=10 AS A FUNCTION OF RADIAL POSITION FOR GR/Re=100
### TABLE VII

Centreline Temperature for Gas Jet (Pr=1)

<table>
<thead>
<tr>
<th>Z (=z/DRe)</th>
<th>0</th>
<th>10^{-1}</th>
<th>10^{0}</th>
<th>10^{1}</th>
<th>10^{2}</th>
<th>10^{3}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0500</td>
<td>0.7751</td>
<td>0.7750</td>
<td>0.7741</td>
<td>0.7698</td>
<td>0.7651</td>
<td>0.7376</td>
</tr>
<tr>
<td>0.1000</td>
<td>0.5432</td>
<td>0.5427</td>
<td>0.5400</td>
<td>0.5331</td>
<td>0.5350</td>
<td>0.5241</td>
</tr>
<tr>
<td>0.1500</td>
<td>0.4120</td>
<td>0.4113</td>
<td>0.4072</td>
<td>0.4004</td>
<td>0.4051</td>
<td>0.4008</td>
</tr>
<tr>
<td>0.2000</td>
<td>0.3309</td>
<td>0.3297</td>
<td>0.3248</td>
<td>0.3190</td>
<td>0.3247</td>
<td>0.3231</td>
</tr>
<tr>
<td>0.2500</td>
<td>0.2762</td>
<td>0.2748</td>
<td>0.2693</td>
<td>0.2647</td>
<td>0.2705</td>
<td>0.2701</td>
</tr>
<tr>
<td>0.3000</td>
<td>0.2368</td>
<td>0.2353</td>
<td>0.2296</td>
<td>0.2260</td>
<td>0.2317</td>
<td>0.2319</td>
</tr>
<tr>
<td>0.3500</td>
<td>0.2072</td>
<td>0.2056</td>
<td>0.1999</td>
<td>0.1970</td>
<td>0.2025</td>
<td>0.2030</td>
</tr>
<tr>
<td>0.4000</td>
<td>0.1846</td>
<td>0.1828</td>
<td>0.1769</td>
<td>0.1746</td>
<td>0.1799</td>
<td>0.1805</td>
</tr>
<tr>
<td>0.4500</td>
<td>0.1663</td>
<td>0.1644</td>
<td>0.1585</td>
<td>0.1568</td>
<td>0.1617</td>
<td>0.1624</td>
</tr>
<tr>
<td>0.5000</td>
<td>0.1512</td>
<td>0.1493</td>
<td>0.1436</td>
<td>0.1422</td>
<td>0.1469</td>
<td>0.1476</td>
</tr>
</tbody>
</table>

### TABLE VIII

Centreline Temperature for Liquid Jet (Pr=10)

<table>
<thead>
<tr>
<th>Z (=z/DRe)</th>
<th>0</th>
<th>10^{-1}</th>
<th>10^{0}</th>
<th>10^{1}</th>
<th>10^{2}</th>
<th>10^{3}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0500</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.9998</td>
<td>0.9950</td>
</tr>
<tr>
<td>0.1000</td>
<td>0.9985</td>
<td>0.9985</td>
<td>0.9984</td>
<td>0.9978</td>
<td>0.9935</td>
<td>0.9769</td>
</tr>
<tr>
<td>0.1500</td>
<td>0.9879</td>
<td>0.9878</td>
<td>0.9876</td>
<td>0.9864</td>
<td>0.9815</td>
<td>0.9513</td>
</tr>
<tr>
<td>0.2000</td>
<td>0.9632</td>
<td>0.9632</td>
<td>0.9634</td>
<td>0.9632</td>
<td>0.9586</td>
<td>0.9211</td>
</tr>
<tr>
<td>0.2500</td>
<td>0.9275</td>
<td>0.9277</td>
<td>0.9288</td>
<td>0.9312</td>
<td>0.9291</td>
<td>0.8884</td>
</tr>
<tr>
<td>0.3000</td>
<td>0.8858</td>
<td>0.8864</td>
<td>0.8889</td>
<td>0.8941</td>
<td>0.8955</td>
<td>0.8546</td>
</tr>
<tr>
<td>0.3500</td>
<td>0.8424</td>
<td>0.8432</td>
<td>0.8474</td>
<td>0.8551</td>
<td>0.8602</td>
<td>0.8206</td>
</tr>
<tr>
<td>0.4000</td>
<td>0.7994</td>
<td>0.8007</td>
<td>0.8066</td>
<td>0.8163</td>
<td>0.8242</td>
<td>0.7875</td>
</tr>
<tr>
<td>0.4500</td>
<td>0.7582</td>
<td>0.7602</td>
<td>0.7675</td>
<td>0.7786</td>
<td>0.7897</td>
<td>0.7552</td>
</tr>
<tr>
<td>0.5000</td>
<td>0.7206</td>
<td>0.7221</td>
<td>0.7307</td>
<td>0.7426</td>
<td>0.7561</td>
<td>0.7322</td>
</tr>
</tbody>
</table>
FIGURE 8 - PLOT OF CENTERLINE TEMPERATURE FOR THE GAS JET (PR=1) AS A FUNCTION OF AXIAL POSITION

- - - - - - YIH
- - - - - - MOLLENDORF AND GEBHARDT
- - - - - - NUMERICAL

Gr/Re = 0
Gr/Re = 0, 10, 10^0, 10^1, 10^2

Z - AXIAL POSITION
FIGURE 9 - PLOT OF CENTERLINE TEMPERATURE FOR THE LIQUID JET (PR=10) AS A FUNCTION OF AXIAL POSITION

- - - - YIH
- - - - - MOLLENDORF AND GEBHARDT
- - - - - NUMERICAL

Gr/Re=10^2
Gr/Re=10^1
Gr/Re=10^0
Gr/Re=0.10^7
Gr/Re=0.10^6,10^5,10^4,10^3

Z - AXIAL POSITION

T - CENTERLINE
From Tables VII and VIII it is observed that buoyancy has little effect on the centreline temperature. Mollendorf and Gebhardt (7) derive the following equation for the centreline temperature.

\[ T_M = \left( \frac{2Pr+1}{2} \right) Z^{-1} + \frac{b}{g} (2Pr+1)^2 \frac{Gr}{Re} Z n(Pr) \]  

(21)

Figure 9 shows that the temperature predicted by this relationship is inadequate. For small Z the first term in equation (2) dominates and the resultant solution suffers from the same inadequacy as the similarity solution. For large Z the second term dominates and increases without bound. Morton (5) predicts the following relationship for centreline temperatures

\[ T_H = \left( \frac{1}{8E} \right) \left( \frac{1+Pr}{Pr^2} \right) Z^{-1} \]  

(22)

It suggests that the centreline temperature is inversely proportional to the axial position. Their relationship also predicts that T is independent of Gr/Re. Comparison of the numerical values for relative centreline temperatures* as recorded in Tables IX and X agrees with this observation.

The temperature half-widths are recorded in Figures 10 and 11. As in the case of the axial velocity the temperature profile thins. The degree of thinning increases for increasing Gr/Re and Pr.

---

* For a given Gr/Re, the relative centreline temperature is defined as the ratio between the absolute centreline temperature and the corresponding centreline temperature for Gr/Re=1.
TABLE IX.

Relative Centreline Temperatures
for Gas Jet (Pr=1)
at Z (=z/DR_e) = 0.50

<table>
<thead>
<tr>
<th>Gr/Re</th>
<th>Morton</th>
<th>Numerical</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10^{-1})</td>
<td>1.0000</td>
<td>1.0400</td>
</tr>
<tr>
<td>(10^{0})</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>(10^{1})</td>
<td>1.0000</td>
<td>0.9914</td>
</tr>
<tr>
<td>(10^{2})</td>
<td>1.0000</td>
<td>1.0231</td>
</tr>
<tr>
<td>(10^{3})</td>
<td>1.0000</td>
<td>1.0283</td>
</tr>
</tbody>
</table>

TABLE X

Relative Centreline Temperatures
for Liquid Jet (Pr=10)
at Z (=z/DR_e) = 0.50

<table>
<thead>
<tr>
<th>Gr/Re</th>
<th>Morton</th>
<th>Numerical</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10^{-1})</td>
<td>1.0000</td>
<td>0.9884</td>
</tr>
<tr>
<td>(10^{0})</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>(10^{1})</td>
<td>1.0000</td>
<td>1.0162</td>
</tr>
<tr>
<td>(10^{2})</td>
<td>1.0000</td>
<td>1.0353</td>
</tr>
<tr>
<td>(10^{3})</td>
<td>1.0000</td>
<td>1.0022</td>
</tr>
</tbody>
</table>
FIGURE 10 - PLOT OF TEMPERATURE HALF-WIDTH FOR THE GAS JET (PR=1) AS A FUNCTION OF AXIAL POSITION

Gr/Re = 0

Gr/Re = 10^1

Gr/Re = 10^2

Gr/Re = 10^3

Z - AXIAL POSITION
FIGURE 11 - PLOT OF TEMPERATURE HALF-WIDTH FOR THE LIQUID JET (Gr=10) AS A FUNCTION OF AXIAL POSITION
V - CONCLUSIONS AND RECOMMENDATIONS

The effect of thermal buoyancy on laminar vertical axisymmetric jets has been analysed. The results show that the effect is a function of Prandtl, Reynolds and Grashof numbers. The ratio, Gr/Re, is of particular importance because it is a measure of the relative strength of buoyancy and inertial forces. As Gr/Re increases from zero the predominant effect is to increase the axial velocity.

Agreement between the numerical and analytical solutions to the buoyant jet problem is not good. Mollendorf and Gebhardt's perturbation solution approaches and then diverges from the numerical solution. Brand and Lahey's solution may at best agree for large Z. Morton's entrainment model gives the best agreement. But, even this agreement is only good in a relative sense. In general, due to the coupling of momentum and energy equations when buoyancy is present, the only suitable method for studying the resultant flow is a numerical technique such as the one used in this study.

Recommendations are in two directions, experimental and theoretical. Since the author has found little experimental work in this field, it is recommended that the experimental side of this work be done. On the theoretical side we recommend that the extension of this project include turbulent
jets, the effect of negative buoyancy, (i.e. cold jet issuing into a hot medium) and the effect of buoyancy caused by density differences. In the latter the momentum equation would no longer be coupled to the energy equations. Instead it would be coupled to a conservation of species equation. The study of thermal buoyancy in turbulent jets is important because most buoyant flows in nature occur in the turbulent regime.
REFERENCES

23. Trent, D.S. and Welty, J.R.; "Numerical Computation of

A,Am - Coefficients defined in Appendix B
B,Bm - Coefficients defined in Appendix B
C,Cm - Coefficients defined in Appendix B
D,Dm - Coefficients defined in Appendix B
D - Nozzle diameter
E,Em - Coefficients defined in Appendix B
E - Morton's entrainment parameter
F,Fm - Coefficients defined in Appendix B
Gr - Grashof number
H,Hm - Coefficients defined in Appendix B
Pr - Prandtl number
Q1,Q2,Q3 - Coefficients defined in Appendix B
Re - Reynolds number
R - Dimensionless radial coordinate, r/D
T - Dimensionless temperature, (t-ta)/(to-ta)
Vr - Dimensionless radial velocity, (Vr/Vg)Re
Vz - Dimensionless axial velocity, (Vz/Vo)
z - Dimensionless axial coordinate, z/DRa,g
f,h - Represent similarity function defined by Mollendorf and Gebhardt (7). At the centreline f=0.10 and h=0.001 for Pr=10.
p - Absolute pressure
r - Radial coordinate
t - Temperature
k - Thermal conductivity
Cp - Specific heat
v - Velocity
z - Axial Coordinate

Greek letters
ΔR - Step size in R-direction
Δz - Step size in Z direction
β - Expansion coefficient
δ - Boundary layer thickness
λ - Iteration counter
μ - Viscosity
ρ - Density
α - Thermal diffusivity, \( k/\rho C_p \)

Subscripts
a - Denotes ambient conditions
k - Denotes conditions along the outer boundary
M - Denotes free jet central line properties
m - Denotes grid points in a direction normal to the direction of flow
n - Denotes grid points in the direction of flow
o - Denotes nozzle exit properties
R; r - Denotes radial component
rel - Denotes relative velocity or temperature
t - Refers to energy
v - Refers to momentum
Z; z - Denotes axial component in free jet
APPENDIX A

THE DERIVATION OF THE BOUNDARY LAYER EQUATIONS
FOR AN INCOMPRESSIBLE THERMALLY BUOYANT VERTICAL
JET.

Assuming the flow is steady and incompressible, and
radial pressure gradients and viscous dissipation are negligible,
the general equations for the conservation of mass, momentum
and energy are:

Continuity

\[ \frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{\partial v_z}{\partial z} = 0 \]  \hspace{1cm} (A-1)

Momentum

- \( \dot{z} \) direction

\[ \rho v_r \frac{\partial v_z}{\partial r} + \rho v_z \frac{\partial v_z}{\partial z} = \frac{\partial p}{\partial z} + \frac{\partial}{\partial r} \left[ r \left( \frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right) \right] \]
\[ + 2\mu \frac{\partial}{\partial z} \left( \frac{\partial v_z}{\partial z} \right) + \rho g_z \]  \hspace{1cm} (A-2)

- \( r \) direction

\[ \rho v_r \frac{\partial v_r}{\partial r} + \rho v_z \frac{\partial v_r}{\partial z} = \frac{2\mu}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_r}{\partial r} \right) \]
\[ - 2\mu \frac{v_r}{r^2} + \mu \frac{d}{dz} \left( \frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right) \]  \hspace{1cm} (A-3)

Energy

\[ \rho C_p \left( v_r \frac{\partial T}{\partial r} + v_z \frac{\partial T}{\partial z} \right) = k \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} \right] \]  \hspace{1cm} (A-4)

Note also since the jet is vertical the radial gravitational
term, \( \rho g_r \), is zero.
If the fluid is at rest (moving slow relative the speed of sound in the media) and has some mean temperature, the vertical pressure gradient, \( \frac{\partial p}{\partial z} \) is given by

\[
\frac{\partial p}{\partial z} = \bar{\rho}g z
\]

Furthermore, assuming that the Boussinesq approximation \((\Delta \rho/\rho' \ll 1)\) applies, the relationship between the density at any point, \( \rho' \), and the mean density, \( \bar{\rho} \), is

\[
\rho = \bar{\rho} - \bar{\rho}(\overline{T - T})
\]

Substituting equations (A-5) and (A-6) into equation (A-2) yields

\[
\rho v_r \frac{\partial v_z}{\partial r} + \rho v_z \frac{\partial v_z}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} \left[ \nu \mu \left( \frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right) \right]
\]

\[
+ \frac{3}{\partial z} \left( 2 \mu \frac{\partial v_z}{\partial z} \right) - \bar{\rho}(\overline{T - T})
\]

Making the boundary layer assumptions that

- the thickness of the boundary layer is small compared to the length, and
- the velocity along the boundary layer is much larger than the velocity across the boundary layer

the following order of magnitude approximations can be postulated.

\[
r = O(\delta)
\]

\[
z = O(1)
\]

\[
v_r = O(\delta)
\]

\[
v_z = O(1)
\]
These equations are more easily solved if they are rearranged in dimensionless form. By introducing the following dimensionless variables:

\[
\begin{align*}
Z &= (Z/D)/Re \\
R &= R/D \\
V_Z &= V_Z/V_0 \\
V_R &= (V_R/V_0)Re = \frac{\rho V_r D}{\mu} \\
T &= \frac{t - t_a}{t_0 - t_a} \\
Re &= \frac{\rho V_0 D}{\mu} \\
Gr &= \frac{\rho^2 g D^3 (t_0 - t_a)}{\mu^2} \\
Pr &= \frac{C_0 \mu}{k}
\end{align*}
\]

the conservation equations become

Continuity equation -
\[
\frac{\partial V_Z}{\partial Z} + \frac{1}{R} \frac{\partial V_R}{\partial R} = 0
\]
(Momentum equation -
\[
V_Z \frac{\partial V_Z}{\partial Z} + V_R \frac{\partial V_Z}{\partial R} = \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial V_Z}{\partial R} \right) + \frac{Gr}{Re} T
\]
(energy equation -
\[
V_Z \frac{\partial T}{\partial Z} + V_R \frac{\partial T}{\partial R} = \frac{1}{Pr} \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial T}{\partial R} \right)
\]
where \( \delta \) is the boundary layer thickness.

Introducing these order of magnitude approximations in the general conservation equations makes the entire r-direction momentum equation and certain terms of the z-direction momentum and energy equations insignificant and therefore negligible if

\[
\begin{align*}
\mu &= (\delta^2) \\
g\beta &= (1) \\
\frac{k}{c_p} &= (\delta^2)
\end{align*}
\]

Thus, the boundary layer equations are

- Continuity equation

\[ \frac{\partial v_z}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} \left( r v_r \right) = 0 \]  

- Momentum equation

\[ \frac{\partial}{\partial z} \left( 
\frac{\partial v_z}{\partial r} 
\right) + \rho v_z \frac{\partial v_z}{\partial z} = \frac{\mu}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{\rho g \beta}{r} (t - t_a) \]  

- Energy equation

\[ v_z \frac{\partial t}{\partial z} + v_r \frac{\partial t}{\partial r} = a \frac{\partial}{\partial r} \left( \frac{\partial t}{\partial r} \right) \]
APPENDIX B

THE DERIVATION OF THE FREE JET DIFFERENCE EQUATIONS

Tomich (10), in developing a free jet finite difference technique, derived the difference equations and used them in writing the computer program. Their derivation is outlined below.

The dimensionless equations for the conservation of mass, momentum and energy in a thermally buoyant, incompressible free jet are

\[ \frac{\partial}{\partial z} (V_z) + \frac{1}{R} \frac{\partial V_R}{\partial R} = 0 \quad (B-1) \]

\[ V_z \frac{\partial V_z}{\partial z} + V_R \frac{\partial V_z}{\partial R} = \frac{1}{R} \frac{\partial}{\partial R} \left( \frac{V_z}{R} + \frac{Gr}{Re} T \right) \quad (B-2) \]

\[ V_z \frac{\partial T}{\partial z} + V_R \frac{\partial T}{\partial R} = \frac{1}{Pr} \frac{\partial}{\partial R} \left( \frac{T}{R} \right) \quad (B-3) \]

By expanding the above equations B-1, B-2, and B-3, we obtain, respectively

\[ \frac{\partial}{\partial z} V_z + \frac{1}{R} V_R + \frac{\partial V_R}{\partial R} = 0 \quad (V-4) \]

\[ \frac{V_z}{R} \frac{\partial V_z}{\partial z} + V_R \frac{\partial V_z}{\partial R} = \frac{1}{R} \frac{\partial V_z}{\partial R} + \frac{\partial^2 V_z}{\partial R^2} + \frac{Gr}{Re} T \quad (B-5) \]

\[ \frac{V_z}{R} \frac{\partial T}{\partial z} + V_R \frac{\partial T}{\partial R} = \frac{1}{Pr} \frac{\partial T}{\partial R} + \frac{1}{Pr} \frac{\partial^2 T}{\partial R^2} \quad (B-6) \]
Now, to obtain the difference equations for the finite difference network shown in Figure B-1, the underlined variables in Equations (B-4), (B-5) and (B-6) are replaced by their respective values at a point \((z+\Delta z, r)\). Then, the finite difference approximations for derivatives at a point \((z+\Delta z, r)\) are substituted into these equations.

The following finite difference approximations were used for temperature:

\[
\frac{\partial T}{\partial z} = \frac{T(z+\Delta r, r) - T(z, r)}{\Delta z} \quad \text{(B-7)}
\]

\[
\frac{\partial T}{\partial r} = \frac{T(z+\Delta z, r+\Delta r) - T(z+\Delta z, r-\Delta r)}{2\Delta r} \quad \text{(B-8)}
\]

\[
\frac{\partial^2 T}{\partial r^2} = \frac{T(z+\Delta z, r+\Delta r) - 2T(z+\Delta z, r) + T(z+\Delta z, r-\Delta r)}{(\Delta r)^2} \quad \text{(B-9)}
\]

for velocity:

\[
\frac{\partial V_z}{\partial z} = \frac{V_z(z+\Delta z, r) - V_z(z, r)}{\Delta z} \quad \text{(B-10)}
\]

\[
\frac{\partial V_z}{\partial r} = \frac{V_z(z+\Delta z, r+\Delta r) - V_z(z+\Delta z, r-\Delta r)}{2\Delta r} \quad \text{(B-11)}
\]

\[
\frac{\partial^2 V_z}{\partial r^2} = \frac{V_z(z+\Delta z, r+\Delta r) - 2V_z(z+\Delta z, r) + V_z(z+\Delta z, r-\Delta r)}{(\Delta r)^2} \quad \text{(B-12)}
\]

\[
\frac{\partial V_r}{\partial r} = \frac{V_r(z+\Delta z, r) - V_r(z+\Delta z, r-\Delta r)}{\Delta r} \quad \text{(B-13)}
\]
Figure B-1
The Free Jet Finite Difference Network
After making these substitutions and rearranging, the following difference equations were obtained, for energy:

\[ A \cdot T(Z+\Delta Z, R) + B \cdot T(Z+\Delta Z, R+\Delta R) + C \cdot T(Z+\Delta Z, R-\Delta R) = D \]  \hspace{1cm} (B-14)

where:

\[ A = \frac{V_Z(Z+\Delta Z, R)}{Z} + \frac{2}{Pr (\Delta R)^2} \]  \hspace{1cm} (B-15)

\[ B = \frac{V_R(Z+\Delta Z, R)}{2\Delta R} - \frac{1}{2 Pr R\Delta R} - \frac{1}{Pr (\Delta R)^2} \]  \hspace{1cm} (B-16)

\[ C = \frac{V_R(Z-\Delta Z, R)}{2\Delta R} + \frac{1}{2 Pr R\Delta R} - \frac{1}{Pr (\Delta R)^2} \]  \hspace{1cm} (B-17)

\[ D = \frac{V_Z(Z+\Delta Z, R)}{\Delta R} T(Z, R) \]  \hspace{1cm} (B-18)

for momentum:

\[ E \cdot V_Z(Z+\Delta Z, R) + F \cdot V_Z(Z+\Delta Z, R+\Delta R) + G \cdot V_z(Z+\Delta Z, R-\Delta R) = H \]  \hspace{1cm} (B-19)

where:

\[ E = \frac{V_Z(Z+\Delta Z, R)}{\Delta Z} + \frac{2}{(\Delta R)^2} \]  \hspace{1cm} (B-20)

\[ F = \frac{V_R(Z+\Delta Z, R)}{2\Delta R} - \frac{1}{2 \Delta RR} - \frac{1}{(\Delta R)^2} \]  \hspace{1cm} (B-21)
\[ G = \frac{V_R(Z + \Delta Z, R)}{2 \Delta R} + \frac{1}{2 \Delta RR} - \frac{1}{(\Delta R)^2} \]  

\[ H = \frac{V_z(Z + \Delta Z, R)}{\Delta Z} V_z(Z, R) + \frac{Gr}{Re} T(Z + \Delta Z, R) \]  

Then, the continuity equation was written in an explicit form

\[ V_R(Z + \Delta Z, R) = Q_1(Q_2 + Q_3) \]  

where:

\[ Q_1 = \frac{1}{(\frac{1}{R} + \frac{1}{\Delta R})} \]  

\[ Q_2 = \frac{V_z(Z, R) - V_z(Z + \Delta Z, R)}{\Delta Z} \]  

\[ Q_3 = \frac{V_R(Z + \Delta Z, R - \Delta R)}{\Delta R} \]

Now, in order to write the equations for the points 1, 2, 3, ..., k of the finite difference network of Figure B-1 we start from the axis of symmetry along which the points are labelled with the subscript 1. First we apply the boundary conditions, Equations (9), to the differential equations B-1, B-2, and B-3, then we make the finite difference substitutions. The following difference equations are obtained.
for energy:
\[ A_1 \cdot T(Z+\Delta Z,1) + B_1 \cdot T(Z+\Delta Z,2) = D_1 \]  
where:
\[ A_1 = \frac{V_2(Z+\Delta Z,1)}{\Delta Z} + Pr \frac{4}{(\Delta R)^2} \]  
\[ B_1 = -Pr \frac{4}{(\Delta R)^2} \]  
\[ D_1 = \frac{V_2(Z-\Delta Z,1)}{\Delta Z} T(Z,1) \]

for momentum:
\[ E_1 \cdot V_2(Z+\Delta Z,1) + F_1 \cdot V_2(Z+\Delta Z,2) = H_1 \]
where:
\[ E_1 = \frac{V(Z+\Delta Z,1)}{\Delta Z} + \frac{4}{(\Delta R)^2} \]  
\[ F_1 = -\frac{4}{(\Delta R)^2} \]  
\[ H_1 = \frac{V_2(Z+\Delta Z,1)}{\Delta Z} V(Z,1) + \frac{Gr}{Re} T(Z+Z,1) \]

The difference equations for the rest of the points are directly obtained from the general form, Equations B-14, B-19, and B-24.
APPENDIX C

Algorithm and Computer Programme

The object of the numerical calculation was to determine both temperature and velocity profiles in a thermally buoyant jet. This object was achieved by using the following algorithm and computer programme.
Algorithm

Start

Yes

Continuation

Run?

No

Retrive profile

info from
temporary storage

Read initialization info from
card input

Initialize

temperature and
velocity profiles

n = 1

Prepare step sizes
for next iteration and set
boundary conditions

n = n + 1

Solve energy equation
for T_{n+1}
\[ |T_{n+1}^n - T_n^n| \text{ Less than set tolerance} \]

- Yes
  - Solve momentum equation for \( v_{z,n+1} \)

- No
  - \( |v_{z,n+1} - v_{z,n+1}^\lambda| \text{ Less than set tolerance} \)
    - Yes
      - Set error flag, EP1
    - No
      - Set error flag, EP2

- Is \( \lambda \) greater than limit?
  - Yes
    - Is either error flag set?
      - Yes
        - End
      - No
        - Is printed output required?
          - No
          - Yes
            - No
Calculate temperature and velocity half-widths

Print profiles

Output profile info to temporary storage

Is n greater than n_{max}?

Yes

End

No
COMPUTER PROGRAMME
PROGRAM TST(INPUT, OUTPUT, TAPE5=INPUT, TAPE6=OUTPUT, TAPE3, TAPE2)

THERMALLY BUOYANT AXISYMMETRIC LAMINAR JET

INPUT DATA

TO-JET EXIT TEMPERATURE, DEG. R
TA-AMBIENT TEMPERATURE, DEG. R
U0-JET EXIT VELOCITY, FT./SEC.
DELZ1-FIRST X-DIRECTION FINITE DIFFERENCE STEP SIZE
DELZ2-SECOND X-DIRECTION FINITE DIFFERENCE STEP SIZE
THE FREE JET PROGRAM
DELZ3-THIRD X-DIRECTION FINITE DIFFERENCE STEP SIZE
DELR1-FIRST R-DIRECTION FINITE DIFFERENCE STEP SIZE
DELR2-SECOND R-DIRECTION FINITE DIFFERENCE STEP SIZE
DELR3-THIRD R-DIRECTION FINITE DIFFERENCE STEP SIZE
PRNO-TURBULENT PRANDTL NUMBER
ITL-TOLERENCE FOR ITERATIVE CONVERGENCE
II-NUMBER OF LINES OF THE FIRST X-DIRECTION STEP SIZE PLUS ONE.
IS-NUMBER OF LINES OF THE SECOND X-DIRECTION STEP SIZE PLUS ONE.
IT-NUMBER OF LINES OF THE THIRD X-DIRECTION STEP SIZE PLUS ONE.
JI-NUMBER OF LINES OF FIRST R-DIRECTION STEP SIZE
JS-NUMBER OF LINES OF SECOND R-DIRECTION STEP SIZE
JT-NUMBER OF LINES OF THIRD R-DIRECTION STEP SIZE
KNOZ-X-DIRECTION LINE NUMBER OF NOZZLE
LIMIT-LIMITING NUMBER OF ITERATIONS
OUTPUT INCLUDES VELOCITIES, TEMPERATURES AND DENSITIES
IN DIMENSIONLESS FORM:
DIMENSION OLDVZ(501), OLDVR(501), VZ(501), VR(501), T(501), 1A(501), R(501), C(501), TEST(501), VZN(501), VREN(501)
2.0(501)
DIMENSION VZMISOI, TNQISOI
DIMENSION VRMISOI
DIMENSION OLDT(501)
EXTERNAL TIMT
CALL RECOVR(TIMT, 148, 0)
SS=SECOND(CP)
NPROF=0
ITAPE=0
ITAPE=1
ITAPE = 0 - NEW RUN
ITAPE = 1 - CONTINUATION RUN, DATA STORED ON TAPE2

IF (ITAPE.EQ.0) GO TO 2231
1021 READ (2) TO, TA, U0, DELZ1, DELZ2, DELZ3, DELR1, DELR2, DELR3, PRNO, ITL,
1022 READ (2) II, IS, IT, JI, JS, JT, KNOZ, LIMIT
READ (2) CONST, CPC, GRAS
READ (2) R, DELR,
READ (2) VZ
PROGRAM TST

READ (2) VR
READ (2) T
READ (2) M,NPROF
WRITE (3) TO,TA,U0,DELZ1,DELZ2,DELZ3,DELR1,DELR2,DELR3,PRNO,T0
WRITE (3) T0,IS,IT,JI,JS,JT,KAOZ,LIMIT
WRITE (3) CCNST,CPC,GRAS
WRITE (3) Z*DELR
WRITE (3) VR
WRITE (3) T
WRITE (3) M,NPROF

M=M+1.

CONTINUE

IFIITAPE.EQ.11.GO TO 2241
READ (5,100) TO,TA,U0,DELZ1,DELZ2,DELZ3,DELR1,DELR2,DELR3,PRNO,T0
100 FORMAT (8F10.4)
READ (5,200) II,IS,IT,JI,JS,JT,KAOZ,LIMIT
200 FORMAT (8IS)
READ (5,300) CCNST,CPC,GRAS
300 FORMAT (3F10.5)

CONTINUE

IFIITAPE.EQ.0) GO TO 872
J=JI
DO 871 N=1,J
871 VZ(N)=OLDVZ(N)
VR(N)=OLDVR(N)
COUNT=COUNT+1.

CONTINUE

IFIITAPE.EQ.0) GO TO 872
J=JI
DO 871 N=1,J
871 VZ(N)=OLDVZ(N)
VR(N)=OLDVR(N)
COUNT=COUNT+1.
PROGRAM TST  74/74 OPT=2 ROUND=-*-#/* FTN 4.2+ REL

115  871 OLDT(N)=T(N)
     872 CONTINUE
     J=J1
     KAPPA=DELR2/DELR1
     NU=DELR3/DELR2
     PKAP=DELTZ2/DELTZ1
     OMEGA=DELTZ3/DELTZ1
     POS=1.0
     SET=51
     C
     C. PREPARE STEP SIZES FOR NEXT ITERATION
     C. AND SET BOUNDARY CONDITIONS
     C
     2001 IF (II-M) 1003,2003,3003
     1003 IF (IS-M) 4003,5003,6003
     4003 IF (IT-M) 2000,2000,7003
     3003 ZP1=N-1
     Z=ZP1*DELTZ1
     DELZ=DELTZ1
     DELR=DELR1
     J=J1
     POS=POS+1.0
     GO TO 1001
     2003 DO 13 N=1,JS,KAPPA
     NEW=(N-1)/KAPPA+1
     OLVDZ(NEW)=OLVDZ(N)
     OLDR(NEW)=OLDR(N)
     13 OLDT(NEW)=OLDT(N)
     DO 14 N=NEW+JS
     OLVDZ(N)=-0.0
     NM=N-1
     RPM=N-1
     OLDR(N)=OLDR(NM)*RPM/RP
     14 OLDT(N)=0.0
     6003 ZP1=(II-2)
     ZP2=N-II+1
     Z=ZP1*DELTZ1+ZP2*DELTZ2
     DELZ=DELTZ2
     DELR=DELR2
     J=JS
     POS=POS+PKAP
     GO TO 1001
     5003 DO 15 N=1,JS,NU
     NEW=(N-1)/NU+1
     OLVDZ(NEW)=OLVDZ(N)
     OLDR(NEW)=OLDR(N)
     15 OLDT(NEW)=OLDT(N)
     DO 16 N=NEW+JT
     OLVDZ(N)=0.0
     NM=N-1
     RPM=N-1
     OLDR(N)=OLDR(NM)*RPM/RP
     16 OLDT(N)=0.0
     7003 ZP2=IS-II
PROGRAM 74/74 OPT=2  ROUND=*/* FTN 4.2+ REL

175  ZP3=W-IS+1  
    ZP1=W-I+2  
    Z=ZP1*DELZ1+ZP2*DELZ2+ZP3*DELZ3  
    DELZ=DELZ3  
    DELR=DELR3  
    J=JT  
    POS=POS*OMEGA

1001 INCR=0.0  
    J1=J-1.  
    J2=J-2  
    REYK=CONST*CPC  
    PRK=CONST*CPC/PRNO  
    C1=REYK/(DELR*DELZ)  
    CC1=PRK/(DELR*DELZ)

3001 WRITE (6,50)  
      50 FORMAT 1*,T,N*,RR*,VZM*,VZNO*  
      1 VRM  
      VZ(J)=0.0  
      T(J)=0.0  

C   SOLVE ENERGY EQUATIONS FOR TEMPERATURE
C
8001 A(1)=VZ(1)/DELZ+4.0*CC1  
      B(1)=-4.0*CC1  
      IF(TO-TA) 75,76,75  
    76 D(1)=0.0  
      GO TO 86  
    75 CONTINUE  
      D(1)=VZ(1)/DELZ*OLDT(1)  
    86 CONTINUE  
      DO 18 N=2,J1  
      RP=N-1  
      R=RP*DELZ  
      CC2=PRK/(2.0*R*DELZ)  
      CC3=VR(N)//(2.0*DELZ)  
      CC4=VZ(N)/DELZ  
      NP=N+1  
      NM=N-1  
      C(N)=CC2-CC3-CC1  
      A(N)=CC4+2.0*CC1  
      B(N)=CC3-CC2-CC1  
      IF(TO-TA) 77,78,77  
    78 THETA=0.0  
      GO TO 88  
    77 THETA=OLDT(N)  
    88 CONTINUE  
      18 D(N)=CC4*THETA  
      DO 22 N=1+J1  
    22 TEST(N)=T(N)  
      D(1)=D(1)/A(1)  
      DO 19 N=2+J1  
      N1=N-1.  
      B(N1)=B(N1)/A(N1)  
      A(N)=A(N1)-C(N)*B(N1)  
      D(N)=(D(N)-C(N)*D(N1))/A(N)  
      T(J1)=D(J1)  
      DO 20 N=1+J2  

PROGRAM TST

20 T(NN)=D(NN)-B(NN)*T(NN)

C APPLY WEGSTEIN CONVERGENCE PROMOTION

DO 66 N=1+J
T(N)=PAR*TEST(N)+(1.0-PAR)*T(N)
66 CONTINUE
DO 23 N=1+J

C TEST TEMPERATURE PROFILE FOR CONVERGENCE

DIFF=ABS(TEST(N)-T(N))
IF(DIFF=TOL) 23,23,9001
23 CONTINUE
ERR1=0.0
GO TO 24

9001 ERR1=1.0

C SOLVE MOMENTUM EQUATION FOR AXIAL VELOCITY

24 A(1)=VZ(1)/DELZ+4.0*C1
B(1)=-4.0*C1
D(1)=VZ(1)+OLDVZ(1)/DELZ
D(1)=D(1)+GRAS*(REYK**2)*T(1)
DO 3 N=2+J
RP=N-1
R=RP*DELR
C2=REYK/(2.0*R*DELR)
C3=VZ(N)/(2.0*CELR)
C4=VZ(N)/DELF
C(N)=C2-C3+C1
A(N)=C4+2.0*C1
B(N)=C3-C2-C1
D(N)=C4*OLDVZ(N)
D(N)=D(N)+GRAS*(REYK**2)*T(N)
3 CONTINUE
DO 17 N=1+J
17 TEST(N)=VZ(N).
D(1)=D(1)/A(1)
DO 4 N=2+J
N1=N-1
B(N1)=B(N1)/A(N1)
A(N1)=A(N1)-C(N1)*B(N1)
4 D(N)=D(N)+C(N1)*D(N1))/A(N1)
VZ(J1)=D(J1)
DO 5 N=1+J2
NN=J2*N+1
NN1=NN+1
5 VZ(NN)=D(NN)-B(NN)*VZ(NN)
DO 666 N=1+J

C APPLY WEGSTEIN CONVERGENCE PROMOTION TO AXIAL VELOCITY

666 VZ(N)=PAR*TEST(N)+(1.0-PAR)*VZ(N)
SOLVE CONTINUITY EQUATION EXPLICITLY FOR RADIAL VELOCITY

DO 6 N=2,J
N1=N-1
RP=N1
R=RP*DELR
CE1=(1./R+1./DELR)
CE2=(OLDVZ(N)-VZ(N))/DELZ
CE3=VZ(N1)/DELR
6 VR(N)=CE1*(CE2+CE3)
DO 10 N=1,J
ERROR=ABS(TEST(N)-VZ(N))

TEST AXIAL VELOCITY FOR CONVERGENCE

IF (ERROR-TOL) 10,10,5001
10 CONTINUE

PROCEED TO NEXT AXIAL POSITION IF BOTH TEMPERATURE AND AXIAL VELOCITY HAVE CONVERGED

IF (ERR1) 26,26,5001
26 CONTINUE
IF (POS-SET) 4001,3002,3002

IF POS EQUALS SET OUTPUT PROFILES

3002 POS=1.0
TX=1./T(1)
VZA=1./VZ(1)
DO 5553 N=1,J
XT=T(N)*TX
VX=VZ(N)*VZA
IF (XT*GT.0.35) JBT=N
IF (VX*GT.0.35) JBV=N
IF (XT*GT.0.75) JBBT=N
IF (VX*GT.0.75) JBBV=N
5553 CONTINUE

CALCULATE VELOCITY HALF WIDTH

NCTRF=1
JBF=JBV
IF (JBF.LT.1) JBF=1
JBBF=JBBV
IF (JBF.EQ.JBBF) JBBF=JBF+1
FB=VZA*VZ(JBF)-0.5
RP=JBF-1
RF=RP*DELR
5501 FBB=VZA*VZ(JBBF)-0.5
RP=JBBF-1
RFF=RP*DELR
IF ((FB-FBB).LE.0.0) STOP 261
BVZ=(RFF*FR-RF*FBB)/(FR-FBB)
IF (NCTRF.EQ.1) GO TO 5502
DIF=BVZ-VZ
IF (ABS(DIF).LE.1.E-02) GO TO 5503
PROGRAM TST
74/74 OPT=2 ROUND***

5502 BVZVZ=BVZ
JST=JBKF
JBBF=BVZ/DELR+1.
IF (JBBF.GT.501) JBBF=(501+JST)/2
IF (JBBF.LT.1) JBBF=(1+JST)/2
IF (JBF.EQ.JBBF) JBBF=JBF+1
NCTRF=NCTRF+1
IF (NCTRF.GT.50) STOP 271
GO TO 5501

5503 CONTINUE

555 CALCULATE TEMPERATURE HALF WIDTH

555 NCTRF=1
JBF=JBT
IF (JBF.LT.1) JBF=1
JBBF=JBBT
IF (JBF.EQ.JBBF) JBBF=JBF+1
FB=TX*T(JBF)-0.5
RP=JBF-1
RF=RP*DELR

555 FBK=TX*T(JBBF)-0.5
RP=JBBF-1
RFF=RP*DELR
IF ((FB-FBB).EQ.0.0) STOP 263
BT=(RFF*FB-RF*FBB)/(FB-FBB)
IF (NCTRF.EQ.1) GO TO 5512
DIF=BT-BT
IF (ABS(DIF).LE.1.E-02) GO TO 5513

5512 BT=BT
JST=JBKF
JBBF=BT/DELR+1.
IF (JBBF.GT.501) JBBF=(501+JST)/2
IF (JBBF.LT.1) JBBF=(1+JST)/2
IF (JBF.EQ.JBBF) JBBF=JBF+1
NCTRF=NCTRF+1
IF (NCTRF.GT.50) STOP 273
GO TO 5511

5513 CONTINUE

555 OUTPUT PROFILES

555 NPRT=1
PRINT 1660
1660 FORMAT(1H1,/) PRINT 1661
1661 FORMAT(39X,SSH*** THERMAL BOUYANCY IN LAMINAR VERTICAL JETS)
1***,/) NPROF=NPROF+1
PRINT 1665, NPROF
1665 FORMAT(105X,**PROFILE NUMBER = *,15,/) RE=1./REYK
PRINT 1662,RE,BVZ,GRAS,BT,PRNO
1662 FORMAT(4X,**RE = *,G10.3,60X,**VELOCITY HALF WIDTH = *,
1G15.7,/,11X,**OR = *,G15.7,56X,**TEMPERATURE HALF WIDTH = *,G15.7,
2,11X,**PR = *,F10.2,/) PRINT 1663
DO 7 N=1,J,5
NPRT=NPRT+1
IF(NPRT.LT.461) GO TO 1557
NPRT=1
1557 IF(NPRT.NE.1) GO TO 1558
PRINT 1660
PRINT 1663
1558 CONTINUE
RP=N-1
R=RP*DELR
ZDRE=Z*REYK
RR=R
VZM(N)=VZ(N)
VZNO(N)=VZ(N)/VZM(1)
TNO(N)=T(N)/T(1)
VRM(N)=VR(N)/REYK
WRITE(6,150) T(N),TNO(N),VZM(N),VZNO(N),VRM(N),ZDRE,RR,N,N
150 FORMAT(5F18.8,4X,2F15.8,17,I9)
7 CONTINUE

STORE MOST RECENT PROFILES ON TAPE3

PRINT 1660
4001 REWIND 3
WRITE (3) T0,TA,U0,DELZ1,DELZ2,DELZ3,DELR1,DELR2,DELR3,PRNO,T
WRITE (3) IT,IS,IT,J,JS,JT,KN02,LIMIT
WRITE (3) CONST,CPC,GRAS
WRITE (3) Z,DELR
WRITE (3) VZ
WRITE (3) VR
WRITE (3) T
WRITE (3) M,NPROF
ENDFILE 3
DO 9 N=1,J
OLDVZ(N)=VZ(N)
OLDVR(N)=VR(N)
9 OLDT(N)=T(N)
WRITE (6,999) INCR
999 FORMAT (15)
M=M+1

EXIT IF DOWNSTREAM LIMIT IS MET
PROGRAM TST

IF(IT-M) 3996,3996,3996
3996 CONTINUE
GO TO 2001
5001 INCR=INCR+1
SF=SECOND(CF)
SEC=SF-SS
SS=SF
TMLF=XTIME(Z)-SEC
IF(TMLF.LT_SEC) STOP541
IF (INCR.LT LIMIT) 1009,1009,7001
1009 GO TO 8001
7001 DO 12 N=1,J
RP=N-1
R=RP*DELR
WRITE(6,250) V2(N),VR(N),T(N),M,N,Z,N
250 FORMAT(3F18.8,2I5,2F15.8)
WRITE(6,809) TEST(N)
809 FORMAT(F20.8)
12 CONTINUE
WRITE (6,9999) INCR
9999 FORMAT(15)
GO TO 2000
2000 CONTINUE
3000 CONTINUE
END

RY OF CHANGES MADE BY THE OPTIMIZER
WORDS OF INVARIANT RLIST REMOVED FROM THE LOOP BEGINNING AT LINE 235
WORDS OF INVARIANT RLIST REMOVED FROM THE LOOP BEGINNING AT LINE 255
WORDS OF INVARIANT RLIST REMOVED FROM THE LOOP BEGINNING AT LINE 280
WORDS OF INVARIANT RLIST REMOVED FROM THE LOOP BEGINNING AT LINE 288

TER ALLOCATION
ISTERS ASSIGNED OVER THE LOOP BEGINNING AT LINE 235
ISTERS ASSIGNED OVER THE LOOP BEGINNING AT LINE 280.
SUBROUTINE TIMT(NT, IGAIN, NUL)
DIMENSION NT(17)
INTEGER R0
BO=(NT(1).AND.1778)
IF(BO.NE.8) GO TO 1
IGAIN=0
RETURN
1  IGAIN=1
STOP333
END
APPENDIX D

ADDITIONAL RESULTS

D.1a-e Axial Velocity Profiles for the Gas Jet as a Function of Radial Position

D.2a-d Radial Velocity Profiles for the Liquid Jet as a Function of Radial Position
FIGURE 0-4A - PLOT OF AXIAL VELOCITY PROFILES FOR THE GAS JET (PR=1) AS A FUNCTION OF RADIAL POSITION FOR GV=0
FIGURE D-1B - PLOT OF AXIAL VELOCITY PROFILES FOR THE GAS JET (PR=1) AS A FUNCTION OF RADIAL POSITION FOR GR/RE=1
FIGURE D-1E - PLOT OF AXIAL VELOCITY PROFILES FOR THE
GAS JET (PE=1) AS A FUNCTION OF
RATIO POSITION FOR GR/RE=1000
FIGURE D-2B - PLOT OF RADIAL VELOCITY FOR THE
GAS JET (FF=1) AS A FUNCTION
OF RADIAL POSITION FOR GR/RE=1
FIGURE D-2C - PLOT OF RADIAL VELOCITY FOR THE GAS JET (PP=1) AS A FUNCTION OF RADIAL POSITION FOR GR/AE=10
FIGURE D-20 - PLOT OF RADIAL VELOCITY FOR THE GAS JET (PR=1) AS A FUNCTION OF RADIAL POSITION FOR GR/RE=100.